Hiroshima Math. J.
38 (2008), 349–361

Mathematical treatment of a model for smoldering combustion

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(Received October 26, 2007)
(Revised January 9, 2008)

Abstract. Finger-like smoldering patterns are observed in experiments under micro-gravity. For theoretical understanding of such pattern phenomena, a model of reaction-diffusion system has been proposed. In this paper, we prove the existence and uniqueness of a solution for this reaction-diffusion system. We also consider a large-time behavior of solutions.

1. Introduction

It is reported that thin solid, for an example, paper, cellulose dialysis bags and polyethylene sheets, burning against oxidizing wind develops *finger-like patterns* or *fingering patterns* ([4]). Thin solid is stretched out straight onto the bottom plate and they also control the adjustable vertical gap, denoted by a parameter h, between top and bottom plates. The oxidizing gas is supplied in a uniform laminar flow, opposite to the directions of the front propagation and they control the flow velocity of oxygen, denoted by V_{O2}. When V_{O2} is decreased below a critical value, the smooth front develops a structure which marks the onset of instability. As V_{O2} is decreased further, the peaks are separated by cusp-like minima and a fingering pattern is formed, as shown in Figure 1. Similar phenomena were also observed in a micro-gravity experiment in space (see [2]).

In order to theoretically understand the experimental results above, we propose a phenomenological minimal model described by the following exothermic reaction-diffusion system for the (Kelvin) temperature u ($u = 0$ corresponds to the temperature outside of the experimental device), the density of paper v and the concentration of the inflammable mixed gas w.

2000 *Mathematics Subject Classification*. 35K45, 35K55, 35K57.
Key words and phrases. Reaction-diffusion equation, smoldering combustion, micro-gravity.

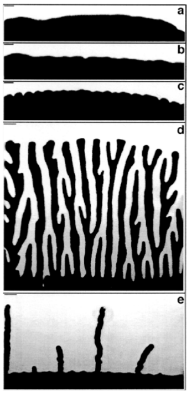


Fig. 1. Instability of combustion front when V_{O2} decreases from a to e. Oxygen flows from downward and the combustion front moves upward (Figure 2 in [4]).

$$
\text{(RD)}\begin{cases}
\dfrac{\partial u}{\partial t} = \mathrm{Le}\Delta u + \lambda' \dfrac{\partial u}{\partial x} + \gamma f(u)vw - g(u), & (x, y) \in I \times \Omega,\ t > 0, \\
\dfrac{\partial v}{\partial t} = -f(u)vw, & (x, y) \in I \times \Omega,\ t > 0, \\
\dfrac{\partial w}{\partial t} = \Delta w + \lambda \dfrac{\partial w}{\partial x} - f(u)vw, & (x, y) \in I \times \Omega,\ t > 0,
\end{cases}
$$

where the constants Le, called Lewis number, and γ are positive constants, λ and λ' are nonnegative constants, I is a bounded interval $(0, l_x)$ or a whole line $(-\infty, \infty)$, $\Omega \subset \mathbf{R}^n$ is a bounded domain, and $\Delta = \partial^2/\partial x^2 + \sum_{i=1}^n \partial^2/\partial y_i^2$ is Laplacian as usual. For the nonlinear terms $f(u)$ and $g(u)$, we are concerned with

$$
f(u) = \begin{cases} A \exp(-B/(u - \theta)), & u > \theta, \\ 0, & 0 \le u \le \theta, \end{cases}
$$

for some constants $A, B > 0$ and $\theta \ge 0$, and $g(u) = au^m$ for $a > 0$ and $m \ge 1$. The function f as above is called *Arrhenius kinetics* and g is *heat radiation* in combustion. Keeping the Arrhenius kinetics and the heat radiation in mind, we assume that the nonlinear terms f and g satisfy the following conditions:

(1) The function f is locally Lipschitz continuous and nonnegative for $u \ge 0$.

(2) The function g is locally Lipschitz continuous for $u \ge 0$, $g(0) = 0$, and positive for $u > 0$. Furthermore g is monotone increasing for $u \ge 0$.

(3) The functions f and g satisfy

$$
\lim_{u \to 0} \frac{f(u)}{u} = 0, \qquad \lim_{u \to 0} \frac{f(u)}{g(u)} = 0, \qquad \lim_{u \to \infty} \frac{f(u)}{g(u)} = 0, \tag{A}
$$

which indicate that $f(0) = 0$. In this paper, we consider the nonlinear terms generalized as above.

Next we consider the boundary and initial conditions to (RD). We suppose that if $I = (0, l_x)$, u and w satisfy

$$\frac{\partial u}{\partial x}(0, y, t) = \frac{\partial u}{\partial x}(l_x, y, t) = 0, \qquad \frac{\partial w}{\partial x}(0, y, t) = 0, \qquad w(l_x, y, t) = w_r > 0$$

for any $y \in \Omega$ and $t > 0$, and if $I = (-\infty, \infty)$,

$$\lim_{|x| \to \infty} u(x, y, t) = 0, \qquad \lim_{x \to \infty} w(x, y, t) = w_r, \qquad \lim_{x \to -\infty} w(x, y, t) = w_l \geq 0$$

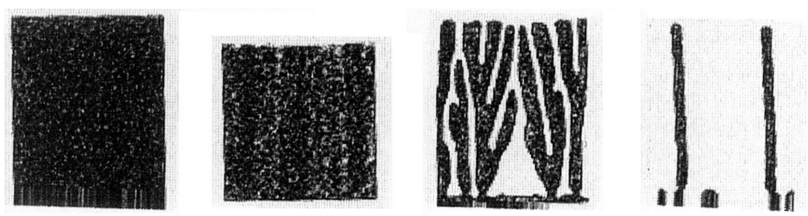
for any $y \in \Omega$ and $t > 0$, where w_r and w_l are some non-negative constants. In both cases we also suppose that u, w satisfy

$$\frac{\partial u}{\partial v}(x, y, t) = 0, \qquad \frac{\partial w}{\partial v}(x, y, t) = 0$$

for $x \in I$, $y \in \partial\Omega$ and $t > 0$, where v is the unit exterior normal vector on $\partial\Omega$. We suppose that the initial functions satisfy

$$u(x, y, 0) = u_0(x, y) \geq 0, \qquad v(x, y, 0) = v_0(x, y) \geq 0,$$
$$w(x, y, 0) = w_0(x, y) \geq 0.$$

We first show some 2-dimensional numerical simulations. We assume the domain to be rectangle, taking $\lambda' = 0$ and λ in (RD) as a controlled parameter. If λ is large, a smooth flame front is observed (see Figure 2 (a)). When λ is decreased, the instability of a smooth flame front occurs (see Figure 2 (b)). As λ is decreased further, a fingering pattern is formed (see



(a) $\lambda = 1.5$ (b) $\lambda = 0.7$ (c) $\lambda = 0.35$ (d) $\lambda = 0.2$

Fig. 2. Spatial patterns of burned paper v ($\mathrm{Le} = 0.3$, $\gamma = 20$, $\lambda' = 0$, $g(u) = 0.25u$, $f(u) = \exp(-1/0.415u)$).

Figures 2 (c), (d)). We can say that numerical simulations suggest that the system (RD) exhibits a qualitative agreement with the experimental results in Figure 1. We thus find that (RD) generates diverse patterns when some parameter is globally varied. This motives us to study qualitative behavior of solutions to (RD). As the first step, the present paper discusses the fundamental problem, namely, we will show the existence and uniqueness of global solution of (RD) and study the asymptotic behavior of the global solution.

This paper is organized as follows: In Section 2, using the upper bound of a solution of (RD) (Lemma 1), we show the global existence and uniqueness of a solution of (RD) (Theorems 1 and 2). In Section 3, we consider the asymptotic behavior of the global solution given in Section 2 (Theorem 3).

2. Existence and uniqueness of a global solution

In this section, we prove the existence and uniqueness of a global solution. We first show the existence and uniqueness of a local solution. For this purpose, we replace w by z such as $w = z + \omega$, where $\omega = \omega(x)$ is a smooth positive function and satisfies $\omega(l_x) = w_r$ and $\omega'(0) = 0$ if $I = (0, l_x)$, or $\omega \to w_r$ as $x \to \infty$ and $\omega \to w_l$ as $x \to -\infty$ if $I = (-\infty, \infty)$. Then we consider the following system derived from (RD) with respect to (u, v, z);

$$\begin{cases} \dfrac{\partial u}{\partial t} = \mathrm{Le}\Delta u + \lambda' \dfrac{\partial u}{\partial x} + \gamma f(u) v(z+\omega) - g(u), & (x,y) \in I \times \Omega,\ t > 0, \\ \dfrac{\partial v}{\partial t} = -f(u) v(z+\omega), & (x,y) \in I \times \Omega,\ t > 0, \\ \dfrac{\partial z}{\partial t} = \Delta z + \lambda \dfrac{\partial z}{\partial x} - f(u) v(z+\omega) + \omega'' + \lambda\omega', & (x,y) \in I \times \Omega,\ t > 0. \end{cases} \tag{2.1}$$

The initial functions u_0, v_0 and z_0 are

$$\begin{gathered} u(x,y,0) = u_0(x,y) \geq 0, \qquad v(x,y,0) = v_0(x,y) \geq 0, \\ z(x,y,0) = w_0(x,y) - \omega(x) \equiv z_0(x,y) \end{gathered} \tag{2.2}$$

for $x \in I$ and $y \in \Omega$. Here we assume $w_0(x,y) \geq 0$ as in Section 1. We suppose that u satisfies

$$\begin{gathered} \frac{\partial u}{\partial x}(0,y,t) = \frac{\partial u}{\partial x}(l_x,y,t) = 0, \qquad \text{if } I = (0, l_x), \\ \lim_{|x| \to \infty} u(x,y,t) = 0, \qquad \text{if } I = (-\infty, \infty) \end{gathered} \tag{2.3}$$

for $y \in \Omega$ and $t > 0$ and z does

$$\begin{aligned} &\frac{\partial z}{\partial x}(0, y, t) = z(l_x, y, t) = 0, \qquad \text{if } I = (0, l_x), \\ &\lim_{|x| \to \infty} z(x, y, t) = 0, \qquad \text{if } I = (-\infty, \infty) \end{aligned} \tag{2.4}$$

for $y \in \Omega$ and $t > 0$. In addition, we suppose that u and z satisfy

$$\frac{\partial u}{\partial \nu}(x, y, t) = 0, \qquad \frac{\partial z}{\partial \nu}(x, y, t) = 0 \tag{2.5}$$

for $x \in I$, $y \in \partial\Omega$ and $t > 0$.

We prove the existence and uniqueness of a local solution of the above system. In the proof, we shall use the standard theory of an analytic semigroup and prove the existence of the following integral equation;

$$\Phi(t) = T(t)\Phi_0 + \int_0^t T(t-s)F(\Phi(s))ds, \tag{2.6}$$

where $\Phi = (u, v, z)^t$, $\Phi_0 = (u_0, v_0, z_0)^t$, $T(t)$ is a semigroup generated by a differential operator A defined by

$$A = \begin{pmatrix} \mathrm{Le}\Delta + \lambda' \dfrac{\partial}{\partial x} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \Delta + \lambda \dfrac{\partial}{\partial x} \end{pmatrix}$$

and

$$F(\Phi) = \begin{pmatrix} \gamma f(u)v(\omega + z) - g(u) \\ -f(u)v(\omega + z) \\ -f(u)v(\omega + z) + \omega'' + \lambda\omega' \end{pmatrix}.$$

We consider the integral equation (2.6) in the functional space X defined by

$$X = L^p(I \times \Omega) \times L^\infty(I \times \Omega) \times L^p(I \times \Omega)$$

for $p > n + 1$. Here the domain of A, denoted by $D(A)$, is defined by

$$D(A) = W_N^{2,p}(I \times \Omega) \times L^\infty(I \times \Omega) \times W_{N,0}^{2,p}(I \times \Omega),$$

where $W_N^{2,p}(I \times \Omega)$ is defined by

$$W_N^{2,p}(I \times \Omega) = \left\{ u \in W^{2,p}(I \times \Omega) \,\middle|\, \frac{\partial u}{\partial \nu} = 0 \right.$$

$$\left. \text{for } x \in I, y \in \partial\Omega \text{ and } u \text{ satisfies (2.3)} \right\}$$

and $W^{2,p}_{N,0}(I \times \Omega)$ is defined by

$$W^{2,p}_{N,0}(I \times \Omega) = \left\{ z \in W^{2,p}(I \times \Omega) \,\middle|\, \frac{\partial z}{\partial \nu} = 0 \right.$$
$$\left. \text{for } x \in I, y \in \partial\Omega \text{ and } z \text{ satisfies (2.4)} \right\}.$$

The functional space $W^{2,p}(I \times \Omega)$ is a usual Sobolev space. Although the domain $I \times \Omega$ may not have the smooth boundary, even if Ω has a smooth boundary, it follows from the standard argument that $-L_u$ and $-L_z$ generate analytic semigroups $T_u(t)$ and $T_z(t)$, respectively (see [1]), where $L_u = -\mathrm{Le}\Delta - \lambda'\partial/\partial x$ and $L_z = -\Delta - \lambda\partial/\partial x$. In fact, these can be expressed by two analytic semigroups because $I \times \Omega$ is a cylindrical domain, which shall be proved in Appendix.

We assume that $u_0 \in D(L_u^\alpha)$, $v_0 \in C^\kappa(I \times \Omega)$ and $z_0 \in D(L_z^\alpha)$ for $1/2 < \alpha < 1$ and $0 < \kappa < 1$. The functional spaces $D(L_u^\alpha)$ and $D(L_z^\alpha)$ are called *fractional spaces* (see Section 2.6 of [3]), and $C^\kappa(I \times \Omega)$ is the Hölder space with a Hölder exponent κ. Then we have the following theorem for existence of a local solution:

Theorem 1. *Assume that $p > n + 1$, $1/2 < \alpha < 1$, $0 < \kappa < 1$, and $\partial\Omega \in C^2$. In addition, suppose that the Hölder continuous function ω has the second order continuous derivatives in $x \in I$ and they are Hölder continuous, belonging to $L^p(I)$. Then, for any $(u_0, v_0, z_0) \in D(L_u^\alpha) \times C^\kappa(I \times \Omega) \times D(L_z^\alpha)$, there exist $T > 0$ and a unique local classical solution (u, v, z) of* (2.1)–(2.5) *for $0 < t < T$.*

Theorem 1 can be shown by a standard argument (see Section 8 of [3]). So we omit the details. In fact, the local solution obtained in Theorem 1 exists globally in time. To prove it, we need to obtain the following a priori estimate:

Lemma 1. *Let (u, v, z) be a solution given in Theorem 1 and set $w = z + \omega$. Then there exists a constant $R > 0$, depending on initial functions u_0, v_0 and w_0, such that for any $(x, y) \in I \times \Omega$, $t > 0$,*

$$0 \le u \le R, \qquad 0 \le v \le R, \qquad 0 \le w \le R.$$

Proof. We first have $0 \le v \le \|v_0\|_{L^\infty(I \times \Omega)}$ because we can solve the second equation (2.1) withe respect to v such as

$$v(x, y, t) = v_0(x, y) \exp\left(-\int_0^t f(u(x, y, s)) w(x, y, s) ds \right). \tag{2.7}$$

Since $\tilde{w} = \max\{\|w_0\|_{L^\infty(I \times \Omega)}, w_r, w_l\}$ is a super-solution of w, we obtain $w \le \tilde{w}$. Setting

$$\tilde{u} = \max\{\|u_0\|_{L^\infty(I\times\Omega)}, \sup\{u > 0 \mid \gamma\|v_0\|_{L^\infty(I\times\Omega)}\tilde{w}f(u) - g(u) > 0\}\},$$

we readily see that $\tilde{u}$ is a super-solution of u, so that we obtain $u \le \tilde{u}$. From the maximum principle, it is easy to see that u and w are nonnegative. Thus the proof is completed.

From the above lemma, the following theorem holds:

Theorem 2. *Let (u, v, z) be a solution given in Theorem 1. Then (u, v, z) exists globally.*

Proof. From Lemma 1 we already know that v is bounded in $t > 0$. In addition, if u and z are Hölder continuous, v is also Hölder continuous because of (2.7). Therefore, in order to prove Theorem 2, it suffices to show that $\|u\|_\alpha \equiv \|u\|_{L^p(I\times\Omega)} + \|L_u^\alpha u\|_{L^p(I\times\Omega)}$ and $\|z\|_\alpha \equiv \|z\|_{L^p(I\times\Omega)} + \|L_z^\alpha z\|_{L^p(I\times\Omega)}$ exist for all $t > 0$.

First of all, we obtain the estimate of $\|u\|_{L^p(I\times\Omega)}$. Since there exist some constants $c_0 > 0$ and $\beta \in (-\infty, \infty)$ such that $\|T_u(t)\| \le c_0 e^{\beta t}$, we have

$$\|u\|_{L^p(I\times\Omega)} \le c_0 e^{\beta t}\|u_0\|_{L^p(I\times\Omega)} + c_1 \int_0^t e^{\beta(t-s)}\|u\|_{L^p(I\times\Omega)} ds,$$

where $c_1 > 0$ is a constant. Hence it follows from Gronwall's inequality that

$$\|u\|_{L^p(I\times\Omega)} \le c_0 e^{\beta t}(1 + e^{c_1 t})\|u_0\|_{L^p(I\times\Omega)} \le 2c_0 e^{(\beta+c_1)t}\|u_0\|_{L^p(I\times\Omega)}.$$

Next we estimate the norm $\|L_u^\alpha u\|_{L^p(I\times\Omega)}$. Since $\|L_u^\alpha T_u(t)\| \le c_0 e^{\beta t}/t^\alpha$ holds for $t > 0$ and $L_u^\alpha T_u(t)u_0 = T_u(t)L_u^\alpha u_0$ for $u_0 \in D(L_u^\alpha)$ (see Theorem 6.13 in Section 2 of [3]), we obtain by using (2.6)

$$\begin{aligned}\|L_u^\alpha u\|_{L^p(I\times\Omega)} &\le c_0 e^{\beta t}\|L_u^\alpha u_0\|_{L^p(I\times\Omega)} + \int_0^t \frac{c_1}{(t-s)^\alpha} e^{\beta(t-s)}\|u\|_{L^p(I\times\Omega)} ds \\ &\le c_0 e^{\beta t}\|L_u^\alpha u_0\|_{L^p(I\times\Omega)} + c_2 t^{1-\alpha} e^{(\beta+c_1)t}\|u_0\|_{L^p(I\times\Omega)}\end{aligned}$$

for a constant $c_2 > 0$. Therefore $\|u\|_\alpha$ exists globally. By using a similar argument, it can also be shown that $\|z\|_\alpha$ exists globally.

3. Asymptotic behavior of u, v and w

In this section we consider the asymptotic behavior of classical solutions of (RD).

Theorem 3. *Set $I = (0, l_x)$ and let (u, v, z) be a solution given in Theorem 1 and $w = z + \omega$. Then u, v and w have the following asymptotic behavior* (i), (ii) *and* (iii)*:*

(i) *For any* $(x, y) \in I \times \Omega$, $\lim_{t\to\infty} u(x, y, t) = 0$.
(ii) *There exists* $v_\infty(x, y) \in L^\infty(I \times \Omega)$ *such that* $\lim_{t\to\infty} v(x, y, t) = v_\infty(x, y)$ *for* $(x, y) \in I \times \Omega$ *and the function* v_∞ *has a positive value at any points* $(x, y) \in I \times \Omega$ *where* $v_0(x, y) > 0$.
(iii) *For any* $(x, y) \in I \times \Omega$, $\lim_{t\to\infty} w(x, y, t) = w_r$.

We need two lemmas to prove Theorem 3. As the first step of the proof of Theorem 3, we prove that the reaction term $f(u)vw$ approaches 0 as $t \to \infty$.

Lemma 2. *Let* (u, v, z) *be a solution given in Theorem* 1 *and set* $w = z + \omega$. *Then it holds that* $f(u)vw \to 0$ *as* $t \to \infty$ *for any* $(x, y) \in I \times \Omega$.

Proof. Since the statement of the lemma is equivalent to $v_t \to 0$, we prove it. It is easy to see that there exists $v_\infty(x, y)$ such that $v(x, y, t) \to v_\infty(x, y)$ as $t \to \infty$ because v decreases monotonically and is nonnegative. Hence, for any $\varepsilon > 0$ and $(x, y) \in I \times \Omega$, there exists $T > 0$ such that for $t > T$,

$$|v(x, y, t) - v_\infty(x, y)| < \varepsilon^2.$$

Then we see that

$$\left|\frac{v(x, y, t+\varepsilon) - v(x, y, t)}{\varepsilon}\right| \le 2\varepsilon,$$

so that we have

$$|v_t(x, y, t + \theta\varepsilon)| \le 2\varepsilon$$

for some $\theta \in (0, 1)$. Thus it follows that

$$\limsup_{t\to\infty} |v_t(x, y, t)| = \limsup_{t\to\infty} |v_t(x, y, t + \theta\varepsilon)| \le 2\varepsilon.$$

Since ε is any small parameter, we have $\lim_{t\to\infty} v_t(x, y, t) = 0$, which completes the proof.

We note that the similar result to Lemma 2 in the case of $I = (-\infty, \infty)$ can be shown.

Now we define a constant M by $M \equiv \sup_{u>0} f(u)/g(u)$. The constant M is well-defined because of the assumption (A).

Lemma 3. *Set* $I = (0, l_x)$ *and let* (u, v, z) *be a solution given in Theorem* 1. *Then it holds that* $u \to 0$ *as* $t \to \infty$ *at any* $(x, y) \in I \times \Omega$.

Proof. From Lemma 2 and the assumption (A), there exists $T > 0$ such that $\gamma f(u)vw \le g(u)/2$ for any $t > T$ and $(x, y) \in I \times \Omega$. Then u satisfies

$$u_t \le \mathrm{Le}\Delta u + \lambda' \frac{\partial u}{\partial x} - \frac{g(u)}{2}$$

for any $t > T$ and $(x, y) \in I \times \Omega$. Now we use a constant R given in Lemma 1 and define $q = q(t)$ by a solution of

$$\begin{cases} q' = -\dfrac{g(q)}{2}, & t > T, \\ q = R, & t = T. \end{cases} \tag{3.1}$$

Using $u \le R$ and applying the comparison principle to u and q, we have $u \le q$ for $t > T$. Since $g(u)$ is positive for $u > 0$ and locally Lipschitz continuous for $u \ge 0$, $q(t) \to 0$ as $t \to \infty$. Hence we find that u approaches 0, which completes the proof.

Now we are in a position to prove the asymptotic behavior (ii), (iii) of Theorem 3. Let $q = q(t)$ be a function given in the proof of Lemma 3. As stated previously, we have $0 \le u \le q$ for any $t > T$ and $(x, y) \in I \times \Omega$. Then it follows from the second equation of (RD) that

$$v_t \ge -f(u)vw \ge -MRg(u)v \ge -MRg(q)v,$$

where R is a constant given in Lemma 1. By using this inequality, we obtain

$$\begin{aligned} v(x, y, t) &\ge v(x, y, T) \exp\left(-MR \int_T^t g(q) ds\right) \\ &= v(x, y, T) \exp(-2MR(q(T) - q(t))) \\ &\ge v(x, y, T) \exp(-2MRq(T)). \end{aligned} \tag{3.2}$$

Here we have an estimate of $v(x, y, T)$ such as

$$\begin{aligned} v(x, y, T) &= v_0(x, y) \exp\left(-\int_0^T f(u)w \, ds\right) \\ &\ge v_0(x, y) \exp\left(-\sup_{0<u<R} f(u)RT\right), \end{aligned} \tag{3.3}$$

because of $w \le R$ and $u \le R$. Therefore it follows from (3.2) and (3.3) that

$$v(x, y, t) \ge v_0(x, y) \exp\left(-\sup_{0<u<R} f(u)RT - 2MRq(T)\right),$$

which implies that $v_\infty(x, y) > 0$ if $v_0(x, y) > 0$.

Next we prove (iii) of our theorem. In the third equation of (RD), we set $\phi = w - w_r$. Then ϕ satisfies the equation

$$\frac{\partial \phi}{\partial t} = \Delta \phi + \lambda \frac{\partial \phi}{\partial x} - f(u)vw, \tag{3.4}$$

and the boundary conditions (2.4) and $\partial\phi/\partial\nu = 0$ for $x \in I$, $y \in \partial\Omega$ and $t > 0$. Using the analytic semigroup $T_z(t)$ generated by $-L_z = \Delta + \lambda\partial/\partial x$, we rewrite (3.4) as the integral equation

$$\phi(t) = T_z(t - \tilde{T})\phi(\tilde{T}) - \int_{\tilde{T}}^{t} T_z(t - s)f(u(s))v(s)w(s)ds$$

for $t > \tilde{T}$, where $\tilde{T} > T$ is sufficiently large and references to the space variable of functions are omitted for notational convenience. In fact, we can show that there exists a constant $c_0 > 0$ such that

$$\|T_z(t)\psi\|_{L^\infty(I\times\Omega)} \le c_0 \min\left\{1, \frac{1}{\sqrt{t}}\right\}\|\psi\|_{L^\infty(I\times\Omega)}$$

for any $\psi \in L^\infty(I \times \Omega)$. Let $\Phi = \Phi(t)$ be a fundamental solution of $\partial/\partial t - \partial^2/\partial x^2 - \lambda\partial/\partial x$. Then we can show that $\|\psi\|_{L^\infty(I\times\Omega)}\Phi(t)$ and $\|\psi\|_{L^\infty(I\times\Omega)}$ are super-solutions of $T_z(t)\psi$. Hence we obtain

$$\begin{aligned}\|T_z(t)\psi\|_{L^\infty(I\times\Omega)} &\le \min\{\|\Phi\|_{L^\infty(I\times\Omega)}, 1\}\|\psi\|_{L^\infty(I\times\Omega)}\\ &\le c_0 \min\left\{1, \frac{1}{\sqrt{t}}\right\}\|\psi\|_{L^\infty(I\times\Omega)}.\end{aligned}$$

Hence it follows that

$$\begin{aligned}\|\phi(t)\|_{L^\infty(I\times\Omega)} &\le \frac{c_0}{\sqrt{t-\tilde{T}}}\|\phi(\tilde{T})\|_{L^\infty(I\times\Omega)} + c_0R^2\int_{\tilde{T}}^{t}\|f(u(s))\|_{L^\infty(I\times\Omega)}ds\\ &\le \frac{c_0}{\sqrt{t-\tilde{T}}}\|\phi(\tilde{T})\|_{L^\infty(I\times\Omega)} + c\int_{\tilde{T}}^{t} g(q(s))ds\\ &= \frac{c_0}{\sqrt{t-\tilde{T}}}\|\phi(\tilde{T})\|_{L^\infty(I\times\Omega)} + 2c(q(\tilde{T}) - q(t))\end{aligned}$$

for a constant $c > 0$, from which we have

$$\limsup_{t\to\infty}\|\phi(t)\|_{L^\infty(I\times\Omega)} \le 2cq(\tilde{T}). \tag{3.5}$$

Since $\tilde{T}$ is any large constant and $q \to 0$ as $t \to \infty$, the right side of (3.5) tends to 0 as $\tilde{T}$ goes to ∞. Hence we have $\|\phi\|_{L^\infty(I\times\Omega)} = \|w - w_r\|_{L^\infty(I\times\Omega)} \to 0$ as $t \to \infty$.

4. Appendix

In this section we show that the differential operators $-L_u$ and $-L_z$ generates analytic semigroups, denoted by $T_u(t)$ and $T_z(t)$, in $L^p(I \times \Omega)$ and can be expressed by two analytic semigroups, as described previously.

First of all, we generally consider the linear parabolic equation in the cylindrical domain

$$\begin{cases} \dfrac{\partial u}{\partial t} = Lu, & x \in I,\ y \in \Omega,\ t > 0, \\ \dfrac{\partial u}{\partial \nu} = 0, & x \in I,\ y \in \partial\Omega,\ t > 0, \\ u(x, y, 0) = \phi(x, y), & x \in I,\ y \in \Omega, \end{cases} \tag{4.1}$$

and also consider the boundary condition of either (2.3) or (2.4), where $Lu = a\Delta u + b_x \partial u/\partial x + b_y \cdot \nabla_y u + cu$, $a > 0$, $b_x \in \mathbf{R}$, $b_y \in \mathbf{R}^n$ and $c \in \mathbf{R}$ are coefficients, and $\nabla_y = (\partial/\partial y_1, \dots, \partial/\partial y_n)$. We consider the above problem in $L^p(I \times \Omega)$ for $p > 1$ and let the domain of L be $W_N^{2,p}(I \times \Omega)$ if we consider (2.3), or $W_{N,0}^{2,p}(I \times \Omega)$ if we consider (2.4). Recall that $W_N^{2,p}(I \times \Omega)$ and $W_{N,0}^{2,p}(I \times \Omega)$ were defined in Section 2. Let $T_x(t)$ be an analytic semigroup generated by

$$L_x = a\frac{\partial^2}{\partial x^2} + b_x \frac{\partial}{\partial x} + c$$

with the boundary condition (2.3) or (2.4) and $T_y(t)$ be also an analytic semigroup generated by

$$L_y = a\Delta_y + b_y \cdot \nabla_y$$

with the homogeneous Neumann boundary condition, where $\Delta_y = \sum_{i=1}^n \partial^2/\partial y_i^2$. It is shown in [3] that L_x and L_y generate $T_x(t)$ and $T_y(t)$, respectively. In addition we let $T(t)$ a one-parameter operator defined by $T(t) = T_x(t)T_y(t)$ for $t \geq 0$. Our purpose in this section is to show that $T(t)$ is an analytic semigroup generated by L.

Proposition 1. *$T(t) = T_x(t)T_y(t)$ is an analytic semigroup generated by L.*

Proof. We first see that

$$T_x(t)T_y(t)\phi = T_y(t)T_x(t)\phi$$

for any $\phi \in L^p(I \times \Omega)$. Since Ω is bounded, $T_y(t)$ can be expanded by the pairs of the eigenvalues and eigenfunctions of L_y, denoted by (μ_i, φ_i) for $i \geq 1$, such as

$$T_y(t) = \sum_{i=1}^{\infty} e^{\mu_i t} \langle \cdot, \varphi_i^* \rangle \varphi_i,$$

where $\langle \cdot , \cdot \rangle$ is the pairing between $L^p(\Omega)$ and the dual space of $L^p(\Omega)$, denoted by $(L^p(\Omega))^*$. Without loss of generality, we assume that $\mathrm{Re}\,\mu_1 \geq \mathrm{Re}\,\mu_2 \geq \cdots \to -\infty$ and $\varphi_i^* \in (L^p(\Omega))^*$ satisfies $\langle \varphi_i, \varphi_i^* \rangle = 1$ for any i. Since $T_y(t)\phi$ belongs to $L^p(I \times \Omega)$, $[T_y(t)\phi](x, y) = \sum_{i=1}^{\infty} e^{\mu_i t} \ \langle \phi(x, \cdot), \varphi_i^* \rangle \varphi_i(y)$ is a $L^p(I)$-function in almost every $y \in \Omega$ by Fubini's theorem. Hence it holds that

$$T_x(t)T_y(t)\phi = \sum_{i=1}^{\infty} e^{\mu_i t}(T_x(t)\langle \phi, \varphi_i^* \rangle)\varphi_i.$$

We can easily check

$$T_x(t)\langle \phi, \varphi_i^* \rangle = \langle T_x(t)\phi, \varphi_i^* \rangle$$

by using a similar expansion of $T_x(t)$ to $T_y(t)$ if I is bounded, or an explicit formula such as

$$[T_x(t)F](x) = \frac{1}{\sqrt{4\pi a t}} \int_{-\infty}^{\infty} \exp\left(-\frac{|x - \xi + b_x t|^2}{4at} + ct \right) F(\xi) d\xi$$

for $F \in L^p(-\infty, \infty)$ if $I = (-\infty, \infty)$. Therefore it follows that

$$T_x(t)T_y(t)\phi = \sum_{i=1}^{\infty} e^{\mu_i t} \langle T_x(t)\phi, \varphi_i^* \rangle \varphi_i = T_y(t)T_x(t)\phi.$$

Since $T_x(t)$ and $T_y(t)$ are commutative and analytic semigroups, it is easy to see that $T(t) = T_x(t)T_y(t)$ is an analytic semigroup.

From the above proposition, $T_u(t)$ and $T_z(t)$ can be expressed by two analytic operators, respectively.

Acknowledgement

K.I. is supported in part by NSF, by the Grant-in-Aid for JSPS Fellows. M.M. is grateful to the Grant-in-Aid for Scientific Research (S) 1814002 for its support. Special thanks go to Professor Eiji Yanagida for many stimulating discussions and continuous encouragement.

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