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On strongly 1-trivial Montesinos knot

Ichiro Torisu

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Abstract. We present a certain family of *strongly* 1-*trivial* Montesinos knots, and show that if a well-known conjecture on Seifert surgery is valid, then the family contains all strongly 1-trivial Montesinos knots.

1. Introduction

Let K be a knot in S^3 and n a positive integer. K is called *strongly n-trivial*, if K admits a diagram containing $n+1$ crossings such that the result of any $0 < m \leq n+1$ crossing changes on these crossings is the trivial knot ([6]). The notion of strong n-triviality of knots appears naturally in the theory of finite type knot invariants. For background, examples and recent studies of strongly n-trivial knots, see [2], [6], [7] and [8]. Note that a strongly n-trivial knot is automatically strongly n'-trivial for all $n' \leq n$ and has unknotting number one. In this paper, we are particularly interested in the case $n = 1$.

In [15], the author proved that a 2-bridge knot $S(\alpha, \beta)$ is strongly 1-trivial if and only if $S(\alpha, \beta)$ is the trivial knot, the trefoil knot or the figure-eight knot (Figure 1).

In this paper, we study strong 1-triviality of *Montesinos knots* via Dehn surgery technique.

Recall that a Montesinos knot $M((\alpha_1, \beta_1), \dots, (\alpha_t, \beta_t))$ is defined to be a knot connecting t rational tangles of slope from $r_1 = \beta_1/\alpha_1$ through $r_t = \beta_t/\alpha_t$ as indicated in Figure 2 (α_i and β_i are coprime integers). For a reference, see [3] and [4]. For an integer q and $\varepsilon_1 = \pm 1$, $\varepsilon_2 = \pm 1$, let $MK_{(q, \varepsilon_1, \varepsilon_2)}$ be the Montesinos knot $M((2q + \varepsilon_1, 2), (q, -1), (2q + \varepsilon_2, 2))$ (see Figure 3).

We first make the following observation:

Observation. For each q, ε_1 and ε_2, the Montesinos knot $MK_{(q, \varepsilon_1, \varepsilon_2)}$ is strongly 1-trivial.

In fact, the crossings $*$ and $**$ for $MK_{(q, \varepsilon_1, \varepsilon_2)}$ as shown in Figure 3 make $MK_{(q, \varepsilon_1, \varepsilon_2)}$ strongly 1-trivial.

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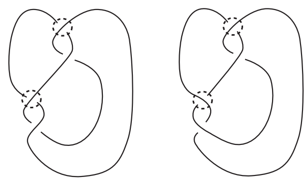

Fig. 1. Strongly 1-trivial 2-bridge knots.

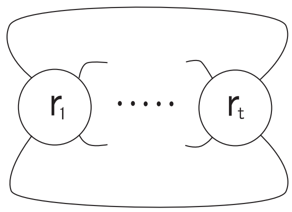

Fig. 2. Montesinos knots.

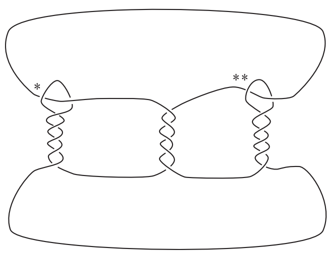

Fig. 3. $MK_{(q,\varepsilon_1,\varepsilon_2)} = M((2q+\varepsilon_1,2),(q,-1),(2q+\varepsilon_2,2))$, where $q=5$ and $\varepsilon_1=\varepsilon_2=+1$.

In this paper, we prove that these are the only Montesinos knots which are strongly 1-trivial, by assuming the following well-known conjecture on Dehn surgery (for example, see [9]).

Conjecture 1. For a knot in S^3 that is neither a torus knot nor a cable of a torus knot, only integral slopes can yield a Seifert fibered space.

Theorem 1. *If Conjecture* 1 *is true, then any strongly* 1-*trivial Montesinos knot is equivalent to* $MK_{(q,\varepsilon_1,\varepsilon_2)}$ *for some* q, ε_1 *and* ε_2.

The proof of this theorem is based on a recent result concerning twisting operation for knots due to M. Aït Nouh, D. Matignon, K. Motegi [1].

Remark 1. (i) $MK_{(0,\varepsilon_1,\varepsilon_2)}$ is the trivial knot and $MK_{(1,1,1)}$ and $MK_{(1,1,-1)}$ are the trefoil knot and the figure-eight knot, respectively.

(ii) It is known that the unknotting number of $M((\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t))$ is greater than one, if $|\alpha_i|\geq 2$ $(i=1,\dots,t)$ and $t\geq 4$ ([12]). See [14], for a conjectural form of the unknotting number 1 Montesinos knots.

2. Proof of Theorem 1

Let $\mathrm{m}_0((\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t))$ be the Seifert fibered space with t singular fibers and orbit space S^2 obtained from the 2-fold branched covering of S^3 along $M((\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t))$. Recall that $\mathrm{m}_0((\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t))$ is obtained by Dehn filling from $S^1\times P$ with surgery coefficients β_i/α_i $(i=1,\dots,t)$, here P is a t-holed 2-sphere (cf. [4]).

For a knot k in S^3 and coprime integers l, s, let $k(l/s)$ denote the 3-manifold obtained by l/s-Dehn surgery on k (cf. [13]). For a 2-component link $k\cup k'$ and two slopes l/s, l'/s', the 3-manifold $k_1\cup k_2(l/s,l'/s')$ is similarly defined.

Using the Montesinos trick (see [10]), the following is proved by an argument similar to that of the proof of Proposition 2.2 in [15].

Proposition 1. *Let K be a Montesinos knot $M((\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t))$. Suppose K is strongly 1-trivial. Then there is a 2-component link $k_1\cup k_2$ in S^3 such that* (i) *k_1 and k_2 are unknotted* (ii) *$k_1\cup k_2(\varepsilon_1/2,\varepsilon_2/2)$ is a Seifert fibered space $\mathrm{m}_0((\alpha_1,\beta_1),\dots,(\alpha_t,\beta_t))$ for some $\varepsilon_i=\pm 1$ $(i=1,2)$.*

The following proposition is well-known ([5], [11], [14, p. 172]).

Proposition 2. (i) *Suppose k is a (p,q)-torus knot. Then, $k(l/2)$ is a Seifert fibered space $\mathrm{m}_0((p,-r),(q,s),(-2pq+l,2))$ for some integers r, s with $ps-rq=1$.*

(ii) *Suppose k is an (m,n)-cable of a (p,q)-torus knot and $k(l/2)$ is a Seifert fibered space Q. Then $l=2mn\pm 1$ and $Q=\mathrm{m}_0((p,-r),(q,s),(-2n^2pq+2mn\pm l,2n^2))$ for some integers r, s with $ps-rq=1$.*

To complete the proof of Theorem 1, we need the following theorem due to M. Aït Nouh, D. Matignon, K. Motegi in [1].

Theorem 2 ([1]). *Let K be a knot in a solid torus V standardly embedded in S^3 and K_n the result of an n-full twist of K along a meridian disk of V. Suppose K is the trivial knot in S^3 and K_n is* (i) *a torus knot or* (ii) *a cable of a torus knot in S^3 for $|n|\geq 2$.*

Then the following holds accordingly, where $\epsilon=\pm 1$ and $\epsilon'=\pm 1$.

(i) *K is an (ϵ,q)-cable of a core of V and K_n is an $(\epsilon+nq,q)$-torus knot, or*

(ii) *K is an (ϵ',q')-cable of an (ϵ,q)-cable of a core of V and K_n is a $(q(\epsilon'+nq'),q')$-cable of an $(\epsilon+nq,q)$-torus knot on the boundary of a neighbourhood of a core of V.*

Proof of Theorem 1. Suppose K is a strongly 1-trivial Montesinos knot $M((\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t))$. By Proposition 1, there is a 2-component link $k_1 \cup k_2$ in S^3 such that (i) k_1 and k_2 are unknotted (ii) $k_1 \cup k_2(\varepsilon_1/2, \varepsilon_2/2)$ is a Seifert fibered space $\mathrm{m}_0((\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t))$ for some $\varepsilon_i = \pm 1$ $(i = 1, 2)$.

Note that the Seifert fibered space $k_1 \cup k_2(\varepsilon_1/2, \varepsilon_2/2) = \mathrm{m}_0((\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t))$ is obtained by a non-integral surgery on the knot k_2 in $k_1(\varepsilon_1/2) = S^3$. By our assumption that Conjecture 1 is valid, this implies that the knot k_2 in $k_1(\varepsilon_1/2)$ is either a torus knot or a cable of a torus knot. Since this knot is obtained from the original knot k_2 in the unknotted solid torus $S^3 - intN(k_1)$ in S^3 by performing $2\varepsilon_1$-full twists (Here $N(k_1)$ represents a regular neighborhood of k_1), we see by Theorem 2 that the knot k_2 in $k_1(\varepsilon_1/2)$ is identified with a knot $K_{2\varepsilon_1}$ described in Theorem 2. Namely, the knot is equal to either (i) the $(\epsilon + 2\varepsilon_1 q, q)$-torus knot or (ii) the $(q(\epsilon' + 2\varepsilon_1 q'), q')$-cable of an $(\epsilon + 2\varepsilon_1 q, q)$-torus knot. Since the linking number of k_1 and $k_2 = K_{2\varepsilon_1}$ is q or qq' accordingly, the (Seifert fibered) space $k_1 \cup k_2(\varepsilon_1/2, \varepsilon_2/2)$ is the result of surgery on the knot k_2 in $k_1(\varepsilon_1/2) = S^3$ with surgery coefficient $\varepsilon_2/2 + 2\varepsilon_1 q^2$ or $\varepsilon_2/2 + 2\varepsilon_1 q^2 q'^2$ accordingly (cf. [13, p. 267]). Suppose the latter occurs. By Proposition 2 (ii), we have $\varepsilon_2 + 4\varepsilon_1 q^2 q'^2 = 2qq'(\epsilon' + 2\varepsilon_1 q') \pm 1$. In this case, easy calculations show that $|q| \leq 1$ and $|q'| \leq 1$, that is to say, k_2 is a torus (in fact, trivial) knot in $k_1(\varepsilon_1/2)$. Therefore the case (ii) is reduced to the case (i). By Proposition 2 (i), the space obtained by $(\varepsilon_2 + 4\varepsilon_1 q^2)/2$-Dehn surgery on the $(\epsilon + 2\varepsilon_1 q, q)$-torus knot is $\mathrm{m}_0((\epsilon + 2\varepsilon_1 q, -2\epsilon\varepsilon_1), (q, \epsilon), (-2(\epsilon + 2\varepsilon_1 q)q + \varepsilon_2 + 4\varepsilon_1 q^2, 2)) = \mathrm{m}_0((-2\epsilon q - \varepsilon_1, 2), (q, \epsilon), (-2\epsilon q + \varepsilon_2, 2))$.

Since $\mathrm{m}_0((\alpha_1, \beta_1), \ldots, (\alpha_t, \beta_t)) = \mathrm{m}_0((-2\epsilon q - \varepsilon_1, 2), (q, \epsilon), (-2\epsilon q + \varepsilon_2, 2))$, the Montesinos knot K is equal to $M((-2\epsilon q - \varepsilon_1, 2), (q, \epsilon), (-2\epsilon q + \varepsilon_2, 2))$, which in turn is equal to $MK_{(-q, -\varepsilon_1, \varepsilon_2)}$ or $MK_{(q, -\varepsilon_1, \varepsilon_2)}$ according as $\epsilon = +1$ or -1. This completes the proof of Theorem 1.

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Ichiro Torisu
University of Occupational and
Environmental Health, Japan
School of Medicine
Yahatanishi-ku Kitakyushu
807-8555*, Japan*
e-mail: torisu@med.uoeh-u.ac.jp