HIROSHIMA MATH. J.
38 (2008), 135–154

Floquet representations and asymptotic behavior of solutions to periodic linear difference equations

Dedicated to Professor Tetsuo Furumochi on his 60th birthday

Toshiki NAITO, Pham Huu Anh NGOC and Jong Son SHIN

(Received January 29, 2007)
(Revised September 6, 2007)

ABSTRACT. We give new representations of solutions for the periodic linear difference equation of the type $x(n+1) = B(n)x(n) + b(n)$, where complex nonsingular matrices $B(n)$ and vectors $b(n)$ are ρ-periodic. These are based on the Floquet multipliers and the Floquet exponents, respectively. By using these representations, asymptotic behavior of solutions is characterized by initial values. In particular, we can characterize necessary and sufficient conditions that the equation has a bounded solution (or a ρ-periodic solution), and the Massera type theorem by initial values.

1. Introduction

Let $\mathbf{C}$ be the set of all complex numbers. Set $\mathbf{N} = \{1, 2, \dots\}$ and $\mathbf{N}_0 = \mathbf{N} \cup \{0\}$. In the present paper we consider the periodic linear difference equation of the form

$$x(n+1) = B(n)x(n) + b(n), \qquad x(0) = w \in \mathbf{C}^p, \quad (n \in \mathbf{N}_0), \tag{1}$$

where $p \times p$ complex matrices $B(n)$ and $b(n) \in \mathbf{C}^p$, $n \in \mathbf{N}_0$, satisfy the conditions $B(n) = B(n+\rho)$ and $b(n) = b(n+\rho)$, $(\rho \geq 2, \rho \in \mathbf{N})$.

We know the variation of constants formula as a representation of solutions for the equation (1), cf. [1]. However, this representation is not proper to obtain asymptotic behavior of solutions to the equation (1).

More recently, T. Naito and J. S. Shin [7] gave new representations and asymptotic behavior of solutions to the linear difference equation of the form

---

The first author was supported by Grant-in-aid for Scientific Research (C), No. 16540141, Japanese Ministry of Education, Culture, Sports, Science and Technology.

The second author is supported by the Japan Society for Promotion of Science (JSPS) ID No. P 05049.

2000 *Mathematics Subject Classification*. Primary: 39A10, 39A11.

Key words and phrases. Periodic linear difference equation, Floquet representation of solution, bounded solution, periodic solution, asymptotic behavior of solution, index of growth order.

$$x(n+1) = Bx(n) + b, \qquad x(0) = w. \tag{2}$$

Refer to [3, 7] for the case where $B = e^{\tau A}$. In particular, by using these representations, the necessary and sufficient conditions that the equation (2) has a bounded solution (or a ρ-periodic solution) are characterized by initial values.

In [8], a representation and asymptotic behavior of solutions for the case where $B(n) = B$ in the equation (1) are investigated in the same direction as above.

The purpose of this paper is to extend the result obtained in [8] to the general equation (1). In the paper, we give two representations of solutions for the equation (1), which are based on the Floquet multipliers and the Floquet exponents, respectively. Using these representations, by initial values we characterize asymptotic properties of solutions to the equation (1). In particular, we obtain the necessary and sufficient conditions that the equation (1) has a bounded solution (or a ρ-periodic solution). Moreover, the Massera type theorem for the equation (1) is characterized by initial values. Finally, we illustrate our results through an example. We emphasize that our approach employed in this paper and in a series of our articles [3, 7, 8] is more valid and effective for the study of the subject in this paper, rather than the one utilizing the Jordan form.

2. Preliminaries

In this section, we state some notations and known results for a general matrix H. Let E be the $p \times p$ unite matrix. For a $p \times p$ complex matrix H we denote by $\sigma(H)$ the set of all eigenvalues of H, and by $h_H(\eta)$ the index of $\eta \in \sigma(H)$. Let $G_H(\eta) = N((H - \eta E)^{h_H(\eta)})$ be the generalized eigenspace corresponding to $\eta \in \sigma(H)$ and $Q_\eta(H) : \mathbf{C}^p \to G_H(\eta)$ the projection corresponding to the direct sum decomposition

$$\mathbf{C}^p = \sum_{\eta \in \sigma(H)} \oplus G_H(\eta).$$

These projections have the following properties:

$$Q_\eta(H)\mathbf{C}^p = G_H(\eta), \qquad HQ_\eta(H) = Q_\eta(H)H, \qquad Q_\nu(H)Q_\eta(H) = 0 \ \ (\nu \neq \eta),$$

$$Q_\eta^2(H) = Q_\eta(H), \qquad E = \sum_{\eta \in \sigma(H)} Q_\eta(H).$$

We make use of the factorial numbers $(n)_k$ such that

$$(n)_k = \begin{cases} 1, & (k = 0), \\ n(n-1)(n-2)\dots(n-k+1), & (k = 1, 2, \dots, n), \\ 0, & (k = n+1, n+2, \dots). \end{cases}$$

Clearly,

$$\frac{(n)_k}{k!} = \binom{n}{k}, \qquad (n)_n = n! \qquad \text{and} \qquad (n)_k = 0 \qquad (k > n).$$

The following result is a fundamental one in the studying the qualitative theory of linear difference equations, which is well known.

Lemma 1. *If* $0 \notin \sigma(H)$, *then*

$$H^n = \sum_{\eta \in \sigma(H)} \sum_{j=0}^{h_H(\eta)-1} \frac{(n)_j}{j!} \eta^{n-j} (H - \eta E)^j Q_\eta(H), \qquad n = 0, 1, 2, \dots. \tag{3}$$

Let $\eta \in \sigma(H)$. *If* $\eta \neq 0$, *then*

$$H^n Q_\eta(H) = \sum_{j=0}^{h_H(\eta)-1} \frac{(n)_j}{j!} \eta^{n-j} (H - \eta E)^j Q_\eta(H).$$

If $\eta = 0$, *then*

$$H^n Q_0(H) = \begin{cases} O & (n \geq h_H(0)) \\ H^n Q_0(H) & (n \leq h_H(0) - 1). \end{cases}$$

For a function $a(z) = (z-1)^{-1}$ $(z \neq 1)$, we have

$$a^{(k)}(z) := \frac{d^k}{dz^k} a(z) = (-1)^k k! (z-1)^{-k-1}. \tag{4}$$

For every $\eta \in \sigma(H)$ $(\eta \neq 1)$, two matrices $Z_\eta(H, m)$ and $Z_\eta(H)$ are given as follows:

$$\begin{aligned} Z_\eta(H, m) &= \sum_{k=0}^{m-1} \frac{a^{(k)}(\eta)}{k!} (H - \eta E)^k \\ &= -\sum_{k=0}^{m-1} \frac{1}{(1-\eta)^{k+1}} (H - \eta E)^k \qquad (m = 1, 2, \dots, h_H(\eta)), \end{aligned}$$

and

$$Z_\eta(H) = Z_\eta(H, h_H(\eta)).$$

Put

$$S_n(H) = \sum_{k=0}^{n-1} H^k \quad (n \geq 1), \qquad S_0(H) = 0.$$

The following lemma can be found in [7].

LEMMA 2 [7]. *Let* $\eta \in \sigma(H)$ *and* $n \in \mathbf{N}_0$.
1) *If* $\eta \neq 1$, *then*

$$S_n(H)Q_\eta(H) = H^n Z_\eta(H)Q_\eta(H) - Z_\eta(H)Q_\eta(H).$$

In particular, if $\eta \neq 1$, $\eta \neq 0$, *then the right side is written as*

$$S_n(H)Q_\eta(H) = \sum_{k=0}^{h_H(\eta)-1} \frac{(n)_k}{k!} \eta^{n-k} (H - \eta E)^k Z_\eta(H)Q_\eta(H) - Z_\eta(H)Q_\eta(H).$$

2) *If* $\eta = 1$, *then*

$$S_n(H)Q_1(H) = \sum_{i=0}^{h_H(1)-1} \binom{n}{i+1} (H - E)^i Q_1(H).$$

3. Floquet representations of solutions of periodic linear difference equations

In this section, two representations of solutions for the equation (1) are given by using the Floquet multipliers and the Floquet exponents, respectively.

3.1. Periodic map. First, we consider the periodic linear homogeneous difference equation of the form

$$x(n+1) = B(n)x(n). \tag{5}$$

Throughout this paper we assume that for all $n \in \mathbf{N}_0$, $B(n)$ is nonsingular. Put

$$\prod_{i=m}^{n-1} B(i) = \begin{cases} B(n-1)B(n-2)\dots B(m) & (n > m) \\ E & (n = m). \end{cases}$$

Then the fundamental matrix $X(n)$ $(n \in \mathbf{N}_0)$ and the fundamental solution $U(n,m)$, $n \geq m$, of the equation (5) are given as

$$X(n) = \prod_{i=0}^{n-1} B(i) \tag{6}$$

and

$$U(n,m) = X(n)X^{-1}(m) = \prod_{i=m}^{n-1} B(i), \tag{7}$$

respectively. Clearly, $U(n,0) = X(n)$. If $n \leq m$, then $U(n,m)$ is defined by

$$U(n,m) = U^{-1}(m,n) = \left(\prod_{i=n}^{m-1} B(i) \right)^{-1}.$$

Lemma 3 [1]. *The fundamental solution $U(n,m)$ $(m, n \in \mathbf{N}_0)$ of the equation (5) has the following properties:*

1) $U(n,n) = E$.
2) $U(n,r)U(r,m) = U(n,m)$.
3) $U(n+\rho, m+\rho) = U(n,m)$.
4) $U^n(m+\rho, m) = U(m+n\rho, m)$.
5) $U(m+n\rho, r) = U^n(m+\rho, m)U(m,r) = U(m,r)U^n(r+\rho, r)$.

For the fundamental matrix $X(n)$ of the equation (5) there are a non-singular matrix $P(n)$ with period ρ and a matrix M such that

$$X(n) = P(n)M^n. \tag{8}$$

Since $U(n,m) = X(n)X^{-1}(m)$, we have $U(n,m) = P(n)M^{n-m}P^{-1}(m)$. Moreover, since $P(n)$ is ρ-periodic, $P^{-1}(n)$ is also ρ-periodic; clearly, $P(\rho) = P(0) = E$, $P^{-1}(\rho) = P^{-1}(0) = E$ and $X(\rho) = M^\rho$.

Next, we define the well known periodic map $V(n)$ $(n \in \mathbf{N}_0)$ by $V(n) = U(n+\rho, n)$. Then it is easy to check the following properties:

$$V(n+\rho) = V(n), \qquad V(n)U(n,m) = U(n,m)V(m),$$

and

$$V(n) = P(n)V(0)P^{-1}(n), \qquad V(0) = M^\rho.$$

In particular, $V(0)$ is called a periodic matrix or a monodromy matrix for the equation (5). Clearly, we have

$$X(m+k\rho) = V^k(m)X(m) = X(m)V^k(0) \qquad (m, k \in \mathbf{N}_0). \tag{9}$$

Let $v \in \sigma(V(0))$. We set $Q_v = Q_v(V(0))$, $Q_v(n) = Q_v(V(n))$ and $h(v) = h_{V(0)}(v)$. Then we have

$$Q_v(n+\rho) = Q_v(n), \qquad Q_v(n)U(n,m) = U(n,m)Q_v(m).$$

LEMMA 4. *Let $v \in \sigma(V(0))$. The periodic map $V(n)$, $n \in \mathbf{N}_0$, has the following spectral properties:*

1) $\sigma(V(n)) = \sigma(V(0))$ *and* $0 \notin \sigma(V(0))$.
2) $h_{V(n)}(v) = h_{V(m)}(v)$ *and*

$$U(n,m)G_{V(m)}(v) = G_{V(n)}(v).$$

In particular,

$$X(n)G_{V(0)}(v) = G_{V(n)}(v).$$

3)

$$\dim G_{V(n)}(v) = \dim G_{V(m)}(v).$$

3.2. A Floquet representation of solutions (I). Every $n \in \mathbf{N}_0$ is expressed by $n = k(n)\rho + m(n)$, $k(n) = \left[\frac{n}{\rho}\right]$, $0 \leq m(n) < \rho - 1$, where for $a \in \mathbf{R}$ the symbol $[a]$ stands for the maximum integer which is not greater than a. Hereafter, throughout this paper we will use this expression for $n \in \mathbf{N}_0$.

First, we give a representation of the solution $x(n; w)$ of the equation (5) with $x(0) = w$. From the relation (9) the fundamental solution $U(n,0)$ of the equation (5) is expressed as $U(n,0) = X(m(n))V^{k(n)}(0)$. Thus the solution $x(n; Q_v w)$ of the equation (5) satisfying the initial condition $Q_v x(0) = Q_v w$, $v \in \sigma(V(0))$, is expressed as

$$x(n; Q_v w) = U(n,0)Q_v w = X(m(n))V^{k(n)}(0)Q_v w.$$

Clearly, $Q_v(n)x(n; w) = x(n; Q_v w)$. By Lemma 1 we have

$$Q_v(n)x(n; w) = X(m(n)) \sum_{j=0}^{h(v)-1} \frac{(k(n))_j}{j!} v^{k(n)-j}(V(0) - vE)^j Q_v w. \tag{10}$$

Next, we consider the periodic linear inhomogeneous difference equation (1). We denote by $x(n; w, b(\cdot))$ the solution of the equation (1).

For every $k \in \mathbf{N}_0$ and $v \in \sigma(V(0))$, we set

$$S_k(b(\cdot)) = \sum_{i=0}^{k-1} U(k, i+1)b(i) \quad (1 \leq k \leq \rho), \qquad S_0(b(\cdot)) = 0$$

and

$$S_{k,v}(b(\cdot)) = \sum_{i=0}^{k-1} U(k, i+1)Q_v(i+1)b(i) \quad (1 \leq k \leq \rho), \qquad S_{0,v}(b(\cdot)) = 0.$$

Lemma 5. *The solution* $x(n) := x(n; w, b(\cdot))$ *of the equation* (1) *is expressed as*

$$x(n) = X(n)w + X(m(n))S_{k(n)}(V(0))S_\rho(b(\cdot)) + S_{m(n)}(b(\cdot)). \tag{11}$$

and

$$x(k(n)\rho) = V^{k(n)}(0)w + S_{k(n)}(V(0))S_\rho(b(\cdot)). \tag{12}$$

Proof. Set $k = k(n)$, $m = m(n)$. By easy calculations, we have

$$x(n) = X(m)V^k(0)w + X(m)\sum_{j=1}^{k}\sum_{i=1}^{\rho} V^{k-j}(0)U(\rho, i)b(i-1) + \sum_{i=1}^{m} U(m, i)b(i-1).$$

Since

$$\sum_{j=1}^{k}\sum_{i=1}^{\rho} V^{k-j}(0)U(\rho, i)b(i-1) = \sum_{j=0}^{k-1} V^j(0)\sum_{i=0}^{\rho-1} U(\rho, i+1)b(i)$$
$$= S_k(V(0))S_\rho(b(\cdot)),$$

we can obtain (11). (12) is obvious. □

From Lemma 5 the component $Q_\nu(n)x(n; w, b(\cdot))$ of solution $x(n; w, b(\cdot))$ is given by

$$Q_\nu(n)x(n; w, b(\cdot))$$
$$= X(n)Q_\nu w + X(m(n))S_{k(n)}(V(0))S_{\rho,\nu}(b(\cdot)) + S_{m(n),\nu}(b(\cdot)). \tag{13}$$

Clearly, the solution $x(n; w, b(\cdot))$ of the equation (1) is expressed as

$$x(n; w, b(\cdot)) = \sum_{\nu\in\sigma(V(0))} Q_\nu(n)x(n; w, b(\cdot)).$$

For each $\nu \in \sigma(V(0))$, we define $Z_\nu(V(0), b(\cdot))$, $\gamma_\nu(w, b(\cdot))$ and $\delta(w, b(\cdot))$ as follows:

$$Z_\nu(V(0), b(\cdot)) = Z_\nu(V(0))S_{\rho,\nu}(b(\cdot)) \qquad (\nu \neq 1),$$
$$\gamma_\nu(w, b(\cdot)) = \gamma_\nu(w, b(\cdot); V(0)) = Q_\nu w + Z_\nu(V(0), b(\cdot)) \qquad (\nu \neq 1),$$
$$\delta(w, b(\cdot)) = \delta(w, b(\cdot); V(0)) = (V(0) - E)Q_1 w + S_{\rho,1}(b(\cdot)) \qquad (\nu = 1).$$

We are now to give a representation of the component $Q_\nu(n)x(n; w, b(\cdot))$ of the solution $x(n; w, b(\cdot))$ to the equation (1). The representation of solutions is based on the Floquet multipliers.

THEOREM 1. *Let* $\nu \in \sigma(V(0))$. *The component* $Q_\nu(n)x(n;w,b(\cdot))$ *of the solution* $x(n;w,b(\cdot))$ *to the equation* (1) *is expressed as follows:*

1) *If* $\nu \neq 1$, *then*

$$\begin{aligned} Q_\nu(n)x(n;w,b(\cdot)) \\ &= X(n)\gamma_\nu(w,b(\cdot)) - X(m(n))Z_\nu(V(0),b(\cdot)) + S_{m(n),\nu}(b(\cdot)) \qquad (14) \\ &= X(m(n))\sum_{j=0}^{h(\nu)-1}\frac{(k(n))_j}{j!}\nu^{k(n)-j}(V(0)-\nu E)^j\gamma_\nu(w,b(\cdot)) \\ &\quad - X(m(n))Z_\nu(V(0),b(\cdot)) + S_{m(n),\nu}(b(\cdot)) \qquad (15) \end{aligned}$$

and

$$Q_\nu(k(n)\rho)x(k(n)\rho;w,b(\cdot)) = V^{k(n)}(0)\gamma_\nu(w,b(\cdot)) - Z_\nu(V(0),b(\cdot)). \qquad (16)$$

2) *If* $\nu = 1$, *then*

$$\begin{aligned} Q_1(n)x(n;w,b(\cdot)) = X(m(n))\sum_{j=0}^{h(1)-1}\frac{(k(n))_{j+1}}{(j+1)!}(V(0)-E)^j\delta(w,b(\cdot)) \\ + X(m(n))Q_1 w + S_{m(n),1}(b(\cdot)) \qquad (17) \end{aligned}$$

and

$$Q_1(k(n)\rho)x(k(n)\rho;w,b(\cdot)) = \sum_{j=0}^{h(1)-1}\frac{(k(n))_{j+1}}{(j+1)!}(V(0)-E)^j\delta(w,b(\cdot)) + Q_1 w.$$

PROOF. Set $N = V(0) - \nu E$ and $k = k(n)$.
Let $\nu \neq 1$. By Lemma 2 we have

$$\begin{aligned} S_k(V(0))S_{\rho,\nu}(b(\cdot)) &= S_k(V(0))Q_\nu S_{\rho,\nu}(b(\cdot)) \\ &= V^k(0)Z_\nu(V(0))S_{\rho,\nu}(b(\cdot)) - Z_\nu(V(0))S_{\rho,\nu}(b(\cdot)). \end{aligned}$$

Hence we get

$$\begin{aligned} V^k(0)Q_\nu w + S_k(V(0))S_\rho(Q_\nu b(\cdot)) \\ &= V^k(0)\big(Q_\nu w + Z_\nu(V(0))S_{\rho,\nu}(b(\cdot))\big) - Z_\nu(V(0))S_{\rho,\nu}(b(\cdot)) \\ &= V^k(0)\gamma_\nu(w,b(\cdot)) - Z_\nu(V(0),b(\cdot)). \end{aligned}$$

Using this relation and Lemma 5, we can obtain the representation (14). The representations (15) and (16) are obvious, because of $m(n) = 0$, $X(0) = E$ and Lemma 1.

Let $v = 1$. Using Lemma 1 and Lemma 2 again, we have

$$
\begin{aligned}
&V^k(0)Q_1 w + S_k(V(0))S_{\rho,1}(b(\cdot)) \\
&= \sum_{j=0}^{h(1)-1} \binom{k}{j} N^j Q_1 w + \sum_{j=0}^{h(1)-1} \binom{k}{j+1} N^j S_{\rho,1}(b(\cdot)) \\
&= \sum_{j=0}^{h(1)-1} \binom{k}{j+1} N^{j+1} Q_1 w + \sum_{j=0}^{h(1)-1} \binom{k}{j+1} N^j S_{\rho,1}(b(\cdot)) + Q_1 w \\
&= \sum_{j=0}^{h(1)-1} \binom{k}{j+1} N^j (N Q_1 w + S_{\rho,1}(b(\cdot))) + Q_1 w \\
&= \sum_{j=0}^{h(1)-1} \binom{k}{j+1} N^j \delta(w, b(\cdot)) + Q_1 w,
\end{aligned}
$$

from which the representation (17) is obtained. Therefore the proof of the theorem is complete. □

Corollary 1. *Let $b(n) = b$ in the equation* (1). *Let $v \in \sigma(V(0))$. The component $Q_v(n)x(n; w, b)$ of solution $x(n; w, b)$ of the equation* (1) *is expressed as follows:*

1) *If $v \neq 1$, then*

$$Q_v(n)x(n; w, b) = X(n)\gamma_v(w, b) - X(m(n))Z_v(V(0), b) + S_{m(n),v}(b).$$

2) *If $v = 1$, then*

$$
\begin{aligned}
Q_1(n)x(n; w, b) &= X(m(n)) \sum_{j=0}^{h(1)-1} \frac{(k(n))_{j+1}}{(j+1)!} (V(0) - E)^j \delta(w, b) \\
&\quad + X(m(n))Q_1 w + S_{m(n),1}(b).
\end{aligned}
$$

We consider the case where $B(n) = B$ $(n = 0, 1, \dots)$ in the equation (1). Then

$$V(n) = V(0) = B^\rho, \qquad U(\rho, i+1) = B^{\rho-i-1} \quad (\rho \geq i+1),$$

from which we have

$$Q_v(n) = Q_v = Q_v(B^\rho) \qquad \text{for } n \in \mathbf{N}_0.$$

Set $\sigma_v(B) = \{\mu \in \sigma(B) \mid v = \mu^\rho\}$ for $v \in \sigma(B^\rho)$ and $Q_\mu = Q_\mu(B)$. Since the relation $G_{B^\rho}(v) = \sum_{\mu \in \sigma_v(B)} \oplus G_B(\mu)$ holds, it follows that for $\mu \in \sigma_v(B)$ the relation $Q_\mu Q_v = Q_\mu$ holds. Thus we obtain

$$Z_\nu(V(0), b(\cdot)) = Z_\nu(B^\rho)\sum_{i=0}^{\rho-1} B^{\rho-i-1}Q_\nu b(i),$$

from which it follows that for $\mu \in \sigma_\nu(B)$

$$Q_\mu Z_\nu(B^\rho, b(\cdot)) = Z_\nu(B^\rho)S_\rho(B, Q_\mu b(\cdot)), \tag{18}$$

where

$$S_k(B, Q_\mu b(\cdot)) = \sum_{i=0}^{k-1} B^{k-i-1}Q_\mu b(i), \quad (1 \le k \le \rho), \qquad S_0(B, Q_\mu b(\cdot)) = 0.$$

Define a function $c(z)$ and numbers $\left\{ \begin{matrix} k \\ m \end{matrix} \right\}_j$ as follows:

$$c(z) := a(z^\rho) = \frac{1}{z^\rho - 1} \qquad (z^\rho \ne 1),$$

and for $k, m, j \in \mathbf{N}$

$$\left\{ \begin{matrix} k \\ m \end{matrix} \right\}_j := k! \sum_{\alpha \in p(k,m,k)} \prod_{i=1}^{k} \frac{((j)_i)^{\alpha_i}}{(\alpha_i!)(i!)^{\alpha_i}},$$

$$\left\{ \begin{matrix} k \\ 0 \end{matrix} \right\}_j = \begin{cases} 0 & (k \ne 0) \\ 1 & (k = 0) \end{cases} \qquad (j \ge 1),$$

where $p(k, m, n)$ stands for the set of all finite sequences $\alpha := (\alpha_1, \alpha_2, \ldots, \alpha_k)$, $\alpha_i \in \mathbf{N}_0$, $(i = 1, 2, \ldots, k)$, such that the conditions

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k = m, \qquad \alpha_1 + 2\alpha_2 + \cdots + k\alpha_k = n$$

are satisfied.

Set

$$B_{k,\mu} = \frac{1}{k!\mu^k}(B - \mu E)^k \qquad (\mu \ne 0).$$

Lemma 6 [8]. *Let $\mu \in \sigma_\nu(B)$ such that $\mu \ne 0$. If $0 \le k \le h(\mu) - 1$, then*

$$\frac{1}{k!\nu^k}(B^\rho - \nu E)^k Q_\mu = \sum_{i=k}^{h(\mu)-1} \left\{ \begin{matrix} i \\ k \end{matrix} \right\}_\rho B_{i,\mu}Q_\mu.$$

If $h(\mu) \le k$, then $(B^\rho - \nu E)^k Q_\mu = 0$.

Then we have that for each $\mu \in \sigma(B)$, $(\mu^\rho = 1)$

$$\delta_\mu(w, b(\cdot); B) := Q_\mu \delta(w, b(\cdot); B^\rho)$$
$$= (B^\rho - E) Q_\mu w + S_\rho(B, Q_\mu b(\cdot))$$
$$= \sum_{i=1}^{h(\mu)-1} (\rho)_i B_{i,\mu} Q_\mu w + S_\rho(B, Q_\mu b(\cdot)). \quad (19)$$

For each $\mu \in \sigma(B)$ such that $\mu^\rho \neq 1$, we set

$$Z_\mu^0(B) = \begin{cases} \displaystyle\sum_{i=0}^{h(\mu)-1} \frac{1}{i!} c^{(i)}(\mu)(B - \mu E)^i & (\mu \neq 0), \\ \displaystyle -\sum_{i=0}^{[(h(\mu)-1)/\rho]} B^{\rho i} & (\mu = 0). \end{cases}$$

Lemma 7 [8]. *Let $\mu \in \sigma_\nu(B)$ and $\nu = \mu^\rho \neq 1$. Then*

$$Z_\nu(B^\rho) Q_\mu = Z_\mu^0(B) Q_\mu.$$

Using this lemma, (18) is reduced to

$$Z_\mu(B, b(\cdot)) := Q_\mu Z_\nu(B^\rho, b(\cdot)) = Z_\mu^0(B) S_\rho(B, Q_\mu b(\cdot)).$$

Thus for $\mu \in \sigma_\nu(B)$, $(\mu^\rho \neq 1)$ we get

$$\gamma_\mu(w, b(\cdot); B) := Q_\mu \gamma_\nu(w, b(\cdot); V(0)) = Q_\mu w + Z_\mu(B, b(\cdot)). \quad (20)$$

Thus we have the following result.

Corollary 2 [8]. *Let $B(n) = B$ in the equation* (1) *and $\mu \in \sigma(B)$. The component $Q_\mu x(n)$ of the solution $x(n) := x(n; w, b(\cdot))$ of the equation* (1) *is expressed as follows:*

1) *If $\mu^\rho \neq 1$, then*

$$Q_\mu x(n) = B^n \gamma_\mu(w, b(\cdot); B) - B^{m(n)} Z_\mu(B, b(\cdot)) + S_{m(n)}(B, Q_\mu b(\cdot)) \quad (21)$$
$$= \mu^n \sum_{j=0}^{h(\mu)-1} (n)_j B_{j,\mu} \gamma_\mu(w, b(\cdot); B) - B^{m(n)} Z_\mu(B, b(\cdot))$$
$$+ S_{m(n)}(B, Q_\mu b(\cdot)).$$

2) *If $\mu^\rho = 1$, then*

$$Q_\mu x(n) = \left(\sum_{j=0}^{h(\mu)-1} \frac{(k(n))_{j+1}}{j+1} \sum_{i=j}^{h(\mu)-1} \begin{Bmatrix} i \\ j \end{Bmatrix}_\rho B_{i,\mu} \right) B^{m(n)} \delta_\mu(w, b(\cdot); B)$$
$$+ B^{m(n)} Q_\mu w + S_{m(n)}(B, Q_\mu b(\cdot)). \quad (22)$$

PROOF. Let $\mu^\rho \neq 1$. Operating Q_μ to (14) we have

$$Q_\mu x(n) = B^n Q_\mu \gamma_v(w, b(\cdot)) - B^{m(n)} Q_\mu Z_v(B^\rho, b(\cdot)) + S_{m(n)}(B, Q_\mu b(\cdot)).$$

In view of (20), we obtain (21).

Let $\mu^\rho = 1$. Operating Q_μ to (17) we have

$$Q_\mu x(n) = B^{m(n)} \sum_{j=0}^{h(\mu)-1} \frac{(k(n))_{j+1}}{(j+1)!} (B^\rho - E)^j Q_\mu \delta(w, b(\cdot))$$

$$+ B^{m(n)} Q_\mu w + S_{m(n)}(B, Q_\mu b(\cdot)).$$

The representation (22) follows from (19) and Lemma 6. Therefore the proof is complete. □

3.3. A Floquet representation of solutions (II). By the relation (8) and the transformation $x(n) = P(n)y(n)$, the equation (1) is reduced to the equation

$$y(n+1) = My(n) + h(n), \qquad y(0) = w, \tag{23}$$

where $h(n) = P^{-1}(n+1)b(n)$. Since $P^{-1}(n)$ is ρ-periodic, $h(n)$ is also ρ-periodic. Hence the solution $x(n; w, b(\cdot))$ of the equation (1) is expressed as

$$x(n; w, b(\cdot)) = \sum_{\mu \in \sigma(M)} P(n) Q_\mu(M) y(n; w, h(\cdot)).$$

Put

$$x_\mu(n; w, b(\cdot)) = P(n) Q_\mu(M) y(n; w, h(\cdot)) (= P(n) Q_\mu(M) P^{-1}(n) x(n; w, b(\cdot)))$$

Applying Corollary 2, we get the following result, which is based on the Floquet exponents.

THEOREM 2. *Let $\mu \in \sigma(M)$. The component $x_\mu(n)$ of the solution $x(n) := x(n; w, b(\cdot))$ of the equation* (1) *is expressed as follows:*

1) *If $\mu^\rho \neq 1$, then*

$$x_\mu(n) = X(n)\gamma_\mu(w, h(\cdot); M) - X(m(n)) Z_\mu(M, h(\cdot))$$

$$+ P(m(n)) S_{m(n)}(M, Q_\mu(M) h(\cdot))$$

$$= P(n) \sum_{j=0}^{h(\mu)-1} (n)_j \mu^n M_{j,\mu} \gamma_\mu(w, h(\cdot); M) - P(m(n)) M^{m(n)} Z_\mu(M, h(\cdot))$$

$$+ P(m(n)) S_{m(n)}(M, Q_\mu(M) h(\cdot)).$$

2) *If* $\mu^{\rho} = 1$, *then*

$$x_{\mu}(n) = \left(P(n) \sum_{j=0}^{h(\mu)-1} \frac{(k(n))_{j+1}}{j+1} \sum_{i=j}^{h(\mu)-1} \left\{ \begin{matrix} i \\ j \end{matrix} \right\}_{\rho} M_{i,\mu} \right) M^{m(n)} \delta_{\mu}(w, h(\cdot); M)$$

$$+ P(m(n)) M^{m(n)} Q_{\mu}(M) w + P(m(n)) S_{m(n)}(M, Q_{\mu}(M) h(\cdot)).$$

4. Asymptotic behavior of solutions

In this section, asymptotic behavior of solutions to the equation (1) is characterized by using representations of solutions obtained in the previous sections.

Let $\sigma(V(0)) = \{v_1, v_2, \dots, v_s\}$. We can describe asymptotic behavior of the solution to the equation (1) by an index of growth order $(d(v_1), d(v_2), \dots, d(v_s))$ for the initial value w defined as follows:

If $v \neq 1$, then $d(v) = 0$ in the case that $\gamma_v(w, b(\cdot)) = 0$; otherwise, $d(v)$ is a positive integer such that

$$(V(0) - vE)^{d(v)-1} \gamma_v(w, b(\cdot)) \neq 0, \qquad (V(0) - vE)^{d(v)} \gamma_v(w, b(\cdot)) = 0.$$

If $v = 1$, then $d(v) = 0$ in the case that $\delta(w, b(\cdot)) = 0$; otherwise, $d(v)$ is a positive integer such that

$$(V(0) - E)^{d(1)-1} \delta(w, b(\cdot)) \neq 0, \qquad (V(0) - E)^{d(1)} \delta(w, b(\cdot)) = 0.$$

Clearly, $d(1) \leq h(1)$.

Using Theorem 1, we obtain the following result.

Theorem 3. *Let* $v \in \sigma(V(0))$, *and* $Q_v(n)x(n)$ *be the component of the solution* $x(n) := x(n; w, b(\cdot))$ *of the equation* (1).

1) *The case where* $|v| > 1$.

(1) *If* $d(v) = 0$, *then* $Q_v(n)x(n)$ *is* ρ*-periodic:*

$$Q_v(k(n)\rho) x(k(n)\rho) = -Z_v(V(0), b(\cdot)).$$

(2) *If* $d(v) \geq 1$, *then* $Q_v(n)x(n)$ *is unbounded:*

$$Q_v(n)x(n) = \frac{\left(\left[\frac{n}{\rho}\right]\right)_{d(v)-1} v^{[n/\rho]}}{(d(v)-1)! v^{d(v)-1}} X\left(n - \left[\frac{n}{\rho}\right]\rho\right) (V(0) - vE)^{d(v)-1} \gamma_v(w, b(\cdot))$$

$$+ o\left(\left(\left[\frac{n}{\rho}\right]\right)_{d(v)-1} v^{[n/\rho]} \right) \qquad (n \to \infty).$$

2) *The case where* $|v| < 1$. *Then* $Q_v(n)x(n)$ *is bounded.*

(1) *If* $d(v) = 0$, *then* $Q_v(n)x(n)$ *is* ρ*-periodic.*

(2) *If* $d(v) \geq 1$, *then* $Q_v(n)x(n)$ *is bounded.*

3) *The case where* $|v| = 1$, $v \neq 1$.

(1) *If* $d(v) = 0$, *then* $Q_v(n)x(n)$ *is* ρ*-periodic.*

(2) *If* $d(v) = 1$, *then* $Q_v(n)x(n)$ *is bounded:*

$$Q_v(n)x(n) = v^{[n/\rho]} X\left(n - \left[\frac{n}{\rho}\right]\rho\right)\gamma_v(w, b(\cdot)) - X\left(n - \left[\frac{n}{\rho}\right]\rho\right) Z_v(V(0), b(\cdot)) + S_{n-[n/\rho]\rho, v}(b(\cdot)).$$

(3) *If* $d(v) \geq 2$, *then* $Q_v(n)x(n)$ *is unbounded:*

$$Q_v(n)x(n) = \frac{\left(\left[\frac{n}{\rho}\right]\right)_{d(v)-1} v^{[n/\rho]}}{(d(v)-1)! v^{d(v)-1}} X\left(n - \left[\frac{n}{\rho}\right]\rho\right)(V(0) - vE)^{d(v)-1}\gamma_v(w, b(\cdot)) + o\left(\left(\left[\frac{n}{\rho}\right]\right)_{d(v)-1}\right) \qquad (n \to \infty).$$

4) *The case where* $v = 1$.

(1) *If* $d(1) = 0$, *then* $Q_1(n)x(n)$ *is* ρ*-periodic:* $Q_1(k(n)\rho)x(k(n)\rho) = Q_1 w$.

(2) *If* $d(1) \geq 1$, *then* $Q_1(n)x(n)$ *is unbounded:*

$$Q_1(n)x(n) = \frac{\left(\left[\frac{n}{\rho}\right]\right)_{d(1)}}{d(1)!} X\left(n - \left[\frac{n}{\rho}\right]\rho\right)(V(0) - vE)^{d(1)-1}\delta(w, b(\cdot)) + o\left(\left(\left[\frac{n}{\rho}\right]\right)_{d(1)}\right) \qquad (n \to \infty).$$

The following results on the boundedness and the ρ-periodicity of solutions for the equation (1) are immediately obtained from Theorem 3.

Theorem 4. *A solution* $x(n; w, b(\cdot))$ *of the equation* (1) *is bounded if and only if the following conditions hold: For every* $v \in \sigma(V(0))$,

1) *if* $|v| > 1$, *then* $\gamma_v(w, b(\cdot)) = 0$;
2) *if* $|v| = 1$, $v \neq 1$, *then* $(V(0) - vE)\gamma_v(w, b(\cdot)) = 0$; *and*
3) *if* $v = 1$, *then* $\delta(w, b(\cdot)) = 0$.

Theorem 5. *The following statements are equivalent:*

1) *A solution* $x(n; w, b(\cdot))$ *of the equation* (1) *is* ρ*-periodic.*
2) *For every* $v \in \sigma(V(0))$,

(1) $\gamma_v(w, b(\cdot)) = 0$, $(v \neq 1)$; *and*

(2) $\delta(w, b(\cdot)) = 0$, $(v = 1)$.

3)

$$(E - V(0))w = S_\rho(b(\cdot)).$$

Proof. The equivalence of 1) and 2) follows from Theorem 3. Using (12), we have that $x(\rho; w, b(\cdot)) = x(0; w, b(\cdot))$ if and only if $(E - V(0))w = S_\rho(b(\cdot))$, which means the equivalence of 1) and 3). $\square$

In particular, we note that for a ρ-periodic solution $x(n; w, b(\cdot))$ of the equation (1) we have

$$Q_v(k(n)\rho)x(k(n)\rho; w, b(\cdot)) = \begin{cases} -Z_v(V(0), b(\cdot)) & (v \neq 1) \\ Q_1 w & (v = 1). \end{cases}$$

Corollary 3. *A bounded solution $x(n; w, b(\cdot))$ of the equation* (1) *is ρ-periodic if and only if $\gamma_v(w, b(\cdot)) = 0$ for all $v \in \sigma(V(0))$ such that $|v| \leq 1$, $v \neq 1$.*

Corollary 4. *If $1 \notin \sigma(V(0))$, then the equation* (1) *has a unique ρ-periodic solution.*

Lemma 8. *The following statements are equivalent: Let $1 \in \sigma(V(0))$.*

1) *There is $Q_1 w$ such that $\delta(w, b(\cdot)) = 0$.*
2) $S_{\rho,1}(b(\cdot)) \in (V(0) - E)G_{V(0)}(1)$.
3) $S_\rho(b(\cdot)) \in R(V(0) - E)$, *the range of $V(0) - E$.*
4) *There is w satisfying the equation*

$$(E - V(0))w = S_\rho(b(\cdot)).$$

Proof. The equivalence of 2) and 3) follows from Lemma 5.5 in [3]. The equivalence of 1) and 4) is given by Theorem 5. The remainder is obvious. $\square$

Finally, we characterize the Massera type theorem (cf. [4]) by initial values. The proof follows from Theorem 4, Theorem 5 and Lemma 8.

Theorem 6. *The following statements are equivalent:*

1) *The equation* (1) *has a solution which is bounded.*
2) $1 \in \sigma(V(0))$ *and there is $Q_1 w$ such that $\delta(w, b(\cdot)) = 0$; or $1 \notin \sigma(V(0))$.*
3) *The equation* (1) *has a ρ-periodic solution.*

5. An example

In this section, we will illustrate our results through an example. We deal with the initial problem to the equation

$$x(n+1) = B(n)x(n) + b(n), \qquad x(0) = w \in \mathbf{C}^3, \qquad (n \in \mathbf{N}_0), \tag{24}$$

where

$$B(n) = \begin{pmatrix} (-1)^n & 1 & 0 \\ 0 & (-1)^{n+1} & 0 \\ 0 & (-1)^{n+1} & 1 \end{pmatrix}, \qquad b(n) = \begin{pmatrix} (-1)^n a \\ 2b + (1 - (-1)^n)a \\ (-1)^n c \end{pmatrix},$$

$$x(n) = \begin{pmatrix} x_1(n) \\ x_2(n) \\ x_3(n) \end{pmatrix}, \qquad w = \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \qquad (a, b, c \in \mathbf{C}).$$

Then the periodic map $V(n)$, $n \in \mathbf{N}_0$ is given by

$$V(n) = U(n+2, n) = B(n+1)B(n) = \begin{pmatrix} -1 & 2(-1)^{n+1} & 0 \\ 0 & -1 & 0 \\ 0 & -1 + (-1)^{n+1} & 1 \end{pmatrix}.$$

Since

$$V(0) = \begin{pmatrix} -1 & -2 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{pmatrix},$$

we have that $\Phi_{V(0)}(v) = |vE - V(0)| = (v+1)^2(v-1)$, that is, $\sigma(V(n)) = \sigma(V(0)) = \{-1, 1\}$. We note that $2 = rank(V(0) + E) > rank(V(0) + E)^2 = 1$. Hence we get $h_{V(0)}(-1) = 2$, $h_{V(0)}(1) = 1$.

Now, we will calculate $Q_{-1}(n)$ and $Q_1(n)$. Since

$$\frac{1}{\Phi_{V(0)}(v)} = \frac{-\frac{1}{4}v - \frac{3}{4}}{(v+1)^2} + \frac{\frac{1}{4}}{v-1},$$

we have

$$\begin{aligned} Q_{-1}(n) &= \left(-\frac{1}{4}V(n) - \frac{3}{4}E\right)(V(n) - E) \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{1}{2} + \frac{1}{2}(-1)^n & 0 \end{pmatrix} \end{aligned}$$

and

$$\begin{aligned} Q_1(n) &= \frac{1}{4}(V(n) + E)^2 \\ &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -\frac{1}{2} + \frac{1}{2}(-1)^{n+1} & 1 \end{pmatrix}. \end{aligned}$$

In particular, if $n = 0$, then we get

$$Q_{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad Q_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix}.$$

Next, we will calculate two quantities $\gamma_{-1}(w, b(\cdot))$ and $\delta(w, b(\cdot))$. We have that

$$\begin{aligned} Z_{-1}(V(0)) &= -\frac{1}{2}E - \frac{1}{2^2}(V(0) + E) \\ &= -\frac{1}{2}E - \frac{1}{4}\begin{pmatrix} 0 & -2 & 0 \\ 0 & 0 & 0 \\ 0 & -2 & 2 \end{pmatrix} \\ &= -\frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix} \end{aligned}$$

and that

$$\begin{aligned} S_{2,-1}(b(\cdot)) &= U(2,1)Q_{-1}(1)b(0) + U(2,2)Q_{-1}(2)b(1) \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} b(0) + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} b(1) \\ &= \begin{pmatrix} -2a + 2b \\ 2a + 4b \\ 2a + 4b \end{pmatrix}. \end{aligned}$$

Therefore we obtain

$$\begin{aligned} \gamma_{-1}(w, b(\cdot)) &= Q_{-1}w + Z_{-1}(V(0), b(\cdot)) = Q_{-1}w + Z_{-1}(V(0))S_{2,-1}(b(\cdot)) \\ &= Q_{-1}w - \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 2 \end{pmatrix}\begin{pmatrix} -2a + 2b \\ 2a + 4b \\ 2a + 4b \end{pmatrix} \\ &= Q_{-1}w - \begin{pmatrix} -2a - b \\ a + 2b \\ a + 2b \end{pmatrix}. \end{aligned}$$

Moreover, we get

$$\begin{aligned} S_{2,1}(b(\cdot)) &= U(2,1)Q_1(1)b(0) + U(2,2)Q_1(2)b(1) \\ &= \begin{pmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} b(0) + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & -1 & 1 \end{pmatrix} b(1) \\ &= \begin{pmatrix} 0 \\ 0 \\ -2a - 2b \end{pmatrix}, \end{aligned}$$

from which it follows that

$$\begin{aligned} \delta(w, b(\cdot)) &= (V(0) - E)Q_1 w + S_{2,1}(b(\cdot)) \\ &= -2 \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{pmatrix} Q_1 w + \begin{pmatrix} 0 \\ 0 \\ -2a - 2b \end{pmatrix}. \end{aligned}$$

Finally, we will check the condition 2) in Lemma 8, that is, $S_{2,1}(b(\cdot)) \in (V(0) - E)G_{V(0)}(1)$. The space $G_{V(0)}(1)$ of solutions of the equation $(V(0) - E)x = 0$ is given by

$$G_{V(0)}(1) = span \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Thus we see that

$$\begin{pmatrix} 0 \\ 0 \\ -2a - 2b \end{pmatrix} \in (V(0) - E)G_{V(0)}(1) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Longleftrightarrow a + b = 0.$$

If $a + b = 0$, then

$$\delta(w, b(\cdot)) = 0 \Longleftrightarrow Q_1 w \in span \left\{ \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\}.$$

Moreover, we have

$$\gamma_{-1}(w, b(\cdot)) = 0 \Longleftrightarrow Q_{-1} w = \begin{pmatrix} -2a - b \\ a + 2b \\ a + 2b \end{pmatrix}.$$

Applying Theorem 5, we obtain the following result.

Proposition 1. *Assume that* $a+b=0$. *Then the following statements are equivalent:*

1) $x(n;w,b(\cdot))$ *is a* 2*-periodic solution of the equation* (24).

2)

$$w \in span\left\{\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} + b\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

We will check the condition 2) in Theorem 4, that is, $(V(0)+E)\gamma_{-1}(w,b(\cdot))=0$. By an easy calculation, we have

$$\gamma_{-1}(w,b(\cdot)) = Q_{-1}w - \begin{pmatrix}-2a-b\\a+2b\\a+2b\end{pmatrix} \in span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\},$$

from which it follows that

$$Q_{-1}w \in span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix}\right\} + \begin{pmatrix}-2a-b\\a+2b\\a+2b\end{pmatrix}.$$

Applying Theorem 4, we obtain the following result.

Proposition 2. *Assume that* $a+b=0$. *Then the following statements are equivalent:*

1) $x(n;w,b(\cdot))$ *is a bounded solution of the equation* (24).

2)

$$w \in span\left\{\begin{pmatrix}1\\0\\0\end{pmatrix},\begin{pmatrix}0\\0\\1\end{pmatrix}\right\} + b\begin{pmatrix}1\\1\\1\end{pmatrix}.$$

Acknowledgement

The authors would like to thank the referee for careful reading of the manuscript and comments.

References

[1] S. N. Elaydi, "An Introduction to Difference equations", Springer-Varlag, New York 1996.

[2] J. Kato, T. Naito and J. S. Shin, Bounded solutions and periodic solutions to linear differential equations in Banach spaces, Proceeding in DEAA, Vietnam, Vietnam J. of Math. **30** (2002), 561–575.

[3] J. Kato, T. Naito and J. S. Shin, A characterization of solutions in linear differential equations with periodic forcing functions, J. Difference Equ. Appl. **11** (2005), 1–19.
[4] J. L. Massera, The existence of periodic solutions of systems of differential equations, Duke Math. J. **17** (1950), 457–475.
[5] K. S. Miller, "An Introduction to the Calculus of Finite Differences and Difference Equations", Dover Publications, New York, 1966.
[6] T. Naito and J. S. Shin, On periodicizing functions, Bull. Korean Math. Soc. **43** (2006), 253–263.
[7] T. Naito and J. S. Shin, Representations of solutions, translation formulae and asymptotic behavior in discrete linear systems and in periodic continuous linear systems, submitted.
[8] T. Naito, P. H. A. Ngoc and J. S. Shin, Representations and asymptotic behavior of solutions to periodic linear difference equations, to appear in Funkcial. Ekvac.

Toshiki Naito
Department of Mathematics
The University of Electro-Communications
Chofu, Tokyo 182-8585 *Japan*
E-mail: naito@e-one.uec.ac.jp

Pham Huu Anh Ngoc
Department of Mathematics
The University of Electro-Communications
Chofu, Tokyo 182-8585 *Japan*
E-mail: phanhngoc@yahoo.com

Jong Son Shin
Department of Mathematics
The University of Electro-Communications
Chofu, Tokyo 182-8585 *Japan*
E-mail: shinjs@jcom.home.ne.jp