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Renormalized solutions of Dirichlet problems for quasilinear elliptic equations with general measure data

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Abstract. We consider quasilinear elliptic equations with lower order term and general measure data. We define renormalized solutions of Dirichlet problems and show the existence of such solutions. We also give uniqueness in some special cases.

Introduction

In this paper, we consider Dirichlet problems for quasilinear elliptic equations with measure data:

$$(\mathrm{E}_\nu) \qquad -\mathrm{div}\,\mathscr{A}(x,\nabla u) + \mathscr{B}(x,u) = \nu$$

on a bounded domain G in the N-space $\mathbf{R}^N$ $(N \geq 2)$, where $\mathscr{A}$ and $\mathscr{B}$ satisfy weighted structure conditions with $p > 1$ as in [8, 9, 10] and ν is a finite signed Radon measure on G.

Existence and uniqueness of solutions with vanishing boundary values for such equations (with structure conditions without weight) have been discussed by many people; [1], [2], [7], [3], [4] and others. These works except [4] treat the case where ν is absolutely continuous with respect to the p-capacity and give uniqueness results by considering "entropy solutions". In [4], the case ν is general (i.e., the case ν is not necessarily absolutely continuous with respect to the p-capacity) is treated and so-called "renormalized" solutions are discussed.

The purpose of this paper is to extend most of the results in [4] in the following three directions:

- We consider equations with the lower order term $\mathscr{B}(x,u)$, while in [4] only the case $\mathscr{B} = 0$ is discussed;
- We consider a weight w, which is p-admissible in the sense of [6], in the structure conditions for $\mathscr{A}$ and $\mathscr{B}$;
- We consider non-vanishing boundary conditions.

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There are five kinds of formulation for the definition of renormalized solutions in [4], which are shown to be equivalent to each other. We adopt one of these formulations with slight modification. With our definition, it becomes clear that renormalized solutions are entropy solutions, so that the uniqueness of entropy solution would immediately imply the uniqueness of renormalized solution.

Because we consider a weight w, our discussions are forced to be based on the weighted Sobolev spaces $H^{1,p}(G;\mu)$ and $H_0^{1,p}(G;\mu)$, where $d\mu = w\,dx$, while the theory in [4] is based on the ordinary Sobolev spaces $W^{1,p}(G)$ and $W_0^{1,p}(G)$.

Boundary conditions will be given by a function $\theta \in H^{1,p}(G;\mu)$. We regard that θ_1 and θ_2 determine the same boundary condition if $\theta_1 - \theta_2 \in H_0^{1,p}(G;\mu)$.

We shall prove the existence of a renormalized solution of (E_ν) with boundary data θ for a general finite signed measure ν. The uniqueness can be shown only in the case ν is absolutely continuous with respect to the (p,μ)-capacity and in the linear case for general finite signed measure.

1. Preliminaries

Throughout this paper, let G be a bounded open set in $\mathbf{R}^N$ $(N \geq 2)$ and we consider a quasi-linear elliptic differential operator

$$Lu = -\mathrm{div}\,\mathscr{A}(x,\nabla u) + \mathscr{B}(x,u)$$

on G. Here, $\mathscr{A} : G \times \mathbf{R}^N \to \mathbf{R}^N$ and $\mathscr{B} : G \times \mathbf{R} \to \mathbf{R}$ satisfy the following conditions for a fixed $1 < p < \infty$ and a *weight* w which is *p-admissible* in the sense of [6]:

(A.1) $x \mapsto \mathscr{A}(x,\xi)$ is measurable on G for every $\xi \in \mathbf{R}^N$ and $\xi \mapsto \mathscr{A}(x,\xi)$ is continuous for a.e. $x \in G$;

(A.2) $\mathscr{A}(x,\xi)\cdot\xi \geq \alpha_1 w(x)|\xi|^p$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in G$ with a constant $\alpha_1 > 0$;

(A.3) $|\mathscr{A}(x,\xi)| \leq \alpha_2 w(x)|\xi|^{p-1}$ for all $\xi \in \mathbf{R}^N$ and a.e. $x \in G$ with a constant $\alpha_2 > 0$;

(A.4) $(\mathscr{A}(x,\xi_1) - \mathscr{A}(x,\xi_2))\cdot(\xi_1 - \xi_2) > 0$ whenever $\xi_1, \xi_2 \in \mathbf{R}^N$, $\xi_1 \neq \xi_2$, for a.e. $x \in G$;

(B.1) $x \mapsto \mathscr{B}(x,t)$ is measurable on G for every $t \in \mathbf{R}$ and $t \mapsto \mathscr{B}(x,t)$ is continuous for a.e. $x \in G$;

(B.2) $|\mathscr{B}(x,t)| \leq \alpha_3 w(x)(|t|^{p-1} + 1)$ for all $t \in \mathbf{R}$ and a.e. $x \in G$ with a constant $\alpha_3 \geq 0$;

(B.3) $t \mapsto \mathscr{B}(x,t)$ is nondecreasing on $\mathbf{R}$ for a.e. $x \in G$.

For the nonnegative measure $\mu : d\mu(x) = w(x)dx$, we consider the weighted Sobolev spaces $H^{1,p}(G;\mu)$, $H_0^{1,p}(G;\mu)$ and $H_{\mathrm{loc}}^{1,p}(G;\mu)$ (see [6] for details). For the notion of (p,μ)-capacity $\mathrm{cap}_{p,\mu}$ and the notion of (p,μ)-quasicontinuity of functions, we refer to [6, Chap. 2 and Chap. 4] (also see [12]). Note that every element in $H^{1,p}_{\mathrm{loc}}(G;\mu)$ has a (p,μ)-quasicontinuous representative ([6, Theorem 4.4]), and so *we shall always assume that functions in $H^{1,p}_{\mathrm{loc}}(G;\mu)$ are (p,μ)-quasicontinuous.*

We shall use the following cut-off functions $T_k : \mathbf{R} \to \mathbf{R}$ for $k > 0$:

$$T_k(t) = \max(\min(t,k), -k).$$

We denote by $Y^p(G;\mu)$ the set of measurable functions u on G such that $|u(x)| < \infty$ for (p,μ)-q.e. $x \in G$ and $T_k(u) \in H^{1,p}_{\mathrm{loc}}(G;\mu)$ for all $k > 0$. Since $\nabla T_{k'}(u) = \nabla T_k(u)$ a.e. on $\{|u| < k\}$ whenever $k' > k$, $Du = \lim_{k\to\infty} \nabla T_k(u)$ is well defined a.e. and measurable in G for $u \in Y^p(G;\mu)$. Obviously, $H^{1,p}_{\mathrm{loc}}(G;\mu) \subset Y^p(G;\mu)$ and $Du = \nabla u$ a.e. for $u \in H^{1,p}_{\mathrm{loc}}(G;\mu)$. $Y^p(G;\mu)$ is not a linear space, but if u, v, $u+v$ are all in $Y^p(G;\mu)$, then $D(u+v) = Du + Dv$ a.e.

Let $\tilde{Y}^p(G;\mu) = Y^p(G;\mu) + H^{1,p}_{\mathrm{loc}}(G;\mu)$. For $u = v + \theta \in \tilde{Y}^p(G;\mu)$ with $v \in Y^p(G;\mu)$ and $\theta \in H^{1,p}_{\mathrm{loc}}(G;\mu)$, we define $Du = Dv + \nabla\theta$. Then Du is defined a.e. independent of the expression $u = v + \theta$, since if $v_1 + \theta_1 = v_2 + \theta_2$ with $v_j \in Y^p(G;\mu)$ and $\theta_j \in H^{1,p}_{\mathrm{loc}}(G;\mu)$, then $Dv_1 = D(v_2 + \theta_2 - \theta_1) = Dv_2 + \nabla(\theta_2 - \theta_1) = (Dv_2 + \nabla\theta_2) - \nabla\theta_1$ a.e.

Lemma 1.1. *If v is a measurable function on G such that $T_k(v) \in H^{1,p}_0(G;\mu)$ for all $k > 0$ and*

$$\lim_{k\to\infty} \frac{1}{k^p} \int_G |\nabla T_k(v)|^p d\mu = 0,$$

then $|v| < \infty$ (p,μ)-q.e. and v is (p,μ)-quasicontinuous in G.

Proof. Let $E_k = \{x \in G \mid |v(x)| \geq k\}$ for $k > 0$. Then, since $|T_k(v)| \in H^{1,p}_0(G;\mu)$ and $|T_k(v)| = k$ on E_k,

$$\frac{1}{k^p} \int_G |\nabla T_k(v)|^p d\mu \geq \mathrm{cap}_{p,\mu}(E_k; G)$$

(cf. [6, Corollary 4.13]; see also [12, p. 11]). Hence, by hypothesis, $\mathrm{cap}_{p,\mu}(E_k; G) \to 0$ $(k \to \infty)$, so that $\mathrm{cap}_{p,\mu}(\bigcap_{k>0} E_k; G) = 0$. This means that $|v| < \infty$ (p,μ)-q.e. in G.

Since $T_k(v)$ is (p,μ)-quasicontinuous in G and $T_k(v) = v$ on $G \setminus E_k$, v is (p,μ)-quasicontinuous in G.

Given a signed Radon measure ν on G, a function $u \in \tilde{Y}^p(G;\mu) \cap L^{p-1}_{\mathrm{loc}}(G;\mu)$ is called a solution of the equation

$$(\mathrm{E}_\nu) \qquad Lu = -\mathrm{div}\, \mathscr{A}(x, \nabla u) + \mathscr{B}(x,u) = \nu$$

in G if $|Du| \in L^{p-1}_{\mathrm{loc}}(G;\mu)$ and

$$\int_G \mathscr{A}(x, Du) \cdot \nabla\varphi \, dx + \int_G \mathscr{B}(x, u)\varphi \, dx = \int_G \varphi \, d\nu \tag{1.1}$$

for all $\varphi \in C_0^\infty(G)$. Note that $\mathscr{A}(x, Du) \in L^1_{\mathrm{loc}}(G; dx)$ by (A.3) and $\mathscr{B}(x, u) \in L^1_{\mathrm{loc}}(G; dx)$ by (B.2), so that the left hand side of (1.1) is well defined.

Recall ([8], etc.) that, for an open set $U \subset G$, $u \in H^{1,p}_{\mathrm{loc}}(G;\mu)$ is said to be $(\mathscr{A}, \mathscr{B})$-harmonic in U if it is a continuous solution of $Lu = 0$ in U. By [8, Theorem 1.1], if u is a solution of $Lu = \nu$ in G and if its restriction to $G \backslash (\mathrm{spt}|\nu|)$ belongs to $H^{1,p}_{\mathrm{loc}}(G \backslash (\mathrm{spt}|\nu|);\mu)$, then u is equal to an $(\mathscr{A}, \mathscr{B})$-harmonic function a.e. in $G \backslash (\mathrm{spt}|\nu|)$. (Here, $\mathrm{spt}|\nu|$ means the support of $|\nu|$.)

A nonnegative measure λ on G is said to be absolutely continuous with respect to the (p, μ)-capacity, if $\lambda(E) = 0$ for every Borel set $E \subset G$ whose (p, μ)-capacity is zero. We shall denote this fact by $\lambda \ll \mathrm{cap}_{p,\mu}$. Note that if $\lambda \in (H_0^{1,p}(G;\mu))^*$, then $\lambda \ll \mathrm{cap}_{p,\mu}$ (cf. [12, Lemma 2.4]).

A nonnegative measure λ on G can be decomposed as $\lambda = \lambda_a + \lambda_s$, where $\lambda_a \ll \mathrm{cap}_{p,\mu}$ and $\lambda_s = \chi_{E_s}\lambda$ with a Borel set $E_s \subset G$ such that $\mathrm{cap}_{p,\mu}(E_s) = 0$ (see [5, Lemma 2.1]). We shall call λ_a the absolutely continuous part of λ and λ_s the singular part of λ (with respect to the (p, μ)-capacity).

Lemma 1.2. *Let ν be a finite signed measure on G and let $u \in H^{1,p}(G;\mu)$ be a solution of the equation $Lu = \nu$. If $|\nu| \in (H_0^{1,p}(G;\mu))^*$, then* (1.1) *holds for all $\varphi \in H_0^{1,p}(G;\mu)$. If $|\nu| \ll \mathrm{cap}_{p,\mu}$, then* (1.1) *holds for all $\varphi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$.*

Proof. If $\varphi_n \in C_0^\infty(G)$ and $\varphi_n \to \varphi$ in $H^{1,p}(G;\mu)$ then, by (A.3) and (B.2),

$$\int_G \mathscr{A}(x, \nabla u) \cdot \nabla\varphi_n \, dx + \int_G \mathscr{B}(x, u)\varphi_n \, dx \to \int_G \mathscr{A}(x, \nabla u) \cdot \nabla\varphi \, dx + \int_G \mathscr{B}(x, u)\varphi \, dx.$$

If $|\nu| \in (H_0^{1,p}(G;\mu))^*$, then $\int_G \varphi_n \, d\nu = \nu(\varphi_n) \to \nu(\varphi) = \int_G \varphi \, d\nu$ (cf. [12, Lemma 2.5]). Hence (1.1) holds.

In case $|\nu| \ll \mathrm{cap}_{p,\mu}$, we take $\varphi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$. Then we can choose $\{\varphi_n\}$ to be uniformly bounded. We may also assume that $\varphi_n \to \varphi$ (p, μ)-quasieverywhere. Thus by Lebesgue's convergence theorem, $\int_G \varphi_n \, d\nu \to \int_G \varphi \, d\nu$, since $\varphi_n \to \varphi$ $|\nu|$-a.e. in G. Thus, (1.1) holds for $\varphi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$.

2. The case $\nu \in (H_0^{1,p}(G;\mu))^*$

By modifying the proof of [12, Corollary 2.7], we obtain

Theorem 2.1. *Let $\theta \in H^{1,p}(G;\mu)$ and $\nu \in (H_0^{1,p}(G;\mu))^*$. Then there exists a unique $u \in H^{1,p}(G;\mu)$ such that $Lu = \nu$ in G and $u - \theta \in H_0^{1,p}(G;\mu)$.*

Proof. The uniqueness follows from (A.4) and (B.3). Note that if u_1 and u_2 are two solutions, then $u_1 - u_2 \in H_0^{1,p}(G;\mu)$.

In order to show the existence, let $X = H_0^{1,p}(G;\mu)$ and consider $Q : X \to X^*$ defined by

$$(Qu, v) = \int_G \mathscr{A}(x, \nabla(u+\theta)) \cdot \nabla v\, dx + \int_G \mathscr{B}(x, u+\theta) v\, dx.$$

Since

$$\begin{aligned}
|(Qu, v)| &\le \alpha_2 \int_G |\nabla(u+\theta)|^{p-1} |\nabla v| d\mu + \alpha_3 \int_G (1 + |u+\theta|^{p-1}) |v| d\mu \\
&\le \alpha_2 \left(\int_G |\nabla(u+\theta)|^p d\mu \right)^{1/p'} \left(\int_G |\nabla v|^p d\mu \right)^{1/p} \\
&\quad + \alpha_3 \left\{ \mu(G)^{1/p'} + \left(\int_G |u+\theta|^p d\mu \right)^{1/p'} \right\} \left(\int_G |v|^p d\mu \right)^{1/p}
\end{aligned}$$

($p' = p/(p-1)$), we see that Q defines a bounded operator $X \to X^*$.

Next, let $u_j \in X$ tend to u in X. As in the proof of [12, Corollary 2.7] (also cf. [13, Lemma 3.3]), we see that

$$\int_G \mathscr{A}(x, \nabla(u_j + \theta)) \cdot \nabla v\, dx \to \int_G \mathscr{A}(x, \nabla(u+\theta)) \cdot \nabla v\, dx$$

and

$$\int_G \mathscr{B}(x, u_j + \theta) v\, dx \to \int_G \mathscr{B}(x, u+\theta) v\, dx$$

as $j \to \infty$ for any $v \in X$. Thus, $Q : X \to X^*$ is demicontinuous.

Finally, to show that Q is a coercive mapping, let $v \in X$. Then, noting that $\mathscr{B}(x, v+\theta)v \ge \mathscr{B}(x,\theta)v$ by (B.3), we have

$$\begin{aligned}
(Qv, v) &= \int_G \mathscr{A}(x, \nabla(v+\theta)) \cdot \nabla v\, dx + \int_G \mathscr{B}(x, v+\theta) v\, dx \\
&\ge \int_G \mathscr{A}(x, \nabla(v+\theta)) \cdot \nabla(v+\theta) dx \\
&\quad - \int_G \mathscr{A}(x, \nabla(v+\theta)) \cdot \nabla\theta\, dx + \int_G \mathscr{B}(x,\theta) v\, dx \\
&\ge \alpha_1 \int_G |\nabla(v+\theta)|^p d\mu \\
&\quad - \alpha_2 \int_G |\nabla(v+\theta)|^{p-1} |\nabla\theta| d\mu - \alpha_3 \int_G (1 + |\theta|^{p-1}) |v| d\mu
\end{aligned}$$

$$\geq c_1 \int_G |\nabla v|^p d\mu - c_2 \int_G |\nabla \theta|^p d\mu$$
$$- c_3 \left(\int_G (1 + |\theta|^p) d\mu\right)^{1/p'} \left(\int_G |v|^p d\mu\right)^{1/p},$$

where c_1, c_2 and c_3 are positive constants depending only on p, α_1, α_2 and α_3; we used Young's and Hölder's inequalities to derive the last inequality. Using this inequality and the Poincaré inequality, we can see that $(Qv, v)/\|v\|_X \to \infty$ as $\|v\|_X \to \infty$, namely, Q is coercive.

Hence, by a general result on nonlinear operators (see [12, Theorem 2.6]), we conclude that there exists $\tilde{u} \in X$ such that $Q\tilde{u} = v$. Then, $u = \tilde{u} + \theta$ is the required solution.

Theorem 2.2. *Let $\theta_1, \theta_2 \in H^{1,p}(G;\mu)$ and $v_1, v_2 \in (H_0^{1,p}(G;\mu))^*$. Let u_j, $j = 1,2$ be the solutions (in $H^{1,p}(G;\mu)$) of $Lu = v_j$ with $u_j - \theta_j \in H_0^{1,p}(G;\mu)$. If $\max(\theta_1 - \theta_2, 0) \in H_0^{1,p}(G;\mu)$ and $v_1 \leq v_2$, then $u_1 \leq u_2$ (p,μ)-quasieverywhere in G.*

Proof. Let $v = \max(u_1 - u_2, 0)$. Since $u_j - \theta_j \in H_0^{1,p}(G;\mu)$, $j = 1,2$ and $\max(\theta_1 - \theta_2, 0) \in H_0^{1,p}(G;\mu)$, $v \in H_0^{1,p}(G;\mu)$. Obviously, $v \geq 0$. Thus, noting that $u_j \in H^{1,p}(G;\mu)$, we have

$$\int_G \mathscr{A}(x, \nabla u_j) \cdot \nabla v \, dx + \int_G \mathscr{B}(x, u_j) v \, dx = v_j(v), \qquad j = 1,2$$

and $v_1(v) \leq v_2(v)$. Hence

$$\int_G (\mathscr{A}(x, \nabla u_1) - \mathscr{A}(x, \nabla u_2)) \cdot \nabla v \, dx + \int_G (\mathscr{B}(x, u_1) - \mathscr{B}(x, u_2)) v \, dx \leq 0$$

or

$$\int_{\{u_1 > u_2\}} (\mathscr{A}(x, \nabla u_1) - \mathscr{A}(x, \nabla u_2)) \cdot (\nabla u_1 - \nabla u_2) dx$$
$$+ \int_{\{u_1 > u_2\}} (\mathscr{B}(x, u_1) - \mathscr{B}(x, u_2))(u_1 - u_2) dx \leq 0.$$

By (A.4) and (B.3), it follows that $\nabla u_1 = \nabla u_2$ a.e. in G, so that $\nabla v = 0$ a.e. in G. Since $v \in H_0^{1,p}(G;\mu)$, this implies that $v = 0$ a.e. in G, i.e., $u_1 \leq u_2$ a.e. in G. Then $u_1 \leq u_2$ (p,μ)-quasieverywhere in G by [6, Theorem 4.12].

Proposition 2.1. *Let v be a finite signed measure on G such that $|v| \in (H_0^{1,p}(G;\mu))^*$ and $\theta \in H^{1,p}(G;\mu)$. If $u \in H^{1,p}(G;\mu)$ is the solution of $Lu = v$ in G such that $u - \theta \in H_0^{1,p}(G;\mu)$, then for $0 \leq a < b < \infty$*

$$\frac{\alpha_1}{p}\int_{\{a\le|u-\theta|<b\}}|\nabla u|^p d\mu + (b-a)\int_{\{|u-\theta|\ge b\}}|\mathscr{B}(x,u)-\mathscr{B}(x,\theta)|dx$$
$$\le \left(\frac{\alpha_2}{\alpha_1}\right)^p\frac{\alpha_1}{p}\int_{\{a\le|u-\theta|<b\}}|\nabla\theta|^p d\mu$$
$$+ (b-a)\left\{\int_{\{|u-\theta|\ge a\}}d|\nu| + \int_{\{|u-\theta|\ge a\}}|\mathscr{B}(x,\theta)|dx\right\}.$$

Proof. Given $0\le a<b<\infty$, let $l(t)=T_{b-a}(t-T_a(t))$. Then, $l(u-\theta)\in H_0^{1,p}(G;\mu)$. Since $u\in H^{1,p}(G;\mu)$, (1.1) holds with $\varphi=l(u-\theta)$ by Lemma 1.2. Thus we have

$$\int_{\{a\le|u-\theta|<b\}}\mathscr{A}(x,\nabla u)\cdot\nabla(u-\theta)dx + \int_{\{|u-\theta|>a\}}\mathscr{B}(x,u)l(u-\theta)dx$$
$$= \int_{\{|u-\theta|>a\}}l(u-\theta)d\nu.$$

Since $(\mathscr{B}(x,u)-\mathscr{B}(x,\theta))l(u-\theta)\ge 0$ and $|l|\le b-a$, it follows that

$$\alpha_1\int_{\{a\le|u-\theta|<b\}}|\nabla u|^p d\mu + (b-a)\int_{\{|u-\theta|\ge b\}}|\mathscr{B}(x,u)-\mathscr{B}(x,\theta)|dx$$
$$\le \alpha_2\int_{\{a\le|u-\theta|<b\}}|\nabla u|^{p-1}|\nabla\theta|d\mu$$
$$+ (b-a)\left\{\int_{\{|u-\theta|\ge a\}}|\mathscr{B}(x,\theta)|dx + \int_{\{|u-\theta|\ge a\}}d|\nu|\right\}.$$

By Young's inequality,

$$\alpha_2\int_{\{a\le|u-\theta|<b\}}|\nabla u|^{p-1}|\nabla\theta|d\mu$$
$$\le \frac{\alpha_1}{p'}\int_{\{a\le|u-\theta|<b\}}|\nabla u|^p d\mu + \frac{\alpha_2}{p}\left(\frac{\alpha_2}{\alpha_1}\right)^{p-1}\int_{\{a\le|u-\theta|<b\}}|\nabla\theta|^p d\mu.$$

From these inequalities we obtain the inequality in the proposition.

Corollary 2.1. *Under the same assumptions as in Proposition* 2.1,

$$\int_{\{|u-\theta|<k\}}|\nabla u|^p d\mu \le \left(\frac{\alpha_2}{\alpha_1}\right)^p\int_G|\nabla\theta|^p d\mu + \frac{pk}{\alpha_1}\left\{|\nu|(G) + \int_G|\mathscr{B}(x,\theta)|dx\right\}$$

for any $k>0$.

Let $p^* = \kappa p(p-1)/[\kappa(p-1)+1]$, where $\kappa > 1$ is the constant appearing in the weighted Sobolev inequality ([6, §1.1]). Note that $p-1 < p^* < p$.

PROPOSITION 2.2. *Under the same assumptions as in Proposition* 2.1, *let*

$$H = \int_G |\nabla\theta|^p d\mu \qquad \text{and} \qquad M = |\nu|(G) + \int_G |\mathscr{B}(x,\theta)|dx.$$

Then

$$\mu(\{|u-\theta| \geq k\}) \leq \frac{C_1}{k^{\kappa(p-1)}}\left(\frac{H}{k} + M\right)^{\kappa} \tag{2.1}$$

and

$$\mu(\{|\nabla(u-\theta)| \geq k\}) \leq \frac{C_2}{k^{p^*}}[(k^{p^*-p}H + M) + (k^{p^*-p}H + M)^{\kappa}] \tag{2.2}$$

for $k > 0$, *where* C_1 *and* C_2 *are positive constants depending only on* p, α_1, α_2, diam G, $\mu(G)$ *and constants appearing in the conditions for the weight* w *(including* κ*).*

PROOF. In this proof, c_j denote positive constants depending only on those given for C_1 and C_2 in the proposition. Since $T_k(u-\theta) \in H_0^{1,p}(G;\mu)$ and G is bounded, the Sobolev inequality implies

$$\left(\int_G |T_k(u-\theta)|^{\kappa p} d\mu\right)^{1/\kappa p} \leq c_1 \left(\int_G |\nabla T_k(u-\theta)|^p d\mu\right)^{1/p}.$$

Hence,

$$\begin{aligned}
\mu(\{|u-\theta| \geq k\}) &\leq \frac{1}{k^{\kappa p}}\int_G |T_k(u-\theta)|^{\kappa p} d\mu \\
&\leq \frac{c_2}{k^{\kappa p}}\left(\int_G |\nabla T_k(u-\theta)|^p d\mu\right)^{\kappa} = \frac{c_2}{k^{\kappa p}}\left(\int_{\{|u-\theta|<k\}} |\nabla u - \nabla\theta|^p d\mu\right)^{\kappa} \\
&\leq \frac{c_3}{k^{\kappa p}}\left(\int_{\{|u-\theta|<k\}} |\nabla u|^p d\mu + \int_{\{|u-\theta|<k\}} |\nabla\theta|^p d\mu\right)^{\kappa} \\
&\leq \frac{c_4}{k^{\kappa p}}(H + kM)^{\kappa},
\end{aligned}$$

where we used Corollary 2.1 to derive the last inequality. This implies (2.1).

To prove (2.2), let

$$\Phi(l,m) = \mu(\{|u-\theta| \geq l, |\nabla(u-\theta)|^p \geq m\})$$

for $l \geq 0$ and $m \geq 0$. Since $\Phi(l,m)$ is nonincreasing both in l and m,

$$\Phi(0,m) \le \frac{1}{m}\int_0^m \{\Phi(0,s) - \Phi(l,s)\}ds + \Phi(l,0).$$

Since $\Phi(0,s) - \Phi(l,s) = \mu(\{|u-\theta| < l, |\nabla(u-\theta)|^p \ge s\})$,

$$\int_0^m \{\Phi(0,s) - \Phi(l,s)\}ds \le \int_{\{|u-\theta|<l\}} |\nabla(u-\theta)|^p d\mu$$
$$\le 2^{p-1}\left(\int_{\{|u-\theta|<l\}} |\nabla u|^p d\mu + \int_{\{|u-\theta|<l\}} |\nabla\theta|^p d\mu\right).$$

Hence, noting that $\Phi(l,0) = \mu(\{|u-\theta| \ge l\})$, we have

$$\mu(\{|\nabla(u-\theta)| \ge k\}) = \Phi(0,k^p)$$
$$\le \frac{2^{p-1}}{k^p}\left(\int_{\{|u-\theta|<l\}} |\nabla u|^p d\mu + H\right) + \mu(\{|u-\theta| \ge l\})$$

for $k > 0$. Thus, by Corollary 2.1 and (2.1),

$$\mu(\{|\nabla(u-\theta)| \ge k\}) \le c_5\left\{\frac{l}{k^p}(H/l + M) + \frac{1}{l^{\kappa(p-1)}}(H/l+M)^\kappa\right\}.$$

Now, choose $l > 0$ so that $l/k^p = 1/l^{\kappa(p-1)}$. Then $l = k^{p-p^*}$ and $l^{\kappa(p-1)} = k^{p^*}$. Hence we obtain (2.2).

Corollary 2.2. *Under the same assumptions as in Proposition* 2.1,

$$\int_G |u-\theta|^q d\mu \le \mu(G) + C_q(H+M)^\kappa \tag{2.3}$$

for $0 < q < \kappa(p-1)$ *and*

$$\int_G |\nabla(u-\theta)|^q d\mu \le \mu(G) + C_q'\{(H+M) + (H+M)^\kappa\} \tag{2.4}$$

for $0 < q < p^*$, *where* H *and* M *are as in Proposition* 2.2 *and* C_q, C_q' *are positive constants depending only on* q *and those on which* C_1 *and* C_2 *in Proposition* 2.2 *depend.*

Proof. Let $0 < q < \kappa(p-1)$. By using (2.1), we have

$$\int_G |u-\theta|^q d\mu \le \int_{\{|u-\theta|\le 1\}} d\mu + \int_1^\infty \mu(\{|u-\theta|^q > t\})dt$$
$$\le \mu(G) + C_1(H+M)^\kappa \int_1^\infty t^{-\kappa(p-1)/q}\, dt,$$

which shows (2.3). Similarly, we obtain (2.4) from (2.2).

3. Convergence results

In this section, we first prove the following theorem:

Theorem 3.1. *Let $\theta \in H^{1,p}(G;\mu)$ be fixed. Let $\{\nu_n\}$ be a sequence of finite signed measures on G such that $|\nu_n| \in (H_0^{1,p}(G;\mu))^*$ for each n and $\sup_n|\nu_n|(G) < \infty$. Let u_n be the solution of $Lu = \nu_n$ such that $u_n - \theta \in H_0^{1,p}(G;\mu)$ for each n. Then there exist a subsequence $\{u_{n_j}\}$ and a function $u \in \tilde{Y}^p(G;\mu)$ such that*

(1) $u_{n_j} \to u$ *a.e. as well as in the measure μ as $j \to \infty$,*
(2) $\nabla u_{n_j} \to Du$ *a.e. as well as in the measure μ as $j \to \infty$,*
(3) $T_k(u - \theta) \in H_0^{1,p}(G;\mu)$ *for all $k > 0$,*
(4) $u \in L^q(G;\mu)$ *for $q > 0$ satisfying $q < \kappa(p-1)$ and $q \le p$,*
(5) $|Du| \in L^r(G;\mu)$ *for $0 < r < p^*$.*

Before proving this theorem, we prepare two lemmas.

Lemma 3.1 ([12, Theorem 2.14]). *Let $\{u_n\}$ be a sequence of functions in $H_0^{1,p}(G;\mu)$ such that $\{\int_G |\nabla u_n|^p d\mu\}$ is bounded. Then there are a subsequence $\{u_{n_j}\}$ and $u \in H_0^{1,p}(G;\mu)$ such that $u_{n_j} \to u$ in $L^p(G;\mu)$.*

Lemma 3.2. *Let $\{\nu_n\}$ be a sequence of signed Radon measures in G such that $|\nu_n| \in (H_0^{1,p}(G;\mu))^*$ and $\sup_n|\nu_n|(G) < \infty$. Let $u_n \in H^{1,p}(G;\mu)$ be the solution of $Lu = \nu_n$ such that $u_n - \theta \in H_0^{1,p}(G;\mu)$ for each n. If $\{u_n\}$ converges in μ, i.e., $\mu(\{|u_n - u_m| > \lambda\}) \to 0$ as $n, m \to \infty$ for all $\lambda > 0$, then $\{\nabla u_n\}$ also converges in μ.*

Proof. Let $\mathscr{A}_x(\xi,\eta) = (\mathscr{A}(x,\xi) - \mathscr{A}(x,\eta)) \cdot (\xi - \eta)$. First, we show

$$\int_{\{|u_n - u_m| \le a\}} \mathscr{A}_x(\nabla u_n, \nabla u_m)dx \le 2aM_0 \tag{3.1}$$

for $a > 0$, where $M_0 = \sup_n|\nu_n|(G)$.

Since $T_a(u_n - u_m) \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$ and u_j is a solution of $Lu = \nu_j$,

$$\int_G \mathscr{A}(x,\nabla u_j) \cdot \nabla T_a(u_n - u_m)dx + \int_G \mathscr{B}(x,u_j)T_a(u_n - u_m)dx = \int_G T_a(u_n - u_m)d\nu_j.$$

Subtracting the above equalities for $j = n$ and m, we have

$$\int_{\{|u_n - u_m| \le a\}} \mathscr{A}_x(\nabla u_n, \nabla u_m)dx + \int_G (\mathscr{B}(x,u_n) - \mathscr{B}(x,u_m))T_a(u_n - u_m)dx$$
$$= \int_G T_a(u_n - u_m)d\nu_n - \int_G T_a(u_n - u_m)d\nu_m.$$

Since $\int_G(\mathscr{B}(x,u_n) - \mathscr{B}(x,u_m))T_a(u_n - u_m)dx \ge 0$, it follows that

$$\int_{\{|u_n-u_m|\le a\}} \mathscr{A}_x(\nabla u_n, \nabla u_m)dx \le a(|\nu_n|(G)+|\nu_m|(G)) \le 2aM_0,$$

which shows (3.1).

Given $\varepsilon > 0$, Proposition 2.2 implies that there exists $k > 0$ such that

$$\mu(\{|\nabla(u_j-\theta)| \ge k\}) < \varepsilon \qquad \text{for all } j. \tag{3.2}$$

For this k and a given $\lambda > 0$, we consider the function

$$f_{k,\lambda}(x) = \inf\{\mathscr{A}_x(\xi,\eta); |\xi - \nabla\theta(x)| \le k, |\eta - \nabla\theta(x)| \le k, |\xi-\eta| \ge \lambda\}.$$

This is a measurable function in G. Since $|\nabla\theta(x)| < \infty$, $\mathscr{A}_x(\xi,\eta) > 0$ if $\xi \ne \eta$ and $(\xi,\eta) \mapsto \mathscr{A}_x(\xi,\eta)$ is continuous for a.e. $x \in G$, we see that $f_{k,\lambda}(x) > 0$ for a.e. $x \in G$. Hence, if we set $E^{(1)}_{\delta,k,\lambda} = \{f_{k,\lambda} < \delta w\}$, then there exists $0 < \delta \le \varepsilon$ such that

$$\mu(E^{(1)}_{\delta,k,\lambda}) < \varepsilon. \tag{3.3}$$

Next, let

$$E^{(2)}_\delta(u_n,u_m) = \{|u_n-u_m| \le \delta^2, \mathscr{A}_x(\nabla u_n, \nabla u_m) \ge \delta w\}.$$

Then by (3.1),

$$\begin{aligned} \mu(E^{(2)}_\delta(u_n,u_m)) &= \int_{E^{(2)}_\delta(u_n,u_m)} w\,dx \\ &\le \frac{1}{\delta}\int_{\{|u_n-u_m|\le\delta^2\}} \mathscr{A}_x(\nabla u_n,\nabla u_m)dx \le 2\delta M_0 \le 2\varepsilon M_0. \end{aligned} \tag{3.4}$$

Now, if $|\nabla(u_n-\theta)(x)| \le k$, $|\nabla(u_m-\theta)(x)| \le k$, $|\nabla u_n(x) - \nabla u_m(x)| > \lambda$, $|u_n(x)-u_m(x)| \le \delta^2$ and $x \notin E^{(2)}_\delta(u_n,u_m)$, then $f_{k,\lambda}(x) \le \mathscr{A}_x(\nabla u_n(x), \nabla u_m(x)) < \delta w(x)$, and hence $x \in E^{(1)}_{\delta,k,\lambda}$. This means that

$$\begin{aligned} \{|\nabla u_n - \nabla u_m| > \lambda\} \subset \{|\nabla u_n - \nabla\theta| \ge k\} \cup \{|\nabla u_m - \nabla\theta| \ge k\} \cup E^{(2)}_\delta(u_n,u_m) \cup E^{(1)}_{\delta,k,\lambda} \\ \cup \{|u_n-u_m| > \delta^2\}, \end{aligned}$$

and hence, in view of (3.2), (3.3) and (3.4)

$$\mu(\{|\nabla u_n - \nabla u_m| > \lambda\}) \le (3+2M_0)\varepsilon + \mu(\{|u_n-u_m| > \delta^2\}).$$

This shows the assertion of the lemma.

Proof of Theorem 3.1. For each $k \in \mathbf{N}$, since

$$\int_G |\nabla T_k(u_n-\theta)|^p d\mu = \int_{\{|u_n-\theta|<k\}} |\nabla u_n - \nabla\theta|^p d\mu,$$

Corollary 2.1 implies that $\{\int_G |\nabla T_k(u_n - \theta)|^p d\mu\}_n$ is bounded. Hence by Lemma 3.1, there exist a subsequence $\{u_n^{(k)}\}$ of $\{u_n\}$ and $v_k \in H_0^{1,p}(G;\mu)$ such that $T_k(u_n^{(k)} - \theta) \to v_k$ in $L^p(G;\mu)$. We may also assume that $T_k(u_n^{(k)} - \theta) \to v_k$ a.e. in G and $\{u_n^{(k+1)}\}$ is a subsequnce of $\{u_n^{(k)}\}$ for each $k \in \mathbf{N}$. We denote the diagonal sequence $\{u_n^{(n)}\}$ again by $\{u_n\}$. Then $T_k(u_n - \theta) \to v_k$ in $L^p(G;\mu)$ as well as a.e. in G.

We show that $\{u_n\}$ is convergent in the measure μ. By Proposition 2.2, given $\varepsilon > 0$ there exists $k > 0$ such that $\mu(\{|u_n - \theta| \geq k\}) < \varepsilon$ for all n. Let $\lambda > 0$. Since $\{T_k(u_n - \theta)\}$ is convergent in $L^p(G;\mu)$ and

$$\mu(\{|T_k(u_n - \theta) - T_k(u_m - \theta)| > \lambda\}) \leq \frac{1}{\lambda^p} \int_G |T_k(u_n - \theta) - T_k(u_m - \theta)|^p d\mu,$$

there is n_0 such that

$$\mu(\{|T_k(u_n - \theta) - T_k(u_m - \theta)| > \lambda\}) < \varepsilon$$

for $n, m \geq n_0$. Hence

$$\begin{aligned}\mu(\{|u_n - u_m| > \lambda\}) &\leq \mu(\{|u_n - \theta| \geq k\}) + \mu(\{|u_m - \theta| \geq k\}) \\ &\quad + \mu(\{|T_k(u_n - \theta) - T_k(u_m - \theta)| > \lambda\}) < 3\varepsilon\end{aligned}$$

for $n, m \geq n_0$, that is $\{u_n\}$ is Cauchy in μ.

Thus there exists a measurable function u such that $u_n \to u$ in μ. By taking a subsequence, we may also assume that $u_n \to u$ a.e. in G. By Corollary 2.2, $\{\int_G |u_n - \theta|^q d\mu\}$ is bounded for $q > 0$ with $q < \kappa(p-1)$, so that $u - \theta \in L^q(G;\mu)$ for such q. Thus $u \in L^q(G;\mu)$ if in addition $q \leq p$.

Since $T_k(u - \theta) = v_k$ a.e., $T_j(v_k) = T_j(T_k(u - \theta)) = T_j(u - \theta) = v_j$ a.e. if $j \leq k$. Since $T_j(v_k)$ and v_j are (p, μ)-quasicontinuous, it follows that $T_j(v_k) = v_j$ (p, μ)-q.e. if $j \leq k$ (cf. [6, Theorem 4.14]). Hence we may assume that $T_k(u - \theta) = v_k$ (p, μ)-q.e., and so $T_k(u - \theta) \in H_0^{1,p}(G;\mu)$.

By Lemma 3.2, $\{\nabla u_n\}$ is also convergent in μ, and hence, by taking further subsequence if necessary, we may assume that $\{\nabla u_n\}$ is convergent a.e. in G. Let $\boldsymbol{g} = \lim_{n\to\infty} \nabla u_n$. By Corollary 2.2, we see that $|\boldsymbol{g} - \nabla\theta| \in L^r(G;\mu)$ and hence $|\boldsymbol{g}| \in L^r(G;\mu)$ for $0 < r < p^*$.

We shall show that $\boldsymbol{g} = Du = \nabla\theta + \lim_{k\to\infty} \nabla T_k(u - \theta)$ a.e. in G. First we remark that $\nabla T_k(u_n - \theta) \to \nabla T_k(u - \theta)$ weakly in $L^p(G;\mu)$ by [6, Theorem 1.32] and $\nabla T_k(u - \theta) = Du - \nabla\theta$ a.e. on $\{|u - \theta| < k\}$. Let G_0 be the set of points $x \in G$ for which $|u(x)| < \infty$, $|\theta(x)| < \infty$, $|\boldsymbol{g}(x)| < \infty$, $|\nabla\theta(x)| < \infty$, $u_n(x) \to u(x)$, $\nabla u_n(x) \to \boldsymbol{g}(x)$, $\nabla T_k(u_n - \theta)(x) = \nabla u_n - \nabla\theta(x)$ whenever $k \in \mathbf{N}$, $k > |u_n(x) - \theta(x)|$ for all $n \in \mathbf{N}$ and $\nabla T_k(u - \theta)(x) = Du(x) - \nabla\theta(x)$ whenever $k \in \mathbf{N}$, $k > |u(x) - \theta(x)|$. Then, $\mu(G \backslash G_0) = 0$. For $\delta > 0$, set

$$E_\delta = \{x \in G_0 : |\boldsymbol{g}(x) - Du(x)| > \delta\}.$$

We claim that $\mu(E_\delta) = 0$ for any $\delta > 0$. Supposing the contrary, let $\mu(E_{\delta_0}) > 0$ for some $\delta_0 > 0$. For $k, m \in \mathbf{N}$, set

$$F_{k,m} = \left\{ x \in E_{\delta_0} : \begin{array}{l} (\nabla T_k(u_n - \theta)(x) - \nabla T_k(u - \theta)(x)) \cdot \dfrac{\boldsymbol{g}(x) - Du(x)}{|\boldsymbol{g}(x) - Du(x)|} \\ \geq \delta_0/2 \qquad \text{for all } n \geq m \end{array} \right\}.$$

Let $x \in E_{\delta_0}$ and take $k \in \mathbf{N}$ such that $|u(x) - \theta(x)| < k$. Then there exists $m_x \in \mathbf{N}$ such that $|u_n(x) - \theta(x)| < k$ and $|\boldsymbol{g}(x) - \nabla u_n(x)| < \delta_0/2$ for all $n \geq m_x$. Thus for $n \geq m_x$

$$\begin{aligned} &(\nabla T_k(u_n - \theta)(x) - \nabla T_k(u - \theta)(x)) \cdot (\boldsymbol{g}(x) - Du(x)) \\ &\quad = ((\nabla u_n(x) - \nabla\theta(x)) - (Du(x) - \nabla\theta(x))) \cdot (\boldsymbol{g}(x) - Du(x)) \\ &\quad = (\nabla u_n - Du) \cdot (\boldsymbol{g}(x) - Du(x)) \\ &\quad \geq |\boldsymbol{g}(x) - Du(x)|^2 - |\boldsymbol{g}(x) - \nabla u_n(x)|\,|\boldsymbol{g}(x) - Du(x)| \\ &\quad > \left(\delta_0 - \frac{\delta_0}{2}\right)|\boldsymbol{g}(x) - Du(x)| = \frac{\delta_0}{2}\,|\boldsymbol{g}(x) - Du(x)|, \end{aligned}$$

namely, $x \in F_{k,m_x}$. Therefore, $E_{\delta_0} = \bigcup_{k=1}^{\infty} \bigcup_{m=1}^{\infty} F_{k,m}$. By our assumption that $\mu(E_{\delta_0}) > 0$, there are $k, m \in \mathbf{N}$ such that $\mu(F_{k,m}) > 0$. Then

$$\int_{F_{k,m}} (\nabla T_k(u_n - \theta) - \nabla T_k(u - \theta)) \cdot \frac{\boldsymbol{g} - Du}{|\boldsymbol{g} - Du|}\, d\mu \geq \frac{\delta_0}{2}\mu(F_{k,m}) > 0$$

for all $n \geq m$, which contradicts the weak convergence of $\{\nabla T_k(u_n - \theta)\}$ in $L^p(G;\mu)$. Thus, $\mu(E_\delta) = 0$ for all $\delta > 0$, which means that $\boldsymbol{g} = Du$ a.e., i.e., $\nabla u_n \to Du$ a.e.

The inequality in Corollary 2.1 with u and v replaced by u_n and v_n, respectively, yields the same inequality with ∇u replaced by Du, by Fatou's lemma. Hence, $|u| < \infty$ (p,μ)-q.e. by Lemma 1.1, and hence $u \in \tilde{Y}^p(G;\mu)$.

The next two lemmas will be used in the proof of Theorem 4.1.

Lemma 3.3. (i) *If $\{\boldsymbol{g}_n\}$ is a sequence of $\mathbf{R}^N$-valued measurable functions on G such that $\{\int_G |\boldsymbol{g}_n|^q d\mu\}$ is bounded for some $q > p - 1$ and $\boldsymbol{g}_n \to \boldsymbol{g}$ in the measure μ, then $\mathscr{A}(x, \boldsymbol{g}_n) \to \mathscr{A}(x, \boldsymbol{g})$ in $L^1(G; dx)$.*

(ii) *If $\{f_n\}$ is a sequence of measurable functions on G such that $\{\int_G |f_n|^q d\mu\}$ is bounded for some $q > p - 1$ and $f_n \to f$ in the measure μ, then $\mathscr{B}(x, f_n) \to \mathscr{B}(x, f)$ in $L^1(G; dx)$.*

Proof. We prove only (i). The proof of (ii) is quite similar.

Let $\varepsilon > 0$ be arbitrarily given. For $j = 1, 2, \ldots,$ set

$$E_{\varepsilon,j} = \{x \in G; |\mathscr{A}(x,\xi) - \mathscr{A}(x,\boldsymbol{g}(x))| < \varepsilon w(x) \text{ whenever } |\xi - \boldsymbol{g}(x)| \le 1/j\}.$$

Then $\{E_{\varepsilon,j}\}_j$ is nondecreasing and by the continuity of $\mathscr{A}(x,\xi)$ in ξ, $\mu(G\backslash\bigcup_j E_{\varepsilon,j}) = 0$. Choose j_0 such that $\mu(G\backslash E_{\varepsilon,j_0}) < \varepsilon$ and let $F_n = \{x \in G; |\boldsymbol{g}_n(x) - \boldsymbol{g}(x)| > 1/j_0\}$. Since $\mu(F_n) \to 0$ $(n \to \infty)$ by assumption, there is n_0 such that $\mu(F_n) < \varepsilon$ for $n \ge n_0$. Let

$$D_{n,\varepsilon} = \{x \in G; |\mathscr{A}(x,\boldsymbol{g}_n(x)) - \mathscr{A}(x,\boldsymbol{g}(x))| \ge \varepsilon w(x)\}.$$

Then

$$\int_{G\backslash D_{n,\varepsilon}} |\mathscr{A}(x,\boldsymbol{g}_n(x)) - \mathscr{A}(x,\boldsymbol{g}(x))|dx \le \varepsilon\mu(G) \tag{3.5}$$

for all n. Since $D_{n,\varepsilon} \subset (G\backslash E_{\varepsilon,j_0}) \cup F_n$,

$$\mu(D_{n,\varepsilon}) \le \mu(G\backslash E_{\varepsilon,j_0}) + \mu(F_n) \le \varepsilon + \varepsilon = 2\varepsilon$$

for $n \ge n_0$. Let $\int_G |\boldsymbol{g}_n|^q d\mu \le M$ for all n. Then $\int_G |\boldsymbol{g}|^q d\mu \le M$. Hence, by (A.3) and Hölder's inequality, we have

$$\begin{aligned}\int_{D_{n,\varepsilon}} &|\mathscr{A}(x,\boldsymbol{g}_n(x)) - \mathscr{A}(x,\boldsymbol{g}(x))|dx \\ &\le \alpha_2 \int_{D_{n,\varepsilon}} (|\boldsymbol{g}_n|^{p-1} + |\boldsymbol{g}|^{p-1})d\mu \\ &\le cM^{(p-1)/q}\mu(D_{n,\varepsilon})^{1-(p-1)/q} \le cM^{(p-1)/q}(2\varepsilon)^{1-(p-1)/q} \end{aligned} \tag{3.6}$$

for $n \ge n_0$ with a constant $c = c(p,\alpha_2) > 0$. Since ε is arbitrary, (3.5) and (3.6) show that

$$\lim_{n\to\infty} \int_G |\mathscr{A}(x,\boldsymbol{g}_n(x)) - \mathscr{A}(x,\boldsymbol{g}(x))|dx = 0.$$

Lemma 3.4. *Let ν be a finite signed measure such that $|\nu| \ll \mathrm{cap}_{p,\mu}$. Let $E_n^+ \subset G$ and $E_n^- \subset G$ $(n = 1,2,\dots)$ be Borel sets in G such that $\{E_n^+\}$, $\{E_n^-\}$ are non-decreasing, $\nu^+(G\backslash\bigcup_n E_n^+) = 0$, $\nu^-(G\backslash\bigcup_n E_n^-) = 0$, $\chi_{E_n^+}\nu^+ \in (H_0^{1,p}(G;\mu))^*$ and $\chi_{E_n^-}\nu^- \in (H_0^{1,p}(G;\mu))^*$ for each n. Set $\nu_n = \chi_{E_n^+}\nu^+ - \chi_{E_n^-}\nu^-$. If $\{f_n\}$ is a bounded sequence in $H_0^{1,p}(G;\mu)$ such that $\{\|f_n\|_\infty\}$ is also bounded and $f_n \to f$ a.e. in G, then*

$$\lim_{n\to\infty} \int_G f_n\, d\nu_n = \int_G f\, d\nu.$$

Proof. Let $\|f_n\|_\infty \le M$ for all n $(M > 0)$. Given $\varepsilon > 0$, there is n_0 such that $\nu^+(G\backslash E_{n_0}^+) < \varepsilon/(3M)$. Then for $n \ge n_0$

$$
\begin{aligned}
&\left|\int_G f_n \chi_{E_n^+}\, dv^+ - \int_G f\, dv^+\right| \\
&\quad \le \left|\int_G (f_n - f)\chi_{E_{n_0}^+}\, dv^+\right| + \left|\int_G f_n(\chi_{E_n^+} - \chi_{E_{n_0}^+}) dv^+\right| + \left|\int_G f(1 - \chi_{E_{n_0}^+}) dv^+\right| \\
&\quad \le \left|\int_G (f_n - f)\chi_{E_{n_0}^+}\, dv^+\right| + 2Mv^+(G \backslash E_{n_0}^+) \\
&\quad \le \left|\int_G (f_n - f)\chi_{E_{n_0}^+}\, dv^+\right| + \frac{2}{3}\varepsilon. \qquad (3.7)
\end{aligned}
$$

Since $\{f_n\}$ is a bounded sequence in $H_0^{1,p}(G;\mu)$ and $f_n \to f$ a.e. in G by assumption, $f_n \to f$ weakly in $H_0^{1,p}(G;\mu)$. Since $\chi_{E_{n_0}^+} v^+ \in (H_0^{1,p}(G;\mu))^*$, there is $n_1 \ge n_0$ such that

$$
\left|\int_G (f_n - f)\chi_{E_{n_0}^+}\, dv^+\right| < \frac{\varepsilon}{3}
$$

for $n \ge n_1$. Thus, in view of (3.7), we have

$$
\int_G f_n \chi_{E_n^+}\, dv^+ \to \int_G f\, dv^+ \qquad (n \to \infty).
$$

Similarly, we have

$$
\int_G f_n \chi_{E_n^-}\, dv^- \to \int_G f\, dv^- \qquad (n \to \infty),
$$

and hence the assertion of the lemma follows.

4. Existence of renormalized solutions

Let $\mathscr{L}$ be the family of all Lipschitz continuous functions l on $\mathbf{R}$ such that $l(t) = l(\infty)$ (const.) for $t \ge M$ and $l(t) = l(-\infty)$ (const.) for $t \le -M$ with some $M = M(l) > 0$.

Denote by $\Lambda(G)$ the family of all bounded locally Lipschitz continuous functions φ in G such that $\nabla\varphi$ is also bounded. We know that $\Lambda(G) \subset H^{1,p}(G;\mu)$ (see [6, Lemma 1.11]). Let $\Lambda^+(G) = \{\varphi \in \Lambda(G); \varphi \ge 0\}$.

We also denote by $Y_0^p(G;\mu)$ the set of $v \in Y^p(G;\mu)$ such that $T_k(v) \in H_0^{1,p}(G;\mu)$ for every $k > 0$.

Lemma 4.1. *Let $l \in \mathscr{L}$, $\varphi \in \Lambda(G)$, $v \in Y_0^p(G;\mu)$ and $\psi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$. If either $l(0) = 0$ or $\varphi \in H_0^{1,p}(G;\mu)$, then $l(v+\psi)\varphi \in H_0^{1,p}(G;\mu) \cap L^\infty(G;\mu)$.*

PROOF. Let $l(t) = \text{const.}$ on $(-\infty, -M]$ as well as on $[M, \infty)$, and let $M' = \|\psi\|_\infty$. Then, $l(v+\psi) = l(T_{M+M'}(v)+\psi)$. Since $T_{M+M'}(v)+\psi \in H_0^{1,p}(G;\mu)$, it follows that $l(v+\psi) \in H^{1,p}(G;\mu) \cap L^\infty(G)$ and $l(v+\psi) \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$ if $l(0) = 0$. Thus we obtain the assertion of the lemma.

We denote by $\mathscr{U}(\theta)$ the set of all $u \in \tilde{Y}^p(G;\mu) \cap L^{p-1}(G;\mu)$ such that $u - \theta \in Y_0^p(G;\mu)$ and $|Du| \in L^{p-1}(G;\mu)$. Note that if $\theta_1, \theta_2 \in H^{1,p}(G;\mu)$ and $\theta_1 - \theta_2 \in H_0^{1,p}(G;\mu)$, then $\mathscr{U}(\theta_1) = \mathscr{U}(\theta_2)$.

LEMMA 4.2. *Let $u \in \mathscr{U}(\theta)$, $l \in \mathscr{L}$, $\psi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$ and $\varphi \in \Lambda(G)$. Then $\mathscr{A}(x, Du) \cdot \nabla(l(u-\theta+\psi)\varphi) \in L^1(G;dx)$.*

PROOF. Let $v = u - \theta$ and let M and M' be as in the proof of Lemma 4.1. Then we have

$$\begin{aligned}&\mathscr{A}(x, Du) \cdot \nabla(l(u-\theta+\psi)\varphi)\\ &\qquad = [\mathscr{A}(x, \nabla T_{M+M'}(v) + \nabla\theta) \cdot \nabla l(v+\psi)]\varphi + [\mathscr{A}(x, Du) \cdot \nabla\varphi] l(v+\psi).\end{aligned}$$

Since $T_{M+M'}(v) + \theta \in H^{1,p}(G;\mu)$ and $l(v+\psi) \in H^{1,p}(G;\mu)$ as in the proof of Lemma 4.1, the first term in the right hand side belongs to $L^1(G;dx)$ by (A.3). The last term also belongs to $L^1(G;dx)$ by (A.3) since $|Du| \in L^{p-1}(G;\mu)$ and $\nabla\varphi$, $l(v+\psi)$ are bounded.

Given $\theta \in H^{1,p}(G;\mu)$ and a finite signed measure ν on G, u is called a *renormalized solution* of $Lu = \nu$ with boundary data θ if $u \in \mathscr{U}(\theta)$ and

$$\begin{aligned}&\int_G \mathscr{A}(x, Du) \cdot \nabla(l(u-\theta+\psi)\varphi)dx + \int_G \mathscr{B}(x,u) l(u-\theta+\psi)\varphi\,dx\\ &\qquad = \int_G l(u-\theta+\psi)\varphi\,d\nu_a + l(\infty)\int_G \varphi\,d\nu_s^+ - l(-\infty)\int_G \varphi\,d\nu_s^- \qquad (4.1)\end{aligned}$$

whenever $l \in \mathscr{L}$, $\psi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$, $\varphi \in \Lambda(G)$ and either $l(0) = 0$ or $\varphi \in H_0^{1,p}(G;\mu)$, where $\nu_a = \nu_a^+ - \nu_a^-$.

The first term of the left hand side of (4.1) is well defined by Lemma 4.2. As to the second term, we note that $\mathscr{B}(x,u) \in L^1(G;dx)$ for $u \in L^{p-1}(G;\mu)$ by (B.2). Since ν_a is finite, $|\nu_a| \ll \mathrm{cap}_{p,\mu}$ and the integrand is bounded, we see that the first term of the right hand side of (4.1) is also well defined. The last two terms are well defined since ν_s^+ and ν_s^- are finite measures and φ is bounded continuous.

By Lemma 4.1 and Lemma 1.2, the solution $u \in H^{1,p}(G;\mu)$ given in Theorem 2.1 is a renormalized solution in case $\nu \in (H_0^{1,p}(G;\mu))^*$.

REMARK. The renormalized solution u is an "entropy solution" in the following sense (cf. [1], [2], [7], [12]): u satisfies

$$\int_G \mathscr{A}(x, Du) \cdot \nabla T_k(u - \theta + \psi)dx + \int_G \mathscr{B}(x, u)T_k(u - \theta + \psi)dx$$
$$= \int_G T_k(u - \theta + \psi)dv_a + k(v_s^+ + v_s^-)(G)$$

for any $k > 0$ and $\psi \in H_0^{1,p}(G; \mu) \cap L^\infty(G)$.

In fact, we obtain the above equality by taking $l = T_k$ and $\varphi = 1$ in (4.1).

In order to prove the existence of renormalized solutions, we prepare some lemmas.

Lemma 4.3. *For any* $\xi, \eta \in \mathbf{R}^N$,

$$\frac{1}{2}[\mathscr{A}(x, \xi) \cdot \xi] - c|\eta|^p w(x) \le \mathscr{A}(x, \xi) \cdot (\xi + \eta) \le 2[\mathscr{A}(x, \xi) \cdot \xi] + c|\eta|^p w(x)$$

for a.e. $x \in \Omega$ *with a constant* $c = c(p, \alpha_1, \alpha_2) > 0$.

Proof. By (A.3), Young's inequality and (A.2),

$$|\mathscr{A}(x, \xi) \cdot \eta| \le \alpha_2 |\xi|^{p-1} |\eta| w(x) \le \frac{1}{2}\alpha_1 |\xi|^p w(x) + c|\eta|^p w(x)$$
$$\le \frac{1}{2}[\mathscr{A}(x, \xi) \cdot \xi] + c|\eta|^p w(x)$$

a.e. with $c = c(p, \alpha_1, \alpha_2) > 0$. From this, the required inequalities immediately follow.

Lemma 4.4 ([12, Lemma 2.12]). *Let* λ *be a finite nonnegative measure on* G. *Then there exists a sequence* $\{\lambda_n\}$ *of nonnegative measures on* G *such that* $\lambda_n(G) \le \lambda(G)$ *and* $\lambda_n \in (H_0^{1,p}(G; \mu))^*$ *for every* n, *and*

$$\int_G \varphi \, d\lambda_n \to \int_G \varphi \, d\lambda$$

for all bounded continuous φ.

Lemma 4.5 ([12, Proof of Theorem 6.1 and Remark 6.3 (ii)]). *Let* λ *be a finite nonnegative measure such that* $\lambda \ll \mathrm{cap}_{p,\mu}$. *Then there exists an increasing sequence* $\{E_n\}$ *of Borel sets in* G *such that* $\lambda(G \setminus \bigcup_n E_n) = 0$ *and* $\chi_{E_n}\lambda \in (H_0^{1,p}(G; \mu))^*$ *for each* n.

Now we prove our main theorem: the existence of renormalized solutions.

Theorem 4.1. *Given* $\theta \in H^{1,p}(G; \mu)$ *and a finite signed measure* v, *there exists a renormalized solution of* $Lu = v$ *with boundary data* θ. *Further, we can take* u *to be* $(\mathscr{A}, \mathscr{B})$*-harmonic in* $G \setminus (\mathrm{spt}|v|)$.

PROOF. We have decompositions $\nu^+ = \nu_a^+ + \nu_s^+$ and $\nu^- = \nu_a^- + \nu_s^-$ with $\nu_a^+ \ll \mathrm{cap}_{p,\mu}$, $\nu_a^- \ll \mathrm{cap}_{p,\mu}$, $\nu_s^+ = \chi_{S^+}\nu^+$ and $\nu_s^- = \chi_{S^-}\nu^-$ with Borel sets $S^+ \subset G$, $S^- \subset G$ such that $\mathrm{cap}_{p,\mu}(S^+) = \mathrm{cap}_{p,\mu}(S^-) = 0$ and $S^+ \cap S^- = \varnothing$. Let $\nu_a = \nu_a^+ - \nu_a^-$ and $\nu_s = \nu_s^+ - \nu_s^-$.

Applying Lemma 4.5 to ν_a^+ and ν_a^-, we choose Borel sets $E_n^+ \subset G$ and $E_n^- \subset G$ such that $\{E_n^+\}$, $\{E_n^-\}$ are nondecreasing, $\nu_a^+(G \setminus \bigcup_n E_n^+) = 0$, $\nu_a^-(G \setminus \bigcup_n E_n^-) = 0$, $\chi_{E_n^+}\nu_a^+ \in (H_0^{1,p}(G;\mu))^*$ and $\chi_{E_n^-}\nu_a^- \in (H_0^{1,p}(G;\mu))^*$ for each n. Set $(\nu_a)_n = \chi_{E_n^+}\nu_a^+ - \chi_{E_n^-}\nu_a^-$.

Applying Lemma 4.4 to ν_s^+ and ν_s^-, we choose nonnegative measures $(\nu_s^+)_n$ and $(\nu_s^-)_n$ in $(H_0^{1,p}(G;\mu))^*$ such that $(\nu_s^+)_n(G) \le \nu_s^+(G)$, $(\nu_s^-)_n(G) \le \nu_s^-(G)$ and

$$\int_G \varphi\, d(\nu_s^+)_n \to \int_G \varphi\, d\nu_s^+, \qquad \int_G \varphi\, d(\nu_s^-)_n \to \int_G \varphi\, d\nu_s^- \qquad (n \to \infty)$$

for all bounded continuous φ. Set $(\nu_s)_n = (\nu_s^+)_n - (\nu_s^-)_n$.

For each n, $\nu_n = (\nu_a)_n + (\nu_s)_n$ is a finite signed measure on G and $|\nu_n| \in (H_0^{1,p}(G;\mu))^*$. Hence, by Theorem 2.1, there is a unique solution $u_n \in H^{1,p}(G;\mu)$ of $Lu = \nu_n$ such that $u_n - \theta \in H_0^{1,p}(G;\mu)$ for each n. Then by Theorem 3.1, there is a subsequence, which we denote by $\{u_n\}$ again, such that $u_n \to u$ a.e. in G with $u \in \mathscr{U}(\theta)$, $u_n \to u$ in the measure μ, $\nabla u_n \to Du$ a.e. in G and $\nabla u_n \to Du$ in the measure μ. We shall show that this u is the required function. We divide the proof into several steps.

We set $v_n = u_n - \theta$ and $v = u - \theta$ for simplicity. Note that $v_n \in H_0^{1,p}(G;\mu)$ and $T_k(v) \in H_0^{1,p}(G;\mu)$ for all $k > 0$.

1st step. If $\varphi \in \Lambda^+(G)$, then

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \frac{1}{k} \int_{\{k<v_n<2k\}} [\mathscr{A}(x,\nabla u_n) \cdot \nabla u_n]\varphi\, dx \le 2\int_G \varphi\, d\nu_s^+, \tag{4.2}$$

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_{\{v_n \ge 2k\}} \varphi\, d(\nu_s^-)_n \le \int_G \varphi\, d\nu_s^+, \tag{4.3}$$

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \frac{1}{k} \int_{\{-2k<v_n<-k\}} [\mathscr{A}(x,\nabla u_n) \cdot \nabla u_n]\varphi\, dx \le 2\int_G \varphi\, d\nu_s^-, \tag{4.4}$$

$$\limsup_{k\to\infty} \limsup_{n\to\infty} \int_{\{v_n \le -2k\}} \varphi\, d(\nu_s^+)_n \le \int_G \varphi\, d\nu_s^-. \tag{4.5}$$

Proof of (4.2) and (4.3): Let $l_k(t) = \max(T_k(t-k)/k, 0)$ for $k > 0$. Then $l_k(v_n) \in H_0^{1,p}(G;\mu)$ and $\nabla l_k(v_n) = (1/k)\nabla v_n \chi_{\{k<v_n<2k\}}$ a.e. Since $l_k(v_n)\varphi \in H_0^{1,p}(G;\mu)$,

$$\int_G \mathscr{A}(x,\nabla u_n) \cdot \nabla(l_k(v_n)\varphi) dx + \int_G \mathscr{B}(x,u_n) l_k(v_n)\varphi\, dx = \int_G l_k(v_n)\varphi\, d\nu_n,$$

so that

$$\frac{1}{k}\int_{\{k<v_n<2k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla v_n]\varphi\,dx+\int_G l_k(v_n)\varphi\,d(\nu_a^-)_n+\int_G l_k(v_n)\varphi\,d(\nu_s^-)_n$$
$$=-\int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi]l_k(v_n)dx-\int_G\mathscr{B}(x,u_n)l_k(v_n)\varphi\,dx$$
$$+\int_G l_k(v_n)\varphi\,d(\nu_a^+)_n+\int_G l_k(v_n)\varphi\,d(\nu_s^+)_n.$$

By Lemma 4.3,

$$\mathscr{A}(x,\nabla u_n)\cdot\nabla v_n\geq\frac{1}{2}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]-c|\nabla\theta|^p w$$

a.e. in G with a constant $c=c(p,\alpha_1,\alpha_2)>0$. Also, by (B.3), $\mathscr{B}(x,u_n)l_k(v_n)\geq\mathscr{B}(x,\theta)l_k(v_n)$ a.e. in G. Hence, from the above equality we obtain

$$\frac{1}{2k}\int_{\{k<v_n<2k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\,dx+\int_{\{v_n\geq 2k\}}\varphi\,d(\nu_s^-)_n$$
$$\leq\frac{c}{k}\int_G|\nabla\theta|^p d\mu+\left|\int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi]l_k(v_n)dx\right|$$
$$+\left|\int_G\mathscr{B}(x,\theta)l_k(v_n)\varphi\,dx\right|+\int_G l_k(v_n)\varphi\,d(\nu_a^+)_n+\int_G l_k(v_n)\varphi\,d(\nu_s^+)_n.\quad(4.6)$$

By Corollary 2.2 and Lemma 3.3, we see that $\mathscr{A}(x,\nabla u_n)\to\mathscr{A}(x,Du)$ in $L^1(G;dx)$ as $n\to\infty$. Since $l_k(v_n)\to l_k(v)$ a.e. in G and $\{|\nabla\varphi|\,l_k(v_n)\}_n$ is uniformly bounded,

$$\lim_{n\to\infty}\int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi]l_k(v_n)dx=\int_G[\mathscr{A}(x,Du)\cdot\nabla\varphi]l_k(v)dx.$$

Noting that $l_k(v)\to 0$ a.e. in G, by Lebesgue's convergence theorem we have

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi]l_k(v_n)dx=0.\quad(4.7)$$

Since $\mathscr{B}(x,\theta)\in L^1(G;dx)$,

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_G\mathscr{B}(x,\theta)l_k(v_n)\varphi\,dx=\lim_{k\to\infty}\int_G\mathscr{B}(x,\theta)l_k(v)\varphi\,dx=0.\quad(4.8)$$

Next, $\{l_k(v_n)\varphi\}_n$ is bounded in $H_0^{1,p}(G;\mu)$ by Proposition 2.1. Since it is also uniformly bounded in G and $l_k(v_n)\varphi\to l_k(v)\varphi$ a.e. in G as $n\to\infty$, Lemma 3.4 implies

$$\lim_{n\to\infty}\int_G l_k(v_n)\varphi\, d(v_a^+)_n = \int_G l_k(v)\varphi\, dv_a^+.$$

We know that $v < \infty$ (p,μ)-q.e, so that $v < \infty$ v_a^+-a.e. in G. Hence $l_k(v) \to 0$ $(k \to \infty)$ v_a^+-a.e. in G, so that

$$\lim_{k\to\infty}\lim_{n\to\infty}\int_G l_k(v_n)\varphi\, d(v_a^+)_n = 0. \tag{4.9}$$

Finally,

$$0 \le \int_G l_k(v_n)\varphi\, d(v_s^+)_n \le \int_G \varphi\, d(v_s^+)_n \to \int_G \varphi\, dv_s^+ \qquad (n\to\infty). \tag{4.10}$$

From (4.6), (4.7), (4.8), (4.9) and (4.10), we obtain (4.2) and (4.3).

(4.4) and (4.5) can be similarly proved.

2nd step. If $\varphi \in \Lambda^+(G) \cap H_0^{1,p}(G;\mu)$ and $0 < k < m$, then

$$\begin{aligned}
&\limsup_{n\to\infty}\left\{\frac{1}{2}\int_{\{|v_n|\le k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\, dx + \int_{\{-m<v_n<k\}}(k - T_k(v_n))\varphi\, d(v_s^+)_n\right\}\\
&\quad \le c\int_G |\nabla\theta|^p\varphi\, d\mu + \frac{4k}{m}\limsup_{n\to\infty}\int_{\{-2m\le v_n<-m\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\, dx\\
&\qquad + 2k\left\{\int_{\{|v|\le 2m\}}|\mathscr{A}(x,Du)\cdot\nabla\varphi|dx + \int_{\{|v|\le 2m\}}|\mathscr{B}(x,u)|\varphi\, dx\right.\\
&\qquad \left. + \int_G \varphi\, d(v_a^- + v_s^-)\right\}
\end{aligned} \tag{4.11}$$

and

$$\begin{aligned}
&\limsup_{n\to\infty}\left\{\frac{1}{2}\int_{\{|v_n|\le k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\, dx + \int_{\{-k<v_n<m\}}(k + T_k(v_n))\varphi\, d(v_s^-)_n\right\}\\
&\quad \le c\int_G |\nabla\theta|^p\varphi\, d\mu + \frac{4k}{m}\limsup_{n\to\infty}\int_{\{m\le v_n<2m\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\, dx\\
&\qquad + 2k\left\{\int_{\{|v|\le 2m\}}|\mathscr{A}(x,Du)\cdot\nabla\varphi|dx + \int_{\{|v|\le 2m\}}|\mathscr{B}(x,u)|\varphi\, dx\right.\\
&\qquad \left. + \int_G \varphi\, d(v_a^+ + v_s^+)\right\}
\end{aligned} \tag{4.12}$$

with a constant $c = c(p, \alpha_1, \alpha_2) > 0$.

Proof of (4.11): Let $h_m(t) = 1 - |T_1((t - T_m(t))/m)|$ and $f_n = (k - T_k(v_n))h_m(v_n)\varphi$ with $\varphi \in \Lambda^+(G) \cap H_0^{1,p}(G;\mu)$. Then, $f_n \in H_0^{1,p}(G;\mu)$, so that

$$\int_G \mathscr{A}(x,\nabla u_n)\cdot\nabla f_n\,dx + \int_G \mathscr{B}(x,u_n) f_n\,dx = \int_G f_n\,dv_n. \tag{4.13}$$

Now

$$\nabla f_n = -\nabla T_k(v_n)h_m(v_n)\varphi + (k - T_k(v_n))h_m'(v_n)\nabla v_n\varphi + (k - T_k(v_n))h_m(v_n)\nabla\varphi.$$

By Lemma 4.3, we have

$$\frac{1}{2}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\chi_{\{|v_n|\le k\}} \le [\mathscr{A}(x,\nabla u_n)\cdot\nabla T_k(v_n)] + c|\nabla\theta|^p w \tag{4.14}$$

a.e. with $c = c(p,\alpha_1,\alpha_2) > 0$. Since $(k - T_k(v_n))h_m'(v_n) = (2k/m)\chi_{\{-2m<v_n<-m\}}$ for $m > k > 0$, by Lemma 4.3 again, we have

$$\begin{aligned}
&[\mathscr{A}(x,\nabla u_n)\cdot\nabla v_n](k - T_k(v_n))h_m'(v_n)\\
&\qquad \le \frac{4k}{m}\{[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n] + c|\nabla\theta|^p w\}\chi_{\{-2m<v_n<-m\}}
\end{aligned} \tag{4.15}$$

a.e. Thus, using (4.14), (4.13) and (4.15), we have

$$\begin{aligned}
0 &\le \frac{1}{2}\int_{\{|v_n|\le k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\,dx + \int_{\{-m<v_n<k\}}(k - T_k(v_n))\varphi\,d(v_s^+)_n\\
&= \frac{1}{2}\int_{\{|v_n|\le k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]h_m(v_n)\varphi\,dx + \int_G f_n\,d(v_s^+)_n\\
&\le \int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla T_k(v_n)]h_m(v_n)\varphi\,dx + \int_G f_n\,d(v_s^+)_n + c\int_G|\nabla\theta|^p\varphi\,d\mu\\
&= -\int_G \mathscr{A}(x,\nabla u_n)\cdot\nabla f_n\,dx + \int_G f_n\,d(v_s^+)_n + c\int_G|\nabla\theta|^p\varphi\,d\mu\\
&\quad + \int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla v_n](k - T_k(v_n))h_m'(v_n)\varphi\,dx\\
&\quad + \int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi](k - T_k(v_n))h_m(v_n)dx\\
&\le \int_G \mathscr{B}(x,u_n)f_n\,dx + \int_G f_n\,d[(v_a^-)_n + (v_s^-)_n] + c\int_G|\nabla\theta|^p\varphi\,d\mu\\
&\quad + \frac{4k}{m}\int_{\{-2m<v_n<-m\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\,dx\\
&\quad + \int_G[\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi](k - T_k(v_n))h_m(v_n)dx.
\end{aligned} \tag{4.16}$$

Since $\mathscr{A}(x,\nabla u_n) \to \mathscr{A}(x,Du)$ in $L^1(G;dx)$,

$$(k - T_k(v_n))h_m(v_n) \to (k - T_k(v))h_m(v)$$

a.e. and $\{(k - T_k(v_n))h_m(v_n)\}_n$ is uniformly bounded,

$$\lim_{n\to\infty} \int_G [\mathscr{A}(x, \nabla u_n) \cdot \nabla\varphi](k - T_k(v_n))h_m(v_n)dx$$
$$= \int_G [\mathscr{A}(x, Du) \cdot \nabla\varphi](k - T_k(v))h_m(v)dx$$
$$\leq 2k \int_{\{|v| \leq 2m\}} |\mathscr{A}(x, Du) \cdot \nabla\varphi| dx.$$

Similarly, since $\mathscr{B}(x, u_n) \to \mathscr{B}(x, u)$ in $L^1(G; dx)$ by Corollary 2.2 and Lemma 3.3, we have

$$\lim_{n\to\infty} \int_G \mathscr{B}(x, u_n) f_n \, dx$$
$$= \int_G \mathscr{B}(x, u)(k - T_k(v))h_m(v)\varphi \, dx \leq 2k \int_{\{|v| \leq 2m\}} |\mathscr{B}(x, u)|\varphi \, dx.$$

Finally

$$0 \leq \int_G f_n \, d[(v_a^-)_n + (v_s^-)_n]$$
$$\leq 2k \int_G \varphi \, d[(v_a^-)_n + (v_s^-)_n] \to 2k \int_G \varphi \, d(v_a^- + v_s^-) \qquad (n \to \infty).$$

Hence we obtain (4.11) from (4.16).

Inequality (4.12) is similarly proved.

3rd step. Let $\varphi \in \Lambda^+(G)$ and $k > 0$. If $\{\phi_j\}$ is a nonincreasing sequence of functions in $C_0^\infty(G)$ such that $0 \leq \phi_j \leq 1$ for each j, $\phi_j \to 0$ (p, μ)-q.e. in G and $\int_G |\nabla\phi_j|^p d\mu \to 0$ as $j \to \infty$, then

$$\limsup_{j\to\infty} \limsup_{n\to\infty} \int_{\{|v_n| \leq k\}} [\mathscr{A}(x, \nabla u_n) \cdot \nabla u_n]\varphi\phi_j \, dx$$
$$\leq 20k \min\left\{\int_G \varphi \, dv_s^+, \int_G \varphi \, dv_s^-\right\}; \tag{4.17}$$

$$\limsup_{j\to\infty} \limsup_{n\to\infty} \int_G |\mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n)|\varphi\phi_j \, dx$$
$$\leq 40k \min\left\{\int_G \varphi \, dv_s^+, \int_G \varphi \, dv_s^-\right\}; \tag{4.18}$$

$$\limsup_{m\to\infty}\limsup_{j\to\infty}\limsup_{n\to\infty}\int_{\{-m<v_n<k\}}(k-T_k(v_n))\phi_j\varphi\,d(v_s^+)_n \le 10k\int_G \varphi\,dv_s^-; \quad (4.19)$$

$$\limsup_{m\to\infty}\limsup_{j\to\infty}\limsup_{n\to\infty}\int_{\{-k<v_n<m\}}(k+T_k(v_n))\phi_j\varphi\,d(v_s^-)_n \le 10k\int_G \varphi\,dv_s^+. \quad (4.20)$$

Proof of (4.17), (4.19) and (4.20): Obviously, $\varphi\phi_j \in \Lambda^+(G)\cap H_0^{1,p}(G;\mu)$. Thus (4.11) with $\varphi\phi_j$ in place of φ holds for each j. By Lebesgue's convergence theorem, $\lim_{j\to\infty}\int_G|\nabla\theta|^p\varphi\phi_j\,d\mu = 0$, $\lim_{j\to\infty}\int_G|\mathscr{B}(x,u)|\varphi\phi_j\,dx = 0$ and $\lim_{j\to\infty}\int_G\varphi\phi_j\,d|v_a| = 0$. Using (4.4), we have

$$\limsup_{m\to\infty}\limsup_{j\to\infty}\limsup_{n\to\infty}\frac{1}{m}\int_{\{-2m<v_n<-m\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\phi_j\,dx$$

$$\le \limsup_{m\to\infty}\limsup_{n\to\infty}\frac{1}{m}\int_{\{-2m<v_n<-m\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\,dx \le 2\int_G\varphi\,dv_s^-.$$

Since

$$\mathscr{A}(x,Du)\chi_{\{|v|\le 2m\}} = \mathscr{A}(x,\nabla T_{2m}(v)+\nabla\theta)\chi_{\{|v|\le 2m\}}$$

a.e. in G, $\mathscr{A}(x,Du)\chi_{\{|v|\le 2m\}}w^{-1/p}\in L^{p'}(G;dx)$. By assumption, $\nabla(\varphi\phi_j)w^{1/p}\to 0$ in $L^p(G;dx)$. Hence,

$$\lim_{j\to\infty}\int_{\{|v|\le 2m\}}|\mathscr{A}(x,Du)\cdot\nabla(\varphi\phi_j)|dx = 0.$$

Thus, (4.11) with $\varphi\phi_j$ in place of φ yields

$$\limsup_{m\to\infty}\limsup_{j\to\infty}\limsup_{n\to\infty}\left\{\frac{1}{2}\int_{\{|v_n|\le k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\phi_j\,dx + \int_{\{-m<v_n<k\}}(k-T_k(v_n))\varphi\phi_j\,d(v_s^+)_n\right\} \le 10k\int_G\varphi\,dv_s^-.$$

Similarly, we obtain

$$\limsup_{m\to\infty}\limsup_{j\to\infty}\limsup_{n\to\infty}\left\{\frac{1}{2}\int_{\{|v_n|\le k\}}[\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\phi_j\,dx + \int_{\{-k<v_n<m\}}(k+T_k(v_n))\varphi\phi_j\,d(v_s^-)_n\right\} \le 10k\int_G\varphi\,dv_s^+.$$

These two inequalities imply (4.17), (4.19) and (4.20).

Proof of (4.18): Since

$$|\mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n)| \le 2[\mathscr{A}(x, \nabla u_n) \cdot \nabla u_n]\chi_{\{|v_n| \le k\}} + c|\nabla\theta|^p w$$

a.e. by Lemma 4.3, (4.17) implies (4.18).

4th step. For $k > 0$,

$$\lim_{n\to\infty} \int_{\{v_n<k\}} d(v_s^+)_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} \int_{\{v_n>-k\}} d(v_s^-)_n = 0. \tag{4.21}$$

Proof: Given $\varepsilon > 0$, choose compact sets $K^+ \subset S^+$ and $K^- \subset S^-$ such that $v_s^+(S^+ \backslash K^+) < \varepsilon$ and $v_s^-(S^- \backslash K^-) < \varepsilon$. Choose $\varphi \in C_0^\infty(G)$ such that $\varphi = 1$ on K^+, $\varphi = 0$ on K^- and $0 \le \varphi \le 1$ in G. Since $\mathrm{cap}_{p,\mu}(K^+) = 0$, we can choose a nonincreasing sequence ϕ_j in $C_0^\infty(G)$ such that $\phi_j = 1$ on K^+, $0 \le \phi_j \le 1$ in G and $\int_G |\nabla\phi_j|^p d\mu \to 0$ $(j \to \infty)$. Since $1 - \varphi\phi_j \in \Lambda^+(G)$,

$$0 \le \limsup_{n\to\infty} \int_{\{v_n<k\}} (1 - \varphi\phi_j) d(v_s^+)_n \le \lim_{n\to\infty} \int_G (1 - \varphi\phi_j) d(v_s^+)_n$$

$$= \int_G (1 - \varphi\phi_j) dv_s^+ \le v_s^+(S^+ \backslash K^+) < \varepsilon \tag{4.22}$$

for every j. On the other hand, since $k + 1 - T_{k+1}(v_n) \ge 1$ on $\{v_n \le k\}$,

$$0 \le \int_{\{v_n<k\}} \varphi\phi_j \, d(v_s^+)_n$$

$$\le \int_{\{-m<v_n<k+1\}} (k + 1 - T_{k+1}(v_n))\varphi\phi_j \, d(v_s^+)_n + \int_{\{v_n \le -m\}} \varphi\phi_j \, d(v_s^+)_n \tag{4.23}$$

if $m > k + 1$. By (4.19)

$$\limsup_{m\to\infty} \limsup_{j\to\infty} \limsup_{n\to\infty} \int_{\{-m<v_n<k+1\}} (k + 1 - T_{k+1}(v_n))\varphi\phi_j \, d(v_s^+)_n$$

$$\le 10(k+1) \int_G \varphi \, dv_s^- \le 10(k+1) v_s^-(S^- \backslash K^-) < 10(k+1)\varepsilon. \tag{4.24}$$

By (4.5)

$$0 \le \limsup_{m\to\infty} \limsup_{j\to\infty} \limsup_{n\to\infty} \int_{\{v_n \le -m\}} \varphi\phi_j \, d(v_s^+)_n$$

$$\le \limsup_{m\to\infty} \limsup_{n\to\infty} \int_{\{v_n \le -m\}} \varphi \, d(v_s^+)_n \le \int_G \varphi \, dv_s^- \le v_s^-(S^- \backslash K^-) < \varepsilon. \tag{4.25}$$

From (4.22), (4.23), (4.24) and (4.25), we obtain

$$0 \le \limsup_{n\to\infty} \int_{\{v_n<k\}} d(v_s^+)_n < \limsup_{j\to\infty} \limsup_{n\to\infty} \int_{\{v_n<k\}} \varphi\phi_j \, d(v_s^+)_n + \varepsilon$$

$$< (10k + 12)\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this shows the first equality in (4.21).

The second equality can be similarly shown.

5th step. If $\varphi \in \Lambda^+(G)$ and $k > 0$, then

$$\left| \int_{\{|v| \le k\}} [\mathscr{A}(x, Du) \cdot Dv]\varphi \, dx + \int_G [\mathscr{A}(x, Du) \cdot \nabla\varphi] T_k(v) dx \right.$$
$$\left. + \int_G \mathscr{B}(x, u) T_k(v)\varphi \, dx - \int_G T_k(v)\varphi \, dv_a \right| \le 5k \int_G \varphi \, d(v_s^+ + v_s^-). \quad (4.26)$$

Proof: For $m > 0$ let $h_m(t)$ be as in Step 2. Then $T_k(v)\varphi h_m(v_n) \in H_0^{1,p}(G;\mu)$ and

$$\nabla(T_k(v)\varphi h_m(v_n)) = (Dv)\chi_{\{|v| \le k\}}\varphi h_m(v_n) + (\nabla\varphi)T_k(v)h_m(v_n)$$
$$+ \frac{1}{m}(\nabla v_n)\{\chi_{\{-2m<v_n<-m\}} - \chi_{\{m<v_n<2m\}}\}T_k(v)\varphi$$

a.e. in G. Hence

$$\int_{\{|v| \le k\}} [\mathscr{A}(x, \nabla u_n) \cdot Dv]\varphi h_m(v_n) dx + \int_G [\mathscr{A}(x, \nabla u_n) \cdot \nabla\varphi] T_k(v) h_m(v_n) dx$$
$$+ \frac{1}{m}\left\{\int_{\{-2m<v_n<-m\}} - \int_{\{m<v_n<2m\}}\right\}[\mathscr{A}(x, \nabla u_n) \cdot \nabla v_n] T_k(v)\varphi \, dx$$
$$+ \int_G \mathscr{B}(x, u_n) T_k(v)\varphi h_m(v_n) dx = \int_G T_k(v)\varphi h_m(v_n) dv_n. \quad (4.27)$$

Since

$$|\mathscr{A}(x, \nabla u_n)| h_m(v_n) w^{-1/p} \le \alpha_2 |\nabla u_n|^{p-1} w^{1/p'} \chi_{\{|v_n|<2m\}}$$

a.e., $\{\mathscr{A}(x, \nabla u_n) h_m(v_n) w^{-1/p}\}_n$ is bounded in $L^{p'}(G; dx)$ by Corollary 2.1. Further $\mathscr{A}(x, \nabla u_n) h_m(v_n) \to \mathscr{A}(x, Du) h_m(v)$ a.e. in G. Hence

$$\mathscr{A}(x, \nabla u_n) h_m(v_n) w^{-1/p} \to \mathscr{A}(x, Du) h_m(v) w^{-1/p}$$

weakly in $L^{p'}(G; dx)$. Since $(Dv)\chi_{\{|v| \le k\}}\varphi w^{1/p} = \nabla T_k(v)\varphi w^{1/p} \in L^p(G; dx)$, it follows that

$$\lim_{n\to\infty} \int_{\{|v| \le k\}} [\mathscr{A}(x, \nabla u_n) \cdot Dv]\varphi h_m(v_n) dx = \int_{\{|v| \le k\}} [\mathscr{A}(x, Du) \cdot Dv]\varphi h_m(v) dx. \quad (4.28)$$

Since $\mathscr{A}(x, \nabla u_n) \to \mathscr{A}(x, Du)$ in $L^1(G; dx)$, $h_m(v_n) \to h_m(v)$ a.e. in G and $\{T_k(v) h_m(v_n) |\nabla\varphi|\}_n$ is uniformly bounded,

$$\lim_{n\to\infty} \int_G [\mathscr{A}(x, \nabla u_n) \cdot \nabla\varphi] T_k(v) h_m(v_n) dx = \int_G [\mathscr{A}(x, Du) \cdot \nabla\varphi] T_k(v) h_m(v) dx. \quad (4.29)$$

Similarly, since $\mathscr{B}(x,u_n) \to \mathscr{B}(x,u)$ in $L^1(G;dx)$, $h_m(v_n) \to h_m(v)$ a.e. in G and $\{T_k(v)h_m(v_n)\varphi\}_n$ is uniformly bounded,

$$\lim_{n\to\infty} \int_G \mathscr{B}(x,u_n)T_k(v)\varphi h_m(v_n)dx = \int_G \mathscr{B}(x,u)T_k(v)\varphi h_m(v)dx. \tag{4.30}$$

Also, using Corollary 2.1 again, we see that $\{T_k(v)\varphi h_m(v_n)\}_n$ is bounded in $H_0^{1,p}(G;\mu)$. Since it is uniformly bounded and tends to $T_k(v)\varphi h_m(v)$ a.e. in G, Lemma 3.4 implies that

$$\lim_{n\to\infty} \int_G T_k(v)\varphi h_m(v_n)d(v_a)_n = \int_G T_k(v)\varphi h_m(v)dv_a. \tag{4.31}$$

Note that $0 \le h_m(v) \le 1$ and $h_m(v) \to 1$ a.e. as well as $|v_a|$-a.e. in G as $m \to \infty$. Thus, combining (4.28), (4.29), (4.30) and (4.31), and letting $m \to \infty$, we have

$$\begin{aligned}
\lim_{m\to\infty}\lim_{n\to\infty} \Bigg\{ &\int_{\{|v|\le k\}} [\mathscr{A}(x,\nabla u_n)\cdot Dv]\varphi h_m(v_n)dx \\
&+ \int_G [\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi]T_k(v)h_m(v_n)dx \\
&+ \int_G \mathscr{B}(x,u_n)T_k(v)\varphi h_m(v_n)dx - \int_G T_k(v)\varphi h_m(v_n)d(v_a)_n \Bigg\} \\
= &\int_{\{|v|\le k\}} [\mathscr{A}(x,Du)\cdot Dv]\varphi\, dx + \int_G [\mathscr{A}(x,Du)\cdot\nabla\varphi]T_k(v)dx \\
&+ \int_G \mathscr{B}(x,u)T_k(v)\varphi\, dx - \int_G T_k(v)\varphi\, dv_a.
\end{aligned} \tag{4.32}$$

By Lemma 4.3,

$$\begin{aligned}
&\left| \left\{ \int_{\{-2m<v_n<-m\}} - \int_{\{m<v_n<2m\}} \right\} [\mathscr{A}(x,\nabla u_n)\cdot\nabla v_n]T_k(v)\varphi\, dx \right| \\
&\qquad \le 2k \int_{\{m<|v_n|<2m\}} [\mathscr{A}(x,\nabla u_n)\cdot\nabla u_n]\varphi\, dx + ck\|\varphi\|_\infty \int_G |\nabla\theta|^p d\mu.
\end{aligned}$$

Hence, by (4.2) and (4.4),

$$\begin{aligned}
&\limsup_{m\to\infty}\limsup_{n\to\infty} \frac{1}{m} \left| \left\{ \int_{\{-2m<v_n<-m\}} - \int_{\{m<v_n<2m\}} \right\} [\mathscr{A}(x,\nabla u_n)\cdot\nabla v_n]T_k(v)\varphi\, dx \right| \\
&\qquad \le 4k \int_G \varphi\, d(v_s^+ + v_s^-).
\end{aligned} \tag{4.33}$$

Since $|\int_G T_k(v)\varphi h_m(v_n)d(v_s)_n| \le k\int_G \varphi\, d((v_s^+)_n + (v_s^-)_n) \to k\int_G \varphi\, d(v_s^+ + v_s^-)$ $(n\to\infty)$, we obtain (4.26) from (4.27), (4.32) and (4.33).

6th step.

$$\lim_{n\to\infty}\int_G \mathscr{A}(x,\nabla u_n)\cdot\nabla T_k(v_n)dx = \int_G \mathscr{A}(x,Du)\cdot\nabla T_k(v)dx \tag{4.34}$$

for every $k>0$.

Proof: First we show

$$\limsup_{n\to\infty}\left|\int_G [\mathscr{A}(x,\nabla u_n)\cdot\nabla T_k(v_n)]\varphi\, dx - \int_G [\mathscr{A}(x,Du)\cdot\nabla T_k(v)]\varphi\, dx\right|$$
$$\le 6k\int_G \varphi\, d(v_s^+ + v_s^-) \tag{4.35}$$

for any $\varphi\in\Lambda^+(G)$.

Since $T_k(v_n)\varphi\in H_0^{1,p}(G;\mu)$,

$$\int_G [\mathscr{A}(x,\nabla u_n)\cdot\nabla T_k(v_n)]\varphi\, dx + \int_G [\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi]T_k(v_n)dx$$
$$+\int_G \mathscr{B}(x,u_n)T_k(v_n)\varphi\, dx = \int_G T_k(v_n)\varphi\, dv_n. \tag{4.36}$$

By the same arguments as those showing (4.29), (4.30) and (4.31), we see

$$\lim_{n\to\infty}\int_G [\mathscr{A}(x,\nabla u_n)\cdot\nabla\varphi]T_k(v_n)dx = \int_G [\mathscr{A}(x,Du)\cdot\nabla\varphi]T_k(v)dx,$$

$$\lim_{n\to\infty}\int_G \mathscr{B}(x,u_n)T_k(v_n)\varphi\, dx = \int_G \mathscr{B}(x,u)T_k(v)\varphi\, dx$$

and

$$\lim_{n\to\infty}\int_G T_k(v_n)\varphi\, d(v_a)_n = \int_G T_k(v)\varphi\, dv_a.$$

Hence from (4.36) we obtain

$$\limsup_{n\to\infty}\left|\int_G [\mathscr{A}(x,\nabla u_n)\cdot\nabla T_k(v_n)]\varphi\, dx + \int_G [\mathscr{A}(x,Du)\cdot\nabla\varphi]T_k(v)dx\right.$$
$$\left.+\int_G \mathscr{B}(x,u)T_k(v)\varphi\, dx - \int_G T_k(v)\varphi\, dv_a\right|$$

$$\leq \limsup_{n\to\infty} \int_G |T_k(v_n)|\varphi \, d((v_s^+)_n + (v_s^-)_n)$$

$$\leq \lim_{n\to\infty} k \int_G \varphi \, d((v_s^+)_n + (v_s^-)_n) = k \int_G \varphi \, d(v_s^+ + v_s^-).$$

Combining this inequality with (4.26), we have (4.35).

Next, given $\varepsilon > 0$, choose compact sets $K^+ \subset S^+$ and $K^- \subset S^-$ such that $v_s^+(S^+ \backslash K^+) < \varepsilon$ and $v_s^-(S^- \backslash K^-) < \varepsilon$. Since $\mathrm{cap}_{p,\mu}(K^+) = \mathrm{cap}_{p,\mu}(K^-) = 0$, we can choose nonincreasing sequences $\{\phi_j^{(+)}\}$ and $\{\phi_j^{(-)}\}$ in $C_0^\infty(G)$ such that $0 \leq \phi_j^{(\pm)} \leq 1$ in G, $\phi_j^{(+)} = 1$ on K^+, $\phi_j^{(-)} = 1$ on K^-, $(\mathrm{spt}\, \phi_1^{(+)}) \cap (\mathrm{spt}\, \phi_1^{(-)}) = \varnothing$, $\phi_j^{(\pm)} \to 0$ (p,μ)-q.e. in G and $\int_G |\nabla \phi_j^{(\pm)}|^p d\mu \to 0$ $(j \to \infty)$. Set $\varphi_j = \phi_1^{(+)} \phi_j^{(+)} + \phi_1^{(-)} \phi_j^{(-)}$. Then, $0 \leq \varphi_j \leq 1$ and $\varphi_j = 1$ on $K^+ \cup K^-$. Hence,

$$\int_G (1 - \varphi_j) d(v_s^+ + v_s^-) \leq v_s^+(S^+ \backslash K^+) + v_s^-(S^- \backslash K^-) < 2\varepsilon,$$

so that by (4.35)

$$\limsup_{n\to\infty} \left| \int_G [\mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n)](1 - \varphi_j) dx \right.$$
$$\left. - \int_G [\mathscr{A}(x, Du) \cdot \nabla T_k(v)](1 - \varphi_j) dx \right| \leq 12k\varepsilon \tag{4.37}$$

for every j. On the other hand, by (4.18) we have

$$\limsup_{j\to\infty} \limsup_{n\to\infty} \left| \int_G [\mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n)] \varphi_j \, dx \right|$$
$$\leq 40k \left\{ \int_G \phi_1^{(+)} \, dv_s^- + \int_G \phi_1^{(-)} \, dv_s^+ \right\}$$
$$\leq 40k (v_s^-(S^- \backslash K^-) + v_s^+(S^+ \backslash K^+)) < 80k\varepsilon. \tag{4.38}$$

Since $\mathscr{A}(x, Du) \cdot \nabla T_k(v) \in L^1(G; dx)$ by Lemma 4.2, Lebesgue's convergence theorem implies

$$\lim_{j\to\infty} \int_G [\mathscr{A}(x, Du) \cdot \nabla T_k(v)] \varphi_j \, dx = 0. \tag{4.39}$$

Now, by (4.37), (4.38) and (4.39)

$$\limsup_{n\to\infty} \left| \int_G \mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n) dx - \int_G \mathscr{A}(x, Du) \cdot \nabla T_k(v) dx \right| \leq 92k\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, this means (4.34).

7th step.

$$\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta) w^{-1/p} \to \mathscr{A}(x, \nabla T_k(v) + \nabla\theta) w^{-1/p}$$

in $L^{p'}(G; dx)$ for every $k > 0$.

Proof: Since

$$(|\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta)| w^{-1/p})^{p'} \le \alpha_2^{p'} |\nabla T_k(v_n) + \nabla\theta|^p w$$
$$\le \alpha_1^{-1}\alpha_2^{p'} [\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta) \cdot (\nabla T_k(v_n) + \nabla\theta)]$$

a.e., using Lemma 4.3 we have

$$(|\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta)| w^{-1/p})^{p'} \le C_1[\mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n)] + C_2 |\nabla\theta|^p w$$

a.e., where $C_1 = 2\alpha_1^{-1}\alpha_2^{p'}$ and $C_2 = C_2(p, \alpha_1, \alpha_2) > 0$. Similarly, we have

$$(|\mathscr{A}(x, \nabla T_k(v) + \nabla\theta)| w^{-1/p})^{p'} \le C_1[\mathscr{A}(x, Du) \cdot \nabla T_k(v)] + C_2 |\nabla\theta|^p w$$

a.e. Hence

$$|\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta) w^{-1/p} - \mathscr{A}(x, \nabla T_k(v) + \nabla\theta) w^{-1/p}|^{p'}$$
$$\le C_1'\{[\mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n)] + [\mathscr{A}(x, Du) \cdot \nabla T_k(v)]\} + C_2' |\nabla\theta|^p w$$

a.e. with $C_1' = 2^{p'} C_1$ and $C_2' = 2^{p'} C_2$. Now, consider the functions

$$f_n = C_1'\{[\mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n)] + [\mathscr{A}(x, Du) \cdot \nabla T_k(v)]\} + C_2' |\nabla\theta|^p w$$
$$- |\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta) w^{-1/p} - \mathscr{A}(x, \nabla T_k(v) + \nabla\theta) w^{-1/p}|^{p'}.$$

Then $f_n \ge 0$ a.e. for each n and $f_n \to 2C_1'[\mathscr{A}(x, Du) \cdot \nabla T_k(v)] + C_2' |\nabla\theta|^p w$ a.e. as $n \to \infty$. Hence by Fatou's lemma

$$2C_1' \int_G \mathscr{A}(x, Du) \cdot \nabla T_k(v) dx + C_2' \int_G |\nabla\theta|^p d\mu \le \liminf_{n\to\infty} \int_G f_n \, dx$$
$$= C_1' \left\{ \lim_{n\to\infty} \int_G \mathscr{A}(x, \nabla u_n) \cdot \nabla T_k(v_n) dx + \int_G \mathscr{A}(x, Du) \cdot \nabla T_k(v) dx \right\}$$
$$+ C_2' \int_G |\nabla\theta|^p d\mu$$
$$- \limsup_{n\to\infty} \int_G |\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta) w^{-1/p} - \mathscr{A}(x, \nabla T_k(v) + \nabla\theta) w^{-1/p}|^{p'} dx.$$

Therefore by (4.34),

$$\limsup_{n\to\infty} \int_G |\mathscr{A}(x, \nabla T_k(v_n) + \nabla\theta) w^{-1/p} - \mathscr{A}(x, \nabla T_k(v) + \nabla\theta) w^{-1/p}|^{p'} dx \le 0.$$

8th step. Let $l \in \mathscr{L}$, $\psi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$, $\varphi \in \Lambda(G)$ and either $l(0) = 0$ or $\varphi \in H_0^{1,p}(G;\mu)$. Let $l(t) = l(\infty)$ for $t \geq M$ and $l(t) = l(-\infty)$ for $t \leq -M$. Let $k = M + \|\psi\|_\infty$. Then $l(v_n + \psi) = l(T_k(v_n) + \psi)$ and $l(v_n + \psi)\varphi \in H_0^{1,p}(G;\mu)$ by Lemma 4.1. Hence

$$\int_G \mathscr{A}(x, \nabla u_n) \cdot \nabla[l(v_n + \psi)\varphi]dx + \int_G \mathscr{B}(x, u_n)l(v_n + \psi)\varphi\, dx$$
$$= \int_G l(v_n + \psi)\varphi\, dv_n. \tag{4.40}$$

Since $|\nabla l(v_n + \psi)| \leq \|l'\|_\infty(|\nabla T_k(v_n)| + |\nabla\psi|)$, Corollary 2.1 implies that the sequence $\{\int_G |\nabla l(v_n + \psi)|^p d\mu\}_n$ is bounded. Also, $\{l(v_n + \psi)\}_n$ is uniformly bounded and $l(v_n + \psi) \to l(v + \psi)$ a.e. Hence $\nabla l(v_n + \psi) \to \nabla l(v + \psi)$ weakly in $L^p(G;\mu)$ (cf. [6, Theorem 1.32]) and hence

$$\nabla l(v_n + \psi)w^{1/p} \to \nabla l(v + \psi)w^{1/p} \qquad \text{weakly in } L^p(G;dx).$$

Since φ is bounded, from the result in the previous step it follows that

$$\lim_{n\to\infty} \int_G [\mathscr{A}(x, \nabla u_n) \cdot \nabla l(v_n + \psi)]\varphi\, dx = \int_G [\mathscr{A}(x, Du) \cdot \nabla l(v + \psi)]\varphi\, dx. \tag{4.41}$$

In the same way as those for the proof of (4.29), (4.30), (4.31), we have

$$\lim_{n\to\infty} \int_G [\mathscr{A}(x, \nabla u_n) \cdot \nabla\varphi]l(v_n + \psi)dx = \int_G [\mathscr{A}(x, Du) \cdot \nabla\varphi]l(v + \psi)dx; \tag{4.42}$$

$$\lim_{n\to\infty} \int_G \mathscr{B}(x, u_n)l(v_n + \psi)\varphi\, dx = \int_G \mathscr{B}(x, u)l(v + \psi)\varphi\, dx; \tag{4.43}$$

$$\lim_{n\to\infty} \int_G l(v_n + \psi)\varphi\, d(v_a)_n = \int_G l(v + \psi)\varphi\, dv_a. \tag{4.44}$$

As to the integral with respect to v_s, we have

$$\left|\int_G l(v_n + \psi)\varphi\, d(v_s^+)_n - l(\infty)\int_G \varphi\, d(v_s^+)_n\right|$$
$$\leq \int_G |l(v_n + \psi) - l(\infty)|\, |\varphi| d(v_s^+)_n$$
$$\leq 2\|l\|_\infty \|\varphi\|_\infty \int_{\{v_n < k\}} d(v_s^+)_n \to 0 \qquad (n \to \infty)$$

by (4.21). Since $\int_G \varphi\, d(v_s^+)_n \to \int_G \varphi\, dv_s^+$, it follows that

$$\lim_{n\to\infty} \int_G l(v_n + \psi)\varphi\, d(v_s^+)_n = l(\infty)\int_G \varphi\, dv_s^+. \tag{4.45}$$

Similarly we have

$$\lim_{n\to\infty}\int_G l(v_n+\psi)\varphi\, d(v_s^-)_n = l(-\infty)\int_G \varphi\, dv_s^-. \tag{4.46}$$

Combining (4.40), (4.41), (4.42), (4.43), (4.44), (4.45) and (4.46), we finally obtain (4.1).

Final step. In order to show that we can take u to be $(\mathscr{A},\mathscr{B})$-harmonic in $G\backslash(\mathrm{spt}|v|)$, we consider the solutions $u_n^{(+)}$ (resp. $u_n^{(-)}$) of $Lu=(v_a)_n^+ + (v_s^+)_n$ (resp. $Lu = -(v_a)_n^- - (v_s^-)_n$) with boundary data θ. By Theorem 2.2, $u_n^{(-)} \le u_n \le u_n^{(+)}$ a.e. in G for all n. By the above arguments, we may assume that $u_n^{(+)} \to u^{(+)}$ and $u_n^{(-)} \to u^{(-)}$ a.e. in G and that $u^{(+)}$ (resp. $u^{(-)}$) is a (renormalized) solution of $Lu = v^+$ (resp. $Lu = v^-$) with boundary data θ. Then $u^{(-)} \le u \le u^{(+)}$ a.e. in G. We can take $u^{(+)}$ to be $(\mathscr{A},\mathscr{B})$-superharmonic in G (cf. the proof of [14, Theorem 3.2 and Lemma 3.4]). Likewise, we can take $u^{(-)}$ to be $(\mathscr{A},\mathscr{B})$-subharmonic in G.

Now, let U and U' be open sets such that $U \Subset U' \Subset G\backslash(\mathrm{spt}|v|)$. Set $R = \mathrm{dist}\,(\partial U', U)/2$. Then, by [14, Theorem 4.1]

$$u^{(+)}(x) \le c\left(\frac{1}{\mu(B(x,R))}\int_{B(x,R)}(\max(u^{(+)},0))^{\gamma} d\mu\right)^{1/\gamma} + R^{p/(p-1)}$$

for all $x \in U$ with $\gamma > p-1$ and a constant $c>0$ which is independent of x. Here, we used the fact that $v^+ = 0$ on U', so that the Wolff potential $W_{p,\mu}^{v^+}(x,2R) = 0$ for every $x\in U$. We know (see, Theorem 3.1 or [14, Theorem 2.3]) that $u^{(+)} \in L^{\gamma}(U';\mu)$ for $\gamma < \min(p, \kappa(p-1))$. By the doubling property for μ, we see that

$$\frac{1}{\mu(B(x,R))}\int_{B(x,R)}(\max(u^{(+)},0))^{\gamma} d\mu \le C\int_{U'} |u^{(+)}|^{\gamma} d\mu < \infty$$

for all $x\in U$ with a constant C independent of x. It follows that $u^{(+)}$ is bounded from above on U.

Similarly, we can show that $u^{(-)}$ is bounded from below on U. Since $u^{(-)} \le u \le u^{(+)}$ a.e., we conclude that u is essentially bounded on U. Let h_θ be the $(\mathscr{A},\mathscr{B})$-harmonic function in G such that $h_\theta - \theta \in H_0^{1,p}(G;\mu)$. Then $u-h_\theta$ is also essentially bounded on U. Hence $u - h_\theta = T_k(u-h_\theta)$ a.e. on U for some $k>0$. Since $T_k(u-h_\theta)\in H_0^{1,p}(G;\mu)$, it follows that $u\in H^{1,p}(U;\mu)$. Thus, (4.1) with $l=1$ and $\varphi \in C_0^\infty(G)$ such that $(\mathrm{spt}\,\varphi)\subset U$ implies that u can be taken to be $(\mathscr{A},\mathscr{B})$-harmonic in U.

5. Some properties of renormalized solutions

Throughout this section, let $\theta \in H^{1,p}(G;\mu)$ and v be a finite signed measure on G.

PROPOSITION 5.1. *Let u be a renormalized solution of $Lu = \nu$ with boundary data θ. If $l \in \mathscr{L}$ and $l(\infty) = l(-\infty) = 0$, then*

$$\int_G \mathscr{A}(x, Du) \cdot \nabla(l(u - \theta + \psi)\varphi)dx + \int_G \mathscr{B}(x, u)l(u - \theta + \psi)\varphi\, dx$$
$$= \int_G l(u - \theta + \psi)\varphi\, d\nu_a \tag{5.1}$$

for $\varphi, \psi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$.

PROOF. By (4.1), (5.1) holds for $\varphi \in \Lambda(G) \cap H_0^{1,p}(G;\mu)$ and $\psi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$. If $\varphi \in H_0^{1,p}(G;\mu) \cap L^\infty(G)$, then we can choose a uniformly bounded sequence $\{\varphi_j\}$ in $C_0^\infty(G)$ such that $\varphi_j \to \varphi$ in $H_0^{1,p}(G;\mu)$ as well as (p,μ)-q.e. in G.

Let $v = u - \theta$. Since $\mathscr{A}(x, Du) \cdot \nabla l(v + \psi) \in L^1(G;dx)$ as in the proof of Lemma 4.2 and $\mathscr{B}(x,u)l(v + \psi) \in L^1(G;dx)$, Lebesgue's convergence theorem implies

$$\lim_{j\to\infty} \int_G [\mathscr{A}(x, Du) \cdot \nabla l(v + \psi)]\varphi_j\, dx = \int_G [\mathscr{A}(x, Du) \cdot \nabla l(v + \psi)]\varphi\, dx$$

and

$$\lim_{j\to\infty} \int_G \mathscr{B}(x, u)l(v + \psi)\varphi_j\, dx = \int_G \mathscr{B}(x, u)l(v + \psi)\varphi\, dx.$$

Also, since $\varphi_j \to \varphi$ $|\nu_a|$-a.e.,

$$\lim_{j\to\infty} \int_G l(v + \psi)\varphi_j\, d\nu_a = \int_G l(v + \psi)\varphi\, d\nu_a.$$

Finally, let $l(t) = 0$ for $|t| \ge M$. Then, for $k > M + \|\psi\|_\infty$,

$$l(v + \psi)\mathscr{A}(x, Du) = l(v + \psi)\mathscr{A}(x, \nabla T_k(v) + \nabla\theta).$$

Since $|\mathscr{A}(x, \nabla T_k(v) + \nabla\theta)|w^{-1/p} \in L^{p'}(G;dx)$ and $\nabla\varphi_j w^{1/p} \to \nabla\varphi w^{1/p}$ in $L^p(G;dx)$,

$$\lim_{j\to\infty} \int_G [\mathscr{A}(x, Du) \cdot \nabla\varphi_j]l(v + \psi)dx = \int_G [\mathscr{A}(x, Du) \cdot \nabla\varphi]l(v + \psi)dx.$$

Hence by letting $j \to \infty$ in (5.1) with φ_j, we obtain (5.1) for this φ.

Let $C_b(G)$ be the set of all bounded continuous functions on G and $C_b^+(G)$ be the set of nonnegative functions in $C_b(G)$.

PROPOSITION 5.2. *Let $u \in \mathscr{U}(\theta)$ be a renormalized solution of $Lu = \nu$ in G and set $v = u - \theta$. Let $j(k) \ge 1$ for every $k > 0$. Then*

(1)

$$\lim_{k\to\infty}\frac{1}{j(k)}\int_{\{k<v<k+j(k)\}}[\mathscr{A}(x,Du)\cdot Dv]\varphi\,dx=\int_G\varphi\,d\nu_s^+ \tag{5.2}$$

and

$$\lim_{k\to\infty}\frac{1}{j(k)}\int_{\{-k-j(k)<v<-k\}}[\mathscr{A}(x,Du)\cdot Dv]\varphi\,dx=\int_G\varphi\,d\nu_s^- \tag{5.3}$$

for $\varphi\in C_b(G)$*;*

(2)

$$\lim_{k\to\infty}\frac{1}{j(k)}\int_{\{k<v<k+j(k)\}}[\mathscr{A}(x,Du)\cdot Du]\varphi\,dx=\int_G\varphi\,d\nu_s^+ \tag{5.4}$$

and

$$\lim_{k\to\infty}\frac{1}{j(k)}\int_{\{-k-j(k)<v<-k\}}[\mathscr{A}(x,Du)\cdot Du]\varphi\,dx=\int_G\varphi\,d\nu_s^- \tag{5.5}$$

for $\varphi\in C_b(G)$.

Proof. First, let $\varphi\in\Lambda(G)$. For each $k>0$, let $l_k(t)=\max(0,T_{j(k)}(t-k)/j(k))$. Then, $l_k\in\mathscr{L}$ and $l_k(0)=0$, so that by (4.1)

$$\frac{1}{j(k)}\int_{\{k<v<k+j(k)\}}[\mathscr{A}(x,Du)\cdot Dv]\varphi\,dx+\int_{\{v>k\}}[\mathscr{A}(x,Du)\cdot\nabla\varphi]l_k(v)dx$$
$$+\int_{\{v>k\}}\mathscr{B}(x,u)l_k(v)\varphi\,dx=\int_{\{v>k\}}l_k(v)\varphi\,d\nu_a+\int_G\varphi\,d\nu_s^+. \tag{5.6}$$

Since $|\mathscr{A}(x,Du)\cdot\nabla\varphi|\in L^1(G;dx)$, $|\mathscr{B}(x,u)|\varphi\in L^1(G;dx)$, $\varphi\in L^1(G;|\nu^a|)$ and $v<\infty$ a.e. as well as $|\nu_a|$-a.e.,

$$\lim_{k\to\infty}\int_{\{v>k\}}|\mathscr{A}(x,Du)\cdot\nabla\varphi|dx=\lim_{k\to\infty}\int_{\{v>k\}}|\mathscr{B}(x,u)\varphi|dx$$
$$=\lim_{k\to\infty}\int_{\{v>k\}}|\varphi|d|\nu_a|=0.$$

Hence, (5.6) implies (5.2) for $\varphi\in\Lambda(G)$. Similarly, we obtain (5.3) for $\varphi\in\Lambda(G)$.

Next, we show

$$\lim_{k\to\infty}\frac{1}{j(k)}\int_{E_k}|\mathscr{A}(x,Du)\cdot\nabla\theta|dx=0, \tag{5.7}$$

where $E_k=\{k<|v|<k+j(k)\}$. By Lemma 4.3 and (A.2), $|Du|^pw\le(2/\alpha_1)[\mathscr{A}(x,Du)\cdot Dv]+c|\nabla\theta|^pw$ a.e. with $c=c(p,\alpha_1,\alpha_2)>0$. Since

$$\lim_{k\to\infty}\int_{\{|v|>k\}}|\nabla\theta|^p d\mu = 0,$$

it follows from (5.2) and (5.3) with $\varphi = 1$ that

$$\limsup_{k\to\infty}\frac{1}{j(k)}\int_{E_k}|Du|^p d\mu \le \frac{2}{\alpha_1}|\nu_s|(G).$$

By (A.3) and Hölder's inequality

$$\frac{1}{j(k)}\int_{E_k}|\mathscr{A}(x,Du)\cdot\nabla\theta|dx$$

$$\le \alpha_2\left(\frac{1}{j(k)}\int_{E_k}|Du|^p d\mu\right)^{(p-1)/p}\left(\frac{1}{j(k)}\int_{\{|v|>k\}}|\nabla\theta|^p d\mu\right)^{1/p}.$$

Hence we have (5.7).

By (5.7), we immediately obtain (5.4) and (5.5) from (5.2) and (5.3) when $\varphi \in \Lambda(G)$. Now, let $\varphi \in C_b^+(G)$. Let $f_k = (1/j(k))[\mathscr{A}(x,Du)\cdot Du]\chi_{\{k<v<k+j(k)\}}$ for simplicity. Note that $f_k \ge 0$. Then (5.4) for $\psi \in C_0^\infty(G)$ implies

$$\begin{aligned}\int_G \varphi\, d\nu_s^+ &= \sup\left\{\int_G \psi\, d\nu_s^+ \,\middle|\, \psi \in C_0^\infty(G), 0 \le \psi \le \varphi\right\}\\ &= \sup\left\{\lim_{k\to\infty}\int_G f_k\psi\, dx \,\middle|\, \psi \in C_0^\infty(G), 0 \le \psi \le \varphi\right\}\\ &\le \liminf_{k\to\infty}\int_G f_k\varphi\, dx. \qquad (5.8)\end{aligned}$$

If $M = \|\varphi\|_\infty$, then applying (5.8) for $M - \varphi$ in place of φ, we have

$$\int_G (M-\varphi)d\nu_s^+ \le \liminf_{k\to\infty}\int_G f_k(M-\varphi)dx.$$

Since (5.4) holds for $\varphi = M$, it follows that

$$\int_G \varphi\, d\nu_s^+ \ge \limsup_{k\to\infty}\int_G f_k\varphi\, dx.$$

This, together with (5.8), shows that (5.4) holds for $\varphi \in C_b^+(G)$, and hence for all $\varphi \in C_b(G)$. Similarly, we see that (5.5) holds for all $\varphi \in C_b(G)$.

Finally we deduce from (5.7) that (5.2) and (5.3) also hold for all $\varphi \in C_b(G)$.

Corollary 5.1. *If $u \in \mathscr{U}(\theta)$ is a renormalized solution of $Lu = \nu$, then*

$$\frac{1}{\alpha_2}\int_G \varphi\, d\nu_s^+ \le \liminf_{k\to\infty}\int_{\{k<v<k+1\}} |Du|^p \varphi\, d\mu$$
$$\le \limsup_{k\to\infty}\int_{\{k<v<k+1\}} |Du|^p \varphi\, d\mu \le \frac{1}{\alpha_1}\int_G \varphi\, d\nu_s^+$$

and

$$\frac{1}{\alpha_2}\int_G \varphi\, d\nu_s^- \le \liminf_{k\to\infty}\int_{\{-k-1<v<-k\}} |Du|^p \varphi\, d\mu$$
$$\le \limsup_{k\to\infty}\int_{\{-k-1<v<-k\}} |Du|^p \varphi\, d\mu \le \frac{1}{\alpha_1}\int_G \varphi\, d\nu_s^-$$

for $\varphi \in C_b^+(G)$*, where* $v = u - \theta$*.*

Corollary 5.2. *If* $u \in \mathscr{U}(\theta)$ *is a renormalized solution of* $Lu = \nu$ *and if* E *is a relatively closed subset of* G *such that* $|\nu_s|(E) = 0$*, then*

$$\lim_{k\to\infty}\int_{\{k<|u-\theta|<k+1\}\cap E} |Du|^p d\mu = 0.$$

Proof. Let $v = u - \theta$ and let S^+ and S^- be as in the proof of Theorem 4.1. We may assume that $E \cap (S^+ \cup S^-) = \varnothing$. Given $\varepsilon > 0$, choose a compact set $K \subset (S^+ \cup S^-)$ such that $|\nu_s|((S^+ \cup S^-)\backslash K) < \varepsilon$ and choose $\psi \in C_0^\infty(G)$ such that $\psi = 1$ on K, $0 \le \psi \le 1$ in G and $\psi = 0$ on E. Applying Corollary 5.1 with $\varphi = 1 - \psi$, we have

$$\limsup_{k\to\infty}\int_{\{k<|v|<k+1\}\cap E} |Du|^p d\mu \le \limsup_{k\to\infty}\int_{\{k<|v|<k+1\}} |Du|^p (1-\psi) d\mu$$
$$\le \frac{2}{\alpha_1}\int_G (1-\psi)\, d|\nu_s| < \frac{2}{\alpha_1}\varepsilon.$$

This shows the required result.

6. Uniqueness results

In this section, we give two types of uniqueness of renormalized solutions. Uniqueness in the general case is not known (cf. [4, Section 10]).

6.1. The case $|\nu| \ll \mathrm{cap}_{p,\mu}$

Theorem 6.1. *Given* $\theta \in H^{1,p}(G;\mu)$ *and a finite signed measure* ν *on* G*, if* $|\nu| \ll \mathrm{cap}_{p,\mu}$*, then the renormalized solution of* $Lu = \nu$ *with boundary data* θ *is unique.*

Proof. Let u_1 and u_2 be renormalized solutions of $Lu = \nu$ with the same boundary data θ. Let $v_i = u_i - \theta$, $i = 1, 2$. By assumption, $\nu_a = \nu$ and $\nu_s = 0$.

Thus, by taking $l = T_k$ for $k > 0$, $\psi = -T_m(v_i)$ or $-T_m(v_j)$ for $m > 0$ and $\varphi = 1$ in (4.1),

$$\int_G \mathscr{A}(x, Du_i) \cdot \nabla T_k(v_i - T_m(v_j))dx + \int_G \mathscr{B}(x, u_i) T_k(v_i - T_m(v_j))dx$$

$$= \int_G T_k(v_i - T_m(v_j))d\nu \qquad (i, j = 1, 2; i \neq j). \tag{6.1}$$

Fix $k > 0$ and for $m > k$ let

$$A_0(m) = \{|u_1 - u_2| < k, |v_1| < m, |v_2| < m\}.$$

Then

$$\int_{A_0(m)} \mathscr{A}(x, Du_i) \cdot \nabla T_k(v_i - T_m(v_j))dx = \int_{A_0(m)} \mathscr{A}(x, Du_i) \cdot (Du_i - Du_j)dx. \tag{6.2}$$

Next, let

$$A_i(m) = \{|v_i - T_m(v_j)| < k, |v_j| \geq m\}$$

and

$$A_i'(m) = \{|v_i - T_m(v_j)| < k, |v_i| \geq m, |v_j| < m\}$$

$(i, j = 1, 2; i \neq j)$. Then

$$\int_{A_1(m)} \mathscr{A}(x, Du_1) \cdot \nabla T_k(v_1 - T_m(v_2))dx = \int_{A_1(m)} \mathscr{A}(x, Du_1) \cdot Dv_1\, dx$$

$$\geq -\int_{A_1(m)} \mathscr{A}(x, Du_1) \cdot \nabla\theta\, dx \tag{6.3}$$

and

$$\int_{A_1'(m)} \mathscr{A}(x, Du_1) \cdot \nabla T_k(v_1 - T_m(v_2))dx$$

$$= \int_{A_1'(m)} \mathscr{A}(x, Du_1) \cdot (Du_1 - Du_2)dx \geq -\int_{A_1'(m)} \mathscr{A}(x, Du_1) \cdot Du_2\, dx. \tag{6.4}$$

Since $\{|v_1 - T_m(v_2)| < k\} = A_0(m) \cup A_1(m) \cup A_1'(m)$ (disjoint union), (6.2), (6.3) and (6.4) imply

$$\int_G \mathscr{A}(x, Du_1) \cdot \nabla T_k(v_1 - T_m(v_2))dx$$

$$\geq \int_{A_0(m)} \mathscr{A}(x, Du_1) \cdot (Du_1 - Du_2)dx$$

$$- \int_{A_1(m)} \mathscr{A}(x, Du_1) \cdot \nabla\theta\, dx - \int_{A_1'(m)} \mathscr{A}(x, Du_1) \cdot Du_2\, dx. \tag{6.5}$$

Similarly, we obtain

$$\int_G \mathscr{A}(x, Du_2) \cdot \nabla T_k(v_2 - T_m(v_1))dx$$

$$\geq \int_{A_0(m)} \mathscr{A}(x, Du_2) \cdot (Du_2 - Du_1)dx$$

$$- \int_{A_2(m)} \mathscr{A}(x, Du_2) \cdot \nabla\theta\, dx - \int_{A_2'(m)} \mathscr{A}(x, Du_2) \cdot Du_1\, dx. \tag{6.6}$$

Now, let $A_0^*(m) = \{|v_1| < m, |v_2| < m\}$. Then

$$\int_{A_0^*(m)} \mathscr{B}(x, u_1) T_k(v_1 - T_m(v_2))dx + \int_{A_0^*(m)} \mathscr{B}(x, u_2) T_k(v_2 - T_m(v_1))dx$$

$$= \int_{A_0^*(m)} (\mathscr{B}(x, u_1) - \mathscr{B}(x, u_2)) T_k(u_1 - u_2)dx \geq 0 \tag{6.7}$$

and

$$\int_{A_0^*(m)} (T_k(v_1 - T_m(v_2)) + T_k(v_2 - T_m(v_1))d\nu$$

$$= \int_{A_0^*(m)} (T_k(u_1 - u_2) + T_k(u_2 - u_1))d\nu = 0. \tag{6.8}$$

Combining (6.1), (6.5), (6.6), (6.7) and (6.8), we obtain

$$\int_{A_0(m)} \{\mathscr{A}(x, Du_1) - \mathscr{A}(x, Du_2)\} \cdot (Du_1 - Du_2)dx$$

$$\leq \int_{A_1(m)} |\mathscr{A}(x, Du_1)|\, |\nabla\theta| dx + \int_{A_2(m)} |\mathscr{A}(x, Du_2)|\, |\nabla\theta| dx$$

$$+ \int_{A_1'(m)} |\mathscr{A}(x, Du_1)|\, |Du_2| dx + \int_{A_2'(m)} |\mathscr{A}(x, Du_2)|\, |Du_1| dx$$

$$+ k \int_{\{|v_1| \geq m\} \cup \{|v_2| \geq m\}} \{|\mathscr{B}(x, u_1)| + |\mathscr{B}(x, u_2)|\} dx$$

$$+ k \int_{\{|v_1| \geq m\} \cup \{|v_2| \geq m\}} d|\nu|. \tag{6.9}$$

Since $A_i(m) \subset \{m - k \leq |v_i| < m + k\}$ and $A_i'(m) \subset \{m \leq |v_i| < m + k\} \cap \{m - k \leq |v_j| < m\}$, as in the proof of Proposition 5.2, we see

$$\lim_{m \to \infty} \int_{A_i(m)} |\mathscr{A}(x, Du_i)|\, |\nabla\theta| dx = 0 \qquad (i = 1, 2)$$

and

$$\lim_{m\to\infty}\int_{A_i'(m)} |\mathscr{A}(x,Du_i)|\,|Du_j|dx = 0 \qquad (i,j=1,2; i\neq j).$$

Since $\mathscr{B}(x,u_i)\in L^1(G;dx)$, $|v_i|<\infty$ $|\nu|$-a.e. and $|\nu|(G)<\infty$, the last two terms in (6.9) also tend to 0 as $m\to\infty$. Therefore, letting $m\to\infty$ in (6.9), we have

$$\int_{\{|u_1-u_2|<k\}} \{\mathscr{A}(x,Du_1)-\mathscr{A}(x,Du_2)\}\cdot(Du_1-Du_2)dx \le 0.$$

Hence, by (A.4), $Du_1=Du_2$ a.e. in $\{|u_1-u_2|<k\}$. Since this holds for all $k>0$, $Du_1=Du_2$ a.e. in G, so that $Dv_1=Dv_2$ a.e. in G. Thus, $\nabla T_k(v_1)=\nabla T_k(v_2)$ a.e. in G for all $k>0$. Since $T_k(v_i)\in H_0^{1,p}(G;\mu)$, $i=1,2$, it follows that $T_k(v_1)=T_k(v_2)$ (p,μ)-q.e. in G for all $k>0$, which shows $v_1=v_2$ or $u_1=u_2$ (p,μ)-q.e. in G.

Remark. The above proof shows the uniqueness of "entropy solutions" (see the Remark after (4.1)) in case $|\nu|\ll \mathrm{cap}_{p,\mu}$.

Theorem 6.2. *Let $\theta_1,\theta_2\in H^{1,p}(G;\mu)$ and ν_j $(j=1,2)$ be finite signed measures on G such that $|\nu_j|\ll \mathrm{cap}_{p,\mu}$. Let u_j be the renormalized solutions of $Lu=\nu_j$ with boundary data θ_j $(j=1,2)$. If $\max(\theta_1-\theta_2,0)\in H_0^{1,p}(G;\mu)$ and $\nu_1\le\nu_2$, then $u_1\le u_2$ (p,μ)-q.e. in G.*

Proof. By Lemma 4.5 we can choose Borel sets $E_n^+\subset G$ and $E_n^-\subset G$ $(n=1,2,\dots)$ such that $\{E_n^+\}$, $\{E_n^-\}$ are nondecreasing, $\nu_2^+(G\setminus\bigcup_n E_n^+)=0$, $\nu_1^-(G\setminus\bigcup_n E_n^-)=0$, $\chi_{E_n^+}\nu_2^+\in(H_0^{1,p}(G;\mu))^*$ and $\chi_{E_n^-}\nu_1^-\in(H_0^{1,p}(G;\mu))^*$ for all n. Then $\nu_1^+(G\setminus\bigcup_n E_n^+)=0$, $\nu_2^-(G\setminus\bigcup_n E_n^-)=0$, $\chi_{E_n^+}\nu_1^+\in(H_0^{1,p}(G;\mu))^*$ and $\chi_{E_n^-}\nu_2^-\in(H_0^{1,p}(G;\mu))^*$ for all n.

Let $\nu_n^{(j)}=\chi_{E_n^+}\nu_j^+-\chi_{E_n^-}\nu_j^-$, and let $u_n^{(j)}$ be the solution of $Lu=\nu_n^{(j)}$ such that $u_n^{(j)}-\theta_j\in H_0^{1,p}(G;\mu)$ $(j=1,2)$. Since $\nu_n^{(1)}\le\nu_n^{(2)}$, $u_n^{(1)}\le u_n^{(2)}$ (p,μ)-q.e. by Theorem 2.2. By Theorem 3.1 and the proof of Theorem 4.1, there exist subsequences $\{u_{n_m}^{(j)}\}_m$, $j=1,2$, which converge to renormalized solutions $u^{(j)}$ of $Lu=\nu_j$ with boundary data θ_j a.e. in G. By the above theorem, $u^{(j)}=u_j$ (p,μ)-q.e. for each $j=1,2$. Obviously, $u^{(1)}\le u^{(2)}$ a.e., so that $u_1\le u_2$ (p,μ)-q.e.

Theorem 6.3. *Let $\theta_1,\theta_2\in H^{1,p}(G;\mu)$ and ν be a finite signed measure on G such that $|\nu|\ll\mathrm{cap}_{p,\mu}$. Let u_j be the renormalized solution of $Lu=\nu$ with boundary data θ_j, $j=1,2$. Then $|u_1-u_2|\le\|\theta_1-\theta_2\|_{\partial G}$ (p,μ)-q.e. in G, where*

$$\|\theta_1-\theta_2\|_{\partial G}=\inf\{\delta>0:\max(|\theta_1-\theta_2|-\delta,0)\in H_0^{1,p}(G;\mu)\}.$$

Proof. We may assume $\|\theta_1 - \theta_2\|_{\partial G} < \infty$. Let $\delta > \|\theta_1 - \theta_2\|_{\partial G}$ and let $u_{1,\delta}$ be the renormalized solution of $Lu = \nu$ with boundary data $\theta_1 + \delta$. Since $\max(\theta_2 - (\theta_1 + \delta), 0) \in H_0^{1,p}(G;\mu)$, Theorem 6.2 implies $u_2 \le u_{1,\delta}$ (p,μ)-q.e. Next, let $\nu_\delta = \nu + (\mathscr{B}(x, u_1 + \delta) - \mathscr{B}(x, u_1))dx$. Then ν_δ is a finite signed measure on G such that $\nu_\delta \ll \mathrm{cap}_{p,\mu}$ and $\nu_\delta \ge \nu$. Since $u_1 + \delta$ is a renormalized solution of $Lu = \nu_\delta$ with boundary data $\theta_1 + \delta$, Theorem 6.2 implies $u_{1,\delta} \le u_1 + \delta$ (p,μ)-q.e. Hence, $u_2 \le u_1 + \delta$ (p,μ)-q.e. Similarly, we see that $u_1 \le u_2 + \delta$ (p,μ)-q.e., and hence $|u_1 - u_2| \le \delta$ (p,μ)-q.e. in G. Hence the conclusion of the theorem holds.

6.2. The linear case

In this subsection, we consider the linear case, namely the case

$$\mathscr{A}(x,\xi) = A(x)\xi \qquad \text{and} \qquad \mathscr{B}(x,t) = b(x)t,$$

where $A(x)$ is a linear operator $\mathbf{R}^N \to \mathbf{R}^N$ (i.e., an $N \times N$-matrix) for each $x \in G$ such that $x \to A(x)$ is measurable in G and

$$\text{(AL)} \qquad A(x)\xi \cdot \xi \ge \alpha_1 w(x)|\xi|^2 \qquad \text{and} \qquad |A(x)\xi| \le \alpha_2 w(x)|\xi|$$

for a.e. $x \in G$ with positive constants α_1 and α_2; $b(x)$ is a measurable function on G such that

$$\text{(BL)} \qquad 0 \le b(x) \le \beta w(x)$$

for a.e. $x \in G$ with a nonnegative constant β. Then $\mathscr{A}$ satisfies (A.1)–(A.4) and $\mathscr{B}$ satisfies (B.1)–(B.3) with $p = 2$. Thus

$$Lu = -\mathrm{div}\, A(x)\nabla u + b(x)u.$$

As in [4], we use the adjoint operator L^* of L:

$$L^*u = -\mathrm{div}\, A(x)^*\nabla u + b(x)u,$$

where $A(x)^*$ is the adjoint operater of $A(x)$ for each $x \in G$.

Lemma 6.1. *Let $\psi \in L^\infty(G)$. Then the solution of $L^*u = \psi w$ belonging to $H_0^{1,2}(G;\mu)$ is bounded continuous.*

Proof. Let $\mathscr{A}(x,\xi) = A(x)^*\xi$ and $\mathscr{B}(x,t) = b(x)t - \psi(x)w(x)$ $(x \in G)$. Then they satisfy (A.1), (A.2), (A.3), (A.4), (B.1), (B.3) and (B.2) with $D = G$ and $\alpha_3(D) = \beta + M$ in [MO1] with $p = 2$. Since $L^*u = \psi w$ can be written as

$$-\mathrm{div}\, \mathscr{A}(x, \nabla u) + \mathscr{B}(x,u) = 0,$$

the existence of the solution u of $L^*u = \psi w$ with $u \in H_0^{1,2}(G;\mu)$ is assured by Theorem 2.1, and by [8, Theorem 1.1] u is continuous.

To show that u is bounded, let

$$\mathscr{A}_1(x,\xi) = \begin{cases} A(x)^*\xi, & \text{if } x \in G \\ w(x)\xi, & \text{if } x \in \mathbf{R}^N \setminus G \end{cases}$$

and

$$\mathscr{B}_1(x) = \begin{cases} \|\psi\|_\infty w(x), & \text{if } x \in G \\ 0, & \text{if } x \in \mathbf{R}^N \setminus G. \end{cases}$$

Then $\mathscr{A}_1$ satisfies (A.1)–(A.4) on $\mathbf{R}^N \times \mathbf{R}^N$ and $\pm\mathscr{B}_1$ satisfies (B.1)–(B.3) on $\mathbf{R}^N \times \mathbf{R}$ (with $p = 2$). Further note that

$$|\mathscr{B}(x,0)| \le \mathscr{B}_1(x) \qquad \text{for } x \in G. \tag{6.10}$$

Let $L^+u = -\mathrm{div}\, \mathscr{A}_1(x,\nabla u) - \mathscr{B}_1(x)$ and $L^-u = -\mathrm{div}\, \mathscr{A}_1(x,\nabla u) + \mathscr{B}_1(x)$.

Take a ball B such that $G \Subset B$. Let $u^{(+)}$ (resp. $u^{(-)}$) be the continuous solution of $L^+u = 0$ (resp. $L^-u = 0$) on B belonging to $H_0^{1,2}(B;\mu)$. Since $\mathscr{B}_1 \ge 0$, the comparison principle (cf. [8, Proposition 1.1]) implies that $u^+ \ge 0$ and $u^- \le 0$ in B and hence $\mathscr{B}(x,u^{(-)}) \le \mathscr{B}(x,0) \le \mathscr{B}(x,u^{(+)})$ for $x \in G$. Thus, by (6.10), $L^*u^{(+)} - \psi w \ge L^+u^{(+)} = 0$ and $L^*u^{(-)} - \psi w \le L^-u^{(-)} = 0$ in G. Hence, again by the comparison principle, $u^{(-)} \le u \le u^{(+)}$ in G. Since $u^{(+)}$ and $u^{(-)}$ are continuous in B, they are bounded on G. Therefore, u is bounded on G.

Theorem 6.4. *Let $\theta \in H^{1,2}(G;\mu)$, v be a (general) finite signed measure on G and let h_θ^L be the solution of $Lu = 0$ in G such that $h_\theta^L - \theta \in H_0^{1,2}(G;\mu)$. If u is a renormalized solution of $Lu = v$ in G with boundary data θ, then*

$$\int_G u\psi\, d\mu = \int_G h_\theta^L \psi\, d\mu + \int_G u_\psi^*\, dv. \tag{6.11}$$

for any $\psi \in L^\infty(G)$, where u_ψ^* denotes the solution of $L^*u = \psi w$ in G belonging to $H_0^{1,2}(G;\mu) \cap C_b(G)$.

Proof. Let $h_m(t) = \max(\min(m + 1 - |t|, 1), 0)$ $(m > 0)$. Then $h_m \in \mathscr{L}$, $h_m(\infty) = h_m(-\infty) = 0$ and $h_m(t) \to 1$ as $m \to \infty$. Let $v = u - \theta$. Since $u_\psi^* \in H_0^{1,2}(G;\mu)$ and it is bounded by Lemma 6.1, by Proposition 5.1 we have

$$\int_G A(x)Du \cdot \nabla[h_m(v)u_\psi^*]dx + \int_G b(x)uh_m(v)u_\psi^*\, dx = \int_G h_m(v)u_\psi^*\, dv_a,$$

so that

$$\int_G [A(x)Du \cdot \nabla u_\psi^*] h_m(v) dx$$

$$- \int_{\{m<v<m+1\}} [A(x)Du \cdot Dv] u_\psi^* \, dx + \int_{\{-m-1<v<-m\}} [A(x)Du \cdot Dv] u_\psi^* \, dx$$

$$+ \int_G b(x) u h_m(v) u_\psi^* \, dx = \int_G h_m(v) u_\psi^* \, d\nu_a. \tag{6.12}$$

Now, let $H_m(t) = \int_0^t h_m(s) ds$ for $m > 0$. Then, $H_m \in \mathscr{L}$ and $H_m(t) \to t$ as $m \to \infty$. Since $H_m(0) = 0$, $H_m(v) \in H_0^{1,2}(G; \mu)$. Hence

$$\int_G A(x)^* \nabla u_\psi^* \cdot \nabla H_m(v) dx + \int_G b(x) u_\psi^* H_m(v) dx = \int_G H_m(v) \psi \, d\mu,$$

so that

$$\int_G [A(x)Dv \cdot \nabla u_\psi^*] h_m(v) dx = - \int_G b(x) u_\psi^* H_m(v) dx + \int_G H_m(v) \psi \, d\mu$$

$$\to - \int_G b(x) u_\psi^* v \, dx + \int_G v \psi \, d\mu \qquad (m \to \infty),$$

where we used the facts that u_ψ^* is bounded and $H_m(v) \to v$ a.e. in G. Also,

$$\int_G [A(x) \nabla \theta \cdot \nabla u_\psi^*] h_m(v) dx \to \int_G [A(x) \nabla \theta \cdot \nabla u_\psi^*] dx \qquad (m \to \infty),$$

since $[A(x) \nabla \theta \cdot \nabla u_\psi^*] \in L^1(G; dx)$ and $h_m(v) \to 1$ a.e. in G. Hence

$$\lim_{m \to \infty} \int_G [A(x)Du \cdot \nabla u_\psi^*] h_m(v) dx$$

$$= \int_G [A(x) \nabla \theta \cdot \nabla u_\psi^*] dx - \int_G b(x) u_\psi^* v \, dx + \int_G v \psi \, d\mu.$$

Thus, letting $m \to \infty$ in (6.12) and using Proposition 5.2, we have

$$\int_G [A(x) \nabla \theta \cdot \nabla u_\psi^*] dx - \int_G b(x) u_\psi^* v \, dx + \int_G v \psi \, d\mu$$

$$- \int_G u_\psi^* \, d\nu_s^+ + \int_G u_\psi^* \, d\nu_s^- + \int_G b(x) u u_\psi^* \, dx = \int_G u_\psi^* \, d\nu_a,$$

which implies

$$\int_G u \psi \, d\mu = - \left\{ \int_G [A(x) \nabla \theta \cdot \nabla u_\psi^*] dx + \int_G b(x) \theta u_\psi^* \, dx - \int_G \theta \psi \, d\mu \right\}$$

$$+ \int_G u_\psi^* \, d\nu. \tag{6.13}$$

Since $Lh_\theta^L = 0$, $u_\psi^* \in H_0^{1,2}(G;\mu)$, $L^* u_\psi^* = \psi w$ and $h_\theta^L - \theta \in H_0^{1,2}(G;\mu)$,

$$\int_G A(x)\nabla h_\theta^L \cdot \nabla u_\psi^* \, dx + \int_G b(x) h_\theta^L u_\psi^* \, dx = 0$$

and

$$\int_G A(x)^* \nabla u_\psi^* \cdot \nabla (h_\theta^L - \theta) dx + \int_G b(x) u_\psi^* (h_\theta^L - \theta) dx = \int_G (h_\theta^L - \theta)\psi \, d\mu.$$

These two equalities imply

$$\int_G [A(x)\nabla\theta \cdot \nabla u_\psi^*] dx + \int_G b(x)\theta u_\psi^* \, dx - \int_G \theta\psi \, d\mu = -\int_G h_\theta^L \psi \, d\mu.$$

Hence (6.13) means (6.11).

THEOREM 6.5. *Let $\theta_1, \theta_2 \in H^{1,2}(G;\mu)$ and v_1, v_2 be finite signed measures on G. Let u_j, $j = 1,2$ be the renormalized solutions of $Lu = v_j$ with boundary data θ_j. If* $\max(\theta_1 - \theta_2, 0) \in H_0^{1,2}(G;\mu)$ *and* $v_1 \le v_2$, *then* $u_1 \le u_2$ $(2,\mu)$-*quasieverywhere in G.*

PROOF. Let ψ be an arbitrary nonnegative bounded measurable function on G and let $h_{\theta_j}^L$ be the solution of $Lu = 0$ such that $h_{\theta_j}^L - \theta_j \in H_0^{1,2}(G;\mu)$ for each $j = 1,2$. By the comparison principle (cf. [8, Proposition 1.1]), we see that $u_\psi^* \ge 0$ and $h_{\theta_1}^L \le h_{\theta_2}^L$ in G. Hence, by Theorem 6.4,

$$\int_G u_1\psi \, d\mu = \int_G h_{\theta_1}^L \psi \, d\mu + \int_G u_\psi^* \, dv_1 \le \int_G h_{\theta_2}^L \psi \, d\mu + \int_G u_\psi^* \, dv_2 = \int_G u_2 \psi \, d\mu.$$

Since this is true for every nonnegative bounded measurable ψ, we conclude that $u_1 \le u_2$ a.e. in G. Since we have assumed that u_1, u_2 are $(2,\mu)$-quasicontinuous, the assertion of the theorem follows.

COROLLARY 6.1. *Given $\theta \in H^{1,2}(G,\mu)$ and a finite signed measure v on G, the renormalized solution of $Lu = v$ with boundary data θ is unique.*

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