

A Riesz decomposition theorem on harmonic spaces without positive potentials

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ABSTRACT. In this paper, we give a new definition of the flux of a superharmonic function defined outside a compact set in a Brelot space without positive potentials. We also give a new notion of potential in a BS space (that is, a harmonic space without positive potentials containing the constants) which leads to a Riesz decomposition theorem for the class of superharmonic functions that have a harmonic minorant outside a compact set. Furthermore, we give a characterization of the local axiom of proportionality in terms of a global condition on the space.

1. Introduction

The Riesz decomposition theorem for positive superharmonic functions states that any positive superharmonic function on a region of hyperbolic type in \mathbf{C} can be expressed uniquely as the sum of a nonnegative potential and a harmonic function. This result is of no interest when the region is parabolic because there are no nonconstant positive superharmonic functions.

By a *Riesz decomposition theorem for a class of superharmonic functions* we mean a unique representation of each member of this class as a global harmonic function plus a member of the class of a special type. For a harmonic space with a Green function, this special class is the class of nonnegative potentials. In this case the most natural class to consider is the *admissible* superharmonic functions, which are the superharmonic functions that have a harmonic minorant outside a compact set. Indeed, the admissible superharmonic functions are precisely the superharmonic functions which can be written uniquely as sums of a nonnegative potential plus a harmonic function. This follows from the fact that every admissible superharmonic function in a harmonic space with a Green function possesses a global harmonic minorant and hence, the function minus its greatest harmonic minorant is a nonnegative potential.

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In this article, we give a Riesz decomposition theorem for the class of the admissible superharmonic functions on a harmonic space without positive potentials. This class was first studied by Anandam in [2]. See also [6] for a reference on the admissible subharmonic functions.

In this paper, we give a new definition of the flux of a function that is superharmonic outside a compact set which is equivalent to the various definitions of flux in the works of Anandam (see [2] through [8]). It also encompasses the definitions of flux in the discrete setting given in [1], [9], and [15]. We then introduce a new class of potentials (called H -potentials) in the axiomatic setting that was first introduced in the recurrent tree and the complex plane settings in [15] and [16]. This new class allows us to obtain a global Riesz decomposition theorem for the class of superharmonic functions that have a harmonic minorant outside a compact set. We also give a characterization of the local axiom of proportionality in terms of a global condition on the space.

Many axiomatic treatments of potential theory have been formulated (for a survey and a historical context, see [13]). In this work we make use of the axiomatic theory of harmonic and superharmonic functions developed by Brelot (see [12]).

DEFINITION 1.1. A **Brelot space** is a connected, locally connected, locally compact but not compact Hausdorff space Ω together with a harmonic structure in the following sense. For each open set $U \subset \Omega$ there is an associated real vector space of real-valued continuous functions on U (which are called **harmonic functions on U**) satisfying the sheaf property, the regularity axiom, and the Harnack property.

The **harmonic support** of a superharmonic function s is the complement of the largest open set on which s is harmonic.

In a Brelot space, the Minimum Principle for superharmonic functions holds: *A nonnegative superharmonic function on a domain U in a Brelot space is either identically zero or positive everywhere on U* ([12], p. 71).

DEFINITION 1.2. A superharmonic function s on a Brelot space Ω is said to be **admissible** if there are a compact set K and a harmonic function h on $\Omega \setminus K$ such that $h(x) \leq s(x)$ for all $x \in \Omega \setminus K$.

Clearly, positive superharmonic functions and superharmonic functions of compact harmonic support are admissible. We shall see in Proposition 2.3 that K may be taken to be K_0 , any fixed compact set independent of the superharmonic function.

Any nonnegative superharmonic function which has a harmonic minorant has a greatest harmonic minorant (see [12], p. 87).

DEFINITION 1.3. A nonnegative superharmonic function on an open subset U of a Brelot space is called a **positive potential** (or briefly, a **potential**) if its greatest harmonic minorant on U is identically zero.

DEFINITION 1.4. A **BH space** is a Brelot space whose sheaf of harmonic functions contains the constants. A **BP space** is a BH space on which there is a positive potential. A **BS space** (short for *espace Brelot sans potentiel positif*) is a BH space on which no positive potential exists.

Any open subset of \mathbf{R}^n for $n \geq 3$ is a BP space, while the complement of any subset of \mathbf{R}^2 of logarithmic capacity zero is a BS space.

DEFINITION 1.5. A BP space is said to satisfy the **axiom of proportionality** if any two potentials with the same one-point harmonic support are proportional.

THEOREM 1.1 ([17], p. 139). *In a Brelot space without potentials all positive superharmonic functions are harmonic and proportional. In particular, in a BS space, every positive superharmonic function must be constant.*

Thus, a Brelot space which possesses positive superharmonic functions which are not harmonic, has potentials.

The following result is the Riesz decomposition theorem for admissible superharmonic functions on a BP space.

THEOREM 1.2 ([6], p. 66). *In a BP space a superharmonic function is admissible if and only if it can be written uniquely as the sum of a potential and a function harmonic on the whole space.*

THEOREM 1.3 ([20], Theorem 16.1, [2], Theorem 3.6, and [14]). *If Ω is a Brelot space with positive potentials and a countable base of neighborhoods or if Ω is a BS space, then for any $x \in \Omega$, there exists a superharmonic function with harmonic support $\{x\}$.*

By Theorem 1.3 and Theorem 1.2, if Ω is a Brelot space with potentials and a countable base of neighborhoods, then for each $x \in \Omega$ there exists a potential with harmonic support $\{x\}$.

2. The flux of a superharmonic function

The concept of flux on a BS space was introduced for the purpose of associating a harmonic function on Ω to a function that is harmonic outside a compact set.

Nakai (see [22] or Theorem 1.20 of [2]) proved that if h is a function defined on a BP space Ω and harmonic on the complement of a compact set K ,

then $h = h_\Omega + b$ outside a compact set, where h_Ω is harmonic on Ω and b is bounded. This is not true if Ω is a BS space, but we shall describe an obstruction, called the *flux of h* , so that when the flux of h is zero, then h is the sum of a global harmonic function and a bounded function.

The concept of flux of a function superharmonic outside a compact subset of a BS space appeared in [8] and [19].

PROPOSITION 2.1 ([3], Theorem 1.17). *Let Ω be a BS space, $K \subset \Omega$ compact not locally polar. Then there exists a function $H \geq 0$ unbounded and harmonic off K . If K is outer regular, then H may be chosen so that it tends continuously to 0 on ∂K .*

DEFINITION 2.1. Let Ω be a BS space, $K \subset \Omega$ a nonempty compact set. A function H harmonic off K is called a **standard for Ω** associated with K if the following is true: given any function h which is harmonic off an arbitrary compact set, there exist a unique function h_Ω harmonic on the whole space and a unique real number α such that $b = h - \alpha H - h_\Omega$ is bounded off a compact set and $\liminf_{x \rightarrow \infty} b(x) = 0$, where the \liminf is taken with respect to the Alexandrov one-point compactification of Ω . This means that for any increasing exhaustion $\{C_n\}$ consisting of compact sets, and for any positive number ε there exists $N \in \mathbf{N}$ such that for all integers $n \geq N$ there is a point $x \in \Omega \setminus C_n$ such that $b(x) < \varepsilon$, and for all $y \in \Omega \setminus C_n$, $b(y) > -\varepsilon$. Note that by the uniqueness of α , H must be unbounded.

By Theorem 9.7 of [10], the function $H(z) = \log|z|$ is a standard for \mathbf{C} associated with the closed unit disk. In this case, b has an actual limit of 0 at infinity because a function which is harmonic and bounded outside a compact set in \mathbf{C} has a limit at infinity. On the other hand, for $\Omega = \mathbf{C} \setminus \{2\}$, H is still a standard for Ω associated with the closed unit disk, but a bounded harmonic function outside a compact set will have two limits as we approach the boundary of Ω in the extended plane (i.e. as $z \rightarrow 2$ and as $z \rightarrow \infty$).

In [8] (see note following Lemma 2) the following result is shown.

THEOREM 2.1. *Let Ω be a BS space and let H be a nonnegative harmonic function defined outside an outer regular compact set K , not identically 0, and tending to 0 at the boundary of K . Then H is a standard for Ω .*

In [11] (Theorem 4.2), we proved that the outer regularity of the compact set is unnecessary.

In the remainder of this section and in the next section, we shall fix a BS space Ω , a nonempty compact subset K_0 which is not locally polar, and a standard H associated with K_0 tending to 0 on ∂K_0 , which for convenience we extend to be identically 0 in the interior of K_0 . Observe that this extension is subharmonic on all of Ω .

The following is the key result which we shall use to calculate the flux of a superharmonic function outside a compact set.

PROPOSITION 2.2 ([8], Proposition 1). *Let s be superharmonic outside some compact set. Then there exist a superharmonic function s_Ω on Ω and a constant β such that $s = s_\Omega + \beta H$ outside a compact set.*

REMARK 2.1. Let s be a function superharmonic outside a compact set. When it is clear from the context that the values of s on any particular compact set are unimportant, we shall use Proposition 2.2 to change s on a compact set so that it is defined globally and it is superharmonic outside K_0 . In particular, whenever it is useful, we shall assume that s is defined globally, is superharmonic outside K_0 and is lower bounded on K_0 .

The decomposition in Proposition 2.2 is not unique because we may increase β by subtracting an appropriate multiple of $-H$ from s_Ω , since $-H$ is globally superharmonic. Thus, it would be interesting to know the following: let $\beta_0 = \inf \beta$ over all β such that $s = s_\Omega + \beta H$ for some s_Ω superharmonic on Ω . When is $\beta_0 > -\infty$? We shall discuss this question at the end of this section.

REMARK 2.2. Let s be an admissible superharmonic function, and let h_1 be a function which is harmonic outside a compact set K and such that $h_1 \leq s$ outside K . By taking the restriction of h_1 to an outer regular compact set containing K , we may assume that K is outer regular. We may extend h_1 to a continuous function on Ω . Let b_1 be a lower bound of s on K and let b_2 be an upper bound of h_1 on K . Then $h = h_1 - |b_2 - b_1| \leq s$ on Ω and h is harmonic outside K . Thus, when we say that s has a harmonic minorant h outside a compact set, we mean that $h \leq s$ globally and h is harmonic outside a compact set.

REMARK 2.3 ([2], p. 133). The difference of the greatest harmonic minorants h_i of a superharmonic function outside a compact set K_i ($i = 1, 2$) is bounded.

DEFINITION 2.2. Let s be superharmonic outside a compact set. Define $A_s = \{\alpha \in \mathbf{R}: \text{there exists } h_\Omega \text{ harmonic on } \Omega \text{ such that } s - \alpha H \geq h_\Omega\}$.

LEMMA 2.1. *If s is superharmonic outside a compact set, then A_s is bounded above. Furthermore, $A_s \neq \emptyset$ if and only if s has a harmonic minorant outside a compact set.*

PROOF. By Proposition 2.2, there exist $\beta \in \mathbf{R}$ and s_Ω superharmonic on Ω such that $s = \beta H + s_\Omega$ outside a compact set. For $\alpha \in A_s$, let h_Ω be harmonic

on Ω such that $s - \alpha H \geq h_\Omega$. If $\alpha > \beta$, then $s_\Omega - h_\Omega$ is a lower bounded superharmonic function (since it is necessarily bounded below on any compact set), hence is constant. But by Proposition 2.1, a constant cannot be bounded below by a positive multiple of H . Thus, $\alpha \leq \beta$ and so A_s is bounded above by β .

Assume $\alpha \in A_s$. Then there exists h_Ω harmonic on Ω such that $s \geq \alpha H + h_\Omega$. Thus s has a harmonic minorant outside a compact set. Conversely, if s has a harmonic minorant h outside a compact set, then $h = \alpha H + h_\Omega + b$ outside a compact set, for some $\alpha \in \mathbf{R}$, h_Ω harmonic on Ω , and b bounded. Then outside a compact set $s - \alpha H \geq h_\Omega + \inf b$, a global harmonic function. Thus $\alpha \in A_s$. \square

DEFINITION 2.3. Let s be a function on Ω superharmonic outside a compact set. Define the **flux of s at infinity** (or simply the **flux of s**) with respect to H by

$$\text{flux}(s) = \sup A_s.$$

By convention, $\text{flux}(s) = -\infty$ if $A_s = \emptyset$. By Lemma 2.1, the flux of s is finite if A_s is nonempty.

The following result ties together our definition of flux with various earlier definitions, as well as compiling many useful properties of flux.

THEOREM 2.2. (a) *If h is harmonic outside a compact set, then the flux of h is the unique constant α of Definition 2.1 such that $h = \alpha H + h_\Omega + b$ outside a compact set with h_Ω harmonic on Ω and b bounded. In particular, $A_h = (-\infty, \alpha]$.*

(b) *If h is bounded harmonic outside a compact set or harmonic everywhere, then the flux of h is zero. If s is superharmonic everywhere, then $\text{flux}(s) \leq 0$.*

(c) *Let s be an admissible superharmonic function on Ω . If h_1 and h_2 are the greatest harmonic minorants of s outside compact sets K_1 and K_2 , respectively, then the flux of h_1 and the flux of h_2 are equal.*

(d) *If s is a function superharmonic outside a compact set K and has a subharmonic minorant on $\Omega \setminus K$ (in particular, if s is admissible), then its flux is equal to the flux of its greatest harmonic minorant on $\Omega \setminus K$. Consequently, admissible superharmonic functions have finite flux.*

(e) *The flux of a nonadmissible superharmonic function s is equal to $-\infty$.*

(f) *Let s be superharmonic outside a compact set, and write $s = s_\Omega + \beta H$ as in Proposition 2.2. Then $\text{flux}(s) = \text{flux}(s_\Omega) + \beta$.*

(g) *The set of functions which are superharmonic outside a compact set is closed under addition and scalar multiplication by a positive number and the flux is linear on that set.*

- (h) If s is superharmonic outside a compact set and $A_s \neq \emptyset$, then $\text{flux}(s) \in A_s$, so that $A_s = (-\infty, \text{flux}(s)]$.
- (j) If s is superharmonic on Ω and $\text{flux}(s) = 0$, then s is harmonic on Ω .

PROOF. We use Proposition 2.2, Lemma 2.1, Remark 2.3, and the following immediate facts:

- (i) There are no nonconstant positive superharmonic functions, hence no nonharmonic superharmonic function can be bounded below on Ω by a harmonic function.
- (ii) If s is superharmonic outside a compact set, then for all $\alpha \in \mathbf{R}$ and $c > 0$, $A_{s+\alpha H} = A_s + \alpha$, and $A_{cs} = cA_s$.
- (iii) (Monotonicity of the flux) If $s_1 \leq s_2$, with s_1, s_2 superharmonic outside a compact set, then $A_{s_1} \subset A_{s_2}$, so that $\text{flux}(s_1) \leq \text{flux}(s_2)$.
- (iv) If h_Ω is harmonic on Ω and $h_\Omega + \gamma H$ is bounded below, then $\gamma \geq 0$.
- (v) If s_1 and s_2 are superharmonic outside a compact set K , then the greatest harmonic minorant of $s_1 + s_2$ is the sum of the greatest harmonic minorants of s_1 and s_2 on K .

Part (a) follows from (iv). Part (b) follows from (a) and Lemma 2.1. Part (c) holds since Remark 2.3 implies that $A_{h_1} = A_{h_2}$.

To prove (d), let h be the greatest harmonic minorant of s on $\Omega \setminus K$. By (c), without loss of generality we may assume that K contains K_0 . By (a), $A_h \neq \emptyset$, and by (iii), $A_h \subset A_s$, so $A_s \neq \emptyset$. Let $\alpha \in A_s$ so that $s - \alpha H \geq h_\Omega$, a function harmonic on Ω . Thus $s \geq \alpha H + h_\Omega$, which is harmonic off K_0 . Then on $\Omega \setminus K$, $s \geq h \geq \alpha H + h_\Omega$, so by (iii) and (a), $\text{flux}(s) \geq \text{flux}(h) \geq \alpha$. Since this is true for all $\alpha \in A_s$, it follows that $\text{flux}(s) = \text{flux}(h)$.

To prove (e), assume there exists $\alpha \in A_s$. Then $s \geq \alpha H + h_\Omega$, where h_Ω is a function harmonic on Ω . Since $\alpha H + h_\Omega$ is harmonic outside K_0 , s is admissible.

Part (f) follows from (ii). To prove (g), let s_1, s_2 be superharmonic outside the same compact set K , and let h_1 and h_2 be their respective greatest harmonic minorants outside K . Then by (v), $h_1 + h_2$ is the greatest harmonic minorant of $s_1 + s_2$ outside K . Thus by (d) and (a), $\text{flux}(s_1 + s_2) = \text{flux}(h_1 + h_2) = \text{flux}(h_1) + \text{flux}(h_2) = \text{flux}(s_1) + \text{flux}(s_2)$. Linearity with respect to multiplication by a positive constant follows from (ii).

Part (h) follows from (d) and (a). Part (j) follows from (h) and (i). \square

REMARK 2.4. The original definitions of flux given by Anandam [8] separately first for harmonic functions outside a compact set, then for global superharmonic functions, and finally for functions superharmonic outside a compact set, are equivalent to ours, by Theorem 2.2, parts (a), (d), (e) and (f).

PROPOSITION 2.3. *A superharmonic function on a BS space Ω is admissible if and only if it has a minorant which is harmonic outside K_0 .*

PROOF. Let s be an admissible superharmonic function on Ω with flux α . Then by part (h) of Theorem 2.2, there exists a harmonic function h_Ω on Ω such that $s \geq \alpha H + h_\Omega$, which is harmonic outside K_0 . The converse is obvious. \square

We now respond to the question raised after Remark 2.1.

PROPOSITION 2.4. *Let s be superharmonic outside a compact set in a BS space Ω . Let B be the set consisting of all $\beta \in \mathbf{R}$ such that $s = s_\Omega + \beta H$ outside a compact set as in Proposition 2.2. Then*

(a) *The set B is an interval unbounded above and $\inf B \geq \text{flux}(s)$. In particular, if s has finite flux, then B is bounded below.*

(b) *When $\Omega = \mathbf{R}$ with the harmonic structure inherited from the Laplace operator, s has finite flux if and only if B is bounded below.*

PROOF. To prove (a) assume $\beta \in B$ and $\gamma > \beta$. If $s = s_\Omega + \beta H$ outside a compact set K , then $s = s'_\Omega + \gamma H$ outside K , where $s'_\Omega = s_\Omega - (\gamma - \beta)H$ which is globally superharmonic. Thus $\gamma \in B$, proving that B is an interval unbounded above. On the other hand, since the flux of a superharmonic function on Ω is less than or equal to 0, if $s = s_\Omega + \beta H$ outside a compact set then $\beta \geq \text{flux}(s)$. Thus, if $\text{flux}(s)$ is finite, B is bounded below.

By part (a), to prove (b) we need to show that if B is bounded below then s has finite flux. Let $K_0 = \{0\}$. The function $H(x) = |x|$ is a standard for K_0 . Let us consider $s(x) = |x| - x^2$ for $x \in \mathbf{R}$, which has flux $-\infty$ since it does not have a harmonic minorant outside a compact set. Since s is smooth on $\mathbf{R} \setminus \{0\}$ and its Laplacian is -2 there, s is superharmonic on $\mathbf{R} \setminus \{0\}$. For $n \in \mathbf{N}$, let $\tilde{s}_n(x) = s(x) + (2n - 1)H(x)$ and observe that for $x \geq n$, $s'(x) \leq 0$ and for $x \leq -n$, $s'(x) \geq 0$, where s' denotes the derivative of s . Thus the function

$$s_n(x) = \begin{cases} \tilde{s}_n(n) & \text{for } |x| \leq n \\ \tilde{s}_n(x) & \text{for } |x| \geq n \end{cases}$$

is globally superharmonic and $s(x) = s_n(x) + \beta_n H(x)$ outside $[-n, n]$, where $\beta_n = -(2n - 1)$. Thus, B is unbounded below. \square

3. Potentials in a BS space and Riesz decomposition of admissible superharmonic functions

In this section we shall present several classes of admissible superharmonic functions which in a BS space play the role analogous to that of positive

potentials in a BP space. We first introduce two operators which we use to define these classes.

DEFINITION 3.1. Let s be superharmonic on $\Omega \setminus K_0$. Let $\mathcal{E} = \{U_n\}$ be an increasing exhaustion consisting of relatively compact regular sets containing K_0 (which exists by [21]). Let $h_n = h_s^{U_n}$, the solution of the Dirichlet problem with boundary values s on ∂U_n . Define

$$D_{\mathcal{E}}s(x) = \lim_{n \rightarrow \infty} h_n(x)$$

if this limit exists locally uniformly, in which case $D_{\mathcal{E}}s$ is harmonic on Ω .

DEFINITION 3.2. Let K_0 be outer regular and let s be superharmonic on $\Omega \setminus K_0$. Define Ds to be the greatest harmonic minorant of s on each component of the complement of K_0 if such minorant exists and $-\infty$ if it does not exist on that component.

We now present different classes of potentials introduced by Anandam in [6] and [7].

DEFINITION 3.3. An admissible superharmonic function s is said to be in the **class** \mathcal{P} if there exists an exhaustion \mathcal{E} such that $D_{\mathcal{E}}(s - \alpha H)$ exists and is constant, where α is the flux of s . If, furthermore, that constant is 0 for some exhaustion \mathcal{E} , s is called a **BS potential**. Define the **class** \mathcal{Q} as the collection of all admissible superharmonic functions s satisfying the property: there exists $s' \in \mathcal{P}$ such that the difference of the greatest harmonic minorants of s and s' outside a compact set is bounded. This class is independent of the choice of the compact set.

OBSERVATION 3.1. *Suppose s is in class \mathcal{P} and has flux α , so that for some exhaustion \mathcal{E} , $D_{\mathcal{E}}(s - \alpha H)$ exists and is constant. By Lemma 2, p. 235 in [4], $s - \alpha H$ is lower bounded. If $D_{\mathcal{E}'}(s - \alpha H)$ exists for some other exhaustion \mathcal{E}' , then $D_{\mathcal{E}'}(s - \alpha H)$ is a lower bounded harmonic function, hence it is also constant.*

Anandam proved the following partial Riesz decomposition theorem for admissible superharmonic functions on a BS space.

PROPOSITION 3.1 ([5], Lemmas 2 and 3). *Any admissible superharmonic function s on a BS space is a sum of a function in the class \mathcal{Q} and a harmonic function. This decomposition is unique up to an additive constant. If s has compact harmonic support, then the element of \mathcal{Q} can be chosen uniquely to be a BS potential.*

One difficulty in working with these classes of potentials is that given an admissible superharmonic function s , there is no procedure for determining

whether s is in such classes. To overcome this difficulty and the lack of uniqueness in the decomposition of Proposition 3.1, we introduce a new class of potentials.

DEFINITION 3.4. An admissible superharmonic function s is an **H -potential** if

$$\liminf_{x \rightarrow \infty} \{Ds(x) - \alpha H(x)\} = 0,$$

where D is the operator of Definition 3.2, α is the flux of s , and the \liminf is taken with respect to the Alexandrov one-point compactification of Ω .

The following theorem shows that the H -potentials are the class of potentials best suited to describe all admissible superharmonic functions.

THEOREM 3.1 (Global Riesz Decomposition Theorem). *In a BS space Ω every admissible superharmonic function can be written uniquely as the sum of an H -potential and a harmonic function.*

PROOF. Let s be an admissible superharmonic function. Since Ds is the greatest harmonic minorant of s outside K_0 , by Theorem 2.2(a) and (d), there exist h_Ω harmonic and b bounded such that $Ds = \alpha H + h_\Omega + b$ outside a compact set, where α is the flux of s . By adding the condition that

$$(1) \quad \liminf_{x \rightarrow \infty} b(x) = 0,$$

we get uniqueness in the above decomposition. Then the function $p = s - h_\Omega$ is admissible superharmonic with flux α and $Dp = Ds - h_\Omega = \alpha H + b$. By (1), $Dp - \alpha H$ has inferior limit 0 at infinity, thus p is an H -potential and $s = p + h_\Omega$, proving the existence of the decomposition.

To prove the uniqueness, assume $p_1 + h_1 = p_2 + h_2$ where p_1 and p_2 are H -potentials and h_1 and h_2 are harmonic on Ω . Then $p_1 - p_2 = h_2 - h_1$, which is globally harmonic. In particular, p_1 and p_2 have the same flux α . By definition of H -potential, $Dp_j = \alpha H + b_j$ where b_j ($j = 1, 2$) is bounded and harmonic outside a compact set with $\liminf 0$ at infinity. Since $h_2 - h_1 = p_1 - p_2 = D(p_1 - p_2) = Dp_1 - Dp_2 = b_1 - b_2$, the function $b_1 - b_2$ can be extended to a global bounded harmonic function. Thus, $b_1 - b_2$ is constant. Since $\liminf b_1 = \liminf b_2 = 0$, $b_1 = b_2$ and hence $h_1 = h_2$. \square

COROLLARY 3.1. *If s is an H -potential with flux α , then $Ds - \alpha H$ is bounded.*

PROOF. By the proof of Theorem 3.1, for any admissible superharmonic function s , the unique global harmonic function h_Ω in the decomposition of Ds as $\alpha H + h_\Omega + b$ (with b bounded) is the **same** global harmonic function in the

decomposition of s as an H -potential plus a global harmonic function. Thus if s is itself an H -potential, then $h_\Omega = 0$, and so $Ds = \alpha H + b$. \square

In [15], where we restricted our attention to recurrent trees, we defined an H -potential to be an admissible superharmonic function s for which $Ds = \alpha H$, where $\alpha = \text{flux}(s)$ and K_0 is taken to be a single point. It can be easily shown that in this setting the two definitions of H -potential are equivalent.

4. Proportionality in BS spaces

DEFINITION 4.1. A BreLOT space Ω satisfies the **local axiom of proportionality** if for each $x \in \Omega$ and each relatively compact neighborhood U of x , all potentials on U with harmonic support at x are proportional.

THEOREM 4.1 (Local Riesz Decomposition Theorem). *If Ω is a BS space satisfying the local axiom of proportionality, then the following properties hold.*

(a) *For all $x \in \Omega$ there exists a unique BS potential p_x with harmonic support $\{x\}$ such that $\text{flux}(p_x) = -1$.*

(b) [18] *For all s superharmonic on Ω there exists a unique Radon measure $\mu \geq 0$ such that for each relatively compact domain $U \subset \Omega$ and for all $x \in U$*

$$s(x) = \int_U p_y(x) d\mu(y) + h(x),$$

where h is harmonic on U and p_y is the unique BS potential with harmonic support y and flux equal to -1 .

PROOF. (a) By Theorem 1.3, given any $x \in \Omega$, there exists a superharmonic function s_Ω with harmonic support $\{x\}$. By Proposition 3.1, there exists a BS potential p with harmonic support $\{x\}$. Since the flux of an admissible superharmonic function which is not harmonic is negative, the function $p_x = -\frac{p}{\text{flux}(p)}$ is superharmonic, and hence, a BS potential with harmonic support $\{x\}$ and flux -1 , proving the existence.

For the uniqueness, assume that p and q are BS potentials on Ω having harmonic support $\{x\}$ and flux -1 . Let U be a relatively compact regular neighborhood of x . Since p and q are superharmonic on U and U is a BP space, there exist potentials p_1 and p_2 on U with harmonic support $\{x\}$, and h_1, h_2 harmonic on U , such that $p = p_1 + h_1$, and $q = p_2 + h_2$. By the local axiom of proportionality, there exists $\lambda > 0$ such that $p_1 = \lambda p_2$. Thus $h = p - \lambda q$ is harmonic on U , but it is also harmonic off x , and therefore is harmonic on Ω . Thus

$$-1 = \text{flux}(p) = \text{flux}(h + \lambda q) = -\lambda,$$

so $\lambda = 1$. This shows that $p - q$ is harmonic on Ω . By Observation 3.1 we deduce that $p - q$ is constant. Since p and q are BS-potentials, that constant must be 0. Thus $p = q$ on Ω . \square

The following result is a superharmonic extension theorem for BH spaces. It was proved by Hervé [20] in the BP case and later by Anandam (Theorem 3.4 of [2]) in the BS case.

THEOREM 4.2. *Let Ω be a BH space and let U be an open subset of Ω . If s is a superharmonic function on U with compact harmonic support K , then there exists a superharmonic function s_Ω on Ω with harmonic support on K such that $s_\Omega - s$ is harmonic in a neighborhood of K .*

The following is a new characterization of the local axiom of proportionality.

THEOREM 4.3. *In a BH space, the following statements are equivalent.*

- (a) *The local axiom of proportionality holds.*
- (b) *For any two superharmonic functions with the same one-point harmonic support, some nonzero linear combination of them is harmonic.*

PROOF. Assume that the local axiom of proportionality holds on Ω , and let s_1, s_2 be superharmonic on Ω with harmonic support at $x \in \Omega$. Let U be a relatively compact neighborhood of x . Then $s_1|U$ and $s_2|U$ are superharmonic on U with harmonic support at x , and being lower-semicontinuous, they are bounded below on a relatively compact set, so they have harmonic minorants. Let h_1 and h_2 be the greatest harmonic minorants of $s_1|U$ and $s_2|U$, respectively. Thus $s_1|U - h_1$ and $s_2|U - h_2$ are potentials on U with harmonic support at x , and so for some $\alpha > 0$, $s_1|U - h_1 = \alpha(s_2|U - h_2)$. Hence $(s_1 - \alpha s_2)|U = h_1 - \alpha h_2$ which is harmonic on U . But $s_1 - \alpha s_2$ is harmonic outside x , thus $s_1 - \alpha s_2$ is harmonic on Ω , proving that (b) holds.

Conversely, suppose (b) holds. Let $x \in \Omega$, U a relatively compact neighborhood of x , and let p_1 and p_2 be potentials on U with harmonic support at x . By Theorem 4.2, there exist superharmonic functions s_1 and s_2 on Ω with harmonic support at x such that $s_1 - p_1$ and $s_2 - p_2$ are harmonic on a neighborhood of x . Thus, there exist nonzero $\alpha, \beta \in \mathbf{R}$ such that $\alpha s_1 + \beta s_2$ is harmonic on Ω . Then $\alpha(s_1 - p_1) + \beta(s_2 - p_2)$ is harmonic on U , so $\alpha p_1 + \beta p_2$ is harmonic on U . Letting $\lambda = -\frac{\beta}{\alpha}$, we have that $p_1 - \lambda p_2$ is a harmonic function h on U . Notice that λ cannot be negative since otherwise $-\lambda p_2$ would be a potential on U and the sum of two potentials cannot be harmonic. Thus, $\lambda > 0$ and $p_1 = \lambda p_2 + h$. But the greatest harmonic minorant of p_1 and of p_2 is zero. Hence $h = 0$ and so $p_1 = \lambda p_2$ on U , proving the local axiom of proportionality. \square

EXAMPLE 4.1. *Let Ω be an open subset of \mathbf{R}^n whose harmonic structure is defined by the Laplace operator Δ . Then Ω satisfies part (b) of Theorem 4.3, hence satisfies the local axiom of proportionality, since if s_1 and s_2 are superharmonic functions on Ω with support at $x \in \Omega$, then $\Delta s_2(x)s_1 - \Delta s_1(x)s_2$ is harmonic on Ω .*

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