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Oscillation theorems for damped elliptic differential equations of second order

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Abstract. Oscillation theorems for the damped elliptic differential equation of second order

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j y] + \sum_{i=1}^{N} b_i(x)D_i y + c(x, y) = 0$$

are obtained. The results are extensions of averaging techniques due to Coles and Kamenev, and include earlier known results in literature.

1. Introduction

In the qualitative theory of nonlinear partial differential equations (PDE), one of the important problems is to determine whether or not solutions of the equation under consideration are oscillatory. We are here concerned with the oscillatory properties of solutions of the damped elliptic differential equation of second order

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j y] + \sum_{i=1}^{N} b_i(x)D_i y + c(x, y) = 0 \tag{1.1}$$

in $\Omega(r_0) \subseteq \mathbf{R}^N$, where $x = (x_1, \ldots, x_N) \in \mathbf{R}^N$, $D_i y = \partial y/\partial x_i$ for all i, $\|x\| = \left[\sum_{i=1}^{N} x_i^2\right]^{1/2}$, and $\Omega(r_0) = \{x \in \mathbf{R}^N : \|x\| \geq r_0\}$ for some constant $r_0 > 0$.

Throughout this paper we assume that the following conditions hold.

(A1) $A = (a_{ij}(x))$ is a real symmetric positive definite matrix function with $a_{ij} \in \mathbf{C}_{loc}^{1+\nu}(\Omega(r_0), \mathbf{R})$ for all i, j, and $\nu \in (0, 1)$;

(A2) $b_i \in \mathbf{C}_{loc}^{\nu}(\Omega(r_0), \mathbf{R})$ for all i;

(A3) $c \in \mathbf{C}_{loc}^{\nu}(\Omega(r_0) \times \mathbf{R}, \mathbf{R})$ with $c(x, -y) = -c(x, y)$ for all $x \in \mathbf{R}^N$, $y \in \mathbf{R}$;

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(A4) Suppose that there exist functions $p \in \mathbf{C}_{loc}^{\nu}(\Omega(r_0), \mathbf{R})$ and $f \in \mathbf{C}(\mathbf{R}, \mathbf{R}) \cup \mathbf{C}^1(\mathbf{R} - \{0\}, \mathbf{R})$ with $yf(y) > 0$ and $f'(y) \geq \varepsilon > 0$ whenever $y \neq 0$ such that

$$c(x, y) \geq p(x)f(y) \qquad \text{for all } x \in \Omega(r_0),\ y > 0.$$

In what follows, a solution of Eq. (1.1) is a function of the class $\mathbf{C}_{loc}^{2+\nu}(\Omega(r_0), \mathbf{R})$, which satisfies Eq. (1.1) everywhere on $\Omega(r_0)$. We consider only nontrivial solution of Eq. (1.1) which is defined for all large $\|x\|$ (see [2]). The oscillation is considered in the usual sense, i.e., a solution $y(x)$ of Eq. (1.1) is said to be oscillatory if it has zero on $\Omega(a)$ for every $a \geq r_0$. Equation (1.1) is said to be oscillatory if every solution (if any exists) is oscillatory. Conversely, Equation (1.1) is nonoscillatory if there exists a solution which is not oscillatory.

In the absence of damping, namely, $b_i(x) \equiv 0$ for all i, there have been many papers devoted to this case of Eq. (1.1) (see, for example, [7, 9, 11, 12, 15, 16, 17] and the references therein) such as the semilinear elliptic differential equation

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j y] + p(x)f(y) = 0, \tag{1.2}$$

and the more general case

$$\sum_{i,j=1}^{N} D_i[a_{ij}(x)D_j y] + c(x, y) = 0. \tag{1.3}$$

Some of these known oscillation criteria (for instance, [7, 12, 15, 17]) are, roughly speaking, derived from either the criteria due to Wintner [10] or Kamenev [3] for the 1-dimensional second order linear ordinary differential equation (ODE)

$$y''(t) + p(t)y(t) = 0, \qquad p \in \mathbf{C}([t_0, \infty), \mathbf{R}), \tag{1.4}$$

which respectively state that Eq. (1.4) is oscillatory if

$$\lim_{t \to \infty} \int_{t_0}^{t} p(s)ds = \infty,$$

or,

$$\limsup_{t \to \infty} \frac{1}{t^m} \int_{t_0}^{t} (t - s)^m p(s)ds = \infty \qquad \text{for some } m > 1.$$

In [1], Coles introduced the idea of weighted average to obtain an oscillation criterion for Eq. (1.4) which extended an earlier result of Wintner

[10]. Coles'work was subsequently extended by Macki and Wong [5] through the use of an averaging pair. In a different direction, Phiols [8] introduced the concept of general means and obtained further extensions of Kamenev's type criteria for Eq. (1.4). Recently, Kong [4] established the called interval criteria for Eq. (1.4), which use only the information above the function $p(t)$ on a sequence of intervals approaching ∞.

In 1980, using the N-dimensional vector Riccati transformation, the pioneering work of Noussair and Swanson [7] investigated the oscillation of Eq. (1.3), and established Wintner's type criteria for Eq. (1.3) (see [7], Theorem 4). Recently, Xu [12] obtained Phiols-type theorems for Eq. (1.2), and Zhuang et al [17] extended Kong's results to Eq. (1.2).

However, compared to the undamped equations (1.2) and (1.3), the study of oscillation of the damped equation (1.1) has received considerably less attention in the literature. The partial reasons seem that the Riccati transformation, which plays a key role in the proof of the results for (1.2) and (1.3), is N-dimensional vector function, which prevent simple extension of the existing work for ODE. On the other hand, it is very difficult for us to find a suitable transformation, which like the Strum-Liouville transformation for ODE, to reduce Eq. (1.1) into the undamped equation.

In fact, we note that in many areas of their actually application, models describing these problems are affected by such factors as damping term [cf, [2]]. Therefore it is necessary, either theoretically or practically, to study a type of equation in more general sense–damped elliptic differential equations. But, as far as we know Eq. (1.1) has never been the subject of systematic investigations by the averaging techniques [5, 8] expect for paper [6, 14]. Very recently, under the assumption when the damped functions b_i for all i, are differentiable, Xu [13] and Xu et al [14] extended the Wintner and Philos theorems to Eq. (1.1). Such extensions can also be found in Mařík [6] for the linear equation

$$\triangle y + \sum_{i=1}^{N} b_i(x) D_i y + p(x) y = 0. \tag{1.5}$$

It is therefore natural to ask whether the Wintner and Kamenev theorems can be extended to Eq. (1.1) when the damped functions b_i for all i are not necessarily differentiable. The purpose of this paper is to answer this question in the affirmative. In fact, by the approach in use of the averaging pair technique introduced by Macki and Wong [5], we will obtain Wintner's type theorem for Eq. (1.1). Further, using the integral averaging technique developed by Philos [8], we give some Kamenev's type, and more generally, Philos and Kong's type criteria for Eq. (1.1). By choosing appropriate functions, we shall present several easily verifiable oscillation criteria. In particular, three examples are given to illustrate the significance of our main results. It is worth emphasizing that the obtained theorems here are new even for Eqs. (1.2) and (1.3) and improve the main results in [7, 12, 13, 14, 17].

2. Preliminary results

In order to discuss our main results, we need the following definitions and lemmas.

DEFINITION 2.1. *A pair of functions* (σ, α) *is called an averaging pair if*
(i) $\sigma \in \mathbf{C}([r_0, \infty), [0, \infty))$ *and* $\alpha(r) > 0$, α *is absolutely continuous on every compact subinterval of* $[r_0, \infty)$;
(ii) *for some* $\gamma \in [0, 1)$,

$$\lim_{r\to\infty} \int_{r_0}^{r} \sigma(s) \left(\int_{r_0}^{s} \sigma(u)du \right)^{\gamma} \left(\int_{r_0}^{s} \alpha(u)\sigma^2(u)du \right)^{-1} ds = \infty.$$

Note that conditions in Definition 2.1 imply that $\int_{r_0}^{\infty} \sigma(s)ds = \infty$ (see [5]).

DEFINITION 2.2. *Let* $D = \{(r, s) : r \geq s \geq r_0\}$ *and* $D_0 = \{(r, s) : r > s \geq r_0\}$. *We say that a function* $H = H(r, s) \in \mathbf{C}^1(D, \mathbf{R})$ *belongs to a function class* $\mathscr{H}$, *defined by* $H \in \mathscr{H}$, *if there exist functions* $h_1, h_2 \in \mathbf{C}(D_0, R)$ *satisfying the following conditions:*
(H1) $H(r, r) = 0$ *for* $r \geq r_0$, $H(r, s) > 0$ *on* D_0;
(H2) $\frac{\partial H}{\partial r} = h_1(r, s)H(r, s)$ *and* $\frac{\partial H}{\partial s} = -h_2(r, s)H(r, s)$.

LEMMA 2.1. *For two n-dimensional vectors* $u, v \in \mathbf{R}^N$ *and a positive constant a. Then the following inequality*

$$a\|u\|^2 + \langle u, v\rangle \geq \frac{a}{2}\|u\|^2 - \frac{1}{2a}\|v\|^2 \tag{2.1}$$

holds, where $\langle\, , \rangle$ *denotes the usual scalar product in* $\mathbf{R}^N$.

The proof of Lemma 2.1 is easy and can be omitted.

LEMMA 2.2. *Let* $y = y(x)$ *be a nonoscillatory solution of Eq.* (1.1), *and* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$. *Then N-dimensional vector function* $w(x)$ *defined by*

$$w(x) = \frac{1}{f(y)}(A\nabla y)(x) \tag{2.2}$$

satisfies the following partial Riccati inequality

$$\operatorname{div}(\phi(x)w(x)) \le -\phi(x)p(x) - \frac{\varepsilon\phi(x)}{2\lambda_{\max}(x)}\|w(x)\|^2 + \frac{\lambda_{\max}(x)}{2\varepsilon\phi(x)}\|\phi(x)b(x)A^{-1} - \nabla\phi(x)\|^2, \tag{2.3}$$

where $\lambda_{\max}(x)$ *denotes the largest eigenvalue of the matrix* A *and* $\nabla\phi(x) = (D_1\phi(x), \dots, D_N\phi(x))$.

Proof. A direct computation shows that

$$\operatorname{div} w(x) \le -p(x) - f'(y)(w^T A^{-1} w)(x) - \langle b(x)A^{-1}, w^T(x)\rangle. \tag{2.4}$$

Note that

$$(w^T A^{-1} w)(x) \ge \frac{\|w(x)\|^2}{\lambda_{\max}(x)}.$$

Then (2.4) implies that

$$\operatorname{div} w(x) \le -p(x) - \frac{\varepsilon}{\lambda_{\max}(x)}\|w(x)\|^2 - \langle b(x)A^{-1}, w^T(x)\rangle. \tag{2.5}$$

Multiplying (2.5) by $\phi(x)$, we get

$$\operatorname{div}(\phi(x)w(x)) \le -\phi(x)p(x) - \frac{\varepsilon\phi(x)}{\lambda_{\max}(x)}\|w(x)\|^2 - \langle \phi(x)b(x)A^{-1} - \nabla\phi(x), w^T(x)\rangle. \tag{2.6}$$

By Lemma 2.1, we have

$$\frac{\varepsilon\phi(x)}{\lambda_{\max}(x)}\|w(x)\|^2 + \langle \phi(x)b(x)A^{-1} - \nabla\phi(x), w^T(x)\rangle \ge \frac{\varepsilon\phi(x)}{2\lambda_{\max}(x)}\|w(x)\|^2 - \frac{\lambda_{\max}(x)}{2\varepsilon\phi(x)}\|\phi(x)b(x)A^{-1} - \nabla\phi(x)\|^2. \tag{2.7}$$

Combining (2.6) and (2.7), we obtain (2.3). □

For notational simplicity, let

$$\lambda_\phi(r) = \int_{S_r} \phi(x)\lambda_{\max}(x)d\sigma,$$

$$\rho(r) = \exp\left(-\varepsilon\int^r \frac{\eta(s)}{\lambda_\phi(s)}\,ds\right),$$

$$g(r) = \frac{2}{\varepsilon}\rho(r)\lambda_\phi(r),$$

$$p_\phi(x) = \phi(x)p(x) - \frac{\lambda_{\max}(x)}{2\varepsilon\phi(x)}\|\phi(x)b(x)A^{-1} - \nabla\phi(x)\|^2,$$

$$P(r) = \rho(r)\left\{\int_{S_r} p_\phi(x)dS + \frac{\varepsilon}{2}\frac{\eta^2(r)}{\lambda_\phi(r)} - \eta'(r)\right\},$$

where $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$ and $\eta \in \mathbf{C}^1([r_0, \infty), \mathbf{R})$ are given functions, $S_r = \{x \in \mathbf{R}^N : |x| = r\}$, dS denotes the spherical integral element in $\mathbf{R}^N$.

Lemma 2.3. *Assume that $y = y(x)$ is a solution of Eq.* (1.1) *with $y(x) \neq 0$ for all $|x| \geq \tau$, $\tau \geq r_0$. For $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$ and $\eta \in \mathbf{C}^1([r_0, \infty), \mathbf{R})$, let $Z(r)$ be defined by*

$$Z(r) = \rho(r)\left[\int_{S_r} \langle\phi(x)w(x), v(x)\rangle dS + \eta(r)\right], \qquad r \geq \tau. \tag{2.8}$$

Then $Z(r)$ satisfies the Riccati inequality

$$Z'(r) \leq -P(r) - \frac{1}{g(r)}Z^2(r), \qquad r \geq \tau, \tag{2.9}$$

where $v(x) = x/r$, $r = \|x\| \neq 0$, denotes the outward unit normal to S_r.

Proof. By means of the Green formula in (2.8), and noting that (2.3), we obtain

$$\begin{aligned}Z'(r) &= \frac{\rho'(r)}{\rho(r)}Z(r) + \rho(r)\left\{\int_{S_r} \operatorname{div}(\phi(x)w(x))dS + \eta'(r)\right\}\\ &\leq \frac{\rho'(r)}{\rho(r)}Z(r) - \rho(r)\left\{\frac{\varepsilon}{2}\int_{S_r}\frac{\|\phi(x)w(x)\|^2}{\lambda_{\max}(x)\phi(x)}dS + \int_{S_r} p_\phi(x)dS - \eta'(r)\right\}. \end{aligned}\tag{2.10}$$

An application of the Cauchy-Schwarz inequality gives

$$\left(\int_{S_r}\langle\phi(x)W(x), v(x)\rangle dS\right)^2 \leq \left(\int_{S_r}\phi(x)\lambda_{\max}(x)dS\right)\left(\int_{S_r}\frac{\|\phi(x)W(x)\|^2}{\phi(x)\lambda_{\max}(x)}dS\right),$$

and equivalently,

$$\int_{S_r}\frac{\|\phi(x)W(x)\|^2}{\phi(x)\lambda_{\max}(x)}dS \geq \frac{1}{\lambda_\phi(r)}\left(\frac{Z(r)}{\rho(r)} - \eta(r)\right)^2,$$

which, together with (2.10), implies that (2.9) holds. □

Lemma 2.4. *Assume that $y = y(x)$ is a solution of Eq.* (1.1) *with $y(x) > 0$ for $|x| \in [c, b) \subset [r_0, \infty)$. Further, for $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$, $\eta \in \mathbf{C}^1([r_0, \infty), \mathbf{R})$ and $k \in \mathbf{C}^1([r_0, \infty), \mathbf{R}^+)$, let $Z(r)$ be defined as* (2.8) *on $[c, b)$. Then for $H \in \mathscr{H}$,*

$$\Theta_c^b(P) \le H(b,c)k(c)Z(c) + \frac{1}{4}\Theta_c^b(g(h_2 - k'k^{-1})^2), \tag{2.11}$$

where Θ *is an integral operator in terms of* $H(r,s)$ *and* $k(s)$ *as*

$$\Theta_\tau^r(\beta) = \int_\tau^r H(r,s)\beta(s)k(s)ds, \qquad r > \tau \ge r_0,$$

for $\beta \in \mathbf{C}([r_0,\infty),\mathbf{R})$.

Proof. By Lemma 2.3, $Z(r)$ satisfies (2.9). Applying the operator Θ_c^r to (2.9), and noting that $H \in \mathscr{H}$, we obtain

$$\begin{aligned}\Theta_c^r(P) &\le H(r,c)k(c)Z(c) + \Theta_c^r((-h_2 + k'k^{-1})Z - g^{-1}Z^2)\\ &= H(r,c)k(c)Z(c) - \Theta_c^r\left(\left(g^{-1/2}Z + \frac{1}{2}g^{1/2}(h_2 - k'k^{-1})\right)^2\right)\\ &\quad + \frac{1}{4}\Theta_c^r(g(h_2 - k'k^{-1})^2)\\ &\le H(r,c)k(c)Z(c) + \frac{1}{4}\Theta_c^r(g(h_2 - k'k^{-1})^2).\end{aligned}$$

Letting $r \to b^-$ in the above inequality, we obtain (2.11). □

Under a modification of the proof of Lemma 2.4, we have

Lemma 2.5. *Assume that* $y = y(x)$ *is a solution of Eq.* (1.1) *with* $y(x) > 0$ *for* $|x| \in (a,c] \subset [r_0,\infty)$. *Further, for* $\phi \in \mathbf{C}^1(\Omega(r_0),\mathbf{R}^+)$, $\eta \in \mathbf{C}^1([r_0,\infty),\mathbf{R})$ *and* $k \in \mathbf{C}^1([r_0,\infty),\mathbf{R}^+)$, *let* $Z(r)$ *be defined as* (2.8) *on* $(a,c]$. *Then for* $H \in \mathscr{H}$,

$$\Gamma_a^c(P) \le -H(c,a)k(c)Z(c) + \frac{1}{4}\Gamma_a^c(g(h_1 + k'k^{-1})^2), \tag{2.12}$$

where Γ *is an integral operator in terms of* $H(r,s)$ *and* $k(s)$ *as*

$$\Gamma_\tau^r(\beta) = \int_\tau^r H(s,\tau)\beta(s)k(s)ds, \qquad r > \tau \ge r_0,$$

for $\beta \in \mathbf{C}([r_0,\infty),\mathbf{R})$.

3. Oscillation theorems

First of all, we will use the averaging pair technique to establish the Wintner's type oscillation criteria for Eq. (1.1).

Theorem 3.1. *Suppose that there exist* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$, $\eta \in \mathbf{C}^1([r_0, \infty), \mathbf{R})$, $k \in \mathbf{C}^1([r_0, \infty), \mathbf{R}^+)$ *and an averaging pair* (σ, g) *such that*

$$\lim_{r\to\infty} \Pi_{r_0}^r \left(\int_{r_0}^r P(s)ds \right) = \infty, \tag{3.1}$$

where $\Pi_{r_0}^r$ *is a linear operator in terms of* β *as*

$$\Pi_{r_0}^r(\beta(r)) = \left(\int_{r_0}^r \sigma(s)ds \right)^{-1} \int_{r_0}^r \sigma(s)\beta(s)ds, \qquad r > r_0,$$

for $\beta \in \mathbf{C}([r_0, \infty), \mathbf{R})$. *Then Eq.* (1.1) *is oscillatory.*

Proof. Suppose to the contrary that there exists an nonoscillatory solution $y(x)$ of Eq. (1.1). Without loss of generality we may assume that $y(x) \neq 0$ for $|x| \geq r_0$. Define $Z(r)$ on $[r_0, \infty)$ by (2.8). Then, by Lemma 2.3, $Z(r)$ satisfies (2.9). Integrating both side of (2.9) from r_0 to r, we obtain

$$Z(r) + \int_{r_0}^r P(s)ds + \int_{r_0}^r \frac{Z^2(s)}{g(s)} ds \leq Z(r_0). \tag{3.2}$$

Applying the operator $\Pi_{r_0}^r$ to (3.2), we have

$$\Pi_{r_0}^r(Z(r)) + \Pi_{r_0}^r \left(\int_{r_0}^r \frac{Z^2(s)}{g(s)} ds \right) \leq Z(r_0) - \Pi_{r_0}^r \left(\int_{r_0}^r P(s)ds \right). \tag{3.3}$$

From (3.1), the right-hand side of (3.3) tends to $-\infty$, hence, there exists $r_1 > r_0$ such that for $r \geq r_1$,

$$\Pi_{r_0}^r(Z(r)) + \Pi_{r_0}^r \left(\int_{r_0}^r \frac{Z^2(s)}{g(s)} ds \right) < 0,$$

so that,

$$\left| \int_{r_0}^r \sigma(s)Z(s)ds \right| \geq \int_{r_0}^r \sigma(s) \int_{r_0}^s \frac{Z^2(u)}{g(u)} duds := T(r). \tag{3.4}$$

Then, for $r \geq r_1 > r_0$, we have

$$T(r) \geq \int_{r_1}^r \sigma(s) \int_{r_0}^s \frac{Z^2(u)}{g(u)} duds \geq \left(\int_{r_0}^{r_1} \frac{Z^2(s)}{g(s)} ds \right) \left(\int_{r_1}^r \sigma(s)ds \right). \tag{3.5}$$

From the Schwarz inequality, it follows that

$$T^2(r) \leq \left(\int_{r_0}^r \sigma(s)Z(s)ds \right)^2 \leq \left(\int_{r_0}^r g(s)\sigma^2(s)ds \right) \left(\int_{r_0}^r \frac{Z^2(s)}{g(s)} ds \right). \tag{3.6}$$

Noting that (3.4), (3.5) and (3.6), we obtain

$$\left(\int_{r_0}^{r_1}\frac{Z^2(s)}{g(s)}\,ds\right)^{\gamma}\left(\int_{r_0}^{r}\sigma(s)ds\right)^{\gamma}\le T^2(r)T^{\gamma-2}(r)$$

$$\le\left(\int_{r_0}^{r}g(s)\sigma^2(s)ds\right)\left(\int_{r_0}^{r}\frac{Z^2(s)}{g(s)}\,ds\right)T^{\gamma-2}(r)$$

$$=\frac{1}{\sigma(r)}\left(\int_{r_0}^{r}g(s)\sigma^2(s)ds\right)T^{\gamma-2}(r)T'(r),$$

that is,

$$\left(\int_{r_0}^{r_1}\frac{Z^2(s)}{g(s)}\,ds\right)^{\gamma}\sigma(r)\left(\int_{r_0}^{r}\sigma(s)ds\right)^{\gamma}\left(\int_{r_0}^{r}g(s)\sigma^2(s)ds\right)^{-1}\le T^{\gamma-2}(r)T'(r).$$

Integrating the above inequality from r_1 to r, we get

$$\left(\int_{r_0}^{r_1}\frac{Z^2(s)}{g(s)}\,ds\right)^{\gamma}\int_{r_1}^{r}\sigma(s)\left(\int_{r_0}^{s}\sigma(u)du\right)^{\gamma}\left(\int_{r_0}^{s}g(u)\sigma^2(u)du\right)^{-1}ds$$

$$\le\frac{1}{1-\gamma}\,\frac{1}{T^{1-\gamma}(r_1)}<\infty,$$

which contradicts the fact that (σ, g) is an averaging pair. □

Next, by using the averaging technique, we will establish Kamenev's type oscillation theorem. The following theorems 3.2–3.3 present two criteria for Eq. (1.1) which are the analogue of Philos and Kong's criteria for Eq. (1.4).

Theorem 3.2. *Suppose that there exist $H \in \mathscr{H}$ which the partial derivative $\partial H(r,s)/\partial s$ is nonpositive and continuous on D_0, $\phi \in \mathbf{C}^1(\Omega(r_0),\mathbf{R}^+)$, $\eta \in \mathbf{C}^1([r_0,\infty),\mathbf{R})$ and $k \in \mathbf{C}^1([r_0,\infty),\mathbf{R}^+)$ such that*

$$\limsup_{r\to\infty}\frac{1}{H(r,r_0)}\Theta_{r_0}^{r}\left(P-\frac{1}{4}g(h_2-k'k^{-1})^2\right)=\infty. \tag{3.7}$$

Then Eq. (1.1) is oscillatory.

Proof. Assume that Eq. (1.1) is not oscillatory. Then Eq. (1.1) has a solution $y(x)\neq 0$ for $|x|\in[r_1,\infty)$ where $r_1>r_0$. Let $Z(r)$ be defined by (2.8). It follows from Lemma 2.4 that

$$\Theta_{r_1}^{r}\left(P-\frac{1}{4}g(h_2-k'k^{-1})^2\right)\le H(r,r_1)k(r_1)Z(r_1)\le H(r,r_0)k(r_1)|Z(r_1)|,$$

which further yields that

$$\Theta_{r_0}^{r}\left(P-\frac{1}{4}g(h_2-k'k^{-1})^2\right)=(\Theta_{r_0}^{r_1}+\Theta_{r_1}^{r})\left(P-\frac{1}{4}g(h_2-k'k^{-1})^2\right)$$
$$\leq H(r,r_0)\left[\int_{r_0}^{r_1}k(s)|P(s)|ds+k(r_1)|Z(r_1)|\right].$$

Dividing both sides of the above inequality by $H(r,r_0)$ and taking limsup as $r\to\infty$, we obtain a contradiction to (3.7). □

Theorem 3.3. *Suppose that for each* $T\geq r_0$, *there exist* $H\in\mathscr{H}$, $\phi\in\mathbf{C}^1(\Omega(r_0),\mathbf{R}^+)$, $\eta\in\mathbf{C}^1([r_0,\infty),\mathbf{R})$, $k\in\mathbf{C}^1([r_0,\infty),\mathbf{R}^+)$ *and* $a,b,c\in\mathbf{R}^+$ *with* $T\leq a<c<b$ *such that*

$$\frac{1}{H(c,a)}\Gamma_a^c\left(P-\frac{1}{4}g(h_1+k'k^{-1})^2\right)+\frac{1}{H(b,c)}\Theta_c^b\left(P-\frac{1}{4}g(h_2-k'k^{-1})^2\right)>0. \tag{3.8}$$

Then Eq. (1.1) *is oscillatory.*

Proof. (3.8) implies that both (2.11) and (2.12) do not hold for the given c, and hence every solution of Eq. (1.1) must have a zero either for $|x|\in(a,c]$ or $|x|\in(c,b]$. By virtue of the fact that T is arbitrary, we see that every solution of Eq. (1.1) is oscillatory. □

Remark 3.1. For Eqs. (1.1) and (1.3). Let $\phi(x)=1$, $\gamma=0$ and $\sigma(r)=1/g(r)$ in Theorem 3.1. Then Theorem 3.1 improves Theorem 4 in [7] and Theorem 3.1 in [13]. For Eq. (1.5), let $\gamma=0$, Theorem 3.1 extends Theorem 3.7 in [6].

Remark 3.2. For Eqs. (1.1) and (1.2). Let $\phi(x)=1$, Theorem 3.2 generalizes Theorem 2.1 in [12] and Theorem 3.1 in [14].

Remark 3.3. For Eq. (1.2). Let $\phi(x)=1$, Theorem 3.3 covers Theorem 2 in [17].

Remark 3.4. Theorems 3.1–3.3 will be specialized to a perturbed linear equation

$$\sum_{i,j=1}^{N}D_i[a_{ij}(x)D_jy]+\sum_{i=1}^{N}b_i(x)D_iy+c(x)y+\sum_{i=1}^{m}c_i(x)f_i(y)=0, \tag{3.9}$$

where $c,c_i\in\mathbf{C}(\Omega(r_0),\mathbf{R})$, $f_i\in\mathbf{C}^1([r_0,\infty),\mathbf{R}^+)$ with $f_i'(y)\geq0$ for $y>0$ and all i. Define

$$p(x)=\min\{c(x),c_1(x),\dots,c_m(x)\}\qquad\text{and}\qquad f(y)=y+\sum_i f_i(y).$$

Then,

$$c(x)y + \sum_{i=1}^{m} c_i(x) f_i(y) \geq p(x) f(y),$$

and hence Theorems 3.1–3.3 can be applied.

4. Corollaries and examples

The theorems given in section 3 are presented in form of a high degree of generality. It is possible to obtain new criteria for Eq. (1.1) with the appropriate choices of the functions ϕ, η, k and H. In this section, we will give some interesting corollaries. Finally, we provide three examples to illustrate the significance of our main results.

COROLLARY 4.1. *Suppose that there exists $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$ such that $\lambda_\phi(r) = r^\delta$ and*

$$\lim_{r\to\infty} \frac{1}{r} \int_{r_0}^{r} \int_{r_0}^{s} P_0(u)\,du\,ds = \infty, \tag{4.1}$$

where $\delta \in \mathbf{R}$ and $P_0(r) = \int_{S_r} p_\phi(x)\,dS/r^\delta$. Then Eq. (1.1) is oscillatory.

PROOF. Let $\eta(r) = \delta r^{\delta-1}/\varepsilon$ and $\sigma(r) = 1$. An easy computation shows that

$$\rho(r) = \frac{1}{r^\delta}, \qquad g(r) = \frac{2}{\varepsilon}, \qquad \rho(r)\left(\frac{\varepsilon\eta^2(r)}{2\lambda_\phi(r)} - \eta'(r)\right) = \frac{\delta(2-\delta)}{r^2}.$$

Then, for $\gamma \in [0, 1)$,

$$\lim_{r\to\infty} \int_{r_0}^{r} \sigma(s)\left(\int_{r_0}^{s} \sigma(u)\,du\right)^{\gamma} \left(\int_{r_0}^{s} g(u)\sigma^2(u)\,du\right)^{-1} ds$$
$$= \frac{\varepsilon}{2} \lim_{r\to\infty} \int_{r_0}^{r} \frac{ds}{(s-r_0)^{1-\gamma}} = \infty,$$

and, by (4.1),

$$\lim_{r\to\infty} \Pi_{r_0}^{r}\left(\int_{r_0}^{r} P(s)\,ds\right) = \lim_{r\to\infty}\left[\frac{1}{r}\int_{r_0}^{r}\int_{r_0}^{s} P_0(u)\,du\,ds - \frac{\delta(2-\delta)}{r_0}\right] = \infty.$$

It follows from Theorem 3.1 that Eq. (1.1) is oscillatory. □

As an immediate consequence of Theorem 3.2, we get the following corollary.

Corollary 4.2. *Let the assumptions of Theorem* 3.2 *hold except* (3.7) *is replaced by*

$$\limsup_{r\to\infty} \frac{1}{H(r,r_0)} \Theta_{r_0}^r(P) = \infty, \tag{4.2}$$

and

$$\limsup_{r\to\infty} \frac{1}{H(r,r_0)} \Theta_{r_0}^r(g(h_2 - k'k^{-1})^2) < \infty. \tag{4.3}$$

Then Eq. (1.1) *is oscillatory.*

Corollary 4.3. *Suppose that there exists* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$ *such that*

$$\liminf_{r\to\infty} \Lambda(r)P_1(r) > \frac{1}{2\varepsilon}, \tag{4.4}$$

where

$$\Lambda(r) = \int_{r_0}^r \frac{ds}{\lambda_\phi(s)} \qquad \text{and} \qquad P_1(r) = \int_{\Omega(r)} p_\phi(x)dx.$$

Then Eq. (1.1) *is oscillatory.*

Proof. By (4.4), there exist two numbers $b \ge r_0$ and $\xi > 1/(2\varepsilon)$ such that

$$\Lambda(r)P_1(r) \ge \xi, \quad r > b, \qquad \text{and} \qquad \lim_{r\to\infty} \Lambda(r) = \infty.$$

Let

$$H(r,s) = [\Lambda(r) - \Lambda(s)]^2, \qquad \eta(r) = -\frac{1}{\varepsilon\Lambda(r)}, \qquad k(r) = 1.$$

Then,

$$\rho(r) = \Lambda(r) \qquad \text{and} \qquad h_2(r,s) = \frac{2}{\Lambda(r) - \Lambda(s)} \frac{1}{\lambda_\phi(s)}.$$

Hence,

$$\begin{aligned}\Theta_b^r(P) &= \int_b^r [\Lambda(r) - \Lambda(s)]^2 \Lambda(s) d\left(-P_1(s) + \frac{1}{2\varepsilon}\frac{1}{\Lambda(s)}\right) \\ &= [\Lambda(r) - \Lambda(b)]^2 \left(P_1(b)\Lambda(b) - \frac{1}{2\varepsilon}\right) \\ &\quad + \int_b^r \left(P_1(s)\Lambda(s) - \frac{1}{2\varepsilon}\right)\left(\frac{\Lambda^2(r)}{\Lambda(s)} + 3\Lambda(s) - 4\Lambda(r)\right)\Lambda'(s)ds\end{aligned}$$

$$\geq \left(\xi - \frac{1}{2\varepsilon}\right) \int_b^r \left(\frac{\Lambda^2(r)}{\Lambda(s)} + 3\Lambda(s) - 4\Lambda(r)\right) \Lambda'(s) ds$$
$$= \left(\xi - \frac{1}{2\varepsilon}\right) \left\{ \left(\ln \Lambda(r) - \ln \Lambda(b) - \frac{5}{2}\right) \Lambda^2(r) + 4\Lambda(b)\Lambda(r) - \frac{3}{2}\Lambda^2(b) \right\},$$

and

$$\Theta_b^r(g(h_2 - k'k^{-1})^2) = \frac{4}{\varepsilon}[\Lambda^2(r) - \Lambda^2(b)].$$

It follows from Corollary 4.2 that Eq. (1.1) is oscillatory. □

Corollary 4.4. *Suppose that there exist* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$, $\eta \in \mathbf{C}^1([r_0, \infty), \mathbf{R})$ *and for some* $l > 1$ *such that*

$$\limsup_{r \to \infty} G^{-l}(r) \int_{r_0}^r [G(r) - G(s)]^l P(s) ds = \infty, \tag{4.5}$$

where $G(r) = \int_{r_0}^r 1/g(s) ds$. *Then Eq.* (1.1) *is oscillatory.*

Proof. Let

$$H(r,s) = [G(r) - G(s)]^l \qquad \text{and} \qquad k(r) = 1.$$

Then,

$$h_2(r,s) = \frac{l}{G(r) - G(s)} \frac{1}{g(s)}.$$

Hence,

$$\Theta_b^r(g(h_2 - k'k^{-1})^2) = l^2 \int_b^r [G(r) - G(s)]^{l-2} dG(s)$$
$$= \frac{l^2}{l-1}[G(r) - G(b)]^{l-1}.$$

It follows from Corollary 4.2 that Eq. (1.1) is oscillatory. □

By Theorem 3.3, we have following result.

Corollary 4.5. *Suppose that for any* $T \geq r_0$, *there exist* $H \in \mathscr{H}$, $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$, $\eta \in \mathbf{C}^1([r_0, \infty), \mathbf{R})$ *and* $k \in \mathbf{C}^1([r_0, \infty), \mathbf{R}^+)$ *such that*

$$\limsup_{r \to \infty} \Gamma_T^r \left(P - \frac{1}{4} g(h_1 + k'k^{-1})^2\right) > 0, \tag{4.6}$$

and

$$\limsup_{r\to\infty} \Theta_T^r\left(P - \frac{1}{4}g(h_2 - k'k^{-1})^2\right) > 0. \tag{4.7}$$

Then Eq. (1.1) *is oscillatory.*

Corollary 4.6. *Suppose that for any* $T \geq r_0$, *there exist* $\phi \in \mathbf{C}^1(\Omega(r_0), \mathbf{R}^+)$, $\eta \in \mathbf{C}^1([r_0, \infty), \mathbf{R})$, *and for some* $l > 1$ *such that* $\lim_{r\to\infty} G(r) = \infty$, *and*

$$\limsup_{r\to\infty} \frac{1}{G^{l-1}(r)} \int_T^r [G(s) - G(T)]^l P(s) ds > \frac{l^2}{l-1}, \tag{4.8}$$

and

$$\limsup_{r\to\infty} \frac{1}{G^{l-1}(r)} \int_T^r [G(r) - G(s)]^l P(s) ds > \frac{l^2}{l-1}. \tag{4.9}$$

Then Eq. (1.1) *is oscillatory.*

Proof. Let $k(r) = 1$. Noting that $\lim_{r\to\infty} G(r) = \infty$ and proceeding as the proof of Corollary 4.4, we have

$$\limsup_{r\to\infty} \frac{1}{G^{l-1}(r)} \int_T^r H(r, T) g(s) h_1^2(r, T) ds = \frac{l^2}{l-1}, \tag{4.10}$$

and

$$\limsup_{r\to\infty} \frac{1}{G^{l-1}(r)} \int_T^r H(r, s) g(s) h_2^2(r, s) ds = \frac{l^2}{l-1}. \tag{4.11}$$

Thus, by (4.8) and (4.10), we get

$$\limsup_{r\to\infty} \frac{1}{H(r, T)} \Gamma_T^r\left(P - \frac{1}{4}g(h_1 + k'k^{-1})^2\right) > 0,$$

which infers that (4.6) holds. Similarly, (4.9) and (4.11) implies that (4.7) holds. Hence, by Corollary 4.5, Eq. (1.1) is oscillatory. □

To illustrate the significance of our main results, we provide the following examples.

Example 4.1. Consider the equation (1.1) on $\Omega(1)$ with

$$A(x) = \operatorname{diag}\left(\frac{1}{\|x\|}, \dots, \frac{1}{\|x\|}\right), \qquad b_i \in C^v(\Omega(1), \mathbf{R}^+), \quad i = 1, \dots, N,$$

$$p(x) = \frac{1 + \upsilon \sin\|x\|}{\|x\|^{\varrho}}, \qquad f(y) = y + y^3, \tag{4.12}$$

where $N \geq 2$, $\nu \in (0,1)$, $\upsilon \in \mathbf{R}$, $0 < \varrho \leq 1$, $\varepsilon = 1$, $\lambda_{\max}(x) = 1/\|x\|$ and $b(x) = (b_1(x), \ldots, b_N(x))$ might not be differentiable and satisfies

$$\int_{\Omega(r)} \|b(x)\|^2 dx \leq M r^{N-2}, \qquad (M > 0).$$

For Corollary 4.1, let $\phi(x) = 1$, $\delta = N - 1$. A direct calculation gives that

$$p_\phi(x) = \frac{1 + \upsilon \sin\|x\|}{\|x\|^\varrho} - \frac{1}{2}\|x\| \, \|b(x)\|^2,$$

then,

$$P_0(r) = \frac{\omega_N(1 + \upsilon \sin r)}{r^\varrho} - \frac{1}{2r^{N-2}} \int_{S_r} \|b(x)\|^2 dS,$$

where ω_N denotes the surface area of the unit sphere in $\mathbf{R}^N$, i.e., $\omega_N = 2\pi^{N/2}/\Gamma(N/2)$. Thus, for $0 < \varrho \leq 1$,

$$\begin{aligned} &\lim_{r\to\infty} \frac{1}{r} \int_1^r \int_1^s P_0(u)\,duds \\ &\quad = \lim_{r\to\infty} \left[\frac{\omega_N}{r} \int_1^r \int_1^s \frac{1 + \upsilon \sin u}{u^\varrho}\, duds - \frac{1}{2r} \int_1^r \frac{1}{\tau^{N-2}} \int_{S_\tau} \|b(x)\|^2 dS d\tau \right] = \infty. \end{aligned}$$

Hence, by Corollary 4.1, Eq. (4.12) is oscillatory.

Example 4.2. Consider the equation (1.1) on $\Omega(1)$ with

$$A(x) = \mathrm{diag}\left(\frac{1}{\|x\|^2}, \frac{1}{\|x\|^2} \right), \qquad b(x) = \left(\frac{x_1}{\|x\|^4}, \frac{x_1}{\|x\|^4} \right),$$

$$p(x) = \frac{\mu}{\|x\|^4}, \qquad f(y) = y + y^3, \tag{4.13}$$

where $N = 2$, $\mu > 1/2$, $\varepsilon = 1$ and $\lambda_{\max}(x) = \|x\|^{-2}$.

For Corollary 4.3, let $\phi(x) = \|x\|$. A simple computation yields that

$$\Lambda(r) = \frac{1}{2\pi}(r - 1), \qquad P_1(r) = \frac{2\pi\mu}{r}.$$

Then,

$$\liminf_{r\to\infty} \Lambda(r) P_1(r) = \mu.$$

Hence, by Corollary 4.3, Eq. (4.13) is oscillatory if $\mu > 1/2$.

Example 4.3. Consider the equation (1.1) on $\Omega(1)$ with

$$A(x) = \mathrm{diag}(1,1), \qquad b(x) = \left(\frac{x_1}{\|x\|^3}, \frac{x_2}{\|x\|^3}\right),$$

$$p(x) = \frac{\mu}{|x|^2}, \qquad f(y) = y + y^5, \tag{4.14}$$

where $N = 2$, $\mu > 2$, $\varepsilon = 1$ and $\lambda_{\max}(x) = 1$.

For Corollary 4.6, let $\phi(x) = 1/\|x\|$, $\eta(r) = 0$. It is easy to show that

$$\rho(r) = 1, \qquad G(r) = \frac{1}{4\pi}(r-1), \qquad P(r) = \frac{2\pi\mu}{r^2}.$$

Hence, for $l > 1$,

$$\lim_{r\to\infty} \frac{1}{G^{l-1}(r)} \int_T^r [G(r) - G(T)]^l P(s)ds$$
$$= \frac{\mu}{2} \lim_{r\to\infty} \frac{1}{(r-1)^{l-1}} \int_T^r \frac{(s-T)^l}{s^2} ds = \frac{\mu}{2(l-1)}. \tag{4.15}$$

By using Lemma 3.1 in [4], we have

$$\int_T^r \frac{(r-s)^l}{s^2} ds \geq \int_T^r \frac{(s-T)^l}{s^2} ds. \tag{4.16}$$

From (4.15) and (4.16), for $\mu > 2$, there exists $l > 1$ such that $\mu/(2(l-1)) > l^2/(l-1)$. This means that (4.8) and (4.9) hold for same l. Applying Corollary 4.6, we find that Eq. (4.14) is oscillatory.

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