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Non-existence of positive commutators

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We consider the positivity of the commutator $[H, iA]$ for self-adjoint operators H and A in a complex Hilbert space $\mathscr{H} \neq \{0\}$. Throughout the paper we denote by $(\cdot, \cdot)$ the scalar product on $\mathscr{H}$ and by $\|\cdot\|$ the associated norm.

In the case where H and A are bounded, it is well known from the proof of Putnam's theorem that $[H, iA] \geq \alpha \mathbf{1}$, i.e.,

$$([H, iA]\psi, \psi) \geq \alpha \|\psi\|^2 \qquad \text{for any} \qquad \psi \in \mathscr{H},$$

is impossible for any $\alpha > 0$ (see [1; p. 61]). Our purpose in this paper is to extend the above result to the case where H and A are unbounded. In this case, following Mourre [2], we define the commutator $[H, iA]$ by

$$([H, iA]\psi, \phi) = i(A\psi, H\phi) - i(H\psi, A\phi), \qquad \psi, \phi \in D(A) \cap D(H),$$

where $D(A)$ (resp. $D(H)$) denotes the domain of A (resp. H).

We prove

THEOREM. *Let A and H be self-adjoint operators in $\mathscr{H}$ such that $D(H) \subset D(A)$. Then $[H, iA] \geq \alpha \mathbf{1}$ is impossible for any $\alpha > 0$.*

Before proving the theorem, we give a few remarks.

REMARK 1. It follows from the closed graph theorem that the assumption of the previous result is precisely $D(H) = D(A) = \mathscr{H}$.

REMARK 2. If $D(H)$ and $D(A)$ have no inclusion relations, the conclusion in the theorem fails. For example, if $\mathscr{H} = L^2(\mathbf{R})$, $A = x\cdot$ with domain $D(A) = \{\psi \in \mathscr{H}; x\psi \in \mathscr{H}\}$, $H = -id/dx$ with domain $D(H) = \{\psi \in \mathscr{H}; (d/dx)\psi \in \mathscr{H}\}$, then $[H, iA] = \mathbf{1}$ on $D(A) \cap D(H)$.

PROOF OF THEOREM. In what follows $\mathscr{L}(\mathscr{H})$ denotes the Banach space of all bounded operators on $\mathscr{H}$. Suppose that $[H, iA] \geq \alpha \mathbf{1}$ holds for some $\alpha > 0$. We choose $\phi_0 \in \mathscr{H} \backslash \{0\}$ and set $\phi = (H + i)^{-1}\phi_0$. Then $\phi \in D(H) \backslash \{0\}$ and the map $\mathbf{R} \ni t \mapsto e^{-itH}\phi \in \mathscr{H}$ is continuously differentiable. By the closed theorem, there is a constant $C > 0$ such that

$$\|A\psi\| \leq C(\|H\psi\| + \|\psi\|), \qquad \psi \in D(H). \tag{1}$$

We see from (1) that the map $\boldsymbol{R} \ni t \mapsto Ae^{-itH}\phi \in \mathscr{H}$ is continuous and

$$\sup_{t \in \boldsymbol{R}} \|Ae^{-itH}\phi\| \le C(\|H\phi\| + \|\phi\|) . \tag{2}$$

For $\lambda \in \boldsymbol{R}\backslash\{0\}$ we set $R_\lambda = (A + i\lambda)^{-1}$. Then, $AR_\lambda \in \mathscr{L}(\mathscr{H})$ and its operator norm is bounded by one. Moreover, the map $\boldsymbol{R} \ni t \mapsto AR_\lambda e^{-itH}\phi \in \mathscr{H}$ is continuously differentiable and

$$\begin{aligned}(d/dt)(Ai\lambda R_\lambda e^{-itH}\phi, e^{-itH}\phi) &= -i(Ai\lambda R_\lambda He^{-itH}\phi, e^{-itH}\phi) + i(Ai\lambda R_\lambda e^{-itH}\phi, He^{-itH}\phi)\\ &= -i(He^{-itH}\phi, (A - AR_{-\lambda}A)e^{-itH}\phi)\\ &\quad + i((A - AR_\lambda A)e^{-itH}\phi, He^{-itH}\phi)\\ &= ([H, iA]e^{-itH}\phi, e^{-itH}\phi) + i(He^{-itH}\phi, AR_{-\lambda}Ae^{-itH}\phi)\\ &\quad - i(AR_\lambda Ae^{-itH}\phi, He^{-itH}\phi) .\end{aligned}$$

By assumption,

$$(d/dt)(Ai\lambda R_\lambda e^{-itH}\phi, e^{-itH}\phi) \ge \alpha\|\phi\|^2 + f(t, \lambda) , \tag{3}$$

where $f(t, \lambda) = i(He^{-itH}\phi, AR_{-\lambda}Ae^{-itH}\phi) - i(AR_\lambda Ae^{-itH}\phi, He^{-itH}\phi)$. By integrating both sides of (3) over an interval $[0, t]$, $t > 0$, we obtain

$$(Ai\lambda R_\lambda e^{-itH}\phi, e^{-itH}\phi) - (Ai\lambda R_\lambda\phi, \phi) \ge \alpha t\|\phi\|^2 + \int_0^t f(s, \lambda)\, ds . \tag{4}$$

Since $i\lambda R_\lambda \to \mathbf{1}$ strongly in $\mathscr{L}(\mathscr{H})$ as $|\lambda| \to \infty$, for any $t \ge 0$, $Ai\lambda R_\lambda e^{-itH}\phi \to Ae^{-itH}\phi$ in $\mathscr{H}$ and $f(t, \lambda) \to 0$ in $\boldsymbol{C}$ as $|\lambda| \to \infty$. Moreover, by (2),

$$|f(t, \lambda)| \le 2C\|H\phi\|(\|H\phi\| + \|\phi\|) .$$

Therefore, by Lebesgue's dominated convergence theorem, taking the limit $|\lambda| \to \infty$ in (4), we obtain

$$(Ae^{-itH}\phi, e^{-itH}\phi) - (A\phi, \phi) \ge \alpha t\|\phi\|^2 , \qquad t > 0 . \tag{5}$$

By (2) and (5),

$$C(\|H\phi\| + \|\phi\|)\|\phi\| \ge (A\phi, \phi) + \alpha t\|\phi\|^2 , \qquad t > 0 . \tag{6}$$

Dividing both sides of (6) by t and taking the limit $t \to \infty$ in the resulting inequality, we have $\alpha\|\phi\|^2 = 0$ and therefore $\phi = 0$. This contradicts the fact that $\phi \ne 0$. Q.E.D.

References

[1] H. L. Cycon, R. G. Froese, W. Kirsch and B. Simon, *Schrödinger Operators, with Application to Quantum Mechanics and Global Geometry*, Springer-Verlag, Berlin-Heidelberg-New York, 1987.

[2] E. Mourre, *Absence of singular continuous spectrum for certain self-adjoint operators*, Commun. Math. Phys. **78** (1981), 391–408.

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