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Oscillation criteria for nonlinear differential systems with general deviating arguments of mixed type

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1. Introduction

In this paper we consider the nonlinear differential system with deviating arguments of the form

$$(S_\lambda) \qquad y_i'(t) = p_i(t) y_{i+1}(h_{i+1}(t)), \qquad i = 1, 2, \ldots, n-1,$$

$$y_n' = (-1)^\lambda \sum_{m=1}^{N} a_m(t) f_m(y_1(g_m(t))), \qquad t \geq 0, \qquad n \geq 2, \qquad \lambda \in \{1, 2\},$$

under the following standing assumptions:

(A_1) p_i: $[0, \infty) \to [0, \infty)$, $(i = 1, 2, \ldots, n-1)$ are continuous functions and $\int_0^\infty p_i(t)dt = \infty$, $i = 1, 2, \ldots, n-1$;

(A_2) a_m: $[0, \infty) \to [0, \infty)$, $(m = 1, 2, \ldots, N)$ are continuous functions and are not identically zero on any infinite subinterval of $[0, \infty)$;

(A_3) h_i: $[0, \infty) \to R$, $(i = 2, 3, \ldots, n)$ are continuously differentiable functions with $h_i'(t) > 0$ on $[0, \infty)$, and $\lim_{t\to\infty} h_i(t) = \infty$ for $i = 2, 3, \ldots, n$;

(A_4) g_m: $[0, \infty) \to R$ $(m = 1, 2, \ldots, N)$ are continuous functions and $\lim_{t\to\infty} g_m(t) = \infty$ for $m = 1, 2, \ldots, N$;

(A_5) f_m: $R \to R$ $(m = 1, 2, \ldots, N)$ are continuous functions and $uf_m(u) > 0$ for $u \neq 0$, $m = 1, 2, \ldots, N$.

By a proper solution of the system (S_λ) we mean a solution $y = (y_1, y_2, \ldots, y_n) \in C^1[[T_y, \infty), R]$ which satisfies (S_λ) for all sufficiently large t, and $\sup\left\{\sum_{i=1}^n |y_i(t)|; t \geq T\right\} > 0$ for any $T \geq T_y$. We make the standing hypothesis that the system (S_λ) does possess proper solutions.

A proper solution of (S_λ) is called oscillatory if each of its component has arbitrarily large zeros. A proper solution of (S_λ) is called nonoscillatory (weakly nonoscillatory) on $[T_y, \infty)$ if each of its component (at least one component) is eventually of constant sign on $[T, \infty) \subset [T_y, \infty)$.

In this paper we shall study oscillatory properties of solutions of differential systems (S_λ) with deviating arguments of mixed type, which are in general essentially different from those of ordinary $(h_i(t) \equiv t, i = 2, 3, \ldots, n, g_m(t) \equiv t,$

$m = 1, 2, \dots, N$) and retarded differential systems. The first results on oscillation of certain differential systems generated by deviating argument have been obtained in the papers [3, 5].

In this paper we extend the results from the paper [1] to the system (S_λ). Here we give conditions under which all proper solutions of (S_λ) are oscillatory.

Throught the paper we will use the following notations:

(B_1) $H_2(t) = h_2(t)$, $H_i(t) = h_i(H_{i-1}(t))$, $i = 3, 4, \dots, n$; $H_i^{-1}(t)$ is inverse function to $H_i(t)$, $i = 2, 3, \dots, n$.

(B_2) $\gamma_i(t) = \sup\{s \geq 0; h_i(t) \leq t\}$ for $t \geq 0$, $i = 2, 3, \dots, n$, $\gamma(t) = \max\{\gamma_2(t), \dots, \gamma_n(t)\}$ for $t \geq t_0$.

(B_3) $t_{k-1} = \max\{t_k, \gamma_k(t_k)\}$, $s_k = \max\{s_{k-1}, h_k(s_{k-1})\}$, $k = 2, 3, \dots, n$.

(B_4) Let $g_m(t)$, ($m = 1, 2, \dots, N$) be fixed. We define the subsets $\mathscr{A}_m$ and $\mathscr{R}_m$ of $[0, \infty)$ as follows: $\mathscr{A}_m = \{t \in [0, \infty); g_m(t) > t\}$, $\mathscr{R}_m = \{t \in [0, \infty); g_m(t) < t\}$.

2. Main results

The following three theorems are the main results of this paper.

THEOREM 1. *Let $n \geq 3$, $n + \lambda$ be odd and the assumptions* (A_1)–(A_5) *hold. Let*

$$(A_6) \qquad H_i(t) \leq t,\ i = 2, 3, \dots, n-1,\ H_n(t) \geq t \qquad \text{for} \quad t \geq t_0 > 0\,.$$

Suppose that there are integers j, k and r, $1 \leq j, k, r \leq N$ and some positive numbers K_0, k_0 such that the following conditions are satisfied:

$$(C_1) \qquad \int_{\mathscr{A}_j} a_j(s) \int_{H_n^{-1}(s)}^{g_j(H_n^{-1}(s))} p_1(t) \int_{h_n^{-1}(s)}^{H_{n-1}(t)} p_{n-1}(x_{n-1}) \int_{h_{n-1}^{-1}(x_{n-1})}^{H_{n-2}(t)} p_{n-2}(x_{n-2}) \cdots \times \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2)\, dx_2 \dots dx_{n-2}\, dx_{n-1}\, dt\, ds = \infty\,,$$

$$(C_2) \quad \int_{\mathscr{A}_k} a_k(s) \int_{H_n^{-1}(s)}^{g_k(H_n^{-1}(s))} p_1(t) \int_{t_{n-1}}^{h_n^{-1}(s)} p_{n-1}(x_{n-1}) \cdots \int_{t_{l-1}}^{h_l^{-1}(x_l)} p_{l-1}(x_{l-1}) \times \int_{h_{l-1}^{-1}(x_{l-1})}^{H_{l-2}(t)} p_{l-2}(x_{l-2}) \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2)\, dx_2 \dots dx_{l-2}\, dx_{l-1} \dots dx_{n-1}\, dt\, ds = \infty$$

for any l: $3 \leq l \leq n-1$, $n + l + \lambda$ is odd,

$$(C_3) \qquad \int_{\mathscr{A}_r} a_r(s) \int_{t_0}^{H_n^{-1}(s)} p_1(t) \int_{H_{n-1}(s)}^{h_n^{-1}(s)} p_{n-1}(x_{n-1}) \int_{H_{n-2}(s)}^{h_{n-1}^{-1}(x_{n-1})} p_{n-2}(x_{n-2}) \cdots \times \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)\, dx_2 \dots dx_{n-2}\, dx_{n-1}\, dt\, ds = \infty\,.$$

(A_7) *$g_m(t)$, $m = j, k, r$, are nondecreasing functions for $t \geq t_0$; $f_m(x)$, $m = j, k, r$, are increasing functions for $|x| \geq K_0$, in addition $f_r(x)$ is increasing for $|x| \leq k_0$, and*

$$(C_4)\qquad \int_{K_0}^{\infty} \frac{dx}{f_m(x)} < \infty\,, \qquad \int_{-K_0}^{-\infty} \frac{dx}{f_m(x)} < \infty\,, \qquad m = j, k, r\,,$$

$$(C_5)\qquad \int_{0}^{k_0} \frac{dx}{f_r(x)} < \infty\,, \qquad \int_{0}^{-k_0} \frac{dx}{f_r(x)} < \infty\,.$$

Then all proper solutions of (S_λ) *with $n + \lambda$ odd are oscillatory.*

THEOREM 2. *Let $n \geq 3$, $n + \lambda$ be even and the assumptions* (A_1)–(A_6) *hold. Suppose that there are integers j, k, r: $1 \leq j, k, r \leq N$ and some positive numbers K_0, k_0 such that* (C_1), (C_2), (A_7), (C_4) *and* (C_5) *hold.*

In addition suppose that

$$(C_6)\qquad \int_{\mathscr{R}_r} a_r(s) \int_{g_r(H_n^{-1}(s))}^{H_n^{-1}(s)} p_1(t) \int_{H_{n-1}(t)}^{h_n^{-1}(s)} p_{n-1}(x_{n-1}) \int_{H_{n-2}(t)}^{h_{n-1}^{-1}(x_{n-1})} p_{n-2}(x_{n-2}) \cdots$$
$$\times \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)\, dx_2 \ldots dx_{n-2}\, dx_{n-1}\, dt\, ds = \infty\,.$$

Then all proper solutions $y = (y_1, \ldots, y_n)$ of (S_λ) *with $n + \lambda$ even are either oscillatory or $y_i(t)$, $i = 1, 2, \ldots, n$, monotonically tend to zero as $t \to \infty$.*

THEOREM 3. *Let the condition $H_n(t) \geq t$ in* (A_6) *be replaced by $H_n(t) \equiv t$ on $[0, \infty)$. Let additional assumptions of Theorem 2 hold. Then all proper solutions of* (S_λ) *with $n + \lambda$ even are oscillatory.*

REMARK 1. Let $h_2(t) \equiv \cdots \equiv h_n(t) \equiv t$ on $[0, \infty)$, $p_i(t) > 0$ for $i = 1, 2, \ldots, n-1$, $t \geq 0$. Then the system (S_λ) is equivalent to the n-th order scalar differential equation

$$(E_\lambda)\qquad \left(\frac{1}{p_{n-1}(t)} \cdots \left(\frac{1}{p_2(t)}\left(\frac{1}{p_1(t)} y'(t)\right)'\right)' \cdots\right)' = (-1)^\lambda \sum_{m=0}^{N} a_m(t) f_m(y(g_m(t)))\,,$$

and the conditions (C_i), $i = 1, 2, 3, 6$, imply the following ones:

$$(C_1')\qquad \int_{\mathscr{A}_j} a_j(s) \int_{s}^{g_j(s)} p_{n-1}(x_{n-1}) \int_{x_{n-1}}^{g_j(s)} p_{n-2}(x_{n-2}) \ldots$$
$$\times \int_{x_2}^{g_j(s)} p_1(x_1)\, dx_1 \ldots dx_{n-2}\, dx_{n-1}\, ds = \infty\,,$$

$$(C_2')\quad \int_{\mathscr{A}_k} a_k(s)\int_{t_0}^{s} p_{n-1}(x_{n-1})\dots\int_{t_0}^{x_l} p_{l-1}(x_{l-1})\int_{s}^{g_k(s)} p_{l-2}(x_{l-2})\int_{x_{l-2}}^{g_k(s)} p_{l-3}(x_{l-3})\dots$$

$$\times\int_{x_2}^{g_k(s)} p_1(x_1)\,dx_1\dots dx_{l-3}\,dx_{l-2}\,dx_{l-1}\dots dx_{n-1}\,ds=\infty\,,$$

$$(C_3')\quad \int_{\mathscr{A}_r} a_r(s)\int_{t_0}^{s} p_{n-1}(x_{n-1})\int_{t_0}^{x_{n-1}} p_{n-2}(x_{n-2})\dots$$

$$\times\int_{t_0}^{x_2} p_1(x_1)\,dx_1\dots dx_{n-2}\,dx_{n-1}\,ds=\infty\,,$$

$$(C_6')\quad \int_{\mathscr{R}_r} a_r(s)\int_{g_r(s)}^{s} p_{n-1}(x_{n-1})\int_{g_r(s)}^{x_{n-1}} p_{n-2}(x_{n-2})\dots$$

$$\times\int_{g_r(s)}^{x_2} p_1(x_1)\,dx_1\dots dx_{n-2}\,dx_{n-1}\,ds=\infty\,.$$

The following corollaries are immediate consequences of Theorem 1 and Theorem 3.

COROLLARY 1. *Let* $n\geq 3$, $n+\lambda$ *be odd and the assumptions* (A_1), (A_2), (A_4), (A_5) *hold. Suppose that there are integers* j, k *and* r: $1\leq j, k, r\leq N$ *and some positive numbers* K_0, k_0 *such that the conditions* (C_1'), (C_2'), (C_3'), (A_7), (C_4), (C_5) *are satisfied. Then all proper solutions of* (E_λ) *with* $n+\lambda$ *odd are oscillatory.*

COROLLARY 2. *Let* $n\geq 3$, $n+\lambda$ *be even and the assumptions* (A_1), (A_2), (A_4), (A_5) *hold. Suppose that there are integers* j, k *and* r: $1\leq j, k, r\leq N$ *and some positive numbers* K_0, k_0 *such that the conditions* (C_1'), (C_2'), (C_6'), (A_7), (C_4) *and* (C_5) *are satisfied. Then all proper solutions of* (E_λ) *with* $n+\lambda$ *even are oscillatory.*

3. Proofs of theorems

To obtain main results we need the following lemmas.

LEMMA 1 [2]. *Let the conditions* (A_1)–(A_5) *hold and let* $y=(y_1,\dots,y_n)$ *be a regular nonoscillatory solution of* (S_λ) *on the interval* $[0,\infty)$.

I) *Then there exist* $t_0\geq 0$ *and an integer* $l\in\{1,2,\dots,n\}$ *with* $n+\lambda+l$ *odd or* $l=n$ *such that for* $t\geq t_0$

$$(N_l)\quad y_i(t)y_1(t)>0\,,\qquad i=1,2,\dots,l\,,$$

$$(-1)^{l+i}y_i(t)y_1(t)>0\,,\qquad i=l+1,\dots,n\,.$$

II) *In addition let* $\lim_{t\to\infty} y_l(t) = L_l$, $0 \le L_l \le \infty$. *Then*

$$(1) \qquad l > 1\,, \quad L_l > 0 \Rightarrow \lim_{t\to\infty} |y_i(t)| = \infty\,, \quad i = 1, 2, \dots, l-1\,;$$

$$l < n\,, \quad L_l < \infty \Rightarrow \lim_{t\to\infty} y_i(t) = 0\,, \quad i = l+1, \dots, n\,.$$

LEMMA 2 [4]. *Let the conditions* (A_1)–(A_5) *hold. Let* $y = (y_1, \dots, y_n)$ *be a regular solution of* (S_λ) *such that* $y_k(t) \neq 0$ *on* $[t_0, \infty)$ *for some* $k \in \{1, 2, \dots, n\}$.

Then there exists a $t_1 > t_0$ *such that each component* y_i *of* y *is on* $[t_1, \infty)$ *different from zero, monotone and the limit* $\lim_{t\to\infty} y_i(t) = L_i$ *exists (finite or infinite).*

LEMMA 3. *Let the conditions* (A_1)–(A_5) *hold. Let* $y = (y_1, \dots, y_n)$ *be a regular solution of* (S_λ) *on* $[t_0, \infty)$. *Then there exist a* $t_1 \ge t_0$ *and an integer* $l \in \{1, 2, \dots, n\}$ *with* $n + l + \lambda$ *odd or* $l = n$, *such that*

$$(2_l) \qquad |y_l(t)| \ge \int_t^{s_l} p_l(x_l) \int_{h_{l+1}(x)}^{s_{l+1}} p_{l+1}(x_{l+1}) \dots \int_{h_{n-2}(x_{n-3})}^{s_{n-2}} p_{n-2}(x_{n-2})$$
$$\times \int_{h_{n-1}(x_{n-2})}^{s_{n-1}} p_{n-1}(x_{n-1}) |y_n(h_n(x_{n-1}))| \, dx_{n-1}\, dx_{n-2} \dots dx_{l+1}\, dx_l$$

for $t_1 \le t \le s_l$;

$$(3_i) \qquad |y_i(t)| \ge \int_{t_i}^{t} p_i(x_i) \int_{t_{i+1}}^{h_{i+1}(x_i)} p_{i+1}(x_{i+1}) \dots \int_{t_{l-2}}^{h_{l-2}(x_{l-3})} p_{l-2}(x_{l-2})$$
$$\times \int_{t_{l-1}}^{h_{l-1}(x_{l-2})} p_{l-1}(x_{l-1}) |y_l(h_l(x_{l-1}))| \, dx_{l-1}\, dx_{l-2} \dots dx_{i+1}\, dx_i$$

for $t \ge t_{l-1}$, $i = 1, 2, \dots, l-1$.

PROOF. Integrating the k-th equation of (S_λ), $k = l, l+1, \dots, n-1$, from t to $s_k (t < s_k)$ and using (N_l), we get

$$(4_k) \qquad |y_k(t)| \ge \int_t^{s_k} p_k(x) |y_{k+1}(h_{k+1}(x))| \, dx\,.$$

Putting (4_{n-1}) into (4_{n-2}) and it into (4_{n-3}), using nearby (B_3), then repeating this method $n - l - 3$ times, we have (2_l).

Integrating the k-th equation of (S_λ), $k = 1, 2, \dots, l$, from t_k to t and using (N_l), we get

$$(5_k) \qquad |y_k(t)| \ge \int_{t_k}^{t} p_k(x) |y_{k+1}(h_{k+1}(x))| \, dx\,, \qquad t \ge t_k\,.$$

Putting (5_{l-1}) into (5_{l-2}) and it into (5_{l-3}), using nearby (B_3), then repeating this method $l-k-3$ times, we get (3_i).

PROOF OF THEOREM 1. Suppose that (S_λ) has a weakly nonoscillatory regular solution $y=(y_1,\dots,y_n)$. Then by Lemma 2 y is nonoscillatory. Without loss of generality we may suppose that $y_1(g_m(t))>0$ for $m=1,2,\dots,N$, $t\ge t_0>0$. Then the n-th equation of (S_λ) implies that $(-1)^\lambda y_n'(t)\ge 0$ for $t\ge \bar{t}_0$ and it is not identically zero on any infinite interval of $[\bar{t}_0,\infty)$. Then by Lemma 1 and Lemma 3 there exist a $t_1\ge \bar{t}_0$ and an even integer l: $2\le l\le n$, or $l=n$ such that (N_l), (1), (2), (3_i) hold for $t\ge t_1$. Let $T_0\ge t_1$ be so large that $g_m(t)\ge t_1$ for $t\ge T_0$, $1\le m\le N$.

1. Let $l=n$. Replacing t with $h_2(t)$ and l with n in (3_2), we get

$$y_2(h_2(t))\ge\int_{t_2}^{h_2(t)}p_2(x_2)\int_{t_3}^{h_3(x_2)}p_3(x_3)\dots\int_{t_{n-1}}^{h_{n-1}(x_{n-2})}p_{n-1}(x_{n-1})\times|y_n(h_n(x_{n-1}))|\,dx_{n-1}\dots dx_3\,dx_2\qquad\text{for } t\ge\gamma(t_{n-1})\,. \tag{6}$$

Integrating the n-th equation of (S_λ) from t_n $(\ge T_0)$ to $h_n(t)$ and using (N_l), (A_2), (A_5), we have

$$|y_n(h_n(t))|\ge\int_{t_n}^{h_n(t)}\sum_{m=1}^{N}a_m(s)f_m(y_1(g_m(s)))\,ds\ge\int_{t_n}^{h_n(t)}a_j(s)f_j(y_1(g_j(s)))\,ds\,.$$

Putting the last inequality into (6), we obtain

$$y_2(h_2(t))\ge\int_{t_2}^{h_2(t)}p_2(x_2)\dots\int_{t_{n-1}}^{h_{n-1}(x_{n-2})}p_{n-1}(x_{n-1})\int_{t_n}^{h_n(x_{n-1})}\times a_j(x_n)f_j(y_1(g_j(x_n)))\,dx_n\,dx_{n-1}\dots dx_2\,. \tag{7}$$

Interchanging the order of integration in (7), we get

$$y_2(h_2(t))\ge\int_{t_n}^{H_n(t)}a_j(x_n)f_j(y_1(g_j(x_n)))\int_{h_n^{-1}(x_n)}^{H_{n-1}(t)}p_{n-1}(x_{n-1})\dots\int_{h_n^{-1}(x_3)}^{H_2(t)}p_2(x_2)\,dx_2\dots\times dx_{n-1}\,dx_n\qquad\text{for } t\ge\gamma(t_n)\,. \tag{8}$$

Take any $T\ge t_n$ and let $T_j=\sup_{t_n\le t\le T}\max\{g_j(t),t\}$. Multiplying (8) by $p_1(t)/f_j(y_1(t))$ and then integrating from t_n to T_j, using the first equation of (S_λ) and the monotonicity of f_j, y_1, g_j and (A_6), we have

$$(9)\quad \int_{t_n}^{T_j} \frac{y_1'(t)\,dt}{f_j(y_1(t))} \ge \int_{t_n}^{T_j} \frac{p_1(t)}{f_j(y_1(t))} \int_{t_n}^{H_n(t)} a_j(x_n) f_j(y_1(g_j(x_n))) \int_{h_n^{-1}(x_n)}^{H_{n-1}(t)} p_{n-1}(x_{n-1}) \dots \times \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2)\,dx_2 \dots dx_{n-1}\,dx_n\,dt$$

$$\ge \int_{t_n}^{H_n(T_j)} a_j(x_n) \int_{H_n^{-1}(x_n)}^{T_j} p_1(t) \frac{f_j(y_1(g_j(x_n)))}{f_j(y_1(t))} \int_{h_n^{-1}(x_n)}^{H_{n-1}(t)} p_{n-1}(x_{n-1}) \dots \times \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2)\,dx_2 \dots dx_{n-1}\,dt\,dx_n$$

$$\ge \int_{t_n}^{H_n(T_j)} a_j(x_n) \int_{H_n^{-1}(x_n)}^{T_j} p_1(t) \frac{f_j(y_1(g_j(H_n^{-1}(x_n))))}{f_j(y_1(t))} \times \int_{h_n^{-1}(x_n)}^{H_{n-1}(t)} p_{n-1}(x_{n-1}) \dots \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2)\,dx_2 \dots dx_{n-1}\,dt\,dx_n\,.$$

Since $l = n \ge 3$, $\lim_{t\to\infty} y_1(t) = \infty$, we may choose t_n so large that $y_1(t) \ge K_0$ for $t \ge t_n$. Because the functions f_j, y_1, g_j are nondecreasing on $[t_0, \infty)$, $f_j(y_1(g_j(H_n^{-1}(x_n))))/f_j(y_1(t)) \ge 1$ for $u = H_n^{-1}(x_n) \le t \le g_j(u)$, $u \in \mathscr{A}_j$.

From (9) we then have

$$(10)\quad \int_{t_n}^{T_j} \frac{y_1'(t)\,dt}{f_j(y_1(t))} \ge \int_{\mathscr{A}_j \cap [t_n, T]} a_j(x_n) \int_{H_n^{-1}(x_n)}^{g_j(H_n^{-1}(x_n))} p_1(t) \int_{h_n^{-1}(x_n)}^{H_{n-1}(t)} p_{n-1}(x_{n-1}) \dots \int_{h_3^{-1}(x_3)}^{H_2(t)} \times p_2(x_2)\,dx_2 \dots dx_{n-1}\,dt\,dx_n\,.$$

Letting $T \to \infty$ in (10) and using (C_4), we get

$$\int_{\mathscr{A}_j \cap [t_n, \infty)} a_j(x_n) \int_{H_n^{-1}(x_n)}^{g_j(H_n^{-1}(x_n))} p_1(t) \int_{h_n^{-1}(x_n)}^{H_{n-1}(t)} p_{n-1}(x_{n-1}) \dots \int_{h_3^{-1}(x_3)}^{H_2(t)} \times p_2(x_2)\,dx_2 \dots dx_{n-1}\,dt\,dx_n < \infty\,,$$

which contradicts (C_1).

2. Let $3 \le l < n$ and k: $1 \le k \le N$ be fixed. Integrating the n-th equation of (S_λ) from $h_n(t)$ to s_n ($s_n > h_n(t)$), using (A_2), (A_5) and (N_l), we have

$$|y_n(h_n(t))| \ge \int_{h_n(t)}^{s_n} a_k(x) f_k(y_1(g_k(x)))\,dx\,.$$

Combining the last inequality with (2_l), with t replaced by $h_l(t)$, we get

$$(11)\quad y_l(h_l(t)) \ge \int_{h_l(t)}^{s_l} p_l(x_l) \dots \int_{h_{n-1}(x_{n-2})}^{s_{n-1}} p_{n-1}(x_{n-1}) \int_{h_n(x_{n-1})}^{s_n} \times a_k(x) f_k(y_1(g_k(x)))\,dx\,dx_{n-1} \dots dx_l\,,$$

where $s_k = \max\{s_{k-1}, h_k(s_{k-1})\}$, $k = l+1, \ldots, n$, s_n is sufficiently large. Putting (11) into (3_2), in which we replace t with $h_2(t)$, we have

$$(12)\qquad y_2(h_2(t)) \geq \int_{t_2}^{h_2(t)} p_2(x_2) \ldots \int_{t_{l-1}}^{h_{l-1}(x_{l-2})} p_{l-1}(x_{l-1}) \int_{h_l(x_{l-1})}^{s_l} p_l(x_l) \ldots \times \int_{h_{n-1}(x_{n-2})}^{s_{n-1}} p_{n-1}(x_{n-1}) \int_{h_n(x_{n-1})}^{s_n} a_k(x) f_k(y_1(g_k(x))) \times dx\, dx_{n-1} \ldots dx_l\, dx_{l-1} \ldots dx_2 .$$

Interchanging the order of an integration in (12) we get

$$(13)\quad y_2(h_2(t)) \geq \int_{\bar{t}_n}^{H_n(t)} a_k(x_n) f_k(y_1(g_k(x_n))) \int_{\bar{t}_{n-1}}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1}) \ldots \int_{\bar{t}_{l-1}}^{h_l^{-1}(x_l)} p_{l-1}(x_{l-1}) \times \int_{h_{l-1}^{-1}(x_{l-1})}^{H_{l-2}(t)} p_{l-2}(x_{l-2}) \ldots \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2) \times dx_2 \ldots dx_{l-2}\, dx_{l-1} \ldots dx_{n-1}\, dx_n ,$$

where $\bar{t}_{l-1} = t_{l-1}$, $\bar{t}_k = h_k(\bar{t}_{k-1})$, $k = l, \ldots, n-1$.

Take any $T \geq \bar{t}_n$ and let $T_k = \sup_{t_n \leq t \leq T} (\max\{g_k(t), t\})$. Multiplying (13) by $p_1(t)/f_k(y_1(t))$, then integrating from $\bar{t}_n$ to T_k, using the first equation of (S_λ) and the monotonicity of g_k, y_1, f_k and (A_6), we have

$$(14)\quad \int_{\bar{t}_n}^{T_k} \frac{y_1'(t)\, dt}{f_k(y_1(t))} \geq \int_{\bar{t}_n}^{T_k} \frac{p_1(t)}{f_1(y_1(t))} \int_{\bar{t}_n}^{H_n(t)} a_k(x_n) f_k(y_1(g_k(x_n))) \int_{\bar{t}_{n-1}}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1}) \ldots \times \int_{\bar{t}_{l-1}}^{h_l^{-1}(x_l)} p_{l-1}(x_{l-1}) \int_{h_{l-1}^{-1}(x_{l-1})}^{H_{l-2}(t)} p_{l-2}(x_{l-2}) \ldots \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2) \times dx_2 \ldots dx_{l-2}\, dx_{l-1} \ldots dx_{n-1}\, dx_n\, dt$$

$$\geq \int_{H_n(\bar{t}_n)}^{H_n(T_k)} a_k(x_n) \int_{H_n^{-1}(x_n)}^{T_k} p_1(t) \frac{f_k(y_1(g_k(H_n^{-1}(x_n))))}{f_k(y_1(t))} \times \int_{\bar{t}_{n-1}}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1}) \ldots \int_{\bar{t}_{l-1}}^{h_l^{-1}(x_l)} p_{l-1}(x_{l-1}) \int_{h_{l-1}^{-1}(x_{l-1})}^{H_{l-2}(t)} p_{l-2}(x_{l-2}) \ldots \times \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2)\, dx_2 \ldots dx_{l-2}\, dx_{l-1} \ldots dx_{n-1}\, dt\, dx_n .$$

Since $l \geq 3$, $\lim_{t\to\infty} y_1(t) = \infty$, we can take $\bar{t}_n$ so large that $y_1(t) \geq K_0$ for $t \geq \bar{t}_n$. Because the functions $f_k(y_1(t))$, $g_k(t)$ are nondecreasing on $[\bar{t}_n, \infty)$, it is easy to see that $f_k(y_1(g_k(H_n^{-1}(x_n))))/f_k(y_1(t)) \geq 1$ for $u = H_n^{-1}(x_n) \leq t \leq g_r(u)$, $u \in \mathscr{A}_k$. The inequality (14) implies

$$(15)\quad \int_{\bar t_n}^{T_k} \frac{y_1'(t)\,dt}{f_k(y_1(t))} \ge \int_{\mathscr{A}_k\cap[\bar t_n, H_n(T_k)]} a_k(x_n) \int_{H_n^{-1}(x_n)}^{g_k(H_n^{-1}(x_n))} p_1(t) \int_{\bar t_{n-1}}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1})\dots$$
$$\times \int_{\bar t_{l-1}}^{h_l^{-1}(x_l)} p_{l-1}(x_{l-1}) \int_{h_{l-1}^{-1}(x_{l-1})}^{H_{l-2}(t)} p_{l-2}(x_{l-2})\dots \int_{h_3^{-1}(x_3)}^{H_2(t)} p_2(x_2)$$
$$\times\, dx_2\dots dx_{l-2}\, dx_{l-1}\dots dx_{n-1}\, dt\, dx_n\,.$$

Letting $T\to\infty$ in (15) and using (C_4) we get a contradiction to (C_2).

3. Let $l=2$ and $r\colon 1\le r\le N$ be fixed. Integrating the n-th equation of (S_λ) over $[h_n(t), s_n]$, using (A_2), (A_5) and (N_2), we have

$$|y_n(h_n(t))| \ge \int_{h_n(t)}^{s_n} a_r(x) f_r(y_1(g_r(x)))\,dx\,.$$

Putting the last inequality into (2_2) and replacing t with $h_2(t)$, we have

$$(16)\quad y_2(h_2(t)) \ge \int_{h_2(t)}^{s_2} p_2(x_2)\dots \int_{h_{n-1}(x_{n-2})}^{s_{n-1}} p_{n-1}(x_{n-1}) \int_{h_n(x_{n-1})}^{s_n} a_r(x) f_r(y_1(g_r(x)))$$
$$\times\, dx\, dx_{n-1}\dots dx_2\,,\qquad t\ge \gamma(T_0)=T_3\,.$$

Interchanging the order of integration in (16), we get

$$(17)\quad y_2(h_2(t)) \ge \int_{H_n(t)}^{\bar s_n} a_r(x_n) f_r(y_1(g_r(x_n))) \int_{H_{n-1}(t)}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1})\dots \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)$$
$$\times\, dx_2\dots dx_{n-1}\, dx_n\,,$$

where $\bar s_n = h_n(h_{n-1}(\dots(h_3(s_2))\dots))$.

Take any $T\ge T_3$ and let $T_r=\sup_{T_3\le t\le T}\max\{g_r(t), t\}$. Multiplying (17) by $p_1(t)/f_r(y_1(t))$, then integrating from T_3 to T_r, using the first equation of (S_λ) and the monotonicity of g_r, y_1, f_r and (A_6), we get

$$(18)\quad \int_{T_3}^{T_r} \frac{y_1'(t)\,dt}{f_r(y_1(t))} \ge \int_{T_3}^{T_r} \frac{p_1(t)}{f_r(y_1(t))} \int_{H_n(t)}^{H_n(T_r)} a_r(x_n) f_r(y_1(g_r(x_n))) \int_{H_{n-1}(t)}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1})\dots$$
$$\times \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)\, dx_2\dots dx_{n-1}\, dx_n\, dt$$
$$\ge \int_{\mathscr{A}_r\cap[H_n(T_3), H_n(T_r)]} a_r(x_n) \int_{T_3}^{H_n^{-1}(x_n)} p_1(t) \frac{f_r(y_1(g_r(H_n^{-1}(x_n))))}{f_r(y_1(t))}$$
$$\times \int_{H_{n-1}(t)}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1})\dots \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)\, dx_2\dots dx_{n-1}\, dt\, dx_n\,.$$

i) Let $\lim_{t\to\infty} y_1(t)=\infty$. Then proceeding analogously as in the corresponding part of the case 2, we obtain a contradiction to (C_3).

ii) Let $\lim_{t\to\infty} y_1(t) = y_{10}$. If $T \to \infty$, then from (18) in view of (C_5) we have

$$\int_{\mathscr{A}_r \cap [H_n(T_3), \infty)} a_r(x_n) \int_{T_3}^{H_n^{-1}(x_n)} p_1(t) \int_{H_{n-1}(t)}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1}) \dots \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)$$
$$\times\, dx_2 \dots dx_{n-1}\, dt\, dx_n \le \int_0^{y_{10}} \frac{du}{f_r(u)} < \infty ,$$

which contradicts (C_3).

The proof of Theorem 1 is complete.

PROOF OF THEOREM 2. Suppose that (S_λ) has a weakly nonoscillatory regular solution $y = (y_1, \dots, y_n)$. Then by Lemma 2 y is nonoscillatory. Proceeding exactly as in the proof of Theorem 1, we find that (N_l), (1), (2_l), (3_i) hold for $t \ge t_1$ and there exists an odd integer l: $1 \le l \le n$ ($n + \lambda$ is even) or $l = n$. Let $T_0 \ge t_1$ be so large that $g_m(t) \ge t_1$ for $t \ge T_0$, $m = 1, 2, \dots, N$.

1. We suppose that $l = n$ and then $3 \le l < n$. The proofs in these cases are the same as in the corresponding parts of the proof of Theorem 1.

2. Now we consider $l = 1$. Let r, $1 \le r \le N$, be fixed. Replacing t in (2_1) with $h_2(t)$ and using (N_1), we obtain

$$(19)\quad -y_2(h_2(t)) \ge \int_{h_2(t)}^{s_2} p_2(x_2) \dots \int_{h_{n-2}(x_{n-3})}^{s_{n-2}} p_{n-2}(x_{n-2}) \int_{h_{n-1}(x_{n-2})}^{s_{n-1}} p_{n-1}(x_{n-1})$$
$$\times\, y_n(h_n(x_{n-1}))\, dx_{n-1}\, dx_{n-2} \dots dx_2 , \qquad T_3 = \gamma(t_1) \le t \le s_2 .$$

Integrating the n-th equation of (S_λ) over $[h_n(t), s_n]$, $s_n = \max\{s_{n-1}, h_{n-1}(s_{n-1})\}$ and using (A_2), (A_5), (N_1), we get

$$|y_n(h_n(t))| \ge \int_{h_n(t)}^{s_n} a_r(x) f_r(y_1(g_r(x)))\, dx .$$

Putting the last inequality into (19) we have

$$(20)\quad -y_2(h_2(t)) \ge \int_{h_2(t)}^{s_2} p_2(x_2) \dots \int_{h_{n-1}(x_{n-2})}^{s_{n-1}} p_{n-1}(x_{n-1}) \int_{h_n(x_{n-1})}^{s_n} a_r(x) f_r(y_1(g_r(x)))$$
$$\times\, dx\, dx_{n-1} \dots dx_2$$
$$\ge \int_{H_n(t)}^{\bar{s}_n} a_r(x_n) f_r(y_1(g_r(x_n))) \int_{H_{n-1}(t)}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1}) \dots \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)$$
$$\times\, dx_2 \dots dx_{n-1}\, dx_n ,$$

where $\bar{s}_n = h_n(h_{n-1}(\dots(h_3(s_2))\dots))$.

Take any $T \geq T_3$ and let $T_r = \sup_{T_3 \leq t \leq T} \max\{g_r(t), t\}$. Multiplying (20) by $p_1(t)/f_r(y_1(g_r(t)))$, then integrating from T_3 to T_r, using the first equation of (S_λ), the monotonicity of g_r, y_1, f_r and (N_1), we get

$$(21)\quad -\int_{T_3}^{T_r} \frac{y_1'(t)\,dt}{f_r(y_1(t))} \geq \int_{T_3}^{T_r} \frac{p_1(t)}{f_r(y_1(t))} \int_{H_n(t)}^{H_n(T_r)} a_r(x_n) f_r(y_1(g_r(x_n))) \int_{H_{n-1}(t)}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1}) \dots$$

$$\times \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)\,dx_2 \dots dx_{n-1}\,dx_n\,dt$$

$$= \int_{\mathscr{R}_r \cap [H_n(T_3), H_n(T_r)]} a_r(x_n) \int_{g_r(H_n^{-1}(x_n))}^{H_n^{-1}(x_n)} p_1(t) \frac{f_r(y_1(g_r(x_n)))}{f_r(y_1(g_r(H_n^{-1}(x_n))))}$$

$$\times \int_{H_{n-1}(t)}^{h_n^{-1}(x_n)} p_{n-1}(x_{n-1}) \dots \int_{H_2(t)}^{h_3^{-1}(x_3)} p_2(x_2)\,dx_2 \dots dx_{n-1}\,dt\,dx_n\,.$$

Since $y_1(t) > 0$, $y_1'(t) < 0$ for $t \geq t_1$, there exists $\lim_{t\to\infty} y_1(t) = b \geq 0$. Let $\lim_{t\to\infty} y_1(t) = b > 0$. Then

$$(22)\qquad \lim_{t\to\infty} \frac{f_r(y_1(g_r(t)))}{f_r(y_1(g_r(H_n^{-1}(t)))} = 1\,.$$

Letting $T \to \infty$ in (21), using (22) and (C_5), we obtain a contradiction to (C_6).

If $\lim_{t\to\infty} y_1(t) = 0$, then by Lemma 2 $y_i(t)$, $i = 1, 2, \dots, n$, tend monotonically to zero for $t \to \infty$.

The proof of Theorem 2 is complete.

PROOF OF THEOREM 3. To prove Theorem 3, in addition to the proof of Theorem 2 we must show that $\lim_{t\to\infty} y_1(t) = 0$ is impossible.

Let $H_n(t) \equiv t$. Then $f_r(y_1(g_r(t)))/f_r(y_1(g_r(H_n^{-1}(t)))) \equiv 1$. If $\lim_{t\to\infty} y_1(t) = 0$, we may choose $\bar{T}_3$ so large that $|y_1(t)| \leq k_0$ for $t \geq \bar{T}_3$.

Letting $T \to \infty$, from (21) in view of (C_5) we get a contradiction to (C_6) with $H_n^{-1}(t) \equiv t$.

The proof of Theorem 3 is complete.

References

[1] A. F. Ivanov, Y. Kitamura, T. Kusano and V. N. Shevelo, Oscillatory solutions of functional differential equations generated by deviation of arguments of mixed type, Hiroshima Math. J. **12** (1982), 645–655.

[2] P. Marušiak, Oscillation theorems for nonlinear differential systems with general deviating arguments, Hiroshima Math. J. **16** (1986), 651–663.

[3] P. Marušiak and V. N. Shevelo, On the relation between boundedness and oscillation of

solutions of many-dimensional differential systems with deviating arguments, Czechoslovak Math. J. **37** (1987), 559–566.

[4] V. Šeda, On nonlinear differential systems with deviating arguments, Czechoslovak Math. J. **36** (1986), 450–466.

[5] N. V. Varech and V. N. Shevelo, Asymptotic properties of components of solutions of certain many-dimensional systems with deviating arguments (in Russian). In: Differential-functional equations and their applications, Inst. Math. Ukrainian Academy of Sciences, Kiev (1985), 108–124.

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