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Some monomials in the universal Wu classes

Dedicated to Professor Shôrô Araki on his sixtieth birthday

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§1. Introduction

Let BO be the space which classifies stable real vector bundles. Then, its mod 2 cohomology $H^*(BO; Z_2)$ is the polynomial algebra over Z_2 on the (universal) Stiefel-Whitney classes $w_i \in H^i(BO; Z_2)$, $i \geqq 1$ (cf. [2], [6]). Moreover, the Steenrod squaring operation on $H^*(BO; Z_2)$ is given by

$$(1.1) \qquad Sq^j w_i = \sum_{t=0}^{j} \binom{i-1-t}{j-t} w_{i+j-t} w_t \qquad \text{for } 0 \leqq j < i,$$

where $\binom{a}{b}$ is the binomial coefficient and $w_0 = 1$ (cf. [7]).

Let $v_i \in H^i(BO; Z_2)$ be the (universal) Wu classes (cf. [1], [4], [5]) defined inductively by

$$(1.2) \qquad v_0 = w_0 = 1 \qquad \text{and} \qquad w_i = \sum_{k=0}^{i} Sq^k v_{i-k}, \quad i \geqq 1.$$

Then, the Wu class v_i is the polynomial

$$v_i = v_i(w_1, w_2, \ldots) \quad \text{with coefficients in } Z_2$$

on the Stiefel-Whitney classes w_j's, which can be described exactly by using (1.1-2) and the properties of the Steenrod operations, but it is not so easy in general to see the explicit form of this polynomial. In [8, Cor.], we find all monomials $w_{i_1} \ldots w_{i_s}$, $i_1 > \cdots > i_s \geqq 1$, which appear in $v_i(w_1, w_2, \ldots)$ with coefficient 1.

The purpose of this paper is to study the monomials of the form w_i^2 or $w_j w_1^k$, $j \geqq 2$, and to prove the following two theorems.

THEOREM 1.3. *In the polynomial $v_i(w_1, w_2, \ldots)$, the monomial w_j^2, $2j = i$, appears with coefficient 1 when and only when*

$$i = a \geqq 2, \quad or \quad i = a + b, \quad a > b \geqq 2,$$

where a and b are all powers of 2.

THEOREM 1.4. *In* $v_i(w_1, w_2, \dots)$, *the monomial* $w_j w_1^{i-j}$, $i \geqq j \geqq 2$, *appears with coefficient* 1 *if and only if*

$$i = a \geqq 2 \quad and \quad a/2 < j \leqq a\,, \quad or$$

$$i = a + b\,, \quad a > b \geqq 1 \quad and \quad b < j \leqq a\,,$$

where a *and* b *are all powers of* 2.

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§2. Proof of Theorem 1.3

The following result on the binomial coefficient is used frequently.

PROPOSITION 2.1 (cf. [3]).

$$\binom{a}{b} \equiv \prod_{i=0}^{s} \binom{a_i}{b_i} \quad \text{mod } 2$$

for $a = \sum_{i=0}^{s} a_i 2^i$ *and* $b = \sum_{i=0}^{s} b_i 2^i$ *with* $0 \leqq a_i,\ b_i \leqq 1$.

On the Steenrod operation $Sq^j\colon H^i(\ ;Z_2) \to H^{i+j}(\ ;Z_2)$, we use the following properties in this paper:

Sq^j is a natural homomorphism with $Sq^0 = id$,

$Sq^j x = 0$ if $j > i$, $= x^2$ if $j = i$, for $x \in H^i(\ ;Z_2)$,

$Sq^j(xy) = \sum_{k=0}^{j} (Sq^k x)(Sq^{j-k} y)$ (the Cartan formula), and

$Sq^j Sq^k = \sum_{s=0}^{[j/2]} \binom{k-1-s}{j-2s} Sq^{j+k-s} Sq^s$ if $0 < j < 2k$ (the Adem relations),

where [ ] is the Gauss symbol.

For a monomial x on w_j's, we say simply that x appears in $A \in H^i(BO; Z_2)$ when the coefficient of x is 1 in the polynomial representing A on w_j's with coefficients in Z_2. Moreover, we mean in this section by the notation

$$A \sim B \quad \text{for} \quad A, B \in H^{2n}(BO; Z_2)\,, \quad n \geqq 1\,,$$

that the monomial w_n^2 does not appear in $A + B$.

LEMMA 2.2. *Let* $s \geqq 3$ *and* $j_1 \geqq \cdots \geqq j_s \geqq 1$. *Then*

$$Sq^i(w_{j_1} \dots w_{j_s}) \sim 0 \quad \textit{for any } i \geqq 0\,.$$

PROOF. The Cartan formula and (1.1) tell us the lemma by the above definition. □

The following result is a special case of [8, Cor.]:

PROPOSITION 2.3. *The monomial of the form* $w_{i-j}w_j$, $i > 2j \geqq 0$ ($w_0 = 1$), *appears in the Wu class* v_i *if and only if*

$$i = a \geqq 1 \quad \text{and} \quad 0 \leqq j < a/2\,, \quad \text{or} \quad i = a + b\,, \quad a > b \geqq 1 \quad \text{and} \quad j = b\,,$$

where a *and* b *are all powers of* 2.

LEMMA 2.4. $Sq^\alpha(w_a w_b) \sim 0$ *for any powers* $a > b \geqq 1$ *of* 2 *and any* $\alpha \geqq 0$.

PROOF. By the Cartan formula, (1.1) and the definition of $\sim$, we have

$$Sq^\alpha(w_a w_b) = \sum_{i=0}^{\alpha} (Sq^i w_a)(Sq^{\alpha-i} w_b) \sim \sum_{i=0}^{\alpha} \binom{a-1}{i}\binom{b-1}{\alpha-i} w_{a+i} w_{b+\alpha-i}\,.$$

Here, if $b > \alpha - i$, then $b + \alpha - i < 2b \leqq a \leqq a + i$, since $a > b$ are powers of 2. □

LEMMA 2.5. (i) *Let* a *be a power of* 2. *Then*

$$Sq^{2\alpha}(w_a^2) \sim w_{a+\alpha}^2 \qquad \text{for } 0 \leqq \alpha < a\,.$$

(ii) *Let* $a > b$ *be powers of* 2. *Then*

$$Sq^{2\alpha}(w_{a+b}^2) \sim \begin{cases} w_{a+b+\alpha}^2 & \text{for } 0 \leqq \alpha < b \quad \text{or} \quad a \leqq \alpha < a + b\,, \\ 0 & \text{for } b \leqq \alpha < a\,. \end{cases}$$

PROOF. Since $Sq^{2\alpha}(w_i^2) = (Sq^\alpha w_i)^2 \sim \binom{i-1}{\alpha} w_{i+\alpha}^2$, we see the lemma by Proposition 2.1. □

LEMMA 2.6. *Let* $a \geqq 2$ *be a power of* 2. *Then*

$$Sq^{2\alpha} P_{2a} \sim \begin{cases} w_{a+\alpha}^2 & \text{for } 1 \leqq \alpha < a/2\,, \\ 0 & \text{for } a/2 \leqq \alpha < a\,, \end{cases}$$

where $P_{2a} = \sum_{i=1}^{a-1} w_{2a-i} w_i$.

PROOF. In $Sq^{2\alpha} P_{2a} = \sum_{i=1}^{a-1} \sum_{j=0}^{2\alpha} (Sq^j w_{2a-i})(Sq^{2\alpha-j} w_i)$, the coefficient of $w_{a+\alpha}^2$ is seen by (1.1) to be equal to

$$c(a, \alpha) = \sum_{i=a-\alpha}^{a-1} \binom{2a-i-1}{\alpha+i-a}\binom{i-1}{\alpha-i+a}\,.$$

Thus it is sufficient to prove that

(2.7) $c(a, \alpha) \equiv 1$ for $1 \leqq \alpha < a/2$, $\equiv 0$ for $a/2 \leqq \alpha < a$,

where $\equiv$ means $\equiv$ mod 2. We can show (2.7) easily when $a = 2, 4$.

We assume (2.7) for $a \geqq 4$ inductively, and study $c(2a, \alpha)$ by putting

$$m_1 = 2a - i - 1, \quad m_2 = \alpha + i - 2a, \quad m_3 = i - a - 1, \quad m_4 = \alpha - i + 2a$$

for $1 \leqq \alpha < 2a$ and $2a - \alpha \leqq i < 2a$.

Case 1: $1 \leqq \alpha < a/2$. In this case, $0 \leqq m_k < a$ for all k. Therefore,

$$\binom{4a-i-1}{\alpha+i-2a} = \binom{2a+m_1}{m_2} \equiv \binom{m_1}{m_2} \equiv \binom{a+m_1}{m_2} = \binom{2a-j-1}{\alpha+j-a},$$

$$\binom{i-1}{\alpha-i+2a} = \binom{a+m_3}{m_4} \equiv \binom{m_3}{m_4} = \binom{j-1}{\alpha-j+a}$$

for $j = i - a$, by Proposition 2.1, because a is a power of 2. Thus,

$$c(2a, \alpha) \equiv c(a, \alpha) \equiv 1 \qquad \text{if } 1 \leqq \alpha < a/2.$$

Case 2: $a/2 \leqq \alpha < a$. In this case, $0 \leqq m_k < a$ hold except for $m_4 < a$.

If $a + \alpha < i < 2a$, then $m_4 < a$ also holds, and the above proof shows that

$$\binom{4a-i-1}{\alpha+i-2a}\binom{i-1}{\alpha-i+2a} \equiv \binom{m_1}{m_2}\binom{m_3}{m_4} = 0,$$

because $m_1 \geqq m_2$ implies $m_3 \leqq m_4 + a - 2\alpha - 2 < m_4$. If $2a - \alpha \leqq i \leqq a + \alpha$, then $0 \leqq m_4 - a < a$ and $\binom{a+m_3}{m_4} \equiv \binom{m_3}{m_4-a}$. Thus we have

$$c(2a, \alpha) \equiv \textstyle\sum_{i=2a-\alpha}^{a+\alpha} d_1(i)d_2(i) \qquad \text{for}$$

$$d_1(i) = \binom{m_1}{m_2} = \binom{2a-i-1}{\alpha+i-2a}, \qquad d_2(i) = \binom{m_3}{m_4-a} = \binom{i-a-1}{\alpha-i+a}.$$

Put $a' = a/2$. Then $d_1(3a'+j) = \binom{a'-j-1}{\alpha-a'+j} = d_2(3a'-j)$ for any integer j, and $d_1(3a') = d_2(3a') \equiv 1$ since a' is a power of 2. Therefore

$$c(2a, \alpha) \equiv 1 \qquad \text{if } a/2 \leqq \alpha < a.$$

Case 3: $a \leqq \alpha < 2a$. In this case, $0 \leqq m_k < 2a$ for $k = 1, 2$. Hence

$$c(2a, \alpha) \equiv \textstyle\sum_{i=2a-\alpha}^{2a-1} \binom{m_1}{m_2}\binom{a+m_3}{m_4} = 0 \qquad \text{if } a \leqq \alpha < 2a,$$

because $m_1 \geqq m_2$ implies $a + m_3 \leqq m_4 + 2a - 2\alpha - 2 < m_4$.

Therefore, (2.7) and the lemma are proved by induction. □

Now, we prove Theorem 1.3 which is trivial for odd i and is restated by the above notation $\sim$ as follows:

(2.8) $$v_{2i} \sim w_i^2 \quad \text{if} \quad i \in N_1 \cup N_2 , \quad \sim 0 \text{ otherwise} ,$$

where $N_1 = \{2^k | k \geqq 0\}$ and $N_2 = \{2^k + 2^l | k > l \geqq 0\}$.

This holds for $i = 1$, because $v_2 = w_2 + w_1^2 \sim w_1^2$. We prove it for $i \geqq 2$ by induction in the following way, where the inductive assumption is denoted simply by (2.8) and Lemma or Proposition 2.n by 2.n.

We note that $\sim$ is an equivalence relation preserved by $+$. Now,

(2.9) in $v_{2i} = w_{2i} + \sum_{k=1}^{2i} Sq^k v_{2i-k} \sim v_i^2 + \sum_{j=i+1}^{2i-1} Sq^{2i-j} v_j$ of (1.2),

$$v_i^2 \sim w_i^2 \text{ if } i \in N_1, \quad \sim 0 \text{ otherwise}; \qquad Sq^{2i-j} v_j \sim 0 \quad \text{if } j \notin N_1 \cup N_2 ,$$

by 2.3, 2.2 and (2.8), and the other terms are seen by 2.4-6 as follows:

Case 1: $i = 2a$ for $a \in N_1$. In this case, $\{j \in N_1 \cup N_2 | i < j < 2i\} = \{2a + t | t \in N_1, t \leqq a\}$, and then 2.2-4, (2.8) and 2.5(ii) show that

$$Sq^{2a-t} v_{2a+t} \sim \begin{cases} Sq^{2a-1}(w_{2a} w_1) \sim 0 & \text{if } t = 1 , \\ Sq^{2a-t}(w_{2a} w_t + w_{a+t/2}^2) \sim 0 & \text{if } 2 \leqq t \leqq a, \, t \in N_1 . \end{cases}$$

Thus $v_{4a} \sim w_{2a}^2$ by (2.9).

Case 2: $i = 2a + b$ for a, $b \in N_1$, $a \geqq b$. In this case, $\{j \in N_1 \cup N_2 | i < j < 2i\} = \{4a, 4a + t | t \in N_1, t \leqq b\} \cup \{2a + s | s \in N_1, 2b \leqq s \leqq a\}$. Then

$$\begin{aligned} Sq^{2b} v_{4a} &\sim Sq^{2b}(w_{4a} + w_{2a}^2 + P_{4a}) \\ &\sim (1 + \varepsilon) w_i^2 \text{ for } \varepsilon = 1 \quad \text{if } b < a , \quad = 0 \quad \text{if } b = a , \end{aligned}$$

by (1.1), 2.5(i) and 2.6. Also $Sq^{2b-t} v_{4a+t} \sim 0$ in the same way, and

$$Sq^{2a+2b-s} v_{2a+s} \sim \begin{cases} 0 & \text{if } s > 2b , \\ Sq^{2a} w_{a+b}^2 \sim w_i^2 & \text{if } s = 2b \leqq a , \end{cases}$$

by 2.5(ii). Thus $v_{4a+2b} \sim w_{2a+b}^2$ by (2.9).

Case 3: $i = 2a + b + r$ for a, $b \in N_1$, $a \geqq b > r$. In this case, $\{j \in N_1 \cup N_2 | i < j < 2i\} = \{4a, 4a + t | t \in N_1, t \leqq 2b\} \cup \{2a + s | 2b \leqq s \leqq a\}$. Then, in the same way, we have

$$Sq^{2b+2r} v_{4a} \sim (1 + \varepsilon) w_i^2 \qquad \text{for the above } \varepsilon ,$$

$$Sq^{2b+2r-t} v_{4a+t} \sim 0 \quad \text{if } t \leqq b , \quad = Sq^{2r} v_{4a+2b} \sim w_i^2 \quad \text{if } t = 2b ,$$

$$Sq^{2a+2b+2r-s} v_{2a+s} \sim 0 \quad \text{if } s > 2b , \quad \sim w_i^2 \quad \text{if } s = 2b \leqq a ;$$

hence $v_{4a+2b+2r} \sim 0$.

Thus, Theorem 1.3 is proved completely. □

§3. Proof of Theorem 1.4

We mean in this section by the notation

$$A \approx B \qquad \text{for } A, B \in H^m(BO; Z_2), \quad m \geqq 2,$$

that the monomial $w_i w_1^{m-i}$ does not appear in $A + B$ for any $2 \leqq i \leqq m$. This is also an equivalence relation preserved by $+$, and moreover satisfies the following

LEMMA 3.1. *If $A \approx B$, then $Sq^i A \approx Sq^i B$ for $i \geqq 0$ and $w_1 A \approx w_1 B$.*

PROOF. The Cartan formula and (1.1) tell us easily that

$$Sq^i(w_{j_1} \dots w_{j_s} w_1^k) \approx 0 \qquad \text{if } s \geqq 2, \quad j_1 \geqq \dots \geqq j_s \geqq 2, \quad k \geqq 0,$$

$$Sq^i(w_1^k) = \binom{k}{i} w_1^{k+i} \approx 0 \qquad \text{if } k \geqq 1, \quad i + k \geqq 2,$$

by the definition of $\approx$. Thus we see the lemma. □

We put

(3.2) $$Q_m(k) = \sum_{i=0}^{k-1} w_{m-i} w_1^i \qquad \text{for any } m > k \geqq 1.$$

LEMMA 3.3. *Let $a \geqq 1$ be a power of 2. Then*

$$Sq^\alpha Q_{2a}(a) \approx Q_{2a+\alpha}(a) \quad \text{for } 0 \leqq \alpha < a, \quad \approx Q_{2a+\alpha}(2a) \quad \text{for } a \leqq \alpha < 2a.$$

PROOF. The lemma holds trivially for $\alpha = 0$, and so does if $a = 1$ since $Q_2(1) = w_2$ and $Sq^1 w_2 = w_3 + w_2 w_1 = Q_3(2)$.

Let $a \geqq 2$. Then we see the lemma for $\alpha = 1$, because

$$\begin{aligned} Sq^1(w_{2a-i} w_1^i) &= w_{2a-i}(Sq^1 w_1^i) + (Sq^1 w_{2a-i}) w_1^i \\ &= i w_{2a-i} w_1^{i+1} + w_{2a-i} w_1^{i+1} + (2a - i - 1) w_{2a-i+1} w_1^i \\ &= w_{2a-i} w_1^{i+1} + w_{2a-i+1} w_1^i \quad \text{if } i \text{ is even}, \quad = 0 \text{ if } i \text{ is odd}. \end{aligned}$$

In general, the Cartan formula and (1.1) imply that

(3.4) $$Sq^\alpha(w_{m-i} w_1^i) = \sum_{j=0}^{\alpha} (Sq^{\alpha-j} w_{m-i}) Sq^j(w_1^i)$$

$$\approx \sum_{j=0}^{\alpha} \sum_{t=0}^{1} \binom{m-i-1-t}{\alpha-j-t} \binom{i}{j} w_{m+\alpha-i-j-t} w_t w_1^{i+j}, \quad \text{by putting } \binom{m-i-2}{-1} = 0.$$

If $\alpha = m-1$, then the coefficient in (3.4) is 0 for $i \neq j$. Hence,

$$Sq^{m-1}(w_{m-i}w_1^i) \approx w_{2m-2i-1}w_1^{2i} + w_{2m-2i-2}w_1^{2i+1}\,, \tag{3.5}$$

$$Sq^{m-1}Q_m(k) \approx \textstyle\sum_{i=0}^{2k-1} w_{2m-1-i}w_1^i = Q_{2m-1}(2k)\,,$$

and the latter for $m = 2a$ and $k = a$ is the lemma for $\alpha = 2a-1$. Moreover,

$$Sq^\alpha Q_{2m}(2k) \approx Sq^\alpha\{w_{2m} + w_{2m-2k}w_1^{2k} + w_1 \cdot Sq^{m-1}Q_m(k)\} \tag{3.6}$$

by (3.2), (3.5) and Lemma 3.1.

In (3.6) for $m = a = 2k$ and $\alpha = 2a-2$, we have

$$Sq^{2a-2}w_{2a} \approx w_{4a-2}\,, \qquad Sq^{2a-2}(w_a w_1^a) \approx w_{2a-2}w_1^{2a} \qquad \text{by (3.4)}\,.$$

Since $Sq^{2a-3}Sq^{a-1} = 0$ by the Adem relation, we have also

$$Sq^{2a-2}\{w_1 \cdot Sq^{a-1}Q_a(a/2)\} = w_1 \cdot Sq^{2a-2}Sq^{a-1}Q_a(a/2) \approx w_1 \cdot Sq^{2a-2}Q_{2a-1}(a)$$

$$\approx w_1 \cdot Q_{4a-3}(2a) = Q_{4a-2}(2a) + w_{4a-2} + w_{2a-2}w_1^{2a}\,,$$

by (3.5) and Lemma 3.1. Thus the lemma holds for $\alpha = 2a-2$ by (3.6).

Therefore the lemma is proved for $\alpha = 0, 1, 2a-2, 2a-1$; in particular, it holds if $a = 2$. Now, assuming the lemma for $a \geqq 2$ inductively, we study $Sq^\alpha Q_{4a}(2a)$ by (3.6) for $m = 2a = 2k$ as follows.

Case 1: $\alpha = 2n$ for $1 \leqq n \leqq 2a-2$. Then, in (3.6), we have

$$Sq^{2n}w_{4a} \approx w_{4a+2n}\,, \qquad Sq^{2n}(w_{2a}w_1^{2a}) \approx \begin{cases} w_{2a+2n}w_1^{2a} & \text{if } n < a\,, \\ w_{2n}w_1^{4a} & \text{if } n \geqq a\,, \end{cases}$$

by (3.4). To study $Sq^{2n}(w_1 \cdot Sq^{2a-1}Q_{2a}(a))$ by the Cartan formula, we note that the Adem relation and the dimensional reason tell us

$$Sq^{2n-\varepsilon}Sq^{2a-1}Q_{2a}(a) = \textstyle\sum_{j=0}^{n-\varepsilon} \binom{2a-2-j}{2n-\varepsilon-2j} Sq^{2a+2n-1-\varepsilon-j}Sq^jQ_{2a}(a)$$

$$= \binom{2a-2-n+\varepsilon}{\varepsilon} Sq^{2a+n-1}Sq^{n-\varepsilon}Q_{2a}(a) \qquad \text{for } \varepsilon = 0, 1\,,$$

which is $(2a-n-1)(Sq^{n-1}Q_{2a}(a))^2 \approx 0$ if $\varepsilon = 1$. Therefore,

$$Sq^{2n}\{w_1 \cdot Sq^{2a-1}Q_{2a}(a)\} \approx w_1 \cdot Sq^{2a+n-1}Sq^nQ_{2a}(a)$$

$$\approx w_1 \cdot Sq^{2a+n-1}Q_{2a+n}(a') \approx \textstyle\sum_{i=1}^{2a'} w_{4a+2n-i}\, w_1^i\,,$$

where $a' = a$ for $1 \leqq n < a$, $a' = 2a$ for $a \leqq n \leqq 2a-2$, by the inductive assumption, (3.5) and Lemma 3.1. By adding these, we have

$$Sq^{2n}Q_{4a}(2a) \approx Q_{4a+2n}(2a')\,, \qquad \text{as desired}\,.$$

Case 2: $\alpha = 2n + 1$ for $1 \leqq n \leqq 2a - 2$. Note that $Sq^{2n+1} = Sq^1 Sq^{2n}$ by the Adem relation. Then the above result implies that

$$Sq^{2n+1} Q_{4a}(2a) \approx Sq^1 Q_{4a+2n}(2a') \approx Q_{4a+2n+1}(2a'),$$

by Lemma 3.1 and the proof for $\alpha = 1$ stated in the first place.

Therefore, Lemma 3.3 is proved by induction. □

LEMMA 3.7. *Let $a \geqq b \geqq 1$ be powers of 2. Then*

$$Sq^\alpha Q_{2a+b}(2a, b) \approx \begin{cases} Q_{2a+b+\alpha}(2a, b) & \text{for } 0 \leqq \alpha < b, \\ Q_{2a+b+\alpha}(2b, b) & \text{for } b \leqq \alpha < 2a, \\ Q_{2a+b+\alpha}(4a, 2b) & \text{for } 2a \leqq \alpha < 2a + b, \end{cases}$$

where $Q_m(k, l) = \sum_{i=l}^{k-1} w_{m-i} w_1^i$ for any $m > k > l \geqq 0$.

PROOF. By definition, we see that $Q_m(k, 0) = Q_m(k)$ and

$$Q_m(k, l) = w_1^n Q_{m-n}(k - n, l - n) = Q_m(k, n) + Q_m(l, n)$$

for $l \geqq n \geqq 0$, where $Q_m(l, l) = 0$.

If $b = a$, then the lemma is proved by Lemma 3.3, because

$$Sq^\alpha Q_{3a}(2a, a) = Sq^\alpha(w_1^a Q_{2a}(a)) = w_1^a \cdot Sq^\alpha Q_{2a}(a) + w_1^{2a} \cdot Sq^{\alpha-a} Q_{2a}(a).$$

In particular, the lemma holds if $a = 1$.

Now, we prove the lemma for $a > b \geqq 1$ by induction on a. Since $Q_{2a+b}(2a, b) = w_1^b Q_{2a}(a) + w_1^a Q_{a+b}(a, b)$, we see $Sq^\alpha Q_{2a+b}(2a, b)$ by adding

$$Sq^\alpha(w_1^b Q_{2a}(a)) = w_1^b \cdot Sq^\alpha Q_{2a}(a) + w_1^{2b} \cdot Sq^{\alpha-b} Q_{2a}(a),$$

$$Sq^\alpha(w_1^a Q_{a+b}(a, b)) = w_1^a \cdot Sq^\alpha Q_{a+b}(a, b) + w_1^{2a} \cdot Sq^{\alpha-a} Q_{a+b}(a, b);$$

and these are seen by Lemmas 3.1, 3.3 and the inductive assumption and by separating into the following cases:

$$0 \leqq \alpha < b, \quad b \leqq \alpha < a, \quad a \leqq \alpha < a + b, \quad a + b \leqq \alpha < 2a, \quad 2a \leqq \alpha < 2a + b.$$

Then, we can certify easily the conclusion for $Sq^\alpha Q_{2a+b}(2a, b)$. □

Now, we prove Theorem 1.4, which is restated by the above notation $\approx$ as follows:

$$(3.8) \qquad v_i \approx \begin{cases} \sum_{j=0}^{a-1} w_{2a-j} w_1^j = Q_{2a}(a) & \text{if } i = 2a \in N_1, \\ \sum_{j=b}^{2a-1} w_{2a+b-j} w_1^j = Q_{2a+b}(2a, b) & \text{if } i = 2a + b \in N_2, \\ 0 & \text{otherwise}, \end{cases}$$

where N_1 and N_2 are the sets given in (2.8).

(3.8) holds for $i=2$, because $v_2 = w_2 + w_1^2 \approx w_2$; and we prove it by induction on i.

Case 1: $i = 2a \in N_1$, $a \geqq 2$. Then, we can study

$$v_i = w_i + \textstyle\sum_{j=1}^{i/2} Sq^j v_{i-j} \qquad \text{of (1.2)}$$

by using the inductive assumption and Lemmas 3.1 and 3.7 as follows:

If $i - j \notin N_1 \cup N_2$, then $Sq^j v_{i-j} \approx 0$.

If $i - j \in N_1$, then $j = a$ and $Sq^a v_a = v_a^2 \approx 0$.

If $i - j \in N_2$, then $i - j = a + t$ for $t \in N_1$ with $1 \leqq t \leqq a/2$, and

$$Sq^j v_{2a-j} = Sq^{a-t} v_{a+t} \approx Sq^{a-t} Q_{a+t}(a, t) \approx Q_{2a}(2t, t)$$

since $t \leqq a - t < a$. Thus $v_{2a} \approx w_{2a} + \sum_{k=1}^{a-1} w_{2a-k} w_1^k = Q_{2a}(a)$, as desired.

Case 2: $i = 2a + b$, $a \geqq b \geqq 1$, a, $b \in N_1$. Then, in the same way as Case 1, we see the following by using also Lemmas 3.3 and 3.7:

If $i - j \in N_1$, then $j = b$, $Sq^b v_{2a} \approx Sq^b Q_{2a}(a)$ and

$$Sq^b Q_{2a}(a) \approx Q_i(a) \quad \text{when} \quad b < a\,, \quad \approx Q_i(2a) \quad \text{when} \quad b = a\,.$$

If $i - j \in N_2$, then either $i - j = 2a + t$ for $t \in N_1$ with $1 \leqq t \leqq b/2$, and

$$Sq^{b-t} v_{2a+t} \approx Sq^{b-t} Q_{2a+t}(2a, t) \approx Q_i(2t, t)\,;$$

or $i - j = a + s$ for $s \in N_1$ with $b \leqq s \leqq a/2$, and

$$Sq^{a+b-s} v_{a+s} \approx Sq^{a+b-s} Q_{a+s}(a, s) \approx \begin{cases} Q_i(2s, s) & \text{for } 2b \leqq s \leqq a/2\,, \\ Q_i(2a, 2b) & \text{for } s = b\,. \end{cases}$$

Thus $v_i \approx w_i + Q_i(2a) + Q_i(b, 1) = Q_i(2a, b)$ if $b = a$, and

$$v_i \approx w_i + Q_i(a) + Q_i(b, 1) + Q_i(a, 2b) + Q_i(2a, 2b) = Q_i(2a, b) \quad \text{if} \quad b < a\,.$$

Case 3: $i = 2a + b + r$, $a \geqq b > r$, a, $b \in N_1$. Then:

If $i - j \in N_1$, then $j = b + r$, $Sq^{b+r} v_{2a} \approx Sq^{b+r} Q_{2a}(a)$ and

$$Sq^{b+r} Q_{2a}(a) \approx Q_i(a) \quad \text{when } b < a\,, \quad \approx Q_i(2a) \quad \text{when } b = a\,.$$

If $i - j \in N_2$, then either $i - j = 2a + t$ for $t \in N_1$ with $1 \leqq t \leqq b$, and

$$Sq^{b+r-t} v_{2a+t} \approx Sq^{b+r-t} Q_{2a+t}(2a, t) \approx \begin{cases} Q_i(2t, t) & \text{for } t < b\,, \\ Q_i(2a, b) & \text{for } t = b\,; \end{cases}$$

or $i - j = a + s$ for $s \in N_1$ with $b \leqq s \leqq a/2$, and

$$Sq^{a+b+r-s} v_{a+s} \approx Sq^{a+b+r-s} Q_{a+s}(a, s) \approx \begin{cases} Q_i(2s, s) & \text{for } 2b \leqq s \leqq a/2\,, \\ Q_i(2a, 2b) & \text{for } s = b\,. \end{cases}$$

Thus $v_i \approx w_i + Q_i(2a) + Q_i(b, 1) + Q_i(2a, b) = 0$ if $b = a$, and

$$v_i \approx w_i + Q_i(a) + Q_i(b, 1) + Q_i(2a, b) + Q_i(a, 2b) + Q_i(2a, 2b) = 0$$

if $b < a$.

Thus, Theorem 1.4 is proved completely. □

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