HIROSHIMA MATH. J.
20 (1990), 57–62

Subideals of the join of Lie algebras

Dedicated to the memory of Professor Sigeaki Tôgô

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(Received January 20, 1989)

Introduction

Chao and Stitzinger [4] proved that a common subideal of two subalgebras of a finite-dimensional soluble Lie algebra is a subideal of their join. Aldosray [1, 2] showed a similar theorem for a finite-dimensional or a soluble-by-finite Lie algebra over a field of characteristic zero and Tôgô [8] did it for a weak subideal. Recently Stonehewer [6, 7] has shown that a common subnormal subgroup of two permutable subgroups of a $\mathfrak{N}\mathfrak{X} \cap \mathfrak{Y}$-group is subnormal in their product if a common subnormal subgroup of two permutable subgroups of an $\mathfrak{A}\mathfrak{X} \cap \mathfrak{Y}$-group is always subnormal in their product, where $\mathfrak{X} = \mathrm{s}\mathfrak{X}$ and $\mathfrak{Y} = \mathrm{s}\mathfrak{Y} = \mathrm{Q}\mathfrak{Y}$ are classes of groups. The purpose of this paper is to prove some generalizations of these results and an analogue of Stonehewer's result.

In Section 1, we shall show that a common weak subideal of an arbitrary number of subalgebras of a Lie algebra, being finite-codimensional in their join, is a weak subideal of the join (Theorem 3), which is a generalization of Tôgô's theorem and we have the theorem of Chao and Stitzinger as a corollary. In Section 2, we shall show some analogue of Stonehewer's result [7] for Lie algebras (Theorem 6) and have also a similar result for finite-by-soluble Lie algebras to Aldosray [2, Theorem 1] (Theorem 7).

The author wishes to thank Dr. I. Stewart for his encouragement during the preparation of this paper and also the Mathematics Institute of the University of Warwick for its hospitality.

1.

We shall be concerned with Lie algebras over a field $\mathfrak{k}$ which are not necessarily finite-dimensional. Throughout the paper, L will be a Lie algebra over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified.

Notation and terminology will follow Amayo and Stewart [3] and Tôgô [8]. By $H \leq L$, $H \lhd L$, H si L and $H \lhd^m L$ we mean respectively that H is a subalgebra, an ideal, a subideal and an m-step subideal of L. H is a weak subideal of L provided that H is a subalgebra of L and $[L, {}_nH] \subseteq H$ for some integer $n \geqq 0$, and we write H wsi L, more precisely $H \leq^n L$ [5].

We recall the definitions of some classes. $\mathfrak{F}$, $\mathrm{E}\mathfrak{A}$ and $\mathfrak{A}^c$ denote respectively the classes of finite-dimensional, soluble, and soluble of derived length $\leqq c$ ($c \in \mathbf{N}$) Lie algebras over a field $\mathfrak{k}$. Let $\mathfrak{X}$ and $\mathfrak{Y}$ be classes of Lie algebras over $\mathfrak{k}$. $\mathfrak{X}\mathfrak{Y}$ is the class of Lie algebras L having an ideal $I \in \mathfrak{X}$ such that $L/I \in \mathfrak{Y}$, and L is called an $\mathfrak{X}$-by-$\mathfrak{Y}$ Lie algebra.

We recall the following result [8, Corollary of Theorem 2] without proof, being useful for us.

LEMMA 1. *Let L be a soluble Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. Then H* wsi *L if and only if H* si *L.*

PROPOSITION 2. *Let L be a Lie algebra over $\mathfrak{k}$. Let $X \leq H_i$ ($i = 1, \dots, n$) be subalgebras of L such that $\langle H_1, \dots, H_n \rangle = H_1 + \cdots + H_n$. If X* wsi *H_i for any i, then X* wsi *$H_1 + \cdots + H_n$.*

PROOF. Let X be an m-step weak subideal of H_i for each i. Then $[H_i, {}_mX] \subseteq X$ for each i. Therefore $[H_1 + \cdots + H_n, {}_mX] \subseteq X$ and hence X is an m-step weak subideal of $H_1 + \cdots + H_n$.

By Lemma 1 we have immediately the following

COROLLARY. *Let L be a soluble Lie algebra over $\mathfrak{k}$. Let $X \leq H_i$ ($i = 1, \dots, n$) be subalgebras of L such that $\langle H_1, \dots, H_n \rangle = H_1 + \cdots + H_n$. If X* si *H_i for any i, then X* si *$H_1 + \cdots + H_n$.*

We can prove the main theorem in this section.

THEOREM 3. *Let L be a Lie algebra over a field $\mathfrak{k}$. Let $X \leq H_\sigma$ ($\sigma \in I$) be subalgebras of L such that X is of finite codimension in $\langle H_\sigma | \sigma \in I \rangle$. If X is a weak subideal of H_σ for any $\sigma \in I$, then X is a weak subideal of $\langle H_\sigma | \sigma \in I \rangle$.*

PROOF. Let $M = \{a \in L | [a, {}_nX] \subseteq X$ for some $n \in \mathbf{N}\}$. Then M is a subalgebra of L. In fact, let a, $b \in M$. Then there exist s, $t \in \mathbf{N}$ such that $[a, {}_sX] \subseteq X$ and $[b, {}_tX] \subseteq X$. Let $l = \max(s, t)$. By the Jacobi identity and induction on k,

$$[[a, b], {}_kX] \subseteq \sum_{i+j=k} [[a, {}_iX], [b, {}_jX]] .$$

Therefore

$$
\begin{aligned}
[[a, b], {}_{2l-1}X] &\subseteq \sum_{i+j=2l-1} [[a, {}_iX], [b, {}_jX]] \\
&\subseteq \sum_{i=0}^{l-1} ([[a, {}_iX], X^{l-i}] + [[b, {}_iX], X^{l-i}]) \\
&\subseteq X .
\end{aligned}
$$

Hence $[a, b] \in M$ and clearly $a + b \in M$. Thus M is a subalgebra of L. Since X wsi H_σ $(\sigma \in I)$, $H_\sigma \leq M$ for each σ. Then $H = \langle H_\sigma | \sigma \in I \rangle \leq M$. Since X is of finite codimension in H by the assumption, there exist $a_1, \ldots, a_m \in H$ such that $H = X + (a_1, \ldots, a_m)$. Therefore there exists $n_i \in \mathbf{N}$ such that $[a_i, {}_{n_i}X] \subseteq X$ for each i. Let $n = \max_i n_i$. Then $[H, {}_nX] \subseteq X$. Hence $X \leq^n H$.

As a consequence of the theorem, we have immediately the following result ([8, Theorem 7]).

COROLLARY 1. *Let L be a finite-dimensional Lie algebra over* $\mathfrak{k}$. *Let $X \leq H_i$ $(i = 1, \ldots, n)$ be subalgebras of L. If X* wsi *H_i for any i, then X* wsi *$\langle H_1, \ldots, H_n \rangle$.*

We have the following which is the analogue of [1, Corollary 2], without restriction for characteristic of a basic field.

COROLLARY 2. *Let L be a finite-dimensional Lie algebra over* $\mathfrak{k}$. *Let $H_i \leq L$ $(i = 1, \ldots, n)$* and *$L = \langle H_1, \ldots, H_n \rangle$. If H_i* wsi *$\langle H_i, H_j \rangle$ $(i, j = 1, \ldots, n)$, then H_i* wsi *L for any i.*

From Lemma 1 and the theorem, we obtain a generalization of [4, Theorem 6]:

COROLLARY 3. *Let L be a soluble Lie algebra over* $\mathfrak{k}$. *Let $X \leq H_\sigma$ $(\sigma \in I)$ be subalgebras of L. If the codimension of X in $\langle H_\sigma | \sigma \in I \rangle$ is finite and X* si *H_σ for any $\sigma \in I$, then X* si *$\langle H_\sigma | \sigma \in I \rangle$.*

2.

We begin with a key lemma in this section.

LEMMA 4. *Let L be a Lie algebra over a field $\mathfrak{k}$ such that $L = A + X$ with $X \leq L$, $A \lhd L$ and $A \in \mathfrak{A}^c$. If $X \leq^m L$, then $X \lhd^{cm} L$.*

PROOF. We use induction on c. If $c = 1$, then $X \lhd^m L$ by [8, Lemma 3]. Assume that $c > 1$ and the statement is true for $c - 1$. Let $N = A^{(c-1)}$

($\neq 0$). Then N is an abelian ideal of L. Let bars denote subalgebras of L modulo N. Then

$$\bar{L} = \bar{A} + \bar{X}, \qquad \bar{A} \lhd \bar{L}, \qquad \bar{A} \in \mathfrak{A}^{c-1}, \qquad \bar{X} \leq^m \bar{L}.$$

By inductive hypothesis $\bar{X} \lhd^{m(c-1)} \bar{L}$ and so $X + N \lhd^{m(c-1)} L$. Now, N is an abelian ideal of $X + N$ and $X \leq^m X + N$. By [8, Lemma 3] $X \lhd^m X + N$. Therefore $X \lhd^{cm} L$.

REMARK. Under the same condition as the lemma, we can show similarly that X is a λc-step ascendant subalgebra of L if X is λ-step weakly ascendant in L. This is a generalization of [8, Lemma 3].

PROPOSITION 5. *Let L be a Lie algebra over $\mathfrak{k}$ such that $L = H + K = A + X$ with $X, H, K \leq L$, $A \lhd L$ and $A \in \mathfrak{A}^c$. If $X \leq^m H$ and $X \leq^m K$, then $X \lhd^{cm} L$. Especially, if $X \lhd^m H$ and $X \lhd^m K$, then $X \lhd^{cm} L$.*

PROOF. By Proposition 2 $X \leq^m L$ and then $X \lhd^{cm} L$ by Lemma 4.

We now obtain a main theorem which is an analogue of Stonehewer's result [7] for Lie algebras and is better than that in group theory.

THEOREM 6. *Let $\mathfrak{X}$ be a class of Lie algebras over a field $\mathfrak{k}$ and suppose that $L = H + K \in \mathfrak{X}$, H and K subalgebras of L and X a subideal of both H and K always implies that X is subideal in L. Then $L = H + K \in (\mathrm{E}\mathfrak{A})\mathfrak{X}$ and X a subideal of H and K always implies that X is a subideal of L. Moreover suppose that there is an integer $f_1 = f_1(X, m)$ such that $X \lhd^{f_1} L$ whenever $L = H + K \in \mathfrak{X}$ with $X \lhd^m H$ and $X \lhd^m K$. Then there is an integer $f_2 = f_1 + cm$ such that $X \lhd^{f_2} L$ whenever $L = H + K \in \mathfrak{A}^c\mathfrak{X}$ with $X \lhd^m H$ and $X \lhd^m K$.*

PROOF. Let $L = H + K \in \mathfrak{A}^c\mathfrak{X}$, $X \lhd^m H$ and $X \lhd^m K$. Then there exists an ideal N of L such that $N \in \mathfrak{A}^c$ and $L/N \in \mathfrak{X}$. Since $L/N = (H + N)/N + (K + N)/N \in \mathfrak{X}$, $(X + N)/N \lhd^m (H + N)/N$ and $(X + N)/N \lhd^m (K + N)/N$, we have $(X + N)/N \lhd^{f_1} L/N$ by the assumption. Then $X + N \lhd^{f_1} L$. Since $X \leq^m L$ by Proposition 2, $X \leq^m X + N$. Now, N is an ideal of $X + N$ and $N \in \mathfrak{A}^c$. Therefore $X \lhd^{cm} X + N$ by Lemma 4. Hence $X \lhd^{f_2} L$, $f_2 = f_1 + cm$.

By taking $\mathfrak{X} = \mathfrak{F}$ in the theorem, we obtain the following ([2, Theorem 1]) by [1, Theorem].

COROLLARY. *Let L be a soluble-by-finite Lie algebra over a field of characteristic zero and let X, H, K be subalgebras of L such that $L = H + K$ and X si H, X si K. Then X si L. Moreover suppose that there exists an ideal N*

of $L=H+K$ *such that* $N \in \mathfrak{A}^c$ *and* $\dim L/N = n$ $(<\infty)$ *with* $X \lhd^m H$ *and* $X \lhd^m K$. *Then* $X \lhd^f L$, $f = cm + n$.

We can show a similar result for a finite-by-soluble Lie algebra.

THEOREM 7. *Let* L *be a finite-by-soluble Lie algebra over a field of characteristic zero and* X, H, K *be subalgebras of* L *such that* $L=H+K$ *and* X si H, X si K. *Then* X si L. *Moreover suppose that there exists an ideal* F *of* $L=H+K$ *such that* $\dim F = n$ $(<\infty)$ *and* $L/F \in \mathfrak{A}^c$ *with* $X \lhd^m H$ *and* $X \lhd^m K$. *Then* $X \lhd^f L$, *where* $f = f(c, m, n)$.

PROOF. Let $F \lhd L$, $\dim F = n$ $(<\infty)$ and $L/F \in \mathfrak{A}^c$. Let $X \lhd^m H$ and $X \lhd^m K$. Let $S = \sigma(F)$ be the radical of F. Then S is an ideal of L. We write $\bar{B} = (B+S)/S$ for a subalgebra B of L. Then $\bar{L} = \bar{H} + \bar{K}$, $\bar{X} \lhd^m \bar{H}$, $\bar{X} \lhd^m \bar{K}$, $\bar{F} \lhd \bar{L}$, $\dim \bar{F} \le n$ and $\bar{L}/\bar{F} \in \mathfrak{A}^c$. Let $\bar{C} = C_{\bar{L}}(\bar{F})$. Then $\bar{C}$ is an ideal of $\bar{L}$. We claim that $\bar{L} = \bar{F} \oplus \bar{C}$. If $x \in \bar{L}$ then $\mathrm{ad}\, x|_{\bar{F}} \in \mathrm{Der}(\bar{F})$. Since $\bar{F}$ is a finite-dimensional semisimple Lie algebra, there exists $f \in \bar{F}$ such that $\mathrm{ad}\, x|_{\bar{F}} = \mathrm{ad}\, f|_{\bar{F}}$ (cf. [3, Summary 13.1.1]). Then $\mathrm{ad}\,(x-f)|_{\bar{F}} = 0$ and so $x - f \in \bar{C}$. Therefore $\bar{L} \le \bar{F} + \bar{C}$. $\bar{F} \cap \bar{C} = \zeta_1(\bar{F}) = 0$ since $\bar{F}$ is semisimple. Therefore $\bar{L} = \bar{F} \oplus \bar{C}$ and $\dim \bar{L}/\bar{C} \le n$. By [1, Theorem] $\bar{X} + \bar{C} \lhd^n \bar{L}$. $\bar{C} \cong \bar{L}/\bar{F} \in \mathfrak{A}^c$ and $\bar{X} \le^m \bar{X} + \bar{C}$ by Proposition 2. Therefore $\bar{X} \lhd^{cm} \bar{X} + \bar{C}$ by Lemma 4. Thus $\bar{X} \lhd^{cm+n} \bar{L}$ and then $X + S \lhd^{cm+n} L$. Now, S is a soluble ideal of $X + S$, with derived length $\le n$, and $X \le^m X + S$. Again by Lemma 4, $X \lhd^{mn} X + S$. Hence $X \lhd^f L$, where $f = m(c+n) + n$.

REMARK. The theorem does not hold for a field of characteristic $p > 0$ (cf. [1]).

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