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Lie algebras in which every 1-dimensional weak subideal is an ideal

Dedicated to the memory of Professor Shigeaki Tôgô

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Introduction

A Lie algebra L is called a $\mathfrak{C}$-algebra if every 1-dimensional subideal of L is an ideal of L and an (A_∞)-algebra if every nilpotent inner derivation of L is zero. In [2] the authors investigated the properties of $\mathfrak{C}$-algebras and related Lie algebras. On the other hand, in [9] Shimizuike and Tôgô investigated the properties of (not necessarily finite-dimensional) (A_∞)-algebras and related Lie algebras. In terms of weak subideals we easily see that a Lie algebra L is an (A_∞)-algebra if and only if every 1-dimensional weak subideal of L is central in L. Thus it seems to be natural to study an intermediate class of Lie algebras between the class of $\mathfrak{C}$-algebras and that of (A_∞)-algebras.

The purpose of this paper is to investigate the property of $\mathfrak{C}$(wsi)-algebras, that is, Lie algebras in which every 1-dimensional weak subideal is an ideal, and to determine the structure of $\mathfrak{C}$(wsi)-algebras under various circumstances.

In Section 2, we shall show that over any field $\mathfrak{C}$(wsi)-algebras belonging to the class $\mathrm{L}(\mathrm{wsi})\acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A}$ are either abelian or almost-abelian (Theorem 2.2).

In Section 3, we shall show the following results: Let L be a Lie algebra over a field $\mathfrak{k}$ of characteristic zero. If either

(a) L is a serially finite Lie algebra whose locally soluble radical belongs to the class $\acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A}$, or

(b) L is a subideally finite Lie algebra,

then L is a $\mathfrak{C}$(wsi)-algebra if and only if $L = R \oplus S$, where R is an ideal of L which is either abelian or almost-abelian and S is a semisimple (A_∞)-ideal of L (Theorems 3.3 and 3.8). Moreover, when $\mathfrak{k}$ is an algebraically closed field, if L satisfies either the above statement (a) or

(c) L is a weak-subideally finite Lie algebra,

then L is a $\mathfrak{C}$(wsi)-algebra if and only if L is either abelian or almost-abelian (Theorem 3.3 and Proposition 3.5).

In Section 4, we shall give the following examples over any field;

(i) a $\mathfrak{C}$-algebra which is not a $\mathfrak{C}$(wsi)-algebra (Example 1),

(ii) a $\mathfrak{C}$(wsi)-algebra which is not an (A_∞)-algebra (Example 2),

(iii) a serially finite $\mathfrak{C}$(wsi)-algebra which is neither a $\mathfrak{T}$(wsi)-algebra nor a $\mathfrak{C}$(wasc)-algebra (Example 3).

1. Preliminaries

Throughout the paper Lie algebras are not necessarily finite-dimensional over a field $\mathfrak{k}$ of arbitrary characteristic unless otherwise specified. We mostly follow [1] for the use of notations and terminology.

Let L be a Lie algebra over $\mathfrak{k}$ and let H be a subalgebra of L. For an ordinal σ, H is a σ-step ascendant (resp. weakly ascendant) subalgebra of L, denoted by $H \lhd^{\sigma} L$ (resp. $H \leq^{\sigma} L$), if there exists an ascending series (resp. chain) $(H_\alpha)_{\alpha \leq \sigma}$ of subalgebras (resp. subspaces) of L such that

(1) $H_0 = H$ and $H_\sigma = L$,

(2) $H_\alpha \lhd H_{\alpha+1}$ (resp. $[H_{\alpha+1}, H] \subseteq H_\alpha$) for any ordinal $\alpha < \sigma$,

(3) $H_\lambda = \bigcup_{\alpha<\lambda} H_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

H is an ascendant (resp. a weakly ascendant) subalgebra of L, denoted by H asc L (resp. H wasc L), if $H \lhd^{\sigma} L$ (resp. $H \leq^{\sigma} L$) for some ordinal σ. When σ is finite, H is a subideal (resp. weak subideal) of L and denoted by H si L (resp. H wsi L). For a totally ordered set Σ, a series (resp. weak series) from H to L of type Σ is a collection $\{\Lambda_\sigma, V_\sigma : \sigma \in \Sigma\}$ of subalgebras (resp. subspaces) of L such that

(1) $H \subseteq V_\sigma \subseteq \Lambda_\sigma$ for all $\sigma \in \Sigma$,

(2) $L \backslash H = \bigcup_{\sigma \in \Sigma} (\Lambda_\sigma \backslash V_\sigma)$,

(3) $\Lambda_\tau \subseteq V_\sigma$ if $\tau < \sigma$,

(4) $V_\sigma \lhd \Lambda_\sigma$ (resp. $[\Lambda_\sigma, H] \subseteq V_\sigma$) for all $\sigma \in \Sigma$.

H is a serial (resp. weakly serial) subalgebra of L, denoted by H ser L (resp. H wser L), if there exists a series (resp. weak series) from H to L of type Σ for some Σ.

Let $\mathfrak{X}$ be a class of Lie algebras and let Δ be any of the relations $\leq$, $\lhd$, si, asc, ser, wsi, wasc, wser. A Lie algebra L is said to lie in $\mathrm{L}(\Delta)\mathfrak{X}$ if for any finite subset X of L there exists an $\mathfrak{X}$-subalgebra K of L such that $X \subseteq K \Delta L$. In particular we write $\mathrm{L}\mathfrak{X}$ for $\mathrm{L}(\leq)\mathfrak{X}$. When $L \in \mathrm{L}\mathfrak{X}$ (resp. $\mathrm{L}(\mathrm{ser})\mathfrak{X}$, $\mathrm{L}(\mathrm{si})\mathfrak{X}$, $\mathrm{L}(\mathrm{wsi})\mathfrak{X}$), L is called a locally (resp. serially, subideally, weak-subideally) $\mathfrak{X}$-algebra. $\mathfrak{F}$, $\mathfrak{A}$ and $\mathfrak{N}$ are the classes of Lie algebras which are finite-dimensional, abelian and nilpotent respectively. For an ordinal σ, $\acute{\mathrm{E}}_\sigma(\Delta)\mathfrak{X}$ is the class of Lie algebras L having an ascending series $(L_\alpha)_{\alpha \leq \sigma}$ of Δ-subalgebras such that

(1) $L_0 = 0$ and $L_\sigma = L$,

(2) $L_\alpha \lhd L_{\alpha+1}$ and $L_{\alpha+1}/L_\alpha \in \mathfrak{X}$ for any ordinal $\alpha < \sigma$,

(3) $L_\lambda = \bigcup_{\alpha<\lambda} L_\alpha$ for any limit ordinal $\lambda \leq \sigma$.

We define $\acute{\mathrm{E}}(\Delta)\mathfrak{X} = \bigcup_{\sigma>0} \acute{\mathrm{E}}_\sigma(\Delta)\mathfrak{X}$, $\mathrm{E}(\Delta)\mathfrak{X} = \bigcup_{n<\omega} \acute{\mathrm{E}}_n(\Delta)\mathfrak{X}$. In particular we write

$\acute{\text{E}}\mathfrak{X}$ and $\text{E}\mathfrak{X}$ for $\acute{\text{E}}(\leq)\mathfrak{X}$ and $\text{E}(\leq)\mathfrak{X}$ respectively. Thus $\text{E}\mathfrak{A}$ is the class of soluble Lie algebras.

Let H be a subalgebra of L. We denote by $C_L(H)$ the centralizer of H in L. The Hirsch-Plotkin radical $\rho(L)$ of L is the unique maximal locally nilpotent ideal of L. For a locally finite Lie algebra L the locally soluble radical $\sigma(L)$ of L is the unique maximal locally soluble ideal of L.

As in [2] we need the following lemma whose proof is clear.

LEMMA 1.1. *Let L be a Lie algebra and let N be a subspace of L. Then the following statements are equivalent*:

(1) *Every* 1*-dimensional subspace of N is an ideal of L.*

(2) *For any $x \in L$,* ad x *is a scalar transformation on N.*

Moreover, if either (1) *or* (2) *holds, then N is an abelian ideal of L and* $\dim L/C_L(N) \leq 1$.

The proof of the following lemma is similar to that of [1, Lemma 9.1.2(c)].

LEMMA 1.2. *Let Δ be any of the relations* si, wsi, asc, wasc, ser, wser, *and let L be an $\acute{\text{E}}(\Delta)\mathfrak{A}$-algebra. If the subalgebra N of L generated by all nilpotent Δ-subalgebras of L is an ideal of L, then we have $C_L(N) \leq N$.*

Let Δ be any of the relations si, asc, ser, $\lhd^{\sigma}$, wsi, wasc, wser, $\leq^{\sigma}$. $\mathfrak{I}(\Delta)$ is the class of Lie algebras L in which every Δ-subalgebra of L is an ideal of L. $\mathfrak{C}(\Delta)$ is the class of Lie algebras in which every 1-dimensional Δ-subalgebra of L is an ideal of L. In particular we write $\mathfrak{I}$ and $\mathfrak{C}$ for $\mathfrak{I}(\text{si})$ and $\mathfrak{C}(\text{si})$ respectively. Then the following inclusions hold:

$$\begin{array}{ccccccccc} \mathfrak{C}(\text{wser}) & \leq & \mathfrak{C}(\text{wasc}) & \leq & \mathfrak{C}(\leq^{\omega}) & \leq & \mathfrak{C}(\text{wsi}) & \leq & \mathfrak{C} \\ \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} & & \text{\rotatebox{90}{$\leq$}} \\ \mathfrak{I}(\text{wser}) & \leq & \mathfrak{I}(\text{wasc}) & \leq & \mathfrak{I}(\leq^{\omega}) & \leq & \mathfrak{I}(\text{wsi}) & \leq & \mathfrak{I} \end{array} .$$

Since by [13, Theorem 4] $\langle x \rangle$ wasc L if and only if $\langle x \rangle \leq^{\omega} L$ for an element $x \in L$, we remark that $\mathfrak{C}(\text{wasc}) = \mathfrak{C}(\leq^{\omega})$.

According to Singer [11] a Lie algebra L is said to satisfy the condition (A) if any pair of elements x and y of L such that $[x, y, y] = 0$ satisfies $[x, y] = 0$. A derivation δ of a Lie algebra L is said to be n-nilpotent if $L\delta^n = 0$, and nil if for any finite-dimensional subspace V of L there is a positive integer n such that $V\delta^n = 0$. According to Jôichi [5] a Lie algebra L is said to satisfy the condition (A_n) if ad L contains no non-zero n-nilpotent elements ($n \geq 2$). A Lie algebra L is said to satisfy the condition (A_∞) if ad L contains no non-zero nilpotent elements. In [9] Shimizuike and Tôgô introduced the condition (B_∞). A Lie algebra L is said to satisfy the condition (B_∞) if ad L contains no non-zero nil elements. We use the same notation (A) (resp. (A_n), (A_∞), (B_∞)) to express the class of Lie algebras satisfying the condition (A) (resp. (A_n), (A_∞), (B_∞)). These classes of Lie algebras satisfy the following inclusions ([9, Proposition 1]):

$$\mathfrak{A} \leq (A) \leq (B_\infty) \leq (A_\infty) \leq \cdots \leq (A_{n+1}) \leq (A_n) \leq \cdots \leq (A_2)\,.$$

Let L be a Lie algebra and let x be an element of L. Using [13, Lemma 1 and Theorem 4] we have the following

(a) "ad x is n-nilpotent" $\Rightarrow$ "$\langle x\rangle \leq^n L$" $\Rightarrow$ "ad x is $(n+1)$-nilpotent",

(b) "ad x is nilpotent" $\Leftrightarrow$ "$\langle x\rangle$ wsi L",

(c) "ad x is nil" $\Leftrightarrow$ "$\langle x\rangle$ wasc L".

To indicate the relations between (B_∞), (A_∞), (A_n) and $\mathfrak{C}$(wasc), $\mathfrak{C}$(wsi), $\mathfrak{C}(\leq^n)$, we shall introduce the class of Lie algebras $\mathfrak{X}_0$: $L \in \mathfrak{X}_0$ iff every 1-dimensional ideal of L is central in L. Then from (a), (b) and (c) we can show the following

LEMMA 1.3. (1) $(A_{n+1}) \leq \mathfrak{C}(\leq^n) \cap \mathfrak{X}_0 \leq (A_n)$ $(n \geq 2)$.

(2) $(A_\infty) = \mathfrak{C}(\text{wsi}) \cap \mathfrak{X}_0$.

(3) $(B_\infty) = \mathfrak{C}(\text{wasc}) \cap \mathfrak{X}_0$.

2. Generalized soluble $\mathfrak{C}$(wsi)-algebras

In this section we shall show that over any field $\mathfrak{C}$(wsi)-algebras belonging to some class of generalized soluble Lie algebras are either abelian or almost-abelian.

A Lie algebra L is said to be almost-abelian if L is the split extension of an abelian algebra by the 1-dimensional algebra of scalar multiplications. We denote by $\mathfrak{A}_0$ the class of abelian or almost-abelian Lie algebras. First we shall give a characterization of $\mathfrak{A}_0$-algebras.

LEMMA 2.1. (1) *Let L be a Lie algebra over any field $\mathfrak{k}$. Then L is an $\mathfrak{A}_0$-algebra if and only if $[x, y] \in \langle x\rangle + \langle y\rangle$ for any elements x and y of L.*

(2) $\mathfrak{A}_0 \leq \mathfrak{T}(\text{wser})$.

PROOF. (1) Let L be almost-abelian. Then $L = L^2 \dotplus \langle a\rangle$, where L^2 is abelian and ad a is the identity mapping on L^2. For any $x, y \in L$, put $x = u + \alpha a$ and $y = v + \beta a$ $(u, v \in L^2, \alpha, \beta \in \mathfrak{k})$. Then $[x, y] = [u + \alpha a, v + \beta a] = \beta u - \alpha v = \beta x - \alpha y$.

Conversely suppose that $[x, y] \in \langle x\rangle + \langle y\rangle$ for any $x, y \in L$. If L is not abelian, then there are two elements x, y of L satisfying $[x, y] \neq 0$. Put $[x, y] = \alpha x + \beta y$ and assume that $\alpha \neq 0$. Put $z = [x, y]$ and $a = y/\alpha$. Then $[z, a] = z$. Let u be any element of L and put $[u, a] = \alpha_1 u + \beta_1 a$, $[u + z, a] =$

$\alpha_2(u + z) + \beta_2 a$. If u, z, a are linearly independent, then $\alpha_1 = \alpha_2 = 1$, $\beta_1 = \beta_2$. Therefore $u = [u, a] - \beta_1 a \in [L, a] + \langle a \rangle$ and $[u, a, a] = [u, a]$. Otherwise $u \in \langle z \rangle + \langle a \rangle \subseteq [L, a] + \langle a \rangle$ and $[u, a, a] = [u, a]$. Hence $L = [L, a] + \langle a \rangle$ and $\operatorname{ad} a$ is the identity mapping on $[L, a]$. Let $v, w \in [L, a]$. Then by the Jacobi identity $0 = [v, w, a] + [w, a, v] + [a, v, w] = [v, w] + [w, v] - [v, w] = [w, v]$. Hence $[L, a]$ is abelian. This indicates that L is almost-abelian.

(2) Assume that L is an $\mathfrak{A}_0$-algebra. Let H wser L and let $x \in H$, $y \in L$. Then by [3, Lemma 1.2] $H \cap \langle x, y \rangle$ wser $\langle x, y \rangle$. Since $\langle x, y \rangle = \langle x \rangle + \langle y \rangle$ by (1), we have $H \cap \langle x, y \rangle \lhd \langle x, y \rangle$ and so $[x, y] \in H$. Hence $H \lhd L$. That is to say, $L \in \mathfrak{T}(\text{wser})$.

Remark. (i) Since $\mathfrak{A}_0 \leq \mathrm{L}(\mathrm{E}\mathfrak{A} \cap \mathfrak{F})$ Lemma 2.1(2) can be deduced from [6, Lemma 6.3] and [3, Theorem 2.7].

(ii) From Lemma 2.1(2) we see that every almost-abelian Lie algebra lying in the class $\mathrm{L}(\text{wser})\mathfrak{F}$ is finite-dimensional.

Now we shall give the main theorem in this section, which generalizes [2, Proposition 2.1].

Theorem 2.2. *Let Δ be any of the relations* si *and* wsi. *Then over any field we have*

$$\mathrm{L}(\Delta)\acute{\mathrm{E}}(\Delta)\mathfrak{A} \cap \mathfrak{C}(\Delta) = \mathfrak{A}_0 .$$

Proof. Let $L \in \mathrm{L}(\Delta)\acute{\mathrm{E}}(\Delta)\mathfrak{A} \cap \mathfrak{C}(\Delta)$ and let $a, b \in L$. Then there exists a subalgebra H of L such that $H \in \acute{\mathrm{E}}(\Delta)\mathfrak{A}$ and $a, b \in H \Delta L$. Let $N = \{x \in H : \langle x \rangle \Delta H\}$. For every $x \in N$ we have $\langle x \rangle \lhd H$ as $L \in \mathfrak{C}(\Delta)$. Since N is the subalgebra of H generated by all nilpotent Δ-subalgebras, it follows from Lemmas 1.1 and 1.2 that $\dim H/N \leq 1$ and $\operatorname{ad} x$ is a scalar transformation on N for any $x \in H$. Thus $H \in \mathfrak{A}_0$ and $[a, b] \in \langle a \rangle + \langle b \rangle$ by Lemma 2.1(1). By using Lemma 2.1(1) again we conclude that $L \in \mathfrak{A}_0$.

Conversely let $L \in \mathfrak{A}_0$. Then we have $L \in \mathfrak{C}(\Delta)$ by Lemma 2.1(2). This completes the proof.

Before we turn our attention to (A_∞)-algebras, we need the following result, which generalizes [5, Theorem 1(b)].

Lemma 2.3. *Every soluble* (A_3)*-algebra is abelian.*

Proof. Let L be a soluble (A_3)-algebra and let n be an integer ≥ 0 such that $L^{(n)} \neq 0$, $L^{(n+1)} = 0$. Assume that $n \geq 1$. For any element z of $L^{(n)}$, $[L, z, z] \subseteq [L^{(n)}, z] = 0$. Since $L \in (\mathrm{A}_2)$, $[L, z] = 0$. Thus we have $[L^{(n)}, L] = 0$.

Let $x \in L^{(n-1)}$. Then $[L, x, x, x] \subseteq [L^{(n-1)}, x, x] \subseteq [L^{(n)}, x] = 0$. Since $L \in (\mathrm{A}_3)$, $[L, x] = 0$. Hence $L^{(n)} = 0$, which is a contradiction.

COROLLARY 2.4. *Let L be a Lie algebra over any field. If $L \in \mathrm{L}(\mathrm{wsi})\acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A}$, then the following statements are equivalent*:

(1) *L is an (A_∞)-algebra.*
(2) *L is a (B_∞)-algebra.*
(3) *L is an* (A)*-algebra.*
(4) *L is abelian.*

PROOF. It suffices to show that (1)$\Rightarrow$(4). Let L be an (A_∞)-algebra. From Lemmas 1.3(2), 2.3 and Theorem 2.2 it follows that L is abelian.

REMARK. Over any field there exists an $\acute{\mathrm{E}}\mathfrak{A}$-algebra which belongs to (A_∞) but not to $\mathfrak{A}_0$ (Example 3). Therefore in Theorem 2.2 and Corollary 2.4 we can not extend the classes $\acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A}$ and $\acute{\mathrm{E}}(\mathrm{si})\mathfrak{A}$ to the class $\acute{\mathrm{E}}\mathfrak{A}$.

As in Theorem 2.2 we can prove the following result, which generalizes [2, Proposition 2.2].

PROPOSITION 2.5. *Let Δ be any of the relations* asc, wasc, ser, wser. *Then over any field we have*

$$\mathrm{L}(\Delta)\acute{\mathrm{E}}\mathfrak{A} \cap \mathfrak{C}(\Delta) = \mathfrak{A}_0 .$$

COROLLARY 2.6. *Let L be a Lie algebra over any field. If $L \in \mathrm{L}(\mathrm{wasc})\acute{\mathrm{E}}\mathfrak{A}$, then the following statements are equivalent*:

(1) *L is a (B_∞)-algebra.*
(2) *L is an* (A)*-algebra.*
(3) *L is abelian.*

3. Locally finite $\mathfrak{C}$(wsi)-algebras

In this section we shall determine the structure of locally finite $\mathfrak{C}$(wsi)-algebras satisfying some conditions over a field of characteristic zero.

To do this we need the following two lemmas.

LEMMA 3.1. *Let L be a Lie algebra and let Δ be one of the relations* wsi *and* wasc. *Suppose that $L = H \oplus K$, where H and K are $\mathfrak{C}(\Delta)$-ideals of L. If K has no 1-dimensional ideals, then L is a $\mathfrak{C}(\Delta)$-algebra.*

PROOF. The case $\Delta =$ wasc: Let $x \in L$ and put $x = y + z$, $y \in H$, $z \in K$. Suppose that $\langle x \rangle$ wasc L. Then for any $w \in L$, there is a positive integer

n such that $[w, {}_n x] = 0$ by [13, Lemma 1 and Theorem 4]. Therefore $[w, {}_n y] = [w, {}_n z] = 0$. This implies that $\langle y \rangle$ wasc H and $\langle z \rangle$ wasc K. Since $H, K \in \mathfrak{C}(\text{wasc})$, $\langle y \rangle \lhd H$ and $\langle z \rangle \lhd K$. Hence $z = 0$ by hypothesis. Accordingly $\langle x \rangle = \langle y \rangle \lhd H$ and so $\langle x \rangle \lhd L$.

The case of $\Delta = \text{wsi}$ is proved similarly.

REMARK. We can not remove the hypothesis "K has no 1-dimensional ideals" in Lemma 3.1. For instance, let H and K be almost-abelian. Then from Theorem 2.2 we derive that $L = H \oplus K \notin \mathfrak{C}$.

LEMMA 3.2. *Let L be a Lie algebra over any field. Suppose that $L = R \oplus S$, $S = \bigoplus_{\lambda \in \Lambda} S_\lambda \lhd L$, where R is a soluble ideal of L and each S_λ is a non-abelian simple ideal of S.*

(1) *If $H \lhd L$, then $H = (H \cap R) \oplus (H \cap S)$.*

(2) *If $R \in \mathfrak{A}_0$ and $S_\lambda \in \mathfrak{F} \cap (\mathrm{A}_\infty)$ for each λ, then $L \in \mathfrak{I}(\text{wser})$.*

PROOF. (1) Let $H \lhd L$ and let S_1 be the image of H by the projection of L onto S. Then S_1 is an ideal of S containing $S \cap H$. Also $S_1 = \bigoplus_{\lambda \in M} S_\lambda$ for some subset M of Λ by [1, Lemma 13.4.3]. Since $S_1 \leq H + R$, $S_1 = S_1^{(1)} \leq (H + R)^{(1)} \leq H + R^{(1)}$ and $S_1 \leq H + R^{(m)}$ by induction on m. It follows from $R \in \mathrm{E}\mathfrak{A}$ that $S_1 \leq H$. Hence we have $S_1 = S \cap H$. Furthermore since $H \leq R + S_1$, we obtain $H = (H \cap R) \oplus (H \cap S)$.

(2) Assume that H wser L and take $x \in H$ and $y \in L$. Then there exists a subalgebra L_1 of L containing x and y such that $L_1 = R_1 \oplus S_1$, $S_1 = \bigoplus_{\lambda \in M} S_\lambda$, where R_1 is a finite-dimensional subalgebra of R and M is a finite subset of Λ. Put $H_1 = H \cap L_1$. Since H_1 wsi L_1 we have $H_1^\omega \lhd L_1$ using [8, Theorem 2.2]. It follows that $H_1^\omega = (H_1^\omega \cap R_1) \oplus (H_1^\omega \cap S_1)$ from (1) and that $L_1/H_1^\omega \cong (R_1/(R_1 \cap H_1^\omega)) \oplus (S_1/(S_1 \cap H_1^\omega))$. Since $S_1/(S_1 \cap H_1^\omega) \cong \bigoplus_{\lambda \in N} S_\lambda$ for some subset N of M, $L_1/H_1^\omega \in \mathfrak{C}(\text{wsi})$ by [9, Proposition 3] and Lemmas 1.3(2), 2.1, 3.1. Therefore H_1/H_1^ω is an abelian ideal of L_1/H_1^ω by Lemma 1.1. In particular we have $[x, y] \in [H_1, L_1] \subseteq H_1 \subseteq H$. This shows that H is an ideal of L.

Now we shall give the first main theorem in this section, which corresponds to [2, Theorem 2.3].

THEOREM 3.3. *Let L be a serially finite Lie algebra over a field $\mathfrak{k}$ of characteristic zero. Then the following statements are equivalent:*

(1) *L belongs to one of the following classes of Lie algebras:*

$$\mathfrak{I}(\text{wser}), \quad \mathfrak{C}(\text{wser}), \quad \mathfrak{I}(\text{wasc}), \quad \mathfrak{C}(\text{wasc}).$$

(2) *$L = R \oplus S$, where R is an $\mathfrak{A}_0$-ideal of L and S is a semisimple* (A)-*ideal*

of L. In particular L is an $\mathfrak{A}_0$*-algebra in case that* $\mathfrak{k}$ *is an algebraically closed field.*

Moreover, if $\sigma(L) \in \acute{E}(\mathrm{wsi})\mathfrak{A}$*, then the above statements* (1), (2) *are equivalent to the following statement*:

(3) *L belongs to one of the following classes of Lie algebras*:

$$\mathfrak{T}(\mathrm{wsi}), \qquad \mathfrak{C}(\mathrm{wsi}).$$

PROOF. (1) ⇒ (2): Suppose that $L \in \mathfrak{C}(\mathrm{wasc})$. Let $R = \sigma(L)$ and let $N = \rho(R)$. For any $x \in N$ we have $\langle x \rangle \lhd L$ since $\langle x \rangle$ wasc L. By Lemma 1.1 N is an abelian ideal of L and $\dim L/C_L(N) \leq 1$. Since $R^2 \leq N$ by [1, Corollary 13.3.11], for any element x of $C_R(N)$ we have $\langle x \rangle \lhd \langle x \rangle + N \lhd R \lhd L$. Therefore $\langle x \rangle \lhd R$ by $L \in \mathfrak{C}(\mathrm{wasc})$ and so $x \in N$. Hence $C_R(N) = N$ and by Lemma 1.1 R is either abelian or almost-abelian. On the other hand [14, Theorem 2] and [1, Theorem 13.5.7] show that there exists a Levi factor S of L. Furthermore by [1, Theorem 13.4.2] $S = \bigoplus_{\lambda \in \Lambda} S_\lambda$, where each S_λ is a finite-dimensional non-abelian simple ideal of S and therefore $S = S^2 \leq L^2 \leq C_L(N)$. Then we have

$$C_L(N) = C_L(N) \cap (R \dotplus S) = C_R(N) \dotplus S = N \oplus S.$$

Since $S = S^2 = C_L(N)^2$ ch $C_L(N) \lhd L$, it follows that $S \lhd L$. Furthermore $S_\lambda \in \mathfrak{C}(\mathrm{wsi})$ and therefore S satisfies the condition (A) by Lemma 1.3(2), [5, Corollary to Theorem 3] and [9, Proposition 3].

In particular, if $\mathfrak{k} = \bar{\mathfrak{k}}$, then it follows from [12, Theorem 3] that each S_λ is abelian. Hence S must be 0.

(2) ⇒ (1): By [1, Theorem 13.4.2] L satisfies the condition of Lemma 3.2.

(1) ⇒ (3) is trivial.

(3) ⇒ (2) is similar to (1) ⇒ (2) except for putting $N = \{x \in R : \langle x \rangle \text{ wsi } R\}$ and using Lemma 1.2 to show $C_R(N) = N$.

From Theorem 3.3 we can derive the following result, which contains [9, Theorems 11(1) and 13(1)] and the first half of [9, Theorem 12].

COROLLARY 3.4. *Let L be a serially finite Lie algebra over a field* $\mathfrak{k}$ *of characteristic zero. Then the following statements are equivalent*:

(1) *L is an* (A)*-algebra.*

(2) *L is a* (B_∞)*-algebra.*

(3) $L = R \oplus S$*, where R is an abelian ideal of L and S is a semisimple* (A)*-ideal of L. In particular L is abelian in case that* $\mathfrak{k}$ *is an algebraically closed field.*

Moreover, if $\sigma(L) \in \acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A}$, *then the above statements* (1)–(3) *are equivalent to the following statement*:

(4) *L is an* (A_∞)*-algebra.*

PROOF. It suffices to show that (2) ⇒ (3) (resp. (4) ⇒ (3)). Let L be a (B_∞)-algebra (resp. an (A_∞)-algebra). From Lemma 1.3 and Theorem 3.3 it follows that $L = R \oplus S$, where R is an $\mathfrak{A}_0$-ideal of L and S is a semisimple (A)-ideal of L ($S = 0$ in case of $\mathfrak{k} = \bar{\mathfrak{k}}$). Since R is a (B_∞)-algebra (resp. an (A_∞)-algebra) by [9, Proposition 2], R must be abelian by Lemma 2.3.

REMARK. Over any field there exists a locally nilpotent (accordingly serially finite) Lie algebra which belongs to (A_∞) but not to $\acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A} \cup \mathfrak{T}(\mathrm{wsi}) \cup \mathfrak{C}(\mathrm{wasc})$ (Example 3). Therefore we can not remove the hypothesis "$\sigma(L) \in \acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A}$" in the second halves of Theorem 3.3 and Corollary 3.4.

Using the finite-dimensional case in Theorem 3.3 we obtain the following

PROPOSITION 3.5. *Let Δ be any of the relations* wsi, wasc, wser. *Then*:

(1) *Over any field we have*

$$\mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{T}(\Delta) = \mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{T}(\mathrm{wser}),$$

$$\mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{C}(\Delta) = \mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{C}(\mathrm{wser}).$$

(2) *Over a field of characteristic zero we have*

$$\mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{C}(\Delta) = \mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{T}(\mathrm{wser}).$$

(3) *Over an algebraically closed field of characteristic zero we have*

$$\mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{C}(\Delta) = \mathrm{L}(\Delta)\mathfrak{F} \cap \mathfrak{A}_0 .$$

PROOF. Suppose that $L \in \mathrm{L}(\Delta)\mathfrak{F}$. Let H wser L and take any elements $x \in H$ and $y \in L$. Then there exists a finite-dimensional Δ-subalgebra F of L containing x and y. Therefore we have $H \cap F$ wsi F.

(1) Assume that $L \in \mathfrak{T}(\Delta)$. Then we obtain $H \cap F \lhd L$ since $H \cap F \Delta L$. In particular $[x, y] \in H$. Hence $H \lhd L$. This implies that $L \in \mathfrak{T}(\mathrm{wser})$. The second equation of (1) can be proved similarly.

(2) Assume that $L \in \mathfrak{C}(\Delta)$. Then we have $F \in \mathfrak{C}(\Delta)$. By making use of Theorem 3.3 we obtain $F \in \mathfrak{T}(\mathrm{wsi})$ and so $H \cap F \lhd F$. Hence $[x, y] \in H$ and $H \lhd L$. This means that $L \in \mathfrak{T}(\mathrm{wser})$.

(3) Assume that $L \in \mathfrak{C}(\Delta)$. Then for every elements a and b of L there exists an $\mathfrak{A}_0$-subalgebra F of L containing a and b as in the above argument. Therefore Lemma 2.1(1) leads to $L \in \mathfrak{A}_0$.

The following result generalizes the second half of [9, Theorem 12] and [9, Theorem 13(3)].

COROLLARY 3.6. *Let L be a* L(wsi)$\mathfrak{F}$*-algebra over any field* $\mathfrak{k}$. *Then the following statements are equivalent*:

(1) *L is an* (A_∞)*-algebra.*

(2) *L is a* (B_∞)*-algebra.*

In particular, if $\mathfrak{k}$ has characteristic zero, then the above statements (1), (2) *are equivalent to the following statement*:

(3) *L is an* (A)*-algebra.*

Moreover, if $\mathfrak{k}$ is an algebraically closed field, then the above statements (1)–(3) *are equivalent to the following statement*:

(4) *L is abelian.*

PROOF. $(4) \Rightarrow (3) \Rightarrow (2) \Rightarrow (1)$ is trivial.

$(1) \Rightarrow (2)$: Let $L \in (A_\infty)$. Then $L \in \mathfrak{C}(\text{wasc}) \cap \mathfrak{X}_0 = (B_\infty)$ from Lemma 1.3 and Proposition 3.5(1).

Assume that $\operatorname{char} \mathfrak{k} = 0$. $(1) \Rightarrow (3)$: Let $L \in (A_\infty)$ and let $x, y \in L$ with $[x, y, y] = 0$. Then there is a finite-dimensional weak subideal F of L containing x and y. Because of $F \in (A_\infty)$ by [9, Proposition 7], we have $F \in (A)$ by using [5, Corollary to Theorem 3] and so $[x, y] = 0$. Therefore $L \in (A)$.

Moreover, assume that $\mathfrak{k} = \bar{\mathfrak{k}}$. $(1) \Rightarrow (4)$ is shown by Proposition 3.5(3) and Lemma 2.3.

As in Corollary 3.6 we can prove the following result by using [9, Proposition 5].

COROLLARY 3.7. *Let L be a* L(wasc)$\mathfrak{F}$*-algebra over a field* $\mathfrak{k}$ *of characteristic zero. Then the following statements are equivalent*:

(1) *L is a* (B_∞)*-algebra.*

(2) *L is an* (A)*-algebra.*

Moreover, if $\mathfrak{k}$ is an algebraically closed field, then the above statements (1), (2) *are equivalent to the following statement*:

(3) *L is abelian.*

REMARK. In general over an arbitrary field of characteristic zero we have $\mathfrak{A} < (A) < (B_\infty)$ (Example 4).

As a consequence of Theorem 3.3 and Proposition 3.5 we have the following second main theorem in this section.

THEOREM 3.8. *Let L be a Lie algebra over a field of characteristic zero and let $L \in \mathrm{L(wsi)}\mathfrak{F} \cap \mathrm{L(ser)}\mathfrak{F}$ (especially $\mathrm{L(si)}\mathfrak{F}$). Then the following statements are equivalent*:

(1) *L belongs to one of the following classes of Lie algebras*:

$$\mathfrak{T}(\mathrm{wser}), \mathfrak{C}(\mathrm{wser}), \mathfrak{T}(\mathrm{wasc}), \mathfrak{C}(\mathrm{wasc}), \mathfrak{T}(\mathrm{wsi}), \mathfrak{C}(\mathrm{wsi}).$$

(2) *$L = R \oplus S$, where R is an $\mathfrak{A}_0$-ideal of L and S is a semisimple* (A)*-ideal of L.*

The following corollary generalizes [9, Theorem 11(3)] and the second half of [9, Theorem 12].

COROLLARY 3.9. *Let L be a Lie algebra over a field of characteristic zero and let $L \in \mathrm{L(wsi)}\mathfrak{F} \cap \mathrm{L(ser)}\mathfrak{F}$ (especially $\mathrm{L(si)}\mathfrak{F}$). Then the following statements are equivalent*:

(1) *L is an* (A)*-algebra.*

(2) *L is a* (B_∞)*-algebra.*

(3) *L is an* (A_∞)*-algebra.*

(4) *$L = R \oplus S$, where R is an abelian ideal of L and S is a semisimple* (A)*-ideal of L.*

4. Examples

In this section we present some examples. We first show the following lemma.

LEMMA 4.1. (1) *Let L be a Lie algebra. If every non-zero soluble subalgebra of L is* 1*-dimensional, then L satisfies the condition* (A).

(2) *Let L be a* 3*-dimensional simple Lie algebra over any field $\mathfrak{k}$. Then the following statements are equivalent*:

a) *L is non-split, i.e., for any $x \in L$,* ad x *has no non-zero characteristic roots in $\mathfrak{k}$ (cf.* [4, *p.* 14]).

b) *L is an* (A)*-algebra.*

c) *L is a* $\mathfrak{C}(\leq^2)$*-algebra.*

PROOF. (1) Suppose that every non-zero soluble subalgebra of L is 1-dimensional. Let $x, y \in L$ and $[x, y, y] = 0$. If $y = 0$ then $[x, y] = 0$. We assume that $y \neq 0$. Then from $\langle [x, y], y \rangle \in \mathrm{E}\mathfrak{A}$ it follows that $[x, y] \in \langle y \rangle$. Therefore $\langle x, y \rangle \in \mathrm{E}\mathfrak{A}$ and so $x \in \langle y \rangle$. Thus we have $[x, y] = 0$.

(2) a) $\Rightarrow$ b): Let L be non-split. Since L has no 2-dimensional subalgebras, it follows from (1) that L is an (A)-algebra.

b) $\Rightarrow$ c) is clear.

c) $\Rightarrow$ a): Let L be a $\mathfrak{C}(\leq^2)$-algebra. Assume that L is split. Then there exist two elements x and y of L such that $[x, y] = y$ and x, y are linearly independent. Now let $\{x, y, z\}$ be a basis for L. Then $\{[x, y], [y, z], [z, x]\}$ is also a basis for L as $L = L^2$. Suppose that $[y, z] = \alpha x + \beta y + \gamma z$, $[z, x] = \alpha' x + \beta' y + \gamma' z$. Then $0 = [x, y, z] + [y, z, x] + [z, x, y] = (\alpha' - \beta)[x, y] + (1 - \gamma')[y, z] + \gamma[z, x]$. Therefore we obtain $\gamma = 0$. Hence $[z, y, y] = -\alpha y \in \langle y \rangle$. It follows that $\langle y \rangle \leq^2 L$. Since $L \in \mathfrak{C}(\leq^2)$, we have $\langle y \rangle \lhd L$, a contradiction. Hence L is non-split.

REMARK. In Lemma 4.1(1) we can not replace "1-dimensional" by "of dimension ≤ 2" (see Example 4).

EXAMPLE 1. Let L be a 3-dimensional split simple Lie algebra (e.g. [2, Example 4.2]). Then $L \in \mathfrak{X}(\mathrm{ser})$ and $L \notin \mathfrak{C}(\leq^2)$ by Lemma 4.1(2). Hence over any field we have

$$\mathfrak{C}(\mathrm{wsi}) < \mathfrak{C}, \qquad \mathfrak{C}(\mathrm{wasc}) < \mathfrak{C}(\mathrm{asc}), \qquad \mathfrak{C}(\mathrm{wser}) < \mathfrak{C}(\mathrm{ser}),$$

$$\mathfrak{X}(\mathrm{wsi}) < \mathfrak{X}, \qquad \mathfrak{X}(\mathrm{wasc}) < \mathfrak{X}(\mathrm{asc}), \qquad \mathfrak{X}(\mathrm{wser}) < \mathfrak{X}(\mathrm{ser}).$$

EXAMPLE 2. Let L be an almost-abelian Lie algebra. By Lemma 2.1(2) $L \in \mathfrak{X}(\mathrm{wser})$ and by Lemma 2.3 $L \notin (\mathrm{A}_3)$. Hence over any field we have

$$(\mathrm{B}_\infty) < \mathfrak{C}(\mathrm{wasc}), \qquad (\mathrm{A}_\infty) < \mathfrak{C}(\mathrm{wsi}), \qquad (\mathrm{A}_{n+1}) < \mathfrak{C}(\leq^n) \quad (n \geq 2).$$

EXAMPLE 3. Let L be the Lie algebra in [2, Example 4.1], due to Simonjan [10]. Then L is a locally nilpotent Lie algebra which belongs to $(\mathrm{A}_\infty) \cap \acute{\mathrm{E}}\mathfrak{A}$ but not to $\mathfrak{X} \cup \acute{\mathrm{E}}(\mathrm{wsi})\mathfrak{A}$. Hence by [14, Theorem 4] $L \in \mathrm{L}(\mathrm{ser})\mathfrak{F}$. Since every finitely generated subalgebra of L is an ω-step ascendant subalgebra of L, L is not a $\mathfrak{C}(\lhd^\omega)$-algebra. Thus over any field we have

$$(\mathrm{B}_\infty) < (\mathrm{A}_\infty) \quad ([9, \text{p. } 427]), \qquad \mathfrak{C}(\lhd^\omega) < \mathfrak{C},$$

$$\mathfrak{C}(\mathrm{wasc}) < \mathfrak{C}(\mathrm{wsi}), \qquad \mathfrak{X}(\mathrm{wsi}) < \mathfrak{C}(\mathrm{wsi}), \qquad \mathfrak{C}(\mathrm{wsi}) \not\leq \mathfrak{X}.$$

EXAMPLE 4. Let W_0 be a Witt algebra, that is, a Lie algebra over a field of characteristic zero with basis $\{w_0, w_1, w_2, \dots\}$ and multiplication $[w_i, w_j] = (i-j)w_{i+j}$. Then W_0 is not an (A)-algebra, because $[w_0, w_1, w_1] = 0$ but $[w_0, w_1] \neq 0$. Since W_0 has no 1-dimensional weakly ascendant subalgebras, W_0 is a (B_∞)-algebra. We next consider the subalgebra W of W_0 generated by $w_1, w_2, \dots$. Since every non-zero soluble subalgebra of W is 1-dimensional ([7, Corollary to Theorem 1]), it follows from Lemma 4.1(1) that W is an (A)-algebra. In [2, Example 4.4] we observed that W is not a $\mathfrak{X}$-algebra. Since the subalgebra $\langle w_1 \rangle = \bigcap_{n=2}^{\infty} \langle w_1, w_n, w_{n+1}, \dots \rangle$ of W is a serial subalgebra of W, W does not belong to $\mathfrak{C}$(ser). These tell us that over any field of characteristic zero

$$\mathfrak{A} < (\mathrm{A}) < (\mathrm{B}_\infty), \qquad \mathfrak{T}(\mathrm{wasc}) < \mathfrak{C}(\mathrm{wasc}), \qquad \mathfrak{C}(\mathrm{wser}) < \mathfrak{C}(\mathrm{wasc}).$$

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