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Continuability and (non-) oscillatory properties of solutions of generalized Liénard equation

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1. Introduction

In this paper we are concerned with the equation

$$x'' + f(x)x' + g(x) = 0. \tag{1.1}$$

In the last three decades many authors have investigated this equation as well as the equation

$$x'' + f(x)x' + g(x) = e(t). \tag{1.2}$$

They have examined the continuability, boundedness, oscillation and periodicity of the solutions of (1.1) and (1.2). The book of Sansone and Conti [9] contains an almost complete list of papers dealing with these equations as well as a summary of the results published up to 1960. The book Reissig, Sansone and Conti [8] updates this list and summary up to 1962. The list of the papers which appeared between 1962 and 1970 is presented in the paper of John R. Graef [3]. Among the papers which were published in the last twenty years we refer to the following ones: [1], [2], [4]-[6], [10]-[12].

The study of the equations (1.1) and (1.2) was made under two main assumptions:

(1) $f(x) \geqq 0$ for all x, $xg(x) > 0$ for all $x \neq 0$ (see [1], [2]).

(2) $f(x) < 0$ for all $|x|$ small, $xg(x) > 0$ for all $x \neq 0$ (see [3]).

Instead of the damping f Graef made the assumptions concerning the integral $F(x)$ of the damping f such as $xF(x) > 0$ for $|x| \geqq k > 0$, $F(x) \geqq c > 0$ for $x > k$ or $F(x) \leqq -c < 0$ for $x \leqq -k$. Opial [7], examining the existence of periodic solutions, assumes that $xg(x) \neq 0$ for $x \neq 0$ and f and g are odd. In [4] it is assumed that $xg(x) > 0$ for $x \neq 0$ and $F(0) = 0$. Throughout this paper we will assume that

$$xf(x) > 0,\ xg(x) > 0 \text{ for all } x \neq 0; \text{ and } f \text{ and } g \text{ are continuous on } R. \tag{1.3}$$

Our first aim will be to state what kinds of solutions the equation (1.1) can have. We introduce the notation:

$$F(x) = \int_0^x f(t)dt, \quad G(x) = \int_0^x g(t)dt. \tag{1.4}$$

It follows immediately from the assumption (1.3) that

(1.5) $F(x) > 0,\ G(x) > 0$ for all $x \neq 0$, $F(0) = G(0) = 0$, and

$F(x)$ and $G(x)$ are decreasing for $x < 0$ and increasing for $x > 0$.

Using the function F we can replace the equation (1.1) by an equivalent system

$$x' = y - F(x)$$
$$y' = -g(x) \tag{1.6}$$

and the equation (1.2) by

$$x' = y - F(x)$$
$$y' = -g(x) + e(t). \tag{1.7}$$

Under a solution of (1.1) or (1.6) we will always understand an ultimately nontrivial solution.

2. Continuability of the solutions

THEOREM 2.1. *Let $e(t)$ be a piecewise continuous function on R. Let* (1.3) *be satisfied. Then each solution $x(t)$ of* (1.2) *which exists and is positive on $[T-\delta, T)$ is continuable to the right of T.*

PROOF. Assume that the conclusion is not true. Then the solution $(x(t), y(t))$ of (1.7) must satisfy $\lim_{t\to T^-}[|x(t)| + |y(t)|] = \infty$. Assume that $\lim_{t\to T^-} x(t) = \infty$. Denote $V(x, y) = y^2/2 + G(x)$. Then

$$\begin{aligned} V'(t) &= y(t)y'(t) + G_x(x(t))x'(t) \\ &= -g(x(t))F(x(t)) + e(t)y(t), \quad t \in [T-\delta, T), \end{aligned}$$

and integrating between $T-\delta$ and $t \in [T-\delta, T)$, we have

$$V(t) = V(T-\delta) - \int_{T-\delta}^{t} g(x(s))F(x(s))ds + \int_{T-\delta}^{t} e(s)y(s)ds. \tag{2.1}$$

Since $x(s) > 0$ for $T-\delta \leqq s < T$, we have $g(x(s))F(x(s)) > 0$ and

$$V(t) = \frac{1}{2}y^2(t) + G(x(t)) \leqq V(T-\delta) + \int_{T-\delta}^{t} e(s)y(s)ds.$$

Thus

$$\frac{1}{2}y^2(t) < V(T-\delta) + \int_{T-\delta}^{t} e(s)y(s)ds,$$

from which using the inequality $|y| \leqq (y^2 + 1)/2$, we get

$$|y(t)| \leqq \frac{1}{2}(y^2(t) + 1) < \frac{1}{2} + V(T-\delta) + \int_{T-\delta}^{t} |e(s)||y(s)|ds.$$

Then the Gronwall lemma implies

$$|y(t)| \leqq \left(\frac{1}{2} + V(T-\delta)\right) \exp\left(\int_{T-\delta}^{T} |e(s)|ds\right), \quad T-\delta \leqq t < T,$$

so that, $|y(t)|$ is bounded on $[T-\delta, T)$. Then, by integration of the first equation of (1.7), we get

$$(2.2) \qquad x(t) = x(T-\delta) + \int_{T-\delta}^{t} y(s)ds - \int_{T-\delta}^{t} F(x(s))ds, \quad T-\delta \leqq t < T.$$

The second term on the right-hand side is bounded on $[T-\delta, T)$ and $\int_{T-\delta}^{t} F(x(s))ds \geqq 0$ on $[T-\delta, T)$. Assume that $\int_{T-\delta}^{T} F(x(s))ds = \infty$. Then we see from (2.2) that $x(t) < 0$ in a left neighborhood of T, which contradicts the assumption. If $\int_{T-\delta}^{T} F(x(s))ds < \infty$, then from (2.2) follows the boundedness of $x(t)$ on $[T-\delta, T)$, which contradicts the assumption $\lim_{t\to T^-} x(t) = \infty$. Thus, the assumption that $x(t) > 0$ on $[T-\delta, T)$ excludes the possibility $\lim_{t\to T^-} x(t) = \infty$.

Now, assume that $x(t) > 0$ is bounded on $[T-\delta, T)$ and that $y(t)$ is unbounded on $[T-\delta, T)$. Then it follows from (2.1) that

$$V(t) \leqq V(T-\delta) + \int_{T-\delta}^{t} e(s)y(s)ds, \quad T-\delta \leqq t < T.$$

Since $G(x(t)) > 0$ for $T-\delta \leqq t < T$, from the preceding inequality we obtain

$$\frac{1}{2}y^2(t) \leqq V(t) \leqq V(T-\delta) + \int_{T-\delta}^{t} |e(s)||y(s)|ds,$$

which implies that

$$|y(t)| \leqq \frac{1}{2}(y^2(t) + 1) \leqq \frac{1}{2} + V(T-\delta) + \int_{T-\delta}^{t} |e(s)||y(s)|ds.$$

The Gronwall lemma then yields

$$|y(t)| \leqq \left(\frac{1}{2} + V(T-\delta)\right)\exp\left(\int_{T-\delta}^{T} |e(s)|ds\right),$$

which contradicts the assumption that $y(t)$ is unbounded on $[T-\delta, T)$.

REMARK 2.2. It is evident that the Theorem 2.1 holds also for the equation (1.1).

REMARK 2.3. The equation (1.1) under the assumption (1.3) may have a negative solution which is not continuable to the right as the following example shows. Consider the equation

$$x'' + 3xx' + x^3 = 0.$$

Evidently the assumption (1.3) is satisfied. The function $x(t) = (t - \alpha)^{-1}$, $t < \alpha$, is a solution of this equation and $\lim_{t \to \alpha^-} x(t) = -\infty$ and $\lim_{t \to -\infty} x(t) = 0$. On the other hand, $x(t), = (t - \alpha)^{-1}$, $t > \alpha$, is also a solution of this equation for which we have $x(t) > 0$ for $t > \alpha$, $\lim_{t \to \alpha^+} x(t) = \infty$ and $\lim_{t \to \infty} x(t) = 0$.

THEOREM 2.4. *Let the condition* (1.3) *be satisfied. Then every solution of* (1.1) *has only positive local maxima and only negative local minima.*

PROOF. This follows immediately from the equation (1.1). Let $x(t)$ be a solution of (1.1) and let it have a local extremum at the point ξ. Then $x'(\xi) = 0$ and from (1.1) we get $x''(\xi) + g(x(\xi)) = 0$. Thus $\operatorname{sgn} x''(\xi) = -\operatorname{sgn} g(x(\xi)) = -\operatorname{sgn} x(\xi)$.

COROLLARY 2.5. *Let the condition* (1.3) *be satisfied. Let* $x(t)$ *be a solution of* (1.1) *existing on* $[t_0, \infty)$. *If* $x(t)$ *is positive on* $[t_0, \infty)$, *then it has at most one local maximum and no local minimum on* $[t_0, \infty)$; *if* $x(t)$ *is negative on* $[t_0, \infty)$, *then it has at most one local minimum and no local maximum on* $[t_0, \infty)$.

THEOREM 2.6. *Let the condition* (1.3) *be satisfied and suppose that* $F(-\infty) < \infty$. *Then every solution* $x(t)$ *of* (1.1) *which exists and is negative on* $[t_0, T)$, $T < \infty$, *is continuable to the right of* T.

PROOF. Let $x(t)$ be a solution of (1.1) such that $x(t) < 0$ on $[t_0, T)$. Then

$$(x'(t) + F(x(t)))' = -g(x(t)) > 0 \text{ on } [t_0, T),$$

so that $x'(t) + F(x(t))$ is increasing on $[t_0, T)$. Taking into consideration the fact that $F(x(t)) \leqq K < \infty$, we get

$$x'(t_0) + F(x(t_0)) < x'(t) + F(x(t)) \leqq x'(t) + K, \quad t_0 < t < T.$$

Hence by integration

$$(x'(t_0) + F(x(t_0)) - K)(t - t_0) \leqq x(t) - x(t_0), \quad t_0 \leqq t < T.$$

The last two inequalities say that $x(t)$ and $x'(t)$ are bounded from below on $[t_0, T)$. Since $x'(t)$ can be zero at most in one point, there exists a left neighborhood of T in which $x'(t)$ has a constant sign. Hence $\lim_{t \to T^-} x(t)$ exists and since $x(t)$ is negative and bounded from below on $[t_0, T)$ $\lim_{t \to T^-} x(t)$ is finite. Then

$$\lim_{t \to T^-} x'(t) = \lim_{t \to T^-} [x'(t) + F(x(t))] - \lim_{t \to T^-} F(x(t))$$

exists and is finite because both limits on the right hand side exist and are finite. Indeed, $\lim_{t \to T^-} F(x(t)) = F(\lim_{t \to T^-} x(t))$ by continuity of F and

$$\lim_{t \to T^-} [x'(t) + F(x(t))] = x'(t_0) + F(x(t_0)) - \int_{t_0}^{T} g(x(s))ds.$$

Thus, we see that $\lim_{t \to T^-} x(t)$ and $\lim_{t \to T^-} x'(t)$ exist and are finite. Therefore, $x(t)$ is continuable to the right of T.

THEOREM 2.7. *Let the condition* (1.3) *be satisfied and let* $F(-\infty) < \infty$. *Then every solution* $x(t)$ *of* (1.1) *is continuable to the right and can be defined on an infinite interval* $[t_0, \infty)$.

PROOF. This follows from Theorems 2.1 and 2.6.

3. Ultimately positive solutions of the equation (1.1)

THEOREM 3.1. *Let the condition* (1.3) *be satisfied. Let* $x(t)$ *be a positive solution of* (1.1) *on* $[t_0, \infty)$. *Then* $x'(t) + F(t)) > 0$ *for all* $t \in [t_0, \infty)$.

PROOF. Let $x(t)$ be a positive solution of (1.1) on $[t_0, \infty)$. Assume that there exists $t_1 \in [t_0, \infty)$ such that $x'(t_1) + F(x(t_1)) = 0$. Since $[x'(t) + F(x(t))]' = -g(x(t)) < 0$ on $[t_0, \infty)$, $x'(t) + F(x(t))$ is decreasing on $[t_0, \infty)$. Thus, $x'(t) + F(x(t)) \leqq x'(t_2) + F(x(t_2)) < 0$ for all $t \geqq t_2 > t_1$. The function $F(x(t))$ being positive for $t \geqq t_0$, it follows from the preceding inequality that $x'(t) \leqq x'(t_2) + F(x(t_2)) < 0$. Integration on $[t_2, t]$ gives $x(t) \leqq x(t_2) + [x'(t_2) + F(x(t_2))](t - t_2)$. Hence we have $x(t) < 0$ for t large enough, which contradicts the assumption.

COROLLARY 3.2. *Let the condition* (1.3) *be satisfied. Then every solution* $x(t)$ *of* (1.1) *with the initial values* $x(t_0) > 0$, $x'(t_0) < 0$ *such that* $x'(t_0) + F(x(t_0)) \leqq 0$ *has a zero at some* $t_1 > t_0$.

PROOF. This follows immediately from Theorem 3.1.

THEOREM 3.3. *Let the condition* (1.3) *be satisfied and let* $\liminf_{x \to \infty} g(x) > 0$. *Let* $x(t)$ *be a positive solution of* (1.1) *on* $[t_0, \infty)$. *Then there exists* $t_2 \in [t_0, \infty)$ *such that* $x'(t) < 0$ *on* (t_2, ∞). *Thus,* $x(t)$ *is bounded. Moreover,* $\lim_{t \to \infty} x(t) = \lim_{t \to \infty} x'(t) = 0$.

Proof. We will prove the first part of the statement. If $x'(t) < 0$ on $[t_0, \infty)$, there is nothing to prove. Now assume that there exists $t_1 \geqq t_0$ such that $x'(t) > 0$ for all $t \geqq t_1$. Then $x(t)$ increases on $[t_1, \infty)$ and $\lim_{t\to\infty} x(t) = M > 0$. Since $[x'(t) + F(x(t))]' = -g(x(t)) < 0$, $x'(t) + F(x(t))$ decreases on $[t_0, \infty)$ and is positive for $t \geqq t_0$ (see Theorem 3.1). Hence

$$0 < x'(t) + F(x(t)) = x'(t_1) + F(x(t_1)) - \int_{t_1}^{t} g(x(s))ds, \quad t \geqq t_1,$$

and $\int_{t_1}^{\infty} g(x(s))ds < \infty$. Thus $\liminf_{s\to\infty} g(x(s)) = 0$. From the assumption $\liminf_{x\to\infty} g(x) > 0$ and from the positivity and continuity of $g(x)$ for $x > 0$ it follows that the limit $\lim_{t\to\infty} x(t) = M$ cannot be $+\infty$ or a positive number. Thus $\lim_{t\to\infty} x(t) = M = 0$. But this implies that $x'(t)$ cannot be positive for all $t > t_1$. Thus there must exist $t_2 > t_1$ such that $x'(t_2) = 0$. Following Corollary 2.5 $x(t)$ can have at most one local extremum on $[t_0, \infty)$. Hence $x'(t)$ has to have a constant sign on (t_2, ∞), i.e. we must have $x'(t) < 0$ for $t > t_2$.

Now we will prove the second part of the statement. According to the first part of the statement, $x'(t) < 0$ on (t_2, ∞). Thus $x(t)$ decreases on (t_2, ∞) and, therefore, $\lim_{t\to\infty} x(t) = a \geqq 0$. From the relation

$$0 < x'(t) + F(x(t)) = F(x(t_2)) - \int_{t_2}^{t} g(x(s))ds, \quad t \geqq t_2,$$

it follows that $\int_{t_2}^{\infty} g(x(s))ds < \infty$ and hence $\liminf_{s\to\infty} g(x(s)) = 0$. If $a > 0$, then from the continuity of g we get $\liminf_{s\to\infty} g(x(s)) = g(a) > 0$, which is a contradiction. So, the limit $a = \lim_{t\to\infty} x(t)$ must be zero. Then we have

$$0 \leqq L = \lim_{t\to\infty}[x'(t) + F(x(t))] = \lim_{t\to\infty} x'(t)$$

because $x'(t) + F(x(t))$ is positive and monotone on $[t_0, \infty)$ and $\lim_{t\to\infty} F(x(t)) = F(0) = 0$. Since $x'(t) < 0$ on (t_2, ∞) it follows that $L = \lim_{t\to\infty} x'(t) = 0$. This finishes the proof.

The assumption $\liminf_{x\to\infty} g(x) > 0$ in Theorem 3.3 can be replaced by the assumption $G(\infty) = \infty$.

Theorem 3.4. *Let the condition* (1.3) *be satisfied and let* $G(\infty) = \infty$. *Then every ultimately positive solution* $x(t)$ *of* (1.1) *is bounded. Moreover,* $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$, *and* $x'(t) < 0$ *on some interval* (t_2, ∞).

Proof. Let $x(t) > 0$, $t \in [t_1, \infty)$, be a solution of (1.1). Multiplying (1.1) by $2x'(t)$ we have

$$(x'^2(t))' + 2f(x(t))x'^2(t) + 2g(x(t))x'(t) = 0$$

and

$$x'^2(t) + 2\int_{t_1}^{t} f(x(s))x'^2(s)ds + 2G(x(t)) = x'^2(t_1) + 2G(x(t_1)).$$

Therefore, $2G(x(t)) < x'^2(t_1) + 2G(x(t_1))$ for all $t > t_1$, so that $x(t)$ is bounded on $[t_1, \infty)$. Since $x(t) > 0$ on $[t_1, \infty)$ by assumption, $x'(t)$ has a constant sign on some interval $[t_2, \infty) \subset [t_1, \infty)$, and so the limit $\lim_{t\to\infty} x(t) = a \geqq 0$ exists. Suppose that $a > 0$. Then $\int_{t_2}^{\infty} g(x(s))ds = \infty$ and

$$\lim_{t\to\infty} [x'(t) + F(x(t))] = x'(t_2) + F(x(t_2)) - \int_{t_2}^{\infty} g(x(s))ds = -\infty,$$

which contradicts the statement of the Theorem 3.1. This shows that $\lim_{t\to\infty} x(t) = a = 0$. Then from the fact that $x'(t) + F(x(t))$ is positive and decreasing on $[t_1, \infty)$, we see that

$$\begin{aligned}\lim_{t\to\infty} x'(t) &= \lim_{t\to\infty} [x'(t) + F(x(t))] - \lim_{t\to\infty} F(x(t)) \\ &= \lim_{t\to\infty} [x'(t) + F(x(t))] \geqq 0.\end{aligned}$$

The assumption $\lim_{t\to\infty} x'(t) > 0$ leads to the contradiction with the fact that $\lim_{t\to\infty} x(t) = 0$. Consequently, $\lim_{t\to\infty} x'(t) = 0$ and $x'(t) < 0$ on the interval $[t_2, \infty)$.

REMARK 3.5. The assumption $\liminf_{x\to\infty} g(x) > 0$ (resp. $G(\infty) = \infty$) is in a certain sense necessary for the validity of Theorem 3.3 (resp. Theorem 3.4) as the following example shows. Define

$$3f(x) = \begin{cases} x^{-3}, & x \in [1, \infty) \\ x, & x \in [-1, 1) \\ x^{-3}, & x \in (-\infty, -1) \end{cases}, \quad 9g(x) = \begin{cases} x^{-5}, & x \in [1, \infty) \\ x, & x \in [-1, 1) \\ x^{-5}, & x \in (-\infty, -1) \end{cases}.$$

Then, $\liminf_{x\to\infty} g(x) = 0 (G(\infty) < \infty)$. The equation (1.1) has in this case the solution $x(t) = t^{1/3}$, $t > 1$. Thus the statements of Theorem 3.3 (Theorem 3.4) do not hold.

LEMMA 3.6. *Let the condition* (1.3) *be satisfied. Suppose that* $F(x)f(x) \leqq g(x)$ *for all* $x \neq 0$. *Let* $x(t)$ *be a solution of* (1.1) *such that* $x(t) > 0$ *and* $x'(t) < 0$ *on an interval* $[t_0, \infty)$. *Then* $x''(t) > 0$ *on* $[t_0, \infty)$.

PROOF. Assume that there is $t_1 \in [t_0, \infty)$ such that $x''(t_1) = 0$. Then from the equation (1.1) we get $f(x(t_1))x'(t_1) + g(x(t_1)) = 0$. According to Theorem 3.1 it holds that $x'(t) + F(x(t)) > 0$ on $[t_0, \infty)$. Then $x'(t_1) + F(x(t_1)) > 0$ and $F(x(t_1)) > -x'(t_1) = g(x(t_1))/f(x(t_1))$, which contradicts the assumption. Thus $x''(t)$ has a constant sign on $[t_0, \infty)$. But the assumption $x''(t) < 0$ on $[t_0, \infty)$ shows that $x'(t)$ is decreasing on $[t_0, \infty)$, and hence $x'(t) < x'(t_0) < 0$ for $t > t_0$. Integrating this inequality we have $x(t) < x(t_0) + x'(t_0)(t - t_0)$, which implies that $x(t)$ is negative for all sufficiently large t. This contradicts our assumption that $x(t) > 0$ on $[t_0, \infty)$.

LEMMA 3.7. *Let the condition* (1.3) *be satisfied. Let $x(t)$ be a positive solution on $[t_0, \infty)$ of* (1.1) *such that $x''(t) > 0$ on an interval $[t_1, \infty) \subset [t_0, \infty)$. Then $f(x(t))F(x(t)) > g(x(t))$ for all $t \in [t_1, \infty)$.*

PROOF. Let all assumptions of Lemma 3.7 be satisfied. Then from (1.1) we have $0 < x''(t) = -f(x(t))x'(t) - g(x(t))$ for all $t \in [t_1, \infty)$. Hence, $x'(t) < -g(x(t))/f(x(t))$ for $t \in [t_1, \infty)$. According to Theorem 3.1, we have $x'(t) + F(x(t)) > 0$ for all $t \in [t_0, \infty)$, which gives

$$-F(x(t)) < x'(t) < -g(x(t))/f(x(t)).$$

THEOREM 3.8. *Let the condition* (1.3) *be satisfied and suppose that $F(x)f(x) \leqq g(x)$ for all $x \in (-\delta, \delta)$, $x \neq 0$, for some $\delta > 0$. Then the equation* (1.1) *has no solution $x(t)$ such that $x(t) > 0$ and $x'(t) < 0$ on some interval $[t_0, \infty)$.*

PROOF. Assume that (1.1) has a solution $x(t)$ such that $x(t) > 0$ and $x'(t) < 0$ on some interval $[t_0, \infty)$. Then $\lim_{t\to\infty} x(t) = a \geqq 0$. Let $a > 0$. Then $g(a) > 0$ and there exists an interval $(t_2, \infty) \subset [t_0, \infty)$ and a constant $k > 0$ such that $k < g(x(t))$ for all $t \in (t_2, \infty)$. Integrating the second equation of (1.6), we have

$$\begin{aligned} x'(t) + F(x(t)) &= x'(t_2) + F(x(t_2)) - \int_{t_2}^{t} g(x(s))ds \\ &\leqq x'(t_2) + F(x(t_2)) - k(t - t_2), \quad t > t_2. \end{aligned}$$

Hence $x'(t) + F(x(t))$ becomes negative for all t large enough, which contradicts Theorem 3.1. Thus $a = \lim_{t\to\infty} x(t) = 0$. Then there exists an interval $[t_3, \infty) \subset [t_0, \infty)$ such that $0 < x(t) < \delta$ for all $t \in [t_3, \infty)$. For $t \in [t_3, \infty)$ we have the following situation: $x'(t) + F(x(t)) > 0$ and $x'(t) > -F(x(t)) \geqq -g(x(t))/f(x(t))$ because $x(t) > 0$ on $[t_3, \infty)$ (see Theorem 3.1) and $F(x)f(x) \leqq g(x)$ for $x \in (-\delta, \delta)$ by assumption. We then have $x''(t) = -f(x(t))x'(t) - g(x(t)) < 0$ on $[t_3, \infty)$, which leads to a contradiction with the positivity of $x(t)$ on $[t_3, \infty)$.

4. Ultimately negative solutions of the equation (1.1)

THEOREM 4.1. *Let the condition* (1.3) *be satisfied. Let* $x(t)$ *be a solution of* (1.1) *with the initial values* $x(t_0) < 0$ *and* $x'(t_0) \geqq 0$. *Then there exists* $\tau > t_0$ *such that* $x(\tau) = 0$.

PROOF. Assume that the solution $x(t)$ with initial values $x(t_0) < 0$ and $x'(t_0) \geqq 0$ is negative on its interval of existence $[t_0, \alpha)$. Since $x''(t_0) = -f(x(t_0))x'(t_0) - g(x(t_0)) > 0$ there eixsts an interval $[t_0, t_1)$, $t_1 \leqq \alpha$, on which $x''(t) > 0$. Thus, $x'(t)$ is increasing on this interval and therefore $x'(t) > 0$ on (t_0, t_1). Then $x(t)$ is also increasing on this interval. From the fact that $x(t)$ cannot have a negative local maximum on $[t_0, \alpha)$(see Theorem 2.4) it follows that $x'(t) \geqq 0$ on $[t_0, \alpha)$. Therefore $x(t)$ is increasing on $[t_0, \alpha)$ and so the finite limit $\lim_{t\to\alpha^-} x(t) \leqq 0$ exists since $x(t)$ is negative. Then for the function $x'(t) + F(x(t))$ which is increasing on $[t_0, \alpha)$ we get

$$0 < \lim_{t\to\alpha^-} [x'(t) + F(x(t))] = x'(t_0) + F(x(t_0)) - \int_{t_0}^{\alpha} g(x(s))ds.$$

Assume that $\alpha < \infty$. The the limit

$$\lim_{t\to\alpha^-} x'(t) = \lim_{t\to\alpha^-} [x'(t) + F(x(t))] - \lim_{t\to\alpha^-} F(x(t))$$

$$= \lim_{t\to\alpha^-} [x'(t) + F(x(t))] - F(\lim_{t\to\alpha^-} x(t))$$

exists and is finite and nonnegative. It follows that $x(t)$ is continuable to the right of α and therefore α must be ∞. Now, we have the situation that $x(t) < 0$ and $x'(t) \geqq 0$ on $[t_0, \infty)$. Thus, $x''(t) = -f(x(t))x'(t) - g(x(t)) > 0$ on $[t_0, \infty)$. But this leads to a contradiction with the assumption that $x(t) < 0$ on $[t_0, \infty)$.

THEOREM 4.2. *Let the condition* (1.3) *be satisfied. Let* $[t_0, \alpha)$, $\alpha < \infty$, *be the maximal interval of the existence of the solution* $x(t)$ *of* (1.1) *and suppose that* $x(t) < 0$ *and* $x'(t) < 0$ *on* $[t_0, \alpha)$. *Then* $\lim_{t\to\alpha^-} x(t) = -\infty$, *and* $x'(t)$ *is unbounded from below.*

PROOF. Suppose that $\lim_{t\to\alpha^-} x(t) = L$, $-\infty < L < 0$. Then in view of the relation

$$x'(t) + F(x(t)) = x'(t_0) + F(x(t_0)) - \int_{t_0}^{t} g(x(s))ds$$

we see that

$$\lim_{t\to\alpha^-} x'(t) = -F(L) + x'(t_0) + F(x(t_0)) - \int_{t_0}^{\alpha} g(x(s))ds,$$

which is finite. Thus the solution is continuable to the right of α, which contradicts the maximality of the interval $[t_0, \alpha)$. Therefore, $\lim_{t\to\alpha^-} x(t) = L = -\infty$. Then $x'(t)$ must be unbounded from below. In fact, if $-k < x'(t)$ for some $k > 0$, then by integration we get $-k(\alpha - t_0) \leqq x(t) - x(t_0)$, which contradicts the fact that $\lim_{t\to\alpha^-} x(t) = -\infty$.

THEOREM 4.3. *Let the condition* (1.3) *be satisfied. Let* $x(t) < 0$, $t \geqq t_0$, *be a solution of* (1.1). *Then* $\lim_{t\to\infty} x(t) = -\infty$.

PROOF. If $x(t) < 0$, $t \geqq t_0$, is a solution of (1.1), then $x'(t) < 0$ for $t \geqq t_0$. In fact, if there exists $\xi \in [t_0, \infty)$ such that $x'(\xi) \geqq 0$, then from Theorem 4.1 it follows that there exists $\tau > \xi$ such that $x(\tau) = 0$, which gives a contradiction. Hence $x'(t) < 0$ on $[t_0, \infty)$, and so $\lim_{t\to\infty} x(t) = L$, $-\infty \leqq L < 0$, exists. Assume that $L > -\infty$. Then, since $x'(t) + F(x(t))$ is increasing, and $\lim_{t\to\infty} F(x(t)) = F(L) < \infty$, the limit $\lim_{t\to\infty}[x'(t) + F(x(t))] = U \leqq \infty$ exists.

If $U < \infty$, then

$$U = \lim_{t\to\infty}[x'(t) + F(x(t))] = x'(t_0) + F(x(t_0)) - \int_{t_0}^{\infty} g(x(s))ds,$$

which implies $\int_{t_0}^{\infty} [-g(x(s))]ds < \infty$ and $\liminf_{s\to\infty}[-g(x(s))] = 0$. But this is a contradiction, since $\liminf_{t\to\infty}[-g(x(t))] = -g(L) > 0$.

If $U = \infty$, then $\lim_{t\to\infty} x'(t) = \infty$, which gives a contradiction with $x'(t) < 0$. Thus L cannot be finite, i.e. $L = -\infty$.

REMARK 4.4. Theorem 4.3 says nothing about the boundedness of $x'(t)$. It can happen that $\lim_{t\to\infty} x'(t)$ is finite as in the case of the equation $x'' + g(x)x' + g(x) = 0$ which has the solution $x(t) = -t$. But if we assume $F(-\infty) < \infty$ then from the fact that $x'(t) + F(x(t))$ is increasing for $x(t) < 0$ it follows that $\lim_{t\to\infty} x'(t)$ is necessarily finite.

THEOREM 4.5. *Let the condition* (1.3) *be satisfied. Suppose that* $F(-\infty) < \infty$ *and* $\limsup_{x\to-\infty} g(x) < 0$. *Then the equation* (1.1) *has no ultimately negative solution.*

PROOF. Let $x(t)$ be a solution of (1.1) such that $x(t) < 0$ on $[t_1, \infty)$. Then $x'(t) < 0$ on $[t_1, \infty)$ as follows from Theorem 4.1. We then have $F'(x(t)) = f(x(t))x'(t) > 0$ and $[x'(t) + F(x(t))]' = -g(x(t)) > 0$ for all $t \geqq t_1$, so that $F(x(t))$ and $x'(t) + F(x(t))$ are increasing on $[t_1, \infty)$. In view of the assumption

$k = F(-\infty) < \infty$, we have $x'(t) + F(x(t)) \leqq x'(t) + k$, which implies that $x'(t) + F(x(t))$ is bounded from above and

$$\lim_{t\to\infty}[x'(t) + F(x(t))] = x'(t_1) + F(x(t_1)) - \int_{t_1}^{\infty} g(x(s))ds$$

is finite. Thus $\int_{t_1}^{\infty}(-g(x(s)))ds < \infty$ and, therefore, $\liminf_{t\to\infty}(-g(x(t))) = 0 = \liminf_{x\to-\infty}(-g(x))$. But this contradicts the assumption $\limsup_{x\to-\infty} g(x) < 0$.

5. Oscillatory solutions of the equation (1.1)

DEFINITION 5.1. A solution $x(t)$ of the equation (1.1) or (1.2) will be called oscillatory if there exists an increasing sequence $\{t_n\}_{n=1}^{\infty}$ such that $\lim_{n\to\infty} t_n = \infty$ and $x(t_n) = 0$ for $n = 1, 2, \ldots$

THEOREM 5.1. *Let the condition* (1.3) *be satisfied and suppose that* $\liminf_{x\to\infty} g(x) > 0$. *Then every bounded solution* $x(t)$ *of* (1.1) *is either oscillatory or it is ultimately positive and satisfies* $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$.

PROOF. If $x(t)$ is an ultimately positive solution of (1.1) then, according to Theorem 3.3, it is bounded and $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} x'(t) = 0$.

If $x(t)$ is not ultimately positive, it assumes negative values but it cannot be ultimately negative, because in this case it should be unbounded according to Theorem 4.3. But this contradicts the assumption.

THEOREM 5.2. *Let the condition* (1.3) *be satisfied. Suppose moreover that*:
1. $\liminf_{x\to\infty} g(x) > 0$, *or* $G(\infty) = \infty$;
2. $F(x) \leqq g(x)/f(x)$ *for all* $x \in (-\delta, \delta)$, $x \neq 0$, $\delta > 0$;
3. $F(-\infty) < \infty$, $\limsup_{x\to-\infty} g(x) < 0$.

Then all solutions of the equation (1.1) *are oscillatory.*

PROOF. It follows from Theorem 2.7 that every solution $x(t)$ of (1.1) exists on some infinite interval (t_x, ∞). The condition 1 implies that, if $x(t) > 0$ on (T, ∞), according to the Theorem 3.3 (or Theorem 3.4) there exists an interval $[t_1, \infty) \subset (T, \infty)$ on which $x'(t) < 0$. But, according to Theorem 3.8, the condition 2 excludes the existence of such a solution. Thus, conditions 1 and 2 exclude the existence of ultimately positive solutions of (1.1).

According to the Theorem 4.5, the condition 3 excludes the existence of ultimately negative solutions of (1.1).

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