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Totally umbilic hypersurfaces

Dedicated to Professor Yoshihiro Tashiro on his 60th birthday

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Introduction

Let $M_s^m(c)$ be an m-dimensional connected semi-Riemannian manifold of index s and of constant curvature c, which is called an *indefinite space form of index* s and is called simply a *space form*, provided that $s = 0$. The study of hypersurfaces with constant mean curvature of $M^{n+1}(c)$ was initiated by Nomizu and Smyth [10], who proved some results. Later, a compact totally umbilical hypersurface of $M^{n+1}(c)$, $c \geqq 0$, was characterized by Okumura [11] under a certain condition which was given by an inequality between the length of the second fundamental form and the mean curvature. This is also generalized by Hasanis [7] in the complete case.

On the other hand, in connection with the Bernstein-typr problem by Calabi [4], Nishikawa [9] and Cheng and Yau [5], complete connected space-like hypersurfaces with constant mean curvature of a de Sitter space $M_1^{n+1}(c)$, $c > 0$, are recently treated by Akutagawa [2] and Ramanathan [14] independently.

In this paper, complete hypersurfaces with constant mean curvature of a space form of index $s(=0$ or $1)$ are investigated in the two directions. One of the purposes is to give another charaterization of complete totally umbilical hypersurfaces of $M^{n+1}(c)$, $c \geqq 0$. The other is concerned with that of an anti-de Sitter space. In §1, the theory of space-like hypersurfaces of a real space form of index 1 is stated. In §2, a generalization of the theorem due to Okumura [11] and Hasanis [7] is proved. The last section is concerned with space-like hypersurfaces with constant mean curvature of $M_1^{n+1}(c)$, $c \neq 0$.

1. Preliminaries

Let (M', g') be an $(n+1)$-dimensional semi-Riemannian manifold of index $s(=0$ or $1)$. Throughout this paper, manifolds are always assumed to be connected and geometric objects are assumed to be of class C^∞. We choose a local field of orthonormal frames $e_0, e_1, \ldots, e_n$ adapted to the semi-Riemannian metric in M' and let $\omega_0, \omega_1, \ldots, \omega_n$ denote the dual coframes. Suppose that we have $g(e_A, e_B) = \varepsilon_A \delta_{AB}$, $\varepsilon_0 = \pm 1$, $\varepsilon_i = 1$. Here and in the sequel, the following convention on the range of indices is used, unless otherwise stated: $A, B, \ldots = 0, 1, \ldots, n$; $i, j, \ldots = 1, \ldots, n$. The connection forms $\{\omega_{AB}\}$ of M' are characterized by the equations

$$
\begin{aligned}
&d\omega_A + \sum \varepsilon_B \omega_{AB} \wedge \omega_B = 0, \ \omega_{AB} + \omega_{BA} = 0, \\
(1.1) \qquad &d\omega_{AB} + \sum \varepsilon_C \omega_{AC} \wedge \omega_{CB} = \Omega_{AB}, \\
&\Omega_{AB} = (-1/2) \sum \varepsilon_C \varepsilon_D R'_{ABCD} \omega_C \wedge \omega_D,
\end{aligned}
$$

where Ω_{AB}(resp. R'_{ABCD}) denotes the semi-Riemannian curvature form (resp. components of the semi-Riemannian curvature tensor R') of M'. A semi-Riemannian manifold M' is called a *space form of index s* if M' is of index s and of constant sectional curvature. By $M_s^m(c)$ an m-dimensional space form of index s and of constant curvature c is denoted. Then the components R'_{ABCD} of the Riemannian curvature tensor R' for a real space form $M_s^{n+1}(c)$ are given by

$$R'_{ABCD} = c\varepsilon_A \varepsilon_B (\delta_{AD}\delta_{BC} - \delta_{AC}\delta_{BD}).$$

In particular, $M_1^m(c)$ is called a *Lorentz space form.*

Standard models of complete connected Lorentz space forms are given as follows. In an $(n+p)$-dimensional Euclidean space $\boldsymbol{R}^{n+p}$ with a standard basis, a scalar product $\langle , \rangle$ is defined by

$$\langle x, y \rangle = -\sum_{i=1}^{p} x_i y_i + \sum_{p+1}^{n+p} x_j y_j,$$

where $x = (x_1, \ldots, x_{n+p})$ and $y = (y_1, \ldots, y_{n+p})$ are in $\boldsymbol{R}^{n+p}$. This is a scalar product of index p and the space $(\boldsymbol{R}^{n+p}, \langle , \rangle)$ is an indefinite Euclidean space, which is simply denoted by $\boldsymbol{R}_p^{n+p}$. Let $S_1^{n+1}(c)$ be a hypersurface of $\boldsymbol{R}_1^{n+2}$ defined by

$$\langle x, x \rangle = r^2 = 1/c.$$

Then $S_1^{n+1}(c)$ inherits a Lorentz metric from the ambient space $\boldsymbol{R}_1^{n+2}$ with constant curvature c, which is called a *de Sitter space.* On the other hand, let $H_1^{n+1}(c)$ be a hypersurface of $\boldsymbol{R}_2^{n+2}$ defined by

$$\langle x, x \rangle = -r^2 = 1/c.$$

Then $H_1^{n+1}(c)$ induces a Lorentz metric from the ambient space $\boldsymbol{R}_2^{n+2}$ with negative constant curvature c, which is called an *anti-de Sitter space.* For indefinite Riemannian manifolds, refer to O'Neill [13].

Now, let $M' = M_s^{n+1}(c)$ be an $(n+1)$-dimensional space form of index $s(=0$ or $1)$ and of constant curvature c and let M be a hypersurface of $M_0^{n+1}(c)$ or a space-like hypersurface of $M_1^{n+1}(c)$. By restricting the canonical forms ω_A and the connection forms ω_{AB} to the hypersurface M, they are denoted by the same symbol respectively. Then we have

$$\omega_0 = 0, \tag{1.2}$$

and the metric on M induced from the semi-Riemannian metric g' on the ambient space M' under the immersion is given by $g = \sum \omega_i \otimes \omega_i$. Then $\{e_1, \ldots, e_n\}$ becomes a field of orthonormal frames on M with respect to this metric and $\{\omega_1, \ldots, \omega_n\}$ is a field of dual frames on M. From (1.1) and Cartan's lemma it follows that

$$\omega_{0i} = \sum h_{ij}\omega_j, \; h_{ij} = h_{ji}. \tag{1.3}$$

The quadratic form $\alpha = \sum \varepsilon h_{ij}\omega_i\omega_j e_0$ is called the *second fundamental form* on M, where we put $\varepsilon = \varepsilon_0$. That is,

$$\alpha(e_i, e_j) = \varepsilon h_{ij} e_0. \tag{1.4}$$

The connection forms $\{\omega_{ij}\}$ of M are characterized by the structure equations

$$\begin{aligned} &d\omega_i + \sum \omega_{ij} \wedge \omega_j = 0, \; \omega_{ij} + \omega_{ji} = 0, \\ &d\omega_{ij} + \sum \omega_{ik} \wedge \omega_{kj} = \Omega_{ij}, \\ &\Omega_{ij} = (-1/2)\sum R_{ijk\ell}\,\omega_k \wedge \omega_\ell, \end{aligned} \tag{1.5}$$

where Ω_{ij}(reap. $R_{ijk\ell}$) denotes the Riemannian curvature form (resp. components of the Riemannian curvature tensor R) of M. For the semi-Riemannian curvature tensors R' and R of M' and M respectively, it follows from (1.1) and (1.5) that we have the Gauss equation

$$R_{ijk\ell} = c(\delta_{i\ell}\delta_{jk} - \delta_{ik}\delta_{j\ell}) + \varepsilon(h_{i\ell}h_{jk} - h_{ik}h_{j\ell}). \tag{1.6}$$

The components of the Ricci curvature Ric and the scalar curvature r are given by

$$R_{jk} = c(n-1)\delta_{jk} + \varepsilon h h_{jk} - (h_{jk})^2, \tag{1.7}$$

$$r = n(n-1)c + \varepsilon h^2 - h_2, \tag{1.8}$$

where $h = \sum h_{jj}$, $(h_{jk})^2 = \sum \varepsilon h_{jr} h_{rk}$ and $h_2 = \sum (h_{jj})^2$.

Now, components h_{ijk} of the covariant derivative of the second fundamental form of M are given by

$$\sum h_{ijk}\omega_k = dh_{ij} - \sum h_{kj}\omega_{ki} - \sum h_{ik}\omega_{kj}. \tag{1.9}$$

Then, differentiating (1.2) exteriorly, we have the Codazzi equation

$$h_{ijk} = h_{ikj}. \tag{1.10}$$

Similarly components $h_{ijk\ell}$ of the covariant derivative of h_{ijk} are given by

$$\sum h_{ijk\ell}\omega_\ell = dh_{ijk} - \sum h_{\ell jk}\omega_{\ell i} - \sum h_{i\ell k}\omega_{\ell j} - \sum h_{ij\ell}\omega_{\ell k},$$

and by a simple calculation the Ricci formula for the second fundamental form is given by

$$h_{ijk\ell} - h_{ij\ell k} = -\sum h_{rj}R_{rik\ell} - \sum h_{ir}R_{rjk\ell}. \tag{1.11}$$

Making use of this relationship, one can compute the Laplacian of the second fundamental form:

$$\Delta h_{ij} = \sum h_{ijkk} = \sum h_{kkij} + c(nh_{ij} - h\delta_{ij}) + h(h_{ij})^2 - h_2 h_{ij}. \tag{1.12}$$

The Laplacian of the function h_2 may be computed by using (1.6), (1.7) and (1.12):

$$\begin{aligned}(1/2)\Delta h_2 &= \varepsilon\sum h_{ijk}h_{ijk} + \varepsilon\sum h_{ij}h_{kkij} + c(nh_2 - \varepsilon h^2)\\ &\quad + \varepsilon h h_3 - h_2^2,\end{aligned} \tag{1.13}$$

where $h_3 = \sum h_{ij}(h_{ij})^2$.

First of all, a fundamental property for the generalized maximal principle due to Omori [12] and Yau [15] is introduced and then an inequality by Cai [3] is given.

THEOREM 1.1. *Let M be an n-dimensional complete Riemannian manifold whose Ricci curvature is bounded from below. Let F be a C^2-function bounded from above on M, then for any $\varepsilon > 0$, there exists a point p in M such that*

$$\begin{cases} \sup F\text{-}\varepsilon < F(p), \\ |\operatorname{grad} F(p)| < \varepsilon, \\ \Delta F(p) < \varepsilon. \end{cases} \tag{1.14}$$

LEMMA 1.2. *Let $A = (a_{ij})$ be a symmetric $n \times n$ matrix, $n \geqq 2$, and put $A_1 = \operatorname{Tr} A$ and $A_2 = \sum (a_{ij})^2$. Then we have*

$$\begin{aligned}\sum (a_{in})^2 - A_1 a_{nn} &\leqq [n(n-1)A_2 + (n-2)|A_1|\{(n-1)(nA_2 - A_1^2)\}^{1/2}\\ &\quad - 2(n-1)A_1^2]/n^2.\end{aligned}$$

2. Complete hypersurfaces

This section is concerned with complete hypersurfaces with constant mean curvature of a space form. Let $M' = M^{n+1}(c)$ be an $(n+1)$-dimensional space form of constant curvature c and let M be a hypersurface of M'. Then the following formula may be found in [7] and [11]:

$$(1/2)\Delta f^2 \geqq f^2[nc + h^2/n - (n-2)|h|f\{n(n-1)\}^{1/2} - f^2], \tag{2.1}$$

where f denotes a non-negative function defined by $f^2 = h_2 - h^2/n$. So, it is easily seen that if f vanishes identically on M, then M is totally umbilical. By S the length of the second fundamental form is denoted. Namely, we put $S = (h_2)^{1/2}$.

THEOREM 2.1. *Let M be a complete hypersurface with constant mean curvature of an $(n+1)$-dimensional space form $M^{n+1}(c)$, $c \geqq 0$. If the length S satisfies*

$$\sup S^2 < [n\{2(n-1)c + h^2\} - (n-2)|h|\{h^2 + 4(n-1)c\}^{1/2}]/2(n-1), \tag{2.2}$$

then M is totally umbilical.

PROOF. For any positive constant a, a function F defined by $(f^2 + a)^{1/2}$ is smooth and bounded under the assumption of the length S. On the other hand, for any point x and any unit vector v at x we choose a local orthonormal frame $\{e_0, e_1, \ldots, e_n\}$ in M' such that, restricted to M, $e_1, \ldots, e_n$ are tangent to M and $v = e_n$. Then (1.7) gives

$$\mathrm{Ric}(v, v) = (n-1)c + hh_{nn} - \sum (h_{in})^2.$$

According to Lemma 1.2, we have

$$\begin{aligned} \mathrm{Ric}(v, v) \geqq [&n^2(n-1)c - n(n-1)h_2 - (n-2)|h|\{(n-1)(nh_2 - h^2)\}^{1/2} \\ &+ 2(n-1)h^2]/n^2, \end{aligned} \tag{2.3}$$

which yields that the Ricci curvature is bounded from below. This means that Theorem 1.1 due to Omori and Yau can be applied to the function F. Given any positive number ε there exists a point p in M, at which F satisfies (1.14). It follows from these properties that we have

$$\Delta f^2(p) < \varepsilon^2 + \varepsilon F(p) \tag{2.4}$$

by a direct calculation. When ε tends to 0, the right hand side converges to 0, because the function F is bounded. For a convergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ $(m \to \infty)$ there exists a point sequence $\{p_m\}$ so that the sequence $\{F(p_m)\}$ converges to F_0, by taking a subsequence, if necessary. From the definition of the supremum we have $F_0 = \sup F$ and hence the definition of F gives rise to

$$f(p_m) \longrightarrow f_0 = \sup f.$$

(2.1) and (2.4) imply

$$\begin{aligned} f^2(p_m)[nc + h^2/n - (n-2)|h|f(p_m)\{n(n-1)\}^{-1/2} - f^2(p_m)] \\ < (1/2)\Delta f^2(p_m) < \varepsilon_m^2 + \varepsilon_m F(p_m), \end{aligned}$$

from which it follows that

$$f_0^2[nc + h^2/n - (n-2)|h|f_0\{n(n-1)\}^{-1/2} - f_0^2] \leqq 0,$$

as m tends to ∞. By this inequality we have

$$f_0 = 0 \text{ or } f_0 \geqq [n\{h^2 + 4(n-1)c\}^{1/2} - (n-2)|h|]/2\{n(n-1)\}^{1/2}.$$

Under the assumption (2.2) of Theorem 2.1, the restriction above of the supremum of f yields that $f_0 = 0$, which implies that f vanishes identically on M and hence M is totally umbilical. q.e.d.

REMARK 2.1. In the case of $n \geqq 3$, the estimate of the square of the length of the second fundamental form in Theorem 2.1 is better than that of Hasanis [7], provided that c is positive. In fact, we have $\{c + h^2/2(n-1)\}^2 > h^2\{h^2 + 4(n-1)c\}/4(n-1)^2$, and hence

$$\begin{aligned}&[n\{2(n-1)c + h^2\} - (n-2)|h|\{h^2 + 4(n-1)c\}^{1/2}]/2(n-1) - \{2c + h^2/(n-1)\}\\&= (n-2)[\{2(n-1)c + h^2\} - |h|\{h^2 + 4(n-1)c\}^{1/2}]/2(n-1)\\&> 0.\end{aligned}$$

Thus Theorem 2.1 is a generalization of Hasanis' theorem. In his proof the necessity of the restriction $n \geqq 3$ of the dimension should be noticed.

REMARK 2.2. (1) In the case where the ambient space is flat, (2.2) is equivalent to $\sup S^2 < h^2/(n-1)$. This shows that Theorem 2.1 is a generalization of Okumura's theorem [11], in which tha fact is proved when M is compact or when S is constant. Moreover, the estimate is best possible, because the complete hypersurface $M = S^{n-1} \times \boldsymbol{R}$ of $M' = \boldsymbol{R}^{n+1}$ is not umbilical and it satisfies $S^2 = h^2/(n-1)$. (2) In the case where $c > 0$ and $n = 2$, the inequality (2.2) is equivalent to $\sup S^2 < 2c + h^2$, which means that the Gauss curvature is positive. Accordingly, Theorem 2.1 is a generalization of the well known classical theorem.

COROLLARY 2.2. *Under the assumption of Theorem* 2.1, *M is compact, if c is positive.*

PROOF. According to Theorem 2.1, M is totally umbilical. Hence we have $S^2 = h^2/n$. From (2.3) it follows that for any unit vector v at any point x in M, we have

$$\mathrm{Ric}(v, v) \geqq (n-1)c.$$

This means that M is compact by the theorem due to Myers.

3. Complete space-like hypersurfaces

Let M' be an $(n+1)$-dimensional Lorentz space form of constant curvature

c and let M be a space-like hypersurface with constant mean curvature of M'. For the shape operator A we define a symmetric linear transformation P by $P = A - hI/n$, where I denotes the identity transformation. Then we have

$$\mathrm{Tr}\,P = 0, \tag{3.1}$$

$$\mathrm{Tr}\,A^2 = \mathrm{Tr}\,P^2 + h^2/n, \tag{3.2}$$

$$\mathrm{Tr}\,A^3 = \mathrm{Tr}\,P^3 + (3h/n)\mathrm{Tr}\,P^2 + h^3/n^2. \tag{3.3}$$

Now, a non-negative function f is defined by $f^2 = \mathrm{Tr}\,P^2$, i.e., $f^2 = -h_2 - h^2/n$. By virtue of (1.13), we get

$$(1/2)\Delta f^2 \geqq ncf^2 + hh_3 + (f^2 + h^2/n)^2.$$

Substituting (3.2) and (3.3) into the above equation and using the results that $\mathrm{Tr}\,A^2 = -h_2$ and $\mathrm{Tr}\,A^3 = -h_3$, we get

$$(1/2)\Delta f^2 \geqq f^2(nc - h^2/n + f^2) - h\mathrm{Tr}\,P^3. \tag{3.4}$$

Let $a_1, \ldots, a_n$ be real numbers satisfying $\sum a_j = 0$ and $\sum a_j^2 = k^2 (k > 0)$. Then it is seen that we have

$$|\sum a_j^3| \leqq (n-2)\{n(n-1)\}^{-1/2}k^3,$$

cf. Okumura [11, Lemma 2.1].

Since the symmetric linear transformation P satisfies (3.1), the above property can be applied to the eigenvalues of P and hence we have

$$|\mathrm{Tr}\,P^3| \leqq (n-2)\{n(n-1)\}^{-1/2}f^3,$$

from which together with (3.4) it follows that

$$(1/2)\Delta f^2 \geqq f^2[f^2 - (n-2)\{n(n-1)\}^{-1/2}|h|f + (nc - h^2/n)]. \tag{3.5}$$

By S the norm of the second fundamental form is denoted, that is, we put $S = (-h_2)^{1/2} = (\sum h_{ij}h_{ij})^{1/2}$. Making use of this inequality, one finds the following

THEOREM 3.1. *Let M be a complete space-like hypersurface with constant mean curvature of a Lorentz space form $M_1^{n+1}(c)$, $c \leqq 0$. Then the norm S satisfies*

$$h^2/n \leqq S^2 \leqq [n\{h^2 - 2(n-1)c\} + (n-2)|h|\{h^2 - 4(n-1)c\}^{1/2}]/2(n-1). \tag{3.6}$$

PROOF. Given any positive number a, a function F is also defined by $-(f^2 + a)^{-1/2}$. Since M is space-like, the Ricci tensor R_{ij} is given by

$$R_{ij} = (n-1)c\delta_{ij} - hh_{ij} - (h_{ij})^2$$

by (1.6). Let $\lambda_1,\dots,\lambda_n$ be principal curvatures of M. Then the Ricci tensor becomes

$$R_{ij} = \{(n-1)c - h\lambda_i + \lambda_i^2\}\delta_{ij},$$

which yields that the Ricci curvature of M is bounded from below. Since the function F is bounded, we can apply Theorem 1.1 to the function F. So, given any positive number ε there exists a point p at which F satisfies the properties (1.14) in Theorem 1.1. Consequently the following relationship

$$(1/2)F(p)^4 \Delta f^2(p) < 3\varepsilon^2 - F(p)\varepsilon \tag{3.7}$$

can be derived by a simple and direct calculation. For a covergent sequence $\{\varepsilon_m\}$ such that $\varepsilon_m \to 0$ $(m \to \infty)$ there exists a point sequence $\{p_m\}$ such that the sequence $\{F(p_m)\}$ converges to F_0, by taking a subsequence, if necessary. From the definition of the supremum we have $F_0 = \sup F$ and hence the definition of F gives rise to $f(p_m) \to f_0 = \sup f$. On the other hand, it follows from (3.7) that we have

$$(1/2)F(p_m)^4 \Delta f^2(p_m) < 3\,\varepsilon_m^2 - F(p_m)\varepsilon_m, \tag{3.8}$$

and the right hand side converges to 0, because the function F is bounded. Accordingly, for any positive number ε (<2) there is a sufficiently large integer m for which we have

$$F(p_m)^4 \Delta f^2(p_m) < \varepsilon.$$

This relationship and (3.5) yield

$$(2-\varepsilon)f(p_m)^4 - 2(n-2)\{n(n-1)\}^{-1/2}|h|f(p_m)^3$$
$$+ 2(nc - h^2/n - \varepsilon a)f(p_m)^2 - \varepsilon a^2 < 0,$$

which implies that $\{f(p_m)\}$ is bounded. Thus the supremum of F satisfies $F_0 \neq 0$ by the definition of F and by (3.8) we have $\limsup_{m\to\infty} \Delta f^2(p_m) \leqq 0$. This means that the supremum f_0 of the function f satisfies

$$f_0^2[f_0^2 - (n-2)\{n(n-1)\}^{-1/2}|h|f_0 + (cn - h^2/n)] \leqq 0. \tag{3.9}$$

Then the second factor of the left hand side can be regarded as the quadratic equation for f_0, and the constant term is non-positive and the discriminant D is also non-negative, because c is non-positive. Consequently, we have

$$0 \leqq f_0 \leqq [(n-2)\{n(n-1)\}^{-1/2}|h| + D^{1/2}]/2.$$

Since the square of the norm S of the second fundamental form is given by $S^2 = -h_2 = f^2 + h^2/n$, we get the conclusion. q.e.d.

Similar to the hypersurfaces of the space form, the fact that $f_0 = 0$ is equivalent to the result that the function f vansihes identically on M, which means that M is totally umbilical. By taking account of the proof above, the following property is proved. This is due to Akutagawa [2] and Ramanathan [14].

COROLLARY 3.2. *Let M be a complete space-like hypersurface with constant mean curvature of a de Sitter space $S_1^{n+1}(c)$. If $n=2$ and $h^2 \leqq 4c$, or if $n \geqq 3$ and $h^2 < 4(n-1)c$, then M is totally umbilical.*

REMARK 3.1. It is seen by Ishihara [8] that a complete maximal space-like submanifold of $M_p^{n+p}(c)$ is totally geodesic, if c is non-negative.

Now, by means of Corollary 3.2, in the case where the ambient space is an $(n+1)$-dimensional de Sitter space $(n \geqq 3)$, the space-like hypersurfaces satisfying $h^2 \geqq 4(n-1)c$ are next investigated.

THEOREM 3.3. *Let M be a complete space-like hypersurface with constant mean curvature of $S_1^{n+1}(c)$, $n \geqq 3$. If $h^2 \leqq n^2 c$ and if S satisfies*

$$\sup S^2 < [n\{h^2 - 2(n-1)c\} - (n-2)|h|\{h^2 - 4(n-1)c\}^{1/2}]/2(n-1), \tag{3.10}$$

then M is totally umbilical.

PROOF. In order to verify this theorem, it suffices to consider the proof in the case of $h^2 \geqq 4(n-1)c$. Then, by the assumption $h^2 \leqq n^2 c$, the inequality (3.9) gives $f_0 = 0$ or

$$[(n-2)\{n(n-1)\}^{-1/2}|h| - D^{1/2}]/2 \leqq f_0 \leqq \{(n-2)\{n(n-1)\}^{-1/2}|h| + D^{1/2}]/2.$$

Suppose that $f_0 > 0$. By the first inequality we have

$$\begin{aligned} f_0^2 &\geqq [\{n^2 - 2n + 2)h^2 - 2n^2(n-1)c\} - n(n-2)|h|\{h^2 - 4(n-1)c\}^{1/2}]/2n(n-1) \\ &\geqq 0, \end{aligned}$$

where the second equality holds if and only if $h^2 = n^2 c$. This is a contradiction to the inequality (3.10), from which it turns out that $f_0 = 0$. It completes the proof. q.e.d.

REMARK 3.2. In [6], Dajczer and Nomizu gave the following totally umbilical space-like hypersurface of a de Sitter space. For an $n(\geqq 3)$-dimensional Euclidean space $\boldsymbol{R}^n$, the isometric immersion $i:\boldsymbol{R}^n \to S_1^{n+1}(1) \subset \boldsymbol{R}_1^{n+2}$ is given by

$$(x_1, \dots, x_n) \longrightarrow ((x_1^2 + \cdots + x_n^2)/2, x_1, \cdots, x_n, 1 - (x_1^2 + \cdots + x_n^2)/2).$$

Then $\boldsymbol{R}^n$ is a complete space-like hypersurface $S_1^{n+1}(1)$ and it is totally umbilical. Moreover, $h = n$ and $S^2 = n$, and the equality in (3.10) holds.

According to the congruence theorem of Abe, Koike and Yamaguchi [1], one finds the following

COROLLARY 3.4. *Let M be a complete simply connected space-like hypersurface of a de Sitter space* $S_1^{n+1}(1)$. *If the mean curvature is equal to* 1 *and if* sup $S^2 \leqq n$, *then M is congruent to the above example.*

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