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Kirillov-Kostant theory and Feynman path integrals on coadjoint orbits of a certain real semisimple Lie group

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0. Introduction

Alekseev, Faddeev and Shatashvili showed in [1] that any irreducible unitary representation of compact groups can be obtained by path integrals. They computed characters of the representations. We showed in [3] that path integrals give unitary operators of the representation which is constructed by Kirillov-Kostant theory for the Heisenberg group, the affine transformation group on the real line, $SL(2, \mathbf{R})$ ($\cong SU(1, 1)$) and $SU(2)$. For the affine transformation group, we took a real polarization, for $SU(2)$ a complex polarization (but computed without Hamiltonians), and for the Heisenberg group and $SL(2, \mathbf{R})$ both a real polarization and a complex polarization. (For a complex polarization of $SL(2, \mathbf{R})$, we realized it as $SU(1, 1)$ and computed without Hamiltonians.)

In [4] we found that, in order to compute the path integrals with nontrivial Hamiltonians for $SU(1, 1)$ and $SU(2)$ to obtain unitary operators realized by Borel-Weil theory, we have to regularize the Hamiltonian functions, and in [5] we extended the results to the case that the maximal compact subgroup K of a connected semisimple Lie group G has equal rank to the complex rank of G.

In this paper we work with a linear connected noncompact semisimple Lie group G and consider real polarizations.

Let $\mathfrak{g}$ be the Lie algebra of G. We fix a Cartan involution θ of $\mathfrak{g}$ and let the corresponding Cartan decomposition [6] be

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} .$$

Let $\mathfrak{a}$ be a maximal abelian subalgebra of $\mathfrak{p}$ and $\mathfrak{m}$ the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. If we fix a notion of positivity for $\mathfrak{a}$-roots, we can let $\mathfrak{n}$ be the nilpotent subalgebra given as the sum of the root spaces for the positive roots.

In this paper, we explicitly compute the path integrals with Hamiltonians for $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$, to give unitary operators of the representation which is constructed by Kirillov-Kostant theory. When we compute the path integral with the Hamiltonian for $Y \in \mathfrak{n}$, we make the following assumption.

Put

$$\mathscr{C}^0\mathfrak{n} = \mathfrak{n} \qquad \text{and} \qquad \mathscr{C}^{i+1}\mathfrak{n} = [\mathfrak{n}, \mathscr{C}^i\mathfrak{n}] .$$

Then

Assumption $$\mathscr{C}^i\mathfrak{n} = \{0\} \qquad \text{if } i \geq 3 .$$

Lie groups which satisfy the above assumption include $SL(n, k)$ ($n = 2$, 3, 4, $k = \boldsymbol{R}$, $\boldsymbol{C}$) and linear connected semisimple Lie groups of real rank one etc..

For $Y \in \bar{\mathfrak{n}} = \theta\mathfrak{n}$, we have not yet succeeded in computing the path integral explicitly, even under the above assumption.

We also show that one can obtain the formal intertwining operator between the representations which are constructed by Kirillov-Kostant theory with two polarizations, by the path integral. This is a generalization of the results in [3] to our G, which we showed for the Heisenberg group and $SL(2, \mathbf{R})$.

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1 Preliminaries

Let G be a linear connected noncompact semisimple Lie group, $\mathfrak{g}$ its Lie algebra. We fix a Cartan involution θ of $\mathfrak{g}$ and denote the Cartan involution of G corresponding to that of $\mathfrak{g}$, also by θ. Let

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

be the corresponding Cartan decomposition [6], B the Killing form on $\mathfrak{g}$. Since B is nondegenerate, the dual space $\mathfrak{g}^*$ of $\mathfrak{g}$ is identified with $\mathfrak{g}$ by

$$\mathfrak{g}^* \ni \nu \leftrightarrow X_\nu \in \mathfrak{g} , \tag{1.1}$$

where

$$B(X_\nu, X) = \nu(X) \qquad \text{for all } X \in \mathfrak{g} .$$

We also use $\langle \nu, X \rangle$ for $\nu(X)$. Note that B is negative definite on $\mathfrak{k}$ and is positive definite on $\mathfrak{p}$. Moreover $\mathfrak{k}$ and $\mathfrak{p}$ are mutually orthogonal with respect to B.

Let $\mathfrak{a} \subset \mathfrak{p}$ be a maximal abelian subspace, Σ the corresponding set of nonzero restricted roots, and $\mathfrak{m}$ the centralizer $Z_{\mathfrak{k}}(\mathfrak{a})$ of $\mathfrak{a}$ in $\mathfrak{k}$. Fix a Weyl chamber in $\mathfrak{a}$ and let Σ^+ denote the corresponding set of positive restricted roots. Then we have the decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \sum_{\alpha \in \Sigma} \mathfrak{g}_\alpha ,$$

where

$$\mathfrak{g}_0 = \mathfrak{a} \oplus \mathfrak{m} \qquad \text{and} \qquad \mathfrak{g}_\alpha = \{X \in \mathfrak{g};\ [H, X] = \alpha(H)X \ \text{ for } \ H \in \mathfrak{a}\} .$$

This decomposition has the following properties:

$$\text{(i)} \quad \theta\mathfrak{g}_\alpha \subset \mathfrak{g}_{-\alpha}, \tag{1.2}$$

$$\text{(ii)} \quad B(\mathfrak{g}_\alpha, \mathfrak{g}_\beta) = 0 \text{ unless } \alpha + \beta = 0. \tag{1.3}$$

Define

$$\mathfrak{n} = \sum_{\alpha \in \Sigma^+} \mathfrak{g}_\alpha$$

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha ,$$

where $m_\alpha = \dim \mathfrak{g}_\alpha$. Note that $\mathfrak{n}$ is a nilpotent Lie algebra.

Let K, A, N be the analytic subgroups corresponding to $\mathfrak{k}$, $\mathfrak{a}$, $\mathfrak{n}$, respectively, and M the centralizer $Z_K(\mathfrak{a})$ of $\mathfrak{a}$ in K. Then $G = KA\overline{N}$, and $NMA\overline{N}$ is an open subset of G whose complement is of lower dimension and has Haar measure 0, where $\overline{N} = \theta N$.

For any element $\nu \in \mathfrak{a}^*$ we denote by H_ν the element of $\mathfrak{a}$ such that

$$B(H, H_\nu) = \nu(H) \qquad \text{for all } H \in \mathfrak{a} . \tag{1.4}$$

We extend any linear form ν on $\mathfrak{a}$ to a linear form on $\mathfrak{g}$ by defining ν to vanish on the orthogonal complement of $\mathfrak{a}$ with respect to the Killing form.

Let λ be an element of $\mathfrak{a}^*$ which corresponds to a regular element of $\mathfrak{a}$ by (1.4). We denote the coadjoint action of G on $\mathfrak{g}^*$ by Ad^*. Then it is easy to see that the isotropy subgroup

$$G_\lambda = \{g \in G;\ \mathrm{Ad}^*(g)\lambda = \lambda\}$$

at λ equals MA, and its Lie algebra $\mathfrak{g}_\lambda$ equals $\mathfrak{m} \oplus \mathfrak{a}$. As a real polarization we take $\mathfrak{s}_- = \mathfrak{m} \oplus \mathfrak{a} \oplus \bar{\mathfrak{n}}$, where $\bar{\mathfrak{n}} = \theta\mathfrak{n}$. Correspondingly, we put $S_- = MA\overline{N}$.

Then the Lie algebra homomorphism

$$-\sqrt{-1}\lambda : \mathfrak{s}_- \to \sqrt{-1}\mathbf{R} , \qquad X_0 + H + X_- \mapsto -\sqrt{-1}\lambda(H)$$

lifts to the unitary character of S_-:

$$S_- \to U(1)\,, \qquad m \exp H\bar{n} \mapsto e^{-\sqrt{-1}\lambda(H)}\,.$$

We define a one-dimensional representation ξ_λ of S_- by

$$\xi_\lambda : S_- \to \mathbf{C}^\times\,, \qquad m \exp H\bar{n} \mapsto e^{-(\sqrt{-1}\lambda+\rho)(H)}\,.$$

Let L_λ be the C^∞-line bundle over G/S_- associated to the one-dimensional representation ξ_λ of S_-. Then we can identify the space of all C^∞-sections of L_λ with

$$C^\infty(L_\lambda) = \{f \in C^\infty(G);\ f(xb) = \xi_\lambda(b)^{-1} f(x),\ x \in G,\ b \in S_-\}\,.$$

For any $f \in C^\infty(L_\lambda)$ we put

$$\|f\|^2 = \int_K |f(k)|^2 dk\,,$$

where dk is a Haar measure on K. Then $\|\cdot\|$ defines a norm on $C^\infty(L_\lambda)$. We define a Hilbert space V_λ to be the completion of $C^\infty(L_\lambda)$ with respect to the norm. For $g \in G$, $f \in C^\infty(L_\lambda)$ and $x \in G$, we define

$$\pi_\lambda(g) f(x) = f(g^{-1}x)\,.$$

Then one can show that π_λ is an isometry of $C^\infty(L_\lambda)$ into itself, hence π_λ can be uniquely extended to a unitary operator on V_λ, which we also denote by π_λ.

For each $\alpha \in \Sigma^+$ we can find nonzero root vectors $E_{\alpha,i} \in \mathfrak{g}_\alpha$ ($i = 1, \dots, m_\alpha$) such that

$$B(E_{\alpha,i}, \theta E_{\alpha,j}) = -\delta_{ij}\,,$$

using (1.2), (1.3) and the fact that $-B(\cdot, \theta\cdot)$ gives a positive definite inner product on $\mathfrak{g}$, where δ_{ij} is Kronecker's delta. We put $E_{-\alpha,i} = -\theta E_{\alpha,i}$. We introduce differentiable coordinates on $\mathfrak{n}$ and $\bar{\mathfrak{n}}$ as follows:

$$\mathbf{R}^m \to \mathfrak{n}\,, \qquad x = (x_{\alpha,i})_{\alpha \in \Sigma^+, i=1,\dots,m_\alpha} \mapsto \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} x_{\alpha,i} E_{\alpha,i}$$

$$\mathbf{R}^m \to \bar{\mathfrak{n}}\,, \qquad y = (y_{\alpha,i})_{\alpha \in \Sigma^+, i=1,\dots,m_\alpha} \mapsto \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} y_{\alpha,i} E_{-\alpha,i}\,,$$

where $m = \dim \mathfrak{n}$, and put

$$n_x = \exp \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} x_{\alpha,i} E_{\alpha,i} \in N \tag{1.5}$$

$$\bar{n}_y = \exp \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} y_{\alpha,i} E_{-\alpha,i} \in \bar{N}\,. \tag{1.6}$$

We define a map L of $C^\infty(L_\lambda)$ into $C^\infty(N)$ by

$$Lf(n) = f(n) \qquad \text{for } f \in C^\infty(L_\lambda). \tag{1.7}$$

The Haar measures, when suitably normalized, satisfy

$$\int_K f(k)\,dk = \int_N f(\kappa(n))e^{2\rho H(n)}dn \qquad \text{for } f \in C^\infty(L_\lambda),$$

where, $\kappa(\cdot)$ and $H(\cdot)$ denote K-component and logarithm of A-component in the decomposition $KA\bar{N}$, respectively.

Then one can show that

$$\|f\|^2 = \|Lf\|^2,$$

where the norm of the right hand side is with respect to the Haar measure dn on N.

Let $\mathscr{H}_\lambda$ be the completion of the image of $C^\infty(L_\lambda)$ by L. Then one can show that L is extended to an isometry of V_λ onto $\mathscr{H}_\lambda$. We define a representation $(U_\lambda, \mathscr{H}_\lambda)$ of G such that the following diagram commutes for any $g \in G$:

$$\begin{array}{ccc} V_\lambda & \xrightarrow{\ L\ } & \mathscr{H}_\lambda \\ {\scriptstyle \pi_\lambda(g)}\big\downarrow & & \big\downarrow{\scriptstyle U_\lambda(g)} \\ V_\lambda & \xrightarrow{\ L\ } & \mathscr{H}_\lambda . \end{array}$$

For $g \in NMA\bar{N}$, we write as

$$g = n(g)m(g)a(g)\bar{n}(g). \tag{1.8}$$

Then

$$U_\lambda(g)F(x) = e^{(\sqrt{-1}\lambda+\rho)\log a(g^{-1}n_x)}F(n(g^{-1}n_x)) \tag{1.9}$$

for $F \in L(C^\infty(L_\lambda))$.

2 Quantization

We retain the notation of § 1. Moreover, for $x = (x_{\alpha,i})_{\alpha\in\Sigma^+, i=1,\dots,m_\alpha}$, we put

$$X = \sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha} x_{\alpha,i}E_{\alpha,i}. \tag{2.1}$$

In this section we compute the differential representation dU_λ of U_λ and quantize the Hamiltonian functions for $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$.

We decompose $\mathrm{Ad}\,(e^{-X})Y$ as

$$\mathrm{Ad}\,(e^{-X})Y = X_+ + X_0 + H + X_- \tag{2.2}$$

with $X_+ \in \mathfrak{n}$, $X_0 \in \mathfrak{m}$, $H \in \mathfrak{a}$ and $X_- \in \bar{\mathfrak{n}}$.

LEMMA 2.1. *For* $Y \in \mathfrak{g}$ *and* $F \in C_c^\infty(N)$, $dU_\lambda(Y)$ *is given by*

$$dU_\lambda(Y)F(x) = -(\sqrt{-1}\langle \lambda, \mathrm{Ad}\,(n_x)^{-1}Y\rangle + \langle \rho, \mathrm{Ad}\,(n_x)^{-1}Y\rangle)F(x) - \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} c_{\alpha,i}\partial_{\alpha,i}F(x)\,, \tag{2.3}$$

where $x = (x_{\alpha,i})$, $n_x = \exp X$, $\partial_{\alpha,i} = \partial/\partial x_{\alpha,i}$ *and*

$$c_{\alpha,i} = B\left(\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}}X_+, E_{-\alpha,i}\right).$$

PROOF. For $F \in C_c^\infty(N)$, let f be the element of $C^\infty(L_\lambda)$ which corresponds to F under the map L of (1.7).

Then we have

$$\begin{aligned} dU_\lambda(Y)F(x) &= \frac{d}{dt}\bigg|_{t=0} f(\exp(-tY)\exp X) \\ &= \frac{d}{dt}\bigg|_{t=0} f(\exp X \exp(-X)\exp(-tY)\exp X) \\ &= \frac{d}{dt}\bigg|_{t=0} f(\exp X \exp(-t\,\mathrm{Ad}\,(e^{-X})Y)) \\ &= \frac{d}{dt}\bigg|_{t=0} \xi_\lambda(\exp tH) f(\exp X \exp(-tX_+)) \\ &= \frac{d}{dt}\bigg|_{t=0} e^{-(\sqrt{-1}\lambda+\rho)tH} f(\exp X \exp(-tX_+)) \\ &= \left\{-(\sqrt{-1}\lambda + \rho)(H) - \sum_{\alpha,i} B\left(\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}}X_+, E_{-\alpha,i}\right)\partial_{\alpha,i}\right\}F(x)\,. \end{aligned}$$

Here we used the Baker-Campbell-Hausdorff (BCH) formula (see, e.g. [12]):

$$\exp X \exp(-tX_+) = \exp\left(X - t\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}}X_+ + O(t^2)\right).$$

Since we extend a linear form ν on $\mathfrak{a}$ to one on $\mathfrak{g}$ so that it vanishes on the orthogonal complement of $\mathfrak{a}$ with respect to the Killing form, we have

$$-(\sqrt{-1}\lambda + \rho)(H) = -(\sqrt{-1}\lambda + \rho)(\mathrm{Ad}(n_x)^{-1}Y)\,.$$

This completes the proof. ■

Define a 1-form φ by

$$\begin{aligned}\varphi &= \langle \lambda, g^{-1}dg\rangle \\ &= \langle \mathrm{Ad}^*(\bar{n}))\lambda, n(g)^{-1}dn(g)\rangle + \langle \lambda, a(g)^{-1}da(g)\rangle\,,\end{aligned}$$

where d is the exterior derivative on G and $\bar{n} = m(g)a(g)\bar{n}(g)(m(g)a(g))^{-1}$. Since the second term is an exact 1-form, we choose

$$\alpha_{\mathfrak{s}_-} = \langle \mathrm{Ad}^*(\bar{n})\lambda, n(g)^{-1}dn(g)\rangle\,.$$

and parametrize $n(g)$ as $n(g) = \exp X$, where X is of the form (2.1). Let

$$p_{\alpha,i} = \alpha_{\mathfrak{s}_-}(\partial_{\alpha,i})$$

i.e. $p_{\alpha,i}$ is the coefficient of $dx_{\alpha,i}$ in $\alpha_{\mathfrak{s}_-} : \alpha_{\mathfrak{s}_-} = \sum_{\alpha,i} p_{\alpha,i}dx_{\alpha,i}$.

LEMMA 2.2. *For each* $\alpha \in \Sigma^+$ *and* i, $p_{\alpha,i}$ *is given by*

$$p_{\alpha,i} = B\left(\frac{e^{\mathrm{ad}\,X} - 1}{\mathrm{ad}\,X}\mathrm{Ad}\,(\bar{n})H_\lambda, E_{\alpha,i}\right).$$

PROOF. Note that $dX = \sum_{\alpha,i} dx_{\alpha,i}E_{\alpha,i}$. Then we have

$$\begin{aligned}n(g)^{-1}dn(g) &= \exp(-X)d\exp X \\ &= \frac{1 - e^{-\mathrm{ad}\,X}}{\mathrm{ad}\,X}dX \\ &= \sum_{\alpha,i}\sum_{j=0}^{\infty}\frac{(-\mathrm{ad}\,X)^j}{(j+1)!}dx_{\alpha,i}E_{\alpha,i}\,.\end{aligned}$$

Since $B(\cdot,\cdot)$ is ad-invariant, now the statement follows. ■

Recall from [3] that, by definition, the Hamiltonian function is given by

$$H(g:Y) = \langle \mathrm{Ad}^*(g)\lambda, Y\rangle\,.$$

THEOREM 2.3. *For* $Y \in \mathfrak{g}$, *we have*

$$H(g:Y) = \langle \lambda, \mathrm{Ad}\,(n_x)^{-1}Y\rangle + \sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha} c_{\alpha,i}p_{\alpha,i}\,, \tag{2.4}$$

where $g \in NMA\bar{N}$, $n(g) = n_x = \exp X$, $c_{\alpha,i}$ *and* $p_{\alpha,i}$ *as above.*

Before proving the theorem, we prepare a lemma from linear algebra.

LEMMA 2.4. *Let* V *be a vector space over* $\mathbf{R}$ *with a nondegenerate bilinear form* $(\cdot,\cdot)$. *If there exists a basis* $\{u_1, \dots, u_s, v_1, \dots, v_s, w_1, \dots, w_t\}$ *of* V *such that*

$$(u_i, u_j) = 0\,, \qquad (u_i, w_j) = 0\,,$$
$$(v_i, v_j) = 0\,, \qquad (v_i, w_j) = 0\,,$$
$$(u_i, v_j) = \delta_{ij}\,, \qquad (w_i, w_j) = \delta_{ij}\,.$$

Then we have

$$(x, y) = \sum_{i=1}^{s} \{(x, u_i)(v_i, y) + (x, v_i)(u_i, y)\} + \sum_{j=1}^{t} (x, w_j)(w_j, y)$$

for any $x,\ y \in V$.

PROOF. This is an easy calculation. ■

We apply the lemma taking $\{E_{\alpha,i}\}$, $\{E_{-\alpha,i}\}$ and $\{H_i\}$ as $\{u_i\}$, $\{v_i\}$ and $\{w_i\}$, respectively, where $\{H_i\}$ is an orthonormal basis of $\mathfrak{m} \oplus \mathfrak{a}$ with respect to the Killing form such that the first $n(=\dim \mathfrak{a})$ vectors are the orthonormal bases of $\mathfrak{a}$.

PROOF OF THEOREM. By the choice of $E_{\alpha,i}$, X_+ can be written as

$$X_+ = \sum_{\alpha,i} B(\mathrm{Ad}\,(n_x)^{-1} Y, E_{-\alpha,i}) E_{\alpha,i}\,.$$

Then we have

$$\begin{aligned} c_{\alpha,i} &= B\left(\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}} X_+, E_{-\alpha,i}\right) \\ &= \sum_{\beta,j} B\left(\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}} B(\mathrm{Ad}\,(n_x)^{-1} Y, E_{-\beta,j}) E_{\beta,j}, E_{-\alpha,i}\right) \\ &= \sum_{\beta,j} B(\mathrm{Ad}\,(n_x)^{-1} Y, E_{-\beta,j}) B\left(\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}} E_{\beta,j}, E_{-\alpha,i}\right). \end{aligned}$$

Therefore, using Lemma 2.4, we obtain

$$\begin{aligned} \sum_{\alpha,i} c_{\alpha,i} p_{\alpha,i} &= \sum_{\alpha,i,\beta,j} B(\mathrm{Ad}\,(n_x)^{-1} Y, E_{-\beta,j}) B\left(\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}} E_{\beta,j}, E_{-\alpha,i}\right) \\ &\quad \times B\left(\frac{e^{\mathrm{ad}\,X} - 1}{\mathrm{ad}\,X} \mathrm{Ad}\,(\bar{n}) H_\lambda, E_{\alpha,i}\right) \\ &= \sum_{\beta,j} B(\mathrm{Ad}\,(n_x)^{-1} Y, E_{-\beta,j}) B\left(\frac{\mathrm{ad}\,X}{1 - e^{-\mathrm{ad}\,X}} E_{\beta,j}, \frac{e^{\mathrm{ad}\,X} - 1}{\mathrm{ad}\,X} \mathrm{Ad}\,(\bar{n}) H_\lambda\right). \end{aligned}$$

Since B is ad-invariant,

$$B\left(\frac{\mathrm{ad}\,X}{1-e^{-\mathrm{ad}\,X}}E_{\beta,j}, \frac{e^{\mathrm{ad}\,X}-1}{\mathrm{ad}\,X}\mathrm{Ad}\,(\bar{n})H_\lambda\right) = B(E_{\beta,j}, \mathrm{Ad}\,(\bar{n})H_\lambda)\,.$$

Thus, again by Lemma 2.4,

$$\begin{aligned}\sum_{\alpha,i} c_{\alpha,i}p_{\alpha,i} = \sum_{\beta,j} \{&B(\mathrm{Ad}\,(n_x)^{-1}Y, E_{-\beta,j})B(E_{\beta,j}, \mathrm{Ad}\,(\bar{n})H_\lambda)\\ &+ B(\mathrm{Ad}\,(n_x)^{-1}Y, E_{-\beta,j})B(E_{-\beta,j}, \mathrm{Ad}\,(\bar{n})H_\lambda)\}\\ &+ \sum_i^n B(\mathrm{Ad}\,(n_x)^{-1}Y, H_i)B(H_i, \mathrm{Ad}\,(\bar{n})H_\lambda)\\ &- \sum_i^n B(\mathrm{Ad}\,(n_x)^{-1}Y, H_i)B(H_i, \mathrm{Ad}\,(\bar{n})H_\lambda).\end{aligned}$$

The first three terms can be rewritten as

$$B(\mathrm{Ad}\,(n_x)^{-1}Y, \mathrm{Ad}\,(\bar{n})H_\lambda) = B(\mathrm{Ad}\,(g)H_\lambda, Y)\,,$$

and the last term is nothing but

$$-B(H_\lambda, \mathrm{Ad}\,(n_x)^{-1}Y)\,.$$

This completes the proof. ■

Now, using Theorem 2.3, we quantize the Hamiltonian function for $Y \in \mathfrak{m}\oplus\mathfrak{a}$ or $\mathfrak{n}$, replacing $x_{\alpha,i}$ and $\sqrt{-1}p_{\alpha,i}$ in $H(g:Y)$ by $x_{\alpha,i}\times$ (multiplication operator) and $\partial_{\alpha,i}$, respectively, (canonical quantization !) and choosing an operator ordering between $x_{\alpha,i}\times$'s and $\partial_{\alpha,i}$'s.

COROLLARY 2.5. *For $Y \in \mathfrak{m}\oplus\mathfrak{a}$ or $\mathfrak{n}$, we define quantized Hamiltonians $\mathbf{H}(Y)$ as follows*:

(i) *For $Y \in \mathfrak{m}\oplus\mathfrak{a}$,*

$$\mathbf{H}(Y) = \langle\lambda, Y\rangle - \frac{\sqrt{-1}}{2}\sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha}\{c_{\alpha,i}\partial_{\alpha,i} + \partial_{\alpha,i}\circ c_{\alpha,i}\}\,;$$

(ii) *For $Y \in \mathfrak{n}$,*

$$\mathbf{H}(Y) = -\sqrt{-1}\sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha}\partial_{\alpha,i}\circ c_{\alpha,i}\,,$$

where $\circ$ denote the composition of operators. Then the quantized Hamiltonian coincides with $\sqrt{-1}dU_\lambda(Y)$.

PROOF. If $Y \in \mathfrak{m}\oplus\mathfrak{a}$, then $c_{\alpha,i}$ is given by $c_{\alpha,i} = \alpha(Y)x_{\alpha,i}$. Thus (i) follows immediately from the fact that

$$\langle\lambda, (\mathrm{ad}\,X)^jY\rangle = 0 \qquad \text{if } j \geq 1\,.$$

(ii) is obvious if one notes that

$$\langle \lambda, (\mathrm{ad}\, X)^j Y \rangle = 0 \qquad \text{if } j \geq 0$$

and that $\partial_{\alpha,i} c_{\alpha,i} = 0$. The latter follows from the fact that, if $x_{\beta,j}$ is in $c_{\alpha,i}$, then

$$\text{height of } \beta < \text{height of } \alpha . \quad \blacksquare$$

REMARK 2.6. If $Y \in \mathfrak{n}$, since $\partial_{\alpha,i} c_{\alpha,i} = 0$, we obtain

$$\begin{aligned} \mathbf{H}(Y) &= -\sqrt{-1} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} c_{\alpha,i} \partial_{\alpha,i} \\ &= -\frac{\sqrt{-1}}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} \{ c_{\alpha,i} \partial_{\alpha,i} + \partial_{\alpha,i} \circ c_{\alpha,i} \} . \end{aligned}$$

But we do not adopt these quantizations in the present paper.

3 Path integrals

In this section we explicitly compute the path integrals with Hamiltonian functions, but only for $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$.

The path integral is, symbolically, given by

$$\int \mathscr{D}(x, p) \exp \left(\sqrt{-1} \int_0^T \gamma^* \alpha_{\mathfrak{s}_-} - H(g : Y) dt \right)$$

for $Y \in \mathfrak{g}$, where γ denotes certain paths in the phase space [3].

Here we divide the time interval $[0, T]$ into N small intervals

$$\left[\frac{k-1}{N} T, \frac{k}{N} T \right] \qquad (k = 1, \dots, N) .$$

On each small interval $\left[\frac{k-1}{N} T, \frac{k}{N} T \right]$, Corollary 2.5 indicates that we should take the following ordering of Hamiltonian functions $H_k(g : Y)$ with $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$.

(i) For $Y \in \mathfrak{m} \oplus \mathfrak{a}$,

$$H_k(g : Y) = \langle \lambda, Y \rangle + \frac{1}{2} \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} (c_{\alpha,i}^k p_{\alpha,i}^{k-1} + p_{\alpha,i}^{k-1} c_{\alpha,i}^{k-1}) ,$$

where $c_{\alpha,i}^k = \alpha(Y) x_{\alpha,i}^k$.

(ii) For $Y \in \mathfrak{n}$,

$$H_k(g : Y) = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} p_{\alpha,i}^{k-1} c_{\alpha,i}^{k-1} ,$$

where

$$c^k_{\alpha,i} = B\left(\frac{\operatorname{ad} X^k}{e^{\operatorname{ad} X^k} - 1} Y, E_{-\alpha,i}\right)$$

and $X^k = \sum_{\alpha,i} x^k_{\alpha,i} E_{\alpha,i}$.

Now the computation of the path integral.

For $x = (x_{\alpha,i})$, $x' = (x'_{\alpha,i})$ given, let $x^0_{\alpha,i} = x_{\alpha,i}$, $x^N_{\alpha,i} = x'_{\alpha,i}$. We put

$$dx^j = \prod_{\alpha\in\Sigma^+}\prod_{i=1}^{m_\alpha} dx^j_{\alpha,i} \qquad \text{and} \qquad dp^j = \frac{1}{(2\pi)^m}\prod_{\alpha\in\Sigma^+}\prod_{i=1}^{m_\alpha} dp^j_{\alpha,i}$$

for brevity, where $m = \dim \mathfrak{n}$, and put $\tau = \dfrac{T}{N}$. Remark that the Haar measure $dx = \prod_{\alpha\in\Sigma^+}\prod_{i=1}^{m_\alpha} dx_{\alpha,i}$ on N equals the Haar measure dn given in § 1, up to constant multiple.

A. Path integral for $Y \in \mathfrak{m} \oplus \mathfrak{a}$

Recall that if $Y \in \mathfrak{m} \oplus \mathfrak{a}$, then $H_k(g : Y)$ is given by

$$H_k(g : Y) = \langle \lambda, Y \rangle + \frac{1}{2}\sum_{\alpha\in\Sigma^+}\sum_{i=1}^{m_\alpha} p^{k-1}_{\alpha,i}(c^k_{\alpha,i} + c^{k-1}_{\alpha,i}),$$

where $c^k_{\alpha,i} = \alpha(Y)x^k_{\alpha,i}$. Then

$$\begin{aligned}
&\int \mathscr{D}(x, p)\exp\left(\sqrt{-1}\int_0^T \gamma^*\alpha_{\mathfrak{s}_-} - H(g : Y)dt\right)\\
&= \lim_{N\to\infty}\int \mathscr{D}(x, p)\exp\left(\sum_{k=1}^N \int_{((k-1)/N)T}^{(k/N)T} \gamma^*\alpha_{\mathfrak{s}_-} - H_k(g : Y)dt\right)\\
&= \lim_{N\to\infty}\int_{\mathbf{R}^m}\int_{\mathbf{R}^m}\prod_{j=1}^{N-1} dx^j \prod_{j=0}^{N-1} dp^j \times \exp\sqrt{-1}\left(\sum_{k=1}^N\sum_{\alpha,i} p^{k-1}_{\alpha,i}(x^k_{\alpha,i} - x^{k-1}_{\alpha,i})\right)\\
&\quad\times \exp\left(-\sqrt{-1}\tau\sum_{k=1}^N\left(\langle\lambda, Y\rangle + \frac{1}{2}\sum_{\alpha,i}\alpha(Y)p^{k-1}_{\alpha,i}(x^k_{\alpha,i} + x^{k-1}_{\alpha,i})\right)\right)\\
&= e^{-\sqrt{-1}\lambda(Y)T}\lim_{N\to\infty}\int_{\mathbf{R}^m}\cdots\int_{\mathbf{R}^m}\prod_{j=1}^{N-1} dx^j\\
&\quad\times\prod_{k=1}^N\prod_{\alpha,i}\delta\left(x^k_{\alpha,i} - x^{k-1}_{\alpha,i} - \frac{\alpha(Y)}{2}\tau(x^k_{\alpha,i} + x^{k-1}_{\alpha,i})\right)\\
&= e^{-\sqrt{-1}\lambda(Y)T}\lim_{N\to\infty}\int_{\mathbf{R}^m}\cdots\int_{\mathbf{R}^m}\prod_{j=1}^{N-1} dx^j\prod_{\alpha,i}\left|1 - \frac{\alpha(Y)}{2}\tau\right|^{-N}
\end{aligned}$$

$$\times \prod_{k=1}^{N} \prod_{\alpha, i} \delta\left(x_{\alpha,i}^{k} - \frac{(1 + \tau\alpha(Y)/2)}{(1 - \tau\alpha(Y)/2} x_{\alpha,i}^{k-1}\right)$$

$$= e^{-\sqrt{-1}\lambda(Y)T} \lim_{N\to\infty} \prod_{\alpha,i} \left|1 - \frac{\alpha(Y)}{2}\tau\right|^{-N} \times \prod_{\alpha\in\Sigma^+} \delta\left(x_{\alpha,i}^{N} - \frac{(1 + \tau\alpha(Y)/2)}{(1 - \tau\alpha(Y)/2)} x_{\alpha,i}^{0}\right)$$

$$= e^{-\sqrt{-1}\lambda(Y)T} \prod_{\alpha,i} e^{\alpha(Y)/2T} \delta(x'_{\alpha,i} - e^{\alpha(Y)T} x_{\alpha,i})$$

$$= e^{-(\sqrt{-1}\lambda+\rho)(Y)T} \prod_{\alpha,i} \delta(e^{-\alpha(Y)T} x'_{\alpha,i} - x_{\alpha,i}),$$

where $\delta(\cdot)$ is the Dirac delta function on $\mathbf{R}$. This is nothing but the kernel function of the unitary operator $U_\lambda(\exp TY)$ given by (1.9) in §1 with $Y \in \mathfrak{m} \oplus \mathfrak{a}$.

B. Path integral for $Y \in \mathfrak{n}$

Recall that if $Y \in \mathfrak{n}$, then $H_k(g : Y)$ is given by

$$H_k(g : Y) = \sum_{\alpha\in\Sigma^+} \sum_{i=1}^{m_\alpha} p_{\alpha,i}^{k-1} c_{\alpha,i}^{k-1},$$

where

$$c_{\alpha,i}^{k} = B\left(\frac{\operatorname{ad} X^k}{e^{\operatorname{ad} X^k} - 1} Y, E_{-\alpha,i}\right)$$

and $X^k = \sum_{\alpha,i} x_{\alpha,i}^k E_{\alpha,i}$. Now we assume that

$$\mathscr{C}^0\mathfrak{n} \supset \mathscr{C}^1\mathfrak{n} \supset \mathscr{C}^2\mathfrak{n} \supset \mathscr{C}^3\mathfrak{n} = \{0\}, \tag{3.1}$$

where $\mathscr{C}^0\mathfrak{n} = \mathfrak{n}$ and $\mathscr{C}^{i+1}\mathfrak{n} = [\mathfrak{n}, \mathscr{C}^i\mathfrak{n}]$. Therefore

$$\frac{\operatorname{ad} X^k}{e^{\operatorname{ad} X^k} - 1} Y = Y - \frac{1}{2}\operatorname{ad} X^k Y + \frac{1}{12}(\operatorname{ad} X^k)^2 Y. \tag{3.2}$$

We prepare a lemma for the computation of the path integral. Let C^k be the right hand side of (3.2).

LEMMA 3.1. *If* $\{X^k\}_{k\geq 0}$ *satisfy*

$$X^k - X^{k-1} - \tau C^{k-1} = 0 \qquad (k \geq 1),$$

then X^k *is given by*

$$X^k = X + \tau k\left(Y - \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]]\right) - \frac{1}{12}\tau^2 k(k-1)[Y, [X, Y]] \tag{3.3}$$

with $X = X^0$.

PROOF. We prove this lemma by induction on k. It is trivial for $k = 0$. Suppose that (3.3) is true for $k \geq 0$. Then we have

$$[X^k, Y] = [X + k\tau Y - k\tau\frac{1}{2}[X, Y], Y]$$
$$= [X, Y] - \frac{1}{2}k\tau[[X, Y], Y],$$

since $[X, [X, Y]]$ and $[Y, [X, Y]]$ are in the center of $\mathfrak{n}$ by (3.1).

Similarly, we can obtain

$$[X^k, [X^k, Y]] = [X + k\tau Y - k\tau\tfrac{1}{2}[X, Y], [X, Y]]$$
$$= [X, [X, Y]] + k\tau[Y, [X, Y]].$$

Hence

$$X^{k+1} = X + \tau k\left(Y - \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]]\right) - \frac{1}{12}\tau^2 k(k-1)[Y, [X, Y]]$$
$$+ \tau Y - \frac{1}{2}\tau\left([X, Y] - \frac{1}{2}k\tau[[X, Y], Y]\right)$$
$$+ \frac{1}{12}\tau([X, [X, Y]] + k\tau[Y, [X, Y]])$$
$$= X + (k+1)\tau\left(Y - \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]]\right)$$
$$- \frac{1}{12}k(k+1)\tau^2[Y, [X, Y]].$$

This shows (3.3) is true for $k + 1$. ■

Now we go into the computation of the path integral.

$$\int \mathscr{D}(x, p) \exp\left(\sqrt{-1}\int_0^T \gamma^*\alpha_{\mathfrak{s}_-} - H(g : Y)dt\right)$$
$$= \lim_{N\to\infty} \int \mathscr{D}(x, p) \exp \sum_{k=1}^{N} \int_{((k-1)/N)T}^{(k/N)T} \gamma^*\alpha_{\mathfrak{s}_-} - H_k(g : Y)dt\Bigg)$$
$$= \lim_{N\to\infty} \int_{\mathbf{R}^m} \cdots \int_{\mathbf{R}^m} \prod_{j=1}^{N-1} dx^j \prod_{j=0}^{N-1} dp^j$$
$$\times \exp\sqrt{-1}\left(\sum_{k=1}^{N}\sum_{\alpha,i} p_{\alpha,i}^{k-1}(x_{\alpha,i}^k - x_{\alpha,i}^{k-1} - \tau c_{\alpha,i}^{k-1})\right)$$
$$= \lim_{N\to\infty} \int_{\mathbf{R}^m} \cdots \int_{\mathbf{R}^m} \prod_{j=1}^{N-1} dx^j \prod_{k=1}^{N}\prod_{\alpha,i} \delta(x_{\alpha,i}^k - x_{\alpha,i}^{k-1} - \tau c_{\alpha,i}^{k-1}).$$

If we integrate in each x^i, we have

$$X^i - X^{i-1} - \tau C^{i-1} = 0 \qquad (i \geq 1).$$

Then by Lemma 3.1, X^N is given by

$$X^N = X + T\left(Y - \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]]\right) - \frac{1}{12}T^2\frac{N(N-1)}{N^2}[Y, [X, Y]].$$

Letting N tend to ∞, we obtain

$$X' = X + T\left(Y - \frac{1}{2}[X, Y] + \frac{1}{12}[X, [X, Y]]\right) - \frac{1}{12}T^2[Y, [X, Y]].$$

Then by BCH formula one can show that

$$\exp X' = \exp TY \exp X,$$

which shows the whole integral is nothing but the kernel function of the unitary operator $U_\lambda(\exp TY)$ ((1.9)) with $Y \in \mathfrak{n}$.

Thus we have obtained the following theorem.

THEOREM 3.2. (i) *For $Y \in \mathfrak{m} \oplus \mathfrak{a}$, taking the ordering of the Hamiltonian function $H(g : Y)$ ($g \in NMA\bar{N}$) described in this section, the path integral with the Hamiltonian gives the kernel function of the operator $U_\lambda(\exp TY)$.*

(ii) *Assume that the length of the central descending series of $\mathfrak{n}$ is ≤ 3 (see (3.1)). Then for $Y \in \mathfrak{n}$, taking the ordering of the Hamiltonian function $H(g : Y)$ ($g \in NMA\bar{N}$) described in this section, the path integral with the Hamiltonian gives the kernel function of the operator $U_\lambda(\exp TY)$.*

REMARK 3.3. We denote by $|x\rangle$ Dirac delta function on $\mathbf{R}^m$ with support $\{x\}$ and put $|p\rangle(\ ) = e^{\sqrt{-1}p\cdot()}$, where $\cdot$ denotes the standard inner product on $\mathbf{R}^m$. Then, under the ordering in this section, $H_k(g : Y)$ with $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$ satisfies

$$\begin{aligned}\langle x^k | dU_\lambda(Y) | x^{k-1}\rangle &= \int_{\mathbf{R}^m} dp^{k-1}\langle x^k | dU_\lambda(Y) | p^{k-1}\rangle\langle p^{k-1} | x^{k-1}\rangle \\ &= -\sqrt{-1}\int_{\mathbf{R}^m} dp^{k-1} e^{\sqrt{-1}p^{k-1}\cdot(x^k - x^{k-1})} H_k(g : Y). \qquad (3.4)\end{aligned}$$

Therefore, if we proceed modulo τ^2, then, without using the assumption on $\mathfrak{n}$ in Theorem 3.2 (ii), we have

$$\begin{aligned}\langle x' | U_\lambda(\exp TY) | x\rangle &= \lim_{N\to\infty} \langle x' | U_\lambda(\exp(T/N)Y) \cdots U_\lambda(\exp(T/N)Y) | x\rangle \\ &= \lim_{N\to\infty} \int_{\mathbf{R}^m} \cdots \int_{\mathbf{R}^m} \prod_{j=1}^{N-1} dx^j \langle x^N | U_\lambda(\exp(T/N)Y) | x^{N-1}\rangle\end{aligned}$$

$$\times \langle x^{N-1}|\cdots|x^1\rangle\langle x^1|U_\lambda(\exp(T/N)Y)|x^0\rangle$$

$$\sim \lim_{N\to\infty}\int_{\mathbf{R}^m}\cdots\int_{\mathbf{R}^m}\prod_{j=1}^{N-1}dx^j\langle x^N|I+\tau dU_\lambda(Y)|x^{N-1}\rangle$$

$$\times \langle x^{N-1}|\cdots|x^1\rangle\langle x^1|I+\tau dU_\lambda(Y)|x^0\rangle,$$

using (3.4),

$$= \lim_{N\to\infty}\int_{\mathbf{R}^m}\cdots\int_{\mathbf{R}^m}\prod_{j=1}^{N-1}dx^j\prod_{j=0}^{N-1}dp^j\prod_{k=1}^{N}e^{\sqrt{-1}p^{k-1}\cdot(x^k-x^{k-1})}$$

$$\times (1-\sqrt{-1}\tau H_k(g:Y))$$

$$\sim \lim_{N\to\infty}\int_{\mathbf{R}^m}\cdots\int_{\mathbf{R}^m}\prod_{j=1}^{N-1}dx^j\prod_{j=0}^{N-1}dp^j$$

$$\times\left(\exp\left(\sqrt{-1}\sum_{k=1}^{N}p^{k-1}\cdot(x^k-x^{k-1})-\tau H_k(g:Y)\right)\right).$$

4 Intertwining Operator

In this section we take another real polarization and show that the formal intertwining operator between the two representations can be obtained from the path integral.

Let λ be the same element of $\mathfrak{a}^*$ as in §1. We take another real polarization $\mathfrak{s}_+ = \mathfrak{m}\oplus\mathfrak{a}\oplus\mathfrak{n}$. Correspondingly, we put $S_+ = MAN$. Then the Lie algebra homomorphism

$$-\sqrt{-1}\lambda:\mathfrak{s}_+\to\sqrt{-1}\mathbf{R}, \qquad X_0+H+X_+\mapsto -\sqrt{-1}\lambda(H)$$

lifts to the unitary character of S_+:

$$S_+\to U(1), \qquad m\exp Hn\mapsto e^{-\sqrt{-1}\lambda(H)}.$$

We define a one-dimensional representation $\tilde{\xi}_\lambda$ of S_+ by

$$\tilde{\xi}_\lambda: S_+\to\mathbf{C}^\times, \qquad m\exp Hn\mapsto e^{(-\sqrt{-1}\lambda+\rho)(H)}.$$

Let $(\mathscr{H}_{\tilde{\lambda}}, U_{\tilde{\lambda}})$ be the unitary representation of G which is constructed from $\tilde{\xi}_\lambda$ as in §1, instead of ξ_λ. Note that $\tilde{F}\in\mathscr{H}_{\tilde{\lambda}}$ is a function on $\bar{N}$, on which we introduced coordinates by (1.6).

For $g\in\bar{N}MAN$, we write as

$$g=\bar{n}'(g)m'(g)a'(g)n'(g) \tag{4.1}$$

and parametrize $\bar{n}'(g)$ as $\bar{n}'(g)=\bar{n}_y=\exp Y$, where Y is of the form

$$Y = \sum_{\alpha \in \Sigma^+} \sum_{i=1}^{m_\alpha} y_{\alpha,i} E_{-\alpha,i} . \tag{4.2}$$

Then for $g \in G$ and $\tilde{F} \in \mathscr{H}_{\tilde{\lambda}}$ the action is

$$U_{\tilde{\lambda}}(g)\tilde{F}(y) = e^{(\sqrt{-1}\lambda - \rho)\log a'(g^{-1}\bar{n}_y)}\tilde{F}(\bar{n}'(g^{-1}\bar{n}_y)) , \tag{4.3}$$

where $y = (y_{\alpha,i})$ and $\bar{n}_y = \exp \sum_{\alpha \in \Sigma^+} y_{\alpha,i} E_{-\alpha,i}$. If we use the parametrization (4.1), then φ is given by

$$\begin{aligned}\varphi &= \langle \lambda, g^{-1}dg \rangle \\ &= \langle \mathrm{Ad}^* (n')\lambda, \bar{n}'(g)^{-1}d\bar{n}'(g) \rangle + \langle \lambda, a'(g)^{-1}da'(g) \rangle ,\end{aligned}$$

where $n' = m'(g)a'(g)n'(g)(m'(g)a'(g))^{-1}$. Since the second term is an exact 1-form, we choose

$$\alpha_{\mathfrak{s}_+} = \langle \mathrm{Ad}^* (n')\lambda, \bar{n}'(g)^{-1}d\bar{n}'(g) \rangle .$$

Fixing $y' = (y'_{\alpha,i})$ and $y = (y_{\alpha,i})$, we can explicitly compute the path integral with Hamiltonian function for $Y \in \mathfrak{m} \oplus \mathfrak{a}$ or $\mathfrak{n}$, in the same way as in §3.

LEMMA 4.1. *For $g \in NMA\bar{N} \cap \bar{N}MAN$, write g in two ways*:

$$\begin{aligned}g &= n(g)\bar{n}m(g)a(g) \\ &= \bar{n}'(g)n'm'(g)a'(g) .\end{aligned}$$

Then we have

$$\alpha_{\mathfrak{s}_-} - \alpha_{\mathfrak{s}_+} = \langle \lambda, a^{-1}da \rangle ,$$

where $a = a(\bar{n}'(g)^{-1}n(g))$.

PROOF. By hypothesis,

$$n'^{-1}\bar{n}'(g)^{-1}n(g)\bar{n} = m'(g)m(g)^{-1}a'(g)a(g)^{-1} .$$

Putting $n_1 = n(\bar{n}'(g)^{-1}n(g))$, $m = m(\bar{n}'(g)^{-1}n(g))$, and $\bar{n}_1 = \bar{n}(\bar{n}'(g)^{-1}n(g))$, then by the uniqueness of the decomposition $NMA\bar{N}$, we obtain

$$n_1 = n' \qquad \text{and} \qquad \bar{n}_1 = \bar{n}^{-1} .$$

Thus

$$\begin{aligned}&\mathrm{Ad}\,(\bar{n})^{-1}(n(g)^{-1}dn(g)) - \mathrm{Ad}\,(n')^{-1}(\bar{n}'(g)^{-1}d\bar{n}'(g)) \\ &\quad = \mathrm{Ad}\,(\bar{n}_1)(n(g)^{-1}dn(g)) - \mathrm{Ad}\,(n_1)^{-1}(\bar{n}'(g)^{-1}d\bar{n}'(g)) .\end{aligned} \tag{4.4}$$

On the other hand, noting that $ma = n_1^{-1}\bar{n}'(g)^{-1}n(g)\bar{n}_1^{-1}$, we have

$$\begin{aligned}\operatorname{Ad}(a)^{-1}(m^{-1}dm) + a^{-1}da = & -\operatorname{Ad}(ma)^{-1}(n_1^{-1}dn_1) \\ & - \operatorname{Ad}(ma)^{-1}\operatorname{Ad}(n_1)^{-1}(\bar{n}'(g)^{-1}d\bar{n}'(g)) \\ & + \operatorname{Ad}(\bar{n}_1)(n(g)^{-1}dn(g)) - d\bar{n}_1\bar{n}_1^{-1}.\end{aligned}$$

Therefore we obtain

$$\begin{aligned}\langle \lambda, a^{-1}da\rangle = & -\langle \lambda, \operatorname{Ad}(ma)^{-1}(n_1^{-1}dn_1)\rangle - \langle \lambda, \operatorname{Ad}(ma)^{-1}\operatorname{Ad}(n_1)^{-1}(\bar{n}'(g)^{-1}d\bar{n}'(g))\rangle \\ & + \langle \lambda, \operatorname{Ad}(\bar{n}_1)(n(g)^{-1}dn(g))\rangle \\ = & -\langle \operatorname{Ad}^*(ma)\lambda, n_1^{-1}dn_1\rangle - \langle \operatorname{Ad}^*(ma)\lambda, \operatorname{Ad}(n_1)^{-1}(\bar{n}'(g)^{-1}d\bar{n}'(g))\rangle \\ & + \langle \lambda, \operatorname{Ad}(\bar{n}_1)(n(g)^{-1}dn(g))\rangle .\end{aligned}$$

Since each element of MA fixes λ, combining this with (4.4), we obtain the result. ■

We parametrize $n(g) = n_x = \exp X$ and $\bar{n}'(g) = \bar{n}_y = \exp Y$, where X (or Y) is of the form (2.1) (or (4.2), respectively), and fix $x' = (x'_{\alpha,i})$, $x = (x_{\alpha,i})$, $y' = (y'_{\alpha,i})$ and $y = (y_{\alpha,i})$.

Then by Lemma 4.1 we have

$$\begin{aligned}&\int_0^T \gamma^*\alpha_{\mathfrak{s}_-} - H(g:Y)dt - \int_0^T \gamma^*\alpha_{\mathfrak{s}_+} - H(g:Y)dt \\ &\qquad = \langle \lambda, \log a(\bar{n}_{y'}^{-1}n_{x'})\rangle - \langle \lambda, \log a(\bar{n}_y^{-1}n_x)\rangle ,\end{aligned}$$

which implies that

$$\begin{aligned}&\int_0^T \gamma^*\alpha_{\mathfrak{s}_+} - H(g:Y)dt - \langle \lambda, \log a(\bar{n}_y^{-1}n_x)\rangle \\ &\qquad = -\langle \lambda, \log a(\bar{n}_{y'}^{-1}n_{x'})\rangle + \int_0^T \gamma^*\alpha_{\mathfrak{s}_-} - H(g:Y)dt . \qquad (4.5)\end{aligned}$$

Suggested by (4.5) we consider an integral operator with kernel function

$$\exp(-\sqrt{-1}\lambda \log a(\bar{n}_y^{-1}n_x)) .$$

Since this operator does not intertwine U_λ with $U_{\tilde{\lambda}}$, however, we modify the kernel function by multiplying $\exp(\rho \log a(\bar{n}_y^{-1}n_x))$. Thus our kernel function becomes

$$\exp((-\sqrt{-1} + \rho)\log a(\bar{n}_y^{-1}n_x)) . \qquad (4.6)$$

This integral operator coincides with the formal intertwining operator $A(S_+ : S_- : 1 : \sqrt{-1}\lambda)$ given in [9][10], as we shall see soon. The integral operator with kernel function (4.6) is not well-defined in the sense that the integral

$$\int_N e^{(-\sqrt{-1}\lambda+\rho)\log a(\bar{n}_y^{-1}n_x)}F(x)\,dx$$

need not converge for $F \in \mathscr{H}_\lambda$. Knapp and Stein showed in [9][10] that if one regularizes the integral suitably, then the regularized operator, $\mathscr{A}(S_+ : S_- : 1 : \sqrt{-1}\lambda)$ in their notation, is a well-defined intertwining operator and is invertible, i.e., the following diagram commutes for all $g \in G$.

$$\begin{array}{ccc} \mathscr{H}_\lambda & \xrightarrow{\mathscr{A}(S_+:S_-:1:\sqrt{-1}\lambda)} & \mathscr{H}_{\bar{\lambda}} \\ {\scriptstyle U_\lambda(g)}\downarrow & & \downarrow{\scriptstyle U_{\bar{\lambda}}(g)} \\ \mathscr{H}_\lambda & \xrightarrow{\mathscr{A}(S_+:S_-:1:\sqrt{-1}\lambda)} & \mathscr{H}_{\bar{\lambda}} \end{array}$$

THEOREM 4.2. *The path integral with the action defined by* (4.5) *provides the formal intertwining operator* $A(S_+ : S_- : 1 : \sqrt{-1}\lambda)$, *where* $A(S_+ : S_- : 1 : \sqrt{-1}\lambda)$ *is given by*

$$A(S_+ : S_- : 1 : \sqrt{-1}\lambda)f(\bar{n}_y) = \int_N f(\bar{n}_y n_x)\,dx \qquad \text{for } f \in V_\lambda$$

when the indicated integrals are convergent.

For this, we need a lemma.

LEMMA 4.3. *For* $g \in G$ *and* $F \in C_c^\infty(N)$, *the following integral formula holds*:

$$\int_N F(n(g^{-1}n))dn = \int_N F(n)e^{2\rho\log a(gn)}dn\,.$$

For the proof, we refer to [6].

PROOF OF THEOREM 4.2. We decompose $\bar{n}_y n_x$ as

$$\bar{n}_y n_x = n(\bar{n}_y n_x)m(\bar{n}_y n_x)a(\bar{n}_y n_x)\bar{n}(\bar{n}_y n_x)\,.$$

Putting $n_\xi = n(\bar{n}_y n_x)$, we have

$$\begin{aligned} n_x &= \bar{n}_y^{-1}n_\xi m(\bar{n}_y n_x)a(\bar{n}_y n_x)\bar{n}(\bar{n}_y n_x) \\ &= n(\bar{n}_y^{-1}n_\xi)m(\bar{n}_y^{-1}n_\xi)a(\bar{n}_y^{-1}n_\xi)\bar{n}(\bar{n}_y^{-1}n_\xi) \times m(\bar{n}_y n_x)a(\bar{n}_y n_x)\bar{n}(\bar{n}_y n_x) \\ &= n(\bar{n}_y^{-1}n_\xi)m(\bar{n}_y^{-1}n_\xi)m(\bar{n}_y n_x)a(\bar{n}_y^{-1}n_\xi)a(\bar{n}_y n_x) \times (\text{a member of } \bar{N})\,. \end{aligned}$$

Therefore, by the uniqueness of the decomposition we have

$$n_x = n(\bar{n}_y^{-1}n_\xi) \qquad \text{and} \qquad a(\bar{n}_y^{-1}n_\xi)a(\bar{n}_y n_x) = 1\,. \tag{4.7}$$

Then we obtain

$$
\begin{aligned}
& A(S_+ : S_- : 1 : \sqrt{-1}\lambda) f(\bar{n}_y) \\
& = \int_N f(\bar{n}_y n_x) dx \\
& = \int_N e^{(\sqrt{-1}\lambda + \rho) \log a(\bar{n}_y n_x)} f(n(\bar{n}_y n_x)) dx \,,
\end{aligned}
$$

using Lemma 4.3 and (4.7),

$$
\begin{aligned}
& = \int_N e^{-(\sqrt{-1}\lambda + \rho) \log a(\bar{n}_y^{-1} n_\xi)} e^{2\rho \log a(\bar{n}_y^{-1} n_\xi)} f(n_\xi) d\xi \,, \\
& = \int_N e^{(-\sqrt{-1}\lambda + \rho) \log a(\bar{n}_y^{-1} n_\xi)} f(n_\xi) d\xi \\
& = \int_N e^{(-\sqrt{-1}\lambda + \rho) \log a(\bar{n}_y^{-1} n_x)} f(n_x) dx \,. \quad \blacksquare
\end{aligned}
$$

We can compute the path integral for $Y \in \bar{\mathfrak{n}}$ using the polarization given in this section in the same way as in §3.

Thus, considering the composition

$$
\mathscr{A}(S_+ : S_- : 1 : \sqrt{-1}\lambda)^{-1} \circ U_{\tilde{\lambda}}(\exp TY) \circ \mathscr{A}(S_+ : S_- : 1 : \sqrt{-1}\lambda) \,,
$$

we can obtain the unitary operators $U_\lambda(\exp TY)$ for $Y \in \bar{\mathfrak{n}}$ by the path integrals.

Finally, we give a direct proof that $A(S_+ : S_- : \sqrt{-1}\lambda)$ is a formal intertwining operator between U_λ and $U_{\tilde{\lambda}}$.

PROOF. Let F be an element of $C_c^\infty(N)$. By (1.9), we have

$$
\begin{aligned}
& (U_{\tilde{\lambda}}(g) \circ A(S_+ : S_- : 1 : \sqrt{-1}\lambda)) F(y) \\
& = \int dx e^{(-\sqrt{-1}\lambda + \rho) \log a(\bar{n}'(g^{-1}\bar{n}_y)^{-1} n_x)} e^{(\sqrt{-1}\lambda - \rho) \log a'(g^{-1}\bar{n}_y)} F(x) \\
& = \int dx e^{(-\sqrt{-1}\lambda + \rho)\{\log a(\bar{n}'(g^{-1}\bar{n}_y)^{-1} n_x) - \log a'(g^{-1}\bar{n}_y)\}} F(x) \,.
\end{aligned}
$$

On the other hand, by (4.3),

$$
\begin{aligned}
& (A(S_+ : S_- : 1 : \sqrt{-1}\lambda) \circ U_\lambda(g)) F(y) \\
& = \int dx e^{(-\sqrt{-1}\lambda + \rho) \log a(\bar{n}_y^{-1} n_x)} e^{(\sqrt{-1}\lambda + \rho) \log a(g^{-1} n_x)} F(x) \\
& = \int dx e^{(-\sqrt{-1}\lambda + \rho)\{\log a(\bar{n}_y^{-1} n_x) - \log a(g^{-1} n_x)\}} e^{2\rho \log a(g^{-1} n_x)} F(x) \,.
\end{aligned}
$$

Here, if we decompose $g^{-1}n_x$ as

$$g^{-1}n_x = n(g^{-1}n_x)m(g^{-1}n_x)a(g^{-1}n_x)\bar{n}(g^{-1}n_x)$$

and write $n_\xi = n(g^{-1}n_x)$, then

$$\begin{aligned} n_x &= gn_\xi m(g^{-1}n_x)a(g^{-1}n_x)\bar{n}(g^{-1}n_x) \\ &= n(gn_\xi)m(gn_\xi)a(gn_\xi)\bar{n}(gn_\xi)m(g^{-1}n_x)a(g^{-1}n_x)\bar{n}(g^{-1}n_x) \\ &= n(gn_\xi)m(gn_\xi)m(g^{-1}n_x)a(gn_\xi)a(g^{-1}n_x) \times (\text{a member of } \bar{N}) . \end{aligned}$$

Therefore by the uniqueness of the decomposition,

$$a(gn_\xi)a(g^{-1}n_x) = 1 \qquad \text{and} \qquad n_x = n(gn_\xi) .$$

Hence, by Lemma 4.3, we have

$$\begin{aligned} &(A(S_+ : S_- : 1 : \sqrt{-1}\lambda) \circ U_\lambda(g))F(y) \\ &= \int d\xi e^{(-\sqrt{-1}\lambda+\rho)\{\log a(\bar{n}_y^{-1}n(gn_\xi))+\log a(gn_\xi)\}}F(\xi) . \\ &= \int dx e^{(-\sqrt{-1}\lambda+\rho)\{\log a(\bar{n}_y^{-1}n(gn_x))+\log a(gn_x)\}}F(x) . \end{aligned}$$

Now the statement follows from Lemma 4.4 below.

LEMMA 4.4. *Using the notation above, we have*

$$\log a(\bar{n}_y^{-1}n(gn_x)) + \log a(gn_x) = \log a(\bar{n}'(g^{-1}\bar{n}_y)^{-1}n_x) - \log a'(g^{-1}\bar{n}_y) .$$

PROOF. For $x, y \in G$, let x^y denote yxy^{-1}. Then, since

$$g^{-1}\bar{n}_y = \bar{n}'(g^{-1}\bar{n}_y)m'(g^{-1}\bar{n}_y)a'(g^{-1}\bar{n}_y)n'(g^{-1}\bar{n}_y),$$

putting $\bar{n}_y^{-1}n(gn_x) = nma\bar{n}$, we have

$$\begin{aligned} \bar{n}'(g^{-1}\bar{n}_y)^{-1}n_x &= m'(g^{-1}\bar{n}_y)a'(g^{-1}\bar{n}_y)n'(g^{-1}\bar{n}_y)\bar{n}_y^{-1}gn_x \\ &= m'(g^{-1}\bar{n}_y)a'(g^{-1}\bar{n}_y)n'(g^{-1}\bar{n}_y)\bar{n}_y^{-1}n(gn_x)m(gn_x)a(gn_x)\bar{n}(gn_x) , \\ &= (n'(g^{-1}\bar{n}_y)n)^{m'(g^{-1}\bar{n}_y)a'(g^{-1}\bar{n}_y)}m'(g^{-1}\bar{n}_y)a'(g^{-1}\bar{n}_y)ma \\ &\quad \times \bar{n}m(gn_x)a(gn_x)\bar{n}(gn_x) \\ &= (n'(g^{-1}\bar{n}_y)n)^{m'(g^{-1}\bar{n}_y)a'(g^{-1}\bar{n}_y)}m'(g^{-1}\bar{n}_y)a'(g^{-1}\bar{n}_y)ma \\ &\quad \times m(gn_x)a(gn_x)\bar{n}^{(m(gn_x)a(gn_x))^{-1}}\bar{n}(gn_x) . \end{aligned}$$

Hence the uniqueness of the decomposition $NMA\bar{N}$ implies that

$$a(\bar{n}'(g^{-1}\bar{n}_y)^{-1}n_x) = a'(g^{-1}\bar{n}_y)a(\bar{n}_y^{-1}n(gn_x))a(gn_x).$$

Now the statement follows immediately. ■

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