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A kinetic approach to nonlinear degenerate parabolic equations

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Introduction

This paper is concerned with the Cauchy problem for a nonlinear degenerate parabolic equation of the form:

$$\text{(M)} \qquad \begin{cases} u_t + \nabla A(u) = \triangle \beta(u) & (x, t) \in \boldsymbol{R}^N \times (0, \infty) \\ u(x, 0) = u_0(x) \end{cases}$$

where ∇ is the spatial nabla, $\triangle$ is the Laplacian in $\boldsymbol{R}^N$; $A(s) \equiv (A^1(s), \ldots, A^N(s))$, $A^i \in C^1(\boldsymbol{R}^1)$, $i = 1, \ldots, N$, and β is a locally Lipschitz continuous and strictly increasing function on $\boldsymbol{R}^1$. In view of the assumptions on β, the derivative β' may have a countable number of zero points, and so the parabolicity of the above equation may degenerate at those points. A broad class of nonlinear degenerate parabolic equations can be written in the form (M) by choosing suitable functions β and A^i, $i = 1, \ldots, N$. Since this equation is a combination of the single conservation laws and the so-called porous medium equations, the degeneracy of $\beta(u)$ may interact with that of $A(u)$, and so the study of the problem (M) is important from not only the theoretical point of view but also from the point of view of the application to various evolution problems. On the other hand, it does not seem to be known except some special cases that the semigroup approach to the problem (M) is considerably effective. Therefore it is important to make an attempt to treat the problem (M) in an operator theoretic fashion and construct semigroup solutions to (M).

In this paper we discuss the Cauchy problem (M) via the so-called kinetic approach for scalar quasilinear first order equations of conservation type which was proposed by Giga and Miyakawa [10]. This approach is suitable for treating nonlinear evolution problem of this type and their result was extended to a more general form of conservation laws in the subsequent paper [11] by Giga, Miyakawa and Oharu. In both of the two papers cited above, the convergence of approximate solutions is obtained through the compactness theorem for functions of bounded variation. On the other hand, Kobayashi [13] constructed the approximate solutions by using the solutions of the linear Boltzmann equation and established their convergence via the nonlinear semigroup theory. Making careful observations of the limiting procedure, Miyakawa [16] derived a linear viscosity term and Kobayashi [14] treated the porous medium equations in the same spirit. It is interesting to note that in each of the above-mentioned typical cases the nonlinear semigroup theory was efficiently applied to obtain the convergence of the approximate solutions. In this paper we show that this approach is also appropriate for constructing physically right weak solutions to the problem under consideration.

Our problem (M) is a linear combination of the quasilinear first-order hyperbolic conservation laws and the porous medium equations. Therefore, in order to apply the kinetic approach to the problem (M), we necessitate formulating the associated linear Boltzmann equation so that the macroscopic quantity (u in our case) is decomposed into microscopic quantity by means of appropriate kernel functions. The substantial obstruction to be encountered is to solve the uniqueness problem for the stationary problem associated to (M). Concerning the uniqueness of solutions to (M), Vol'pert and Hudjaev [19] have established a uniqueness theorem for a more general case by using their theory of functions of bounded variation. However, the concept of their solution does not fit our approach.

Recently, Yin [21] succeeded in proving the uniqueness of the so-called BV solution of (M) which is more natural and convenient than the solutions treated by Vol'pert and Hudjaev. He showed that the set of jump points of BV solutions is negligible with respect to the N-dimensional Hausdorff measure. Appealing this fact, he removed the difficulty which is caused by the discontinuities. Inspired by his results, we find a dissipative operator associated with (M). It turns out that the kinetic approach is applicable to the nonlinear degenerate parabolic equations of the form (M) in a natural way.

Throughout this paper we employ the same notation as in Miyakawa [16]. Further, some of the results stated here have already obtained in the works [10, 11, 12, 13, 14 and 16] and some other results can be easily derived through slight modifications of well-known results, although we shall give a full description of those results for the sake of completeness.

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1. Formulation and statements of the main result

We first outline the kinetic approach to the Cauchy problem (M). In terms of kinetic theory of gases, the motion of gases can be described in two ways: the macroscopic and microscopic approaches. We regard (M) as a model of macroscopic conservation laws in fluid mechanics, while the corresponding microscopic model is the linear Boltzmann equation in the phase space;

$$\text{(B)}\qquad \begin{cases} f_t + \xi \nabla f = 0 & (x, \xi, t) \in \boldsymbol{R}^N \times \boldsymbol{R}^N \times (0, \infty) \\ f(x, \xi, 0) = f_0(x, \xi) & \end{cases}$$

where the unknown function f stands for the density of particles of gas at the position $x \in \boldsymbol{R}^N$ with velocity $\xi \in \boldsymbol{R}^N$ at time $t \geqq 0$. In order to justify this formulation, we regard $u(x, t)$ as a macroscopic quantity and define a microscopic quantity by

$$f(x, \xi, t) = \int_0^{u(x,t)} \chi(\xi - a(s))\, ds\,,$$

where appearing functions χ and a will be introduced in (1.4) and (1.9) below. Then the following relations can be easily verified:

$$u(x, t) = \int f(x, \xi, t)\, d\xi\,, \tag{1.1}$$

$$A(u(x, t)) - A(0) = \int \xi f(x, \xi, t)\, d\xi\,, \tag{1.2}$$

where $\int [\cdots]\, d\xi$ stands for the usual integral of $[\cdots]$ over $\boldsymbol{R}^N$. Therefore, if f satisfies the linear Boltzmann equation (B) at time t, then

$$h^{-1}[u(x, t+h) - u(x, t)] = \int h^{-1}[f(x, \xi, t+h) - f(x, \xi, t)]\, d\xi$$

$$\to -\int \xi \nabla f(x, \xi, t)\, d\xi = -\nabla A(u(x, t)) \qquad \text{as } h \downarrow 0\,.$$

This means that u satisfies the following hyperbolic equation (H) at time t:

$$\text{(H)}\qquad u_t + \nabla A(u) = 0 \qquad (x, t) \in \boldsymbol{R}^N \times (0, \infty)\,.$$

The relation (1.1) means that the macroscopic quantity $u(x, t)$ is decomposed into microscopic one $f(x, \xi, t)$ in the phase space so that the compatibility condition (1.2) may hold and the flux A may be treated in an appropriate way. The above observation suggests that, by means of a suitable decomposition, the integral with respect to the velocity argument ξ of solutions to the problem (B) with initial value obtained through the operations $u \mapsto f$ and $u_0 \mapsto f(\cdot, \xi, 0)$ may approximate the solution of the problem (M). In fact, we can construct such a new decomposition in the following way.

Let χ be a function with the following properties:

$$\chi \in C_0^\infty(\boldsymbol{R}^N), \qquad \chi \geqq 0 \text{ on } \boldsymbol{R}^N, \qquad \operatorname{supp} \chi \subset \{\xi \in \boldsymbol{R}^N; |\xi| \leqq 1\}, \tag{1.3}$$

$$\chi(\xi) = \chi(|\xi|), \qquad \int \chi(\xi)\, d\xi = 1. \tag{1.4}$$

Let

$$F(w, s) = \begin{cases} 1 & (0 < s \leqq w) \\ -1 & (w \leqq s < 0). \\ 0 & (\text{otherwise}) \end{cases} \tag{1.5}$$

We then consider a family of functions $\{\beta_\varepsilon; \varepsilon > 0\}$ contained in $C^1(\boldsymbol{R}^1)$ which gives a regularization of β in the sense that

$$\beta_\varepsilon'(s) > 0 \quad \text{for } s \in \boldsymbol{R}^1, \qquad \beta_\varepsilon \to \beta \quad \text{as } \varepsilon \downarrow 0 \text{ in } L^\infty_{loc}(\boldsymbol{R}^1), \tag{1.6}$$

$$\sup\{\beta_\varepsilon'(s); |s| \leqq r, \varepsilon > 0\} < \infty \qquad \text{for each } r > 0. \tag{1.7}$$

For the above mentioned functions, we define two functions χ_ε and F_ε by

$$\chi_\varepsilon(\xi, s) = \left(\frac{\varepsilon}{\beta_\varepsilon'(s)^{1/2}}\right)^N \chi\left(\frac{\varepsilon}{\beta_\varepsilon'(s)^{1/2}}\xi\right) \qquad \text{for } (\xi, s) \in \boldsymbol{R}^N \times \boldsymbol{R}^1, \tag{1.8}$$

$$F_\varepsilon(w, \xi) = \int F(w, s)\chi_\varepsilon(\xi - a(s), s)\, ds \qquad \text{for } (w, \xi) \in \boldsymbol{R}^1 \times \boldsymbol{R}^N, \tag{1.9}$$

$$\text{where} \qquad a^i = dA^i/ds, \qquad a(s) = (a^1(s), \ldots, a^N(s)).$$

In the case of hyperbolic conservation laws, it is not necessary to employ the functions $\{\chi_\varepsilon; \varepsilon > 0\}$. Then, the following conditions corresponding to (1.1) and (1.2) are easily checked by means of above relations.

$$w = \int F_\varepsilon(w, \xi)\, d\xi \qquad \text{for } w \in \boldsymbol{R}^1, \tag{D}$$

$$A(w) - A(0) = \int \xi F_\varepsilon(w, \xi)\, d\xi \qquad \text{for } w \in \boldsymbol{R}^1. \tag{C}$$

To verify that the family of functions $\{F_\varepsilon; \varepsilon > 0\}$ gives the desired decomposition, we consider solutions of (B) with initial values of the form $F_\varepsilon(u_0(\cdot), \xi)$, $\varepsilon > 0$. Let $\{U_\xi(t); t \geqq 0\}$ be the family of solution operators to the problem (B). We define a family of nonlinear operators $\{S_h; h > 0\}$ by

$$S_h v = \int U_\xi(h) F_\varepsilon(v(\cdot), \xi)\, d\xi \tag{1.10}$$

for $v \in L^1_{loc}(\boldsymbol{R}^N)$ for which the right-hand side of (1.10) makes sense and ε is a positive number satisfying the following stability condition:

$$\frac{h}{2N\varepsilon^2}\int |\xi|^2\chi(\xi)\,d\xi = 1\,. \tag{1.11}$$

For any $v \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, we can show (see the proof of Lemma 4.3 (ii)) that

$$h^{-1}(S_h v - v) \to \Delta\beta(v) - \nabla A(v) \qquad \text{in } \mathscr{D}'(\boldsymbol{R}^N) \text{ as } h \downarrow 0\,. \tag{1.12}$$

Therefore it is expected that for each $h > 0$ the function

$$u_h(t) = S_h^{[t/h]} u_0 \tag{1.13}$$

approximates the solution of (M), where $[s]$ denotes the greatest integer which does not exceed $s \in \boldsymbol{R}^1$. To ensure the convergence of approximate solution u_h, we employ the approximation theorem for nonlinear semigroups due to Brezis and Pazy [8].

In order to treat the problem (M) via the nonlinear semigroup theory, it is necessary to find a dissipative operator B which is a formal expression $Bv = \Delta\beta(v) - \nabla A(v)$ in an appropriate function space. To this end, we formulate two operators B_0 and B. First we define B_0 in the following way:

(1.14) $v \in D(B_0)$, $w \in B_0 v$ if and only if v, $w \in BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ and

$$\int [\beta(v)\Delta\varphi + A(v)\nabla\varphi]\,dx = \int w\varphi\,dx \qquad \text{for } \forall\varphi \in C_0^\infty(\boldsymbol{R}^N)\,.$$

Obviously, B_0 is a singlevalued operator, hence $B_0 v = \Delta\beta(v) - \nabla A(v)$ holds in $\mathscr{D}'(\boldsymbol{R}^N)$ for $v \in D(B_0)$. The second operator B is defined to be the closure of B_0 in $L^1(\boldsymbol{R}^N)$, i.e., $v \in D(B)$, $w \in Bv$ if and only if there exists a sequence $(v_n)_{n=1}^\infty$ in $D(B_0)$ such that $(v_n, B_0 v_n) \to (v, w)$ in $L^1(\boldsymbol{R}^N)^2$ as $n \to \infty$. In Section 4 we show that the operator B is a densely defined m-dissipative operator in $L^1(\boldsymbol{R}^N)$, and so that, by the generation theorem for nonlinear semigroups due to Crandall and Liggett, there exists a nonlinear semigroup $\{T(t); t \geqq 0\}$ on $L^1(\boldsymbol{R}^N)$ in such a way that for each $v \in L^1(\boldsymbol{R}^N)$

$$(I - \lambda B)^{-[t/\lambda]} v \to T(t)v \qquad \text{in } L^1(\boldsymbol{R}^N) \text{ as } \lambda \downarrow 0 \tag{1.15}$$

uniformly for bounded t.

In order to state our result, we introduce two notions of generalized solution to the Cauchy problem (M).

Definition. For $u_0 \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, a function $u \in L^\infty(\boldsymbol{R}^N \times (0, \infty)) \cap C([0, \infty); L^1(\boldsymbol{R}^N))$ is called a *weak solution* of the problem (M) if $u(\cdot, 0) = u_0$, $\nabla\beta(u) \in L^2(0, T; L^2(\boldsymbol{R}^N)^N)$ for all $T > 0$, and the function u satisfies

$$\int_0^\infty \int [u\varphi_t + A(u)\nabla\varphi]\, dxdt = \int_0^\infty \int \nabla\beta(u)\nabla\varphi\, dxdt\,.$$

for each $\varphi \in C_0^\infty(\boldsymbol{R}^N \times (0, \infty))$. If in addition $u \in BV(\boldsymbol{R}^N \times (0, T))$ for each $T > 0$, then u is said to be a *BV solution* of the problem (M).

REMARK. The above notion of BV solution of (M) is a modified version of the notion proposed by Yin [21] in the sense that the BV solution of Yin belongs to $C([0, \infty); L^1(\boldsymbol{R}^N))$. Since the BV solution in the sense of Yin is unique, it is obvious that our BV solution is also unique.

Our main result in this paper can be now stated as follows.

THEOREM. *Let χ and $\{\beta_\varepsilon; \varepsilon > 0\}$ be any functions satisfying* (1.3), (1.4), (1.6) *and* (1.7). *Suppose that the numbers $h > 0$ and $\varepsilon > 0$ satisfy the stability condition* (1.11). *Then, we have*:

(a) *For each $v \in L^1(\boldsymbol{R}^N)$,*

$$S_h^{[t/h]}v \to T(t)v \qquad \text{in } L^1(\boldsymbol{R}^N) \text{ as } h \downarrow 0 \tag{1.16}$$

where the convergence is uniform for bounded t.

(b) *If $u_0 \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, the function $u(x, t) = [T(t)u_0](x)$ gives a weak solution of the problem* (M).

(c) *If $u_0 \in BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N) \cap \hat{D}(B)$, where $\hat{D}(B)$ denotes the generalized domain of the m-dissipative operator B, the function $u(x, t) = [T(t)u_0](x)$ gives the unique BV solution of the problem* (M).

2. Basic properties of the operators S_h

First we recall the approximation theorem for nonlinear semigroups due to Brezis and Pazy [8] and state it for our convenience.

THEOREM 2.1. *Let X be a real Banach space and $\{C_h; h > 0\}$ a given family of contraction operators on X. If the limit*

$$J(\lambda)v \equiv \lim_{h\downarrow 0} [I - \lambda h^{-1}(C_h - I)]^{-1}v$$

exists for all $v \in X$ and all $\lambda > 0$. Then there exists a m-dissipative operator A in X such that $J(\lambda) = (I - \lambda A)^{-1}$ for all $\lambda > 0$, and

$$C_h^{[t/h]}v \to T(t)v \qquad \text{as } h \downarrow 0 \text{ uniformly for bounded } t$$

for each $v \in \overline{D(A)}$, where $\{T(t); t \geqq 0\}$ is the nonlinear semigroup generated in X by A.

We wish to apply the above theorem to the case where the Banach space X is $L^1(\boldsymbol{R}^N)$ and the operators C_h are S_h. To do this, we prepare some basic estimates concerning the operators S_h and the resolvents $[I - \lambda h^{-1}(S_h - I)]^{-1}$. Here and hereafter, $\|\cdot\|_p$ denotes the usual norm of the Lebesgue space $L^p(\boldsymbol{R}^N)$, $1 \leqq p \leqq \infty$. We represent the duality pairing between the Lebesgue spaces $L^p(\boldsymbol{R}^N)$ and $L^q(\boldsymbol{R}^N)$, $1/p + 1/q = 1$, by $\langle , \rangle$. The same bracket $\langle , \rangle$ will also be used for the case of vector-valued Lebesgue spaces.

LEMMA 2.2. *Let τ_y, $y \in \boldsymbol{R}^N$, be the translation operators defined by $[\tau_y v](x) \equiv v(x + y)$.*

(i) *S_h is a contraction operator on $L^1(\boldsymbol{R}^N)$ and*

$$\|S_h v\|_1 \leqq \|v\|_1 \qquad \textit{for } v \in L^1(\boldsymbol{R}^N).$$

(ii) *$\tau_y S_h = S_h \tau_y$ for $y \in \boldsymbol{R}^N$.*

(iii) *$\|S_h v\|_\infty \leqq \|v\|_\infty$ for $v \in L^\infty(\boldsymbol{R}^N)$.*

PROOF. We first recall the basic properties of the function F defined in (1.5):

$$F(\cdot, s) \text{ is nondecreasing for a fixed } s \in \boldsymbol{R}^1, \tag{2.1}$$

$$F(0, s) = 0, \qquad \operatorname{supp} F(w, \cdot) \subset \{s \in \boldsymbol{R}^1; |s| \leqq |w|\}, \tag{2.2}$$

$$\int f(s)[F(v, s) - F(w, s)]\, ds = \int_w^v f(s)\, ds \qquad \text{for } f \in L^1_{loc}(\boldsymbol{R}^1),\ v,\ w \in \boldsymbol{R}^1. \tag{2.3}$$

In particular, $w = \int F(w, s)\, ds$, $|w| = \int |F(w, s)|\, ds$ and

$$|v - w| = \int |F(v, s) - F(w, s)|\, ds. \tag{2.4}$$

See Kobayashi [12] for the detailed argument. We employ these facts.

(i) For each v, $w \in L^1(\boldsymbol{R}^N)$,

$$\begin{aligned}\int |S_h v - S_h w|\, dx &\leqq \int dx \int |F_\varepsilon(v(\cdot - h\xi), \xi) - F_\varepsilon(w(\cdot - h\xi), \xi)|\, d\xi \\ &= \int d\xi \int |F_\varepsilon(v(\cdot - h\xi), \xi) - F_\varepsilon(w(\cdot - h\xi), \xi)|\, dx \\ &= \int d\xi \int |F_\varepsilon(v, \xi) - F_\varepsilon(w, \xi)|\, dx = \int |v - w|\, dx.\end{aligned}$$

In the derivation of the extreme right-hand side of the above equality we have used the following identity which is a direct consequence of (2.4):

$$|v - w| = \int |F_\varepsilon(v, \xi) - F_\varepsilon(w, \xi)|\, d\xi \qquad \text{for } v,\ w \in \boldsymbol{R}^1 .$$

Since $S_h 0 = 0$ by definition, we obtain $\|S_h v\|_1 \leqq \|v\|_1$ by putting $w = 0$ in the above estimate.

(ii)
$$[\tau_y S_h v](x) = \left(\int F_\varepsilon(v(\cdot - h\xi), \xi)\, d\xi \right)(x + y)$$
$$= \int F_\varepsilon(v(x + y - h\xi), \xi)\, d\xi$$
$$= \int F_\varepsilon([\tau_y v](x - h\xi), \xi)\, d\xi = [S_h \tau_y v](x) .$$

(iii) Since $F_\varepsilon(\cdot, \xi)$ is nondecreasing function for fixed $\xi \in \boldsymbol{R}^N$ by (2.1),

$$F_\varepsilon(-\|v\|_\infty, \xi) \leqq F_\varepsilon(v(x - h\xi), \xi) \leqq F_\varepsilon(\|v\|_\infty, \xi) \qquad \text{a.e. } x \in \boldsymbol{R}^N .$$

Integrating the resultant inequality with respect to ξ over $\boldsymbol{R}^N$, we obtain

$$-\|v\|_\infty \leqq [S_h v](x) \leqq \|v\|_\infty \qquad \text{a.e. } x \in \boldsymbol{R}^N. \quad \blacksquare$$

In what follows, we employ the nonlinear operator B_h on $L^1(\boldsymbol{R}^N)$ defined by

$$B_h = h^{-1}(S_h - I) \qquad \text{for } h > 0 . \tag{2.5}$$

For a fixed h, $\lambda > 0$ and $v \in L^1(\boldsymbol{R}^N)$, the operator

$$\mathscr{F}(v)w \equiv h(h + \lambda)^{-1} v + \lambda(h + \lambda)^{-1} S_h w \qquad \text{for } w \in L^1(\boldsymbol{R}^N) ,$$

defines a strict contraction operator on $L^1(\boldsymbol{R}^N)$ and the subspace $L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ is invariant under $\mathscr{F}(v)$ by Lemma 2.2, the contraction mapping principle asserts that $\mathscr{F}(v)$ admits a unique fixed point in $L^1(\boldsymbol{R}^N)$. Thus the resolvent $(I - \lambda B_h)^{-1}$ of B_h exists and the following lemma follows directly from Lemma 2.2.

LEMMA 2.3. (i) *$(I - \lambda B_h)^{-1}$ is a contraction operator on $L^1(\boldsymbol{R}^N)$ and $\|(I - \lambda B_h)^{-1} v\|_1 \leqq \|v\|_1$ for $v \in L^1(\boldsymbol{R}^N)$.*
(ii) *$\tau_y (I - \lambda B_h)^{-1} = (I - \lambda B_h)^{-1} \tau_y$ for $y \in \boldsymbol{R}^N$.*
(iii) *$\|(I - \lambda B_h)^{-1} v\|_\infty \leqq \|v\|_\infty$ for $v \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$.*

The following lemma plays a central role in ensuring the precompactness in $L^1(\boldsymbol{R}^N)$ of various families of functions which will appear in our subsequent discussions.

LEMMA 2.4. *Let $v \in L^\infty(\boldsymbol{R}^N)$, $k \in \boldsymbol{R}^1$ and $\varphi \in C_0^\infty(\boldsymbol{R}^N)$ with $\varphi \geqq 0$. Then*

(2.6) $$\langle \operatorname{sgn}(v-k)B_h v, \varphi\rangle \leqq \int \langle \operatorname{sgn}(v-k)[F_\varepsilon(v,\xi)-F_\varepsilon(k,\xi)], h^{-1}[U_\xi(h)^* - I]\varphi\rangle\, d\xi\,,$$

where $U_\xi(h)^*$ *is the dual operators in* $L^\infty(\boldsymbol{R}^N)$ *of* $U_\xi(h)$.

PROOF. First we note that $B_h k = 0$ by the definition of S_h. Therefore

(2.7) $$\langle \operatorname{sgn}(v-k)B_h v, \varphi\rangle = \langle \operatorname{sgn}(v-k)[B_h v - B_h k], \varphi\rangle = \int \langle [F_\varepsilon(v,\xi)-F_\varepsilon(k,\xi)], h^{-1}[U_\xi(h)^* - I][\operatorname{sgn}(v-k)\varphi]\rangle\, d\xi\,.$$

Since $|\operatorname{sgn}(v-k)\varphi| \leqq \varphi$, the order-preserving property of the operators $U_\xi(h)^*$ implies that $U_\xi(h)^*[\operatorname{sgn}(v-k)\varphi] \leqq U_\xi(h)^*\varphi$. Using this, the extreme right-hand side of (2.7) does not exceed

$$h^{-1}\int \langle |F_\varepsilon(v,\xi)-F_\varepsilon(k,\xi)|, U_\xi(h)^*\varphi\rangle\, d\xi - h^{-1}\int \langle [F_\varepsilon(v,\xi)-F_\varepsilon(k,\xi)], \operatorname{sgn}(v-k)\varphi\rangle\, d\xi\,.$$

On the other hand, $F_\varepsilon(w,\xi)$ is nondecreasing in w and $|F_\varepsilon(v,\xi)-F_\varepsilon(k,\xi)| = \operatorname{sgn}(v-k)[F_\varepsilon(v,\xi)-F_\varepsilon(k,\xi)]$, we obtain the desired estimate (2.6). ∎

3. Dissipative operator B_0

In order to investigate the properties of the operator B_0 defined by (1.14), we prepare some fundamental facts on the functions of bounded variation. For the basic notions and results we refer to [1], [2], [20] and [22].

We denote by $|E|$ the Lebesgue measure of a Lebesgue measurable set $E \subset \boldsymbol{R}^N$ and by $\mathscr{H}^{N-1}(E)$ the $(N-1)$-dimensional Hausdorff measure of a Borel set $E \subset \boldsymbol{R}^N$. We define an extended real valued seminorm $\|D\cdot\|_M$ on $L^1(\boldsymbol{R}^N)$ by

(3.1)
$$\|Dv\|_M = \sup\left\{\int v \operatorname{div}\varphi\, dx;\ \varphi = (\varphi^1,\dots,\varphi^N) \in C_0^\infty(\boldsymbol{R}^N;\boldsymbol{R}^N),\ \max_i \|\varphi^i\|_\infty \leqq 1\right\}.$$

We say that $v \in L^1(\boldsymbol{R}^N)$ is a *function of bounded variation* in $\boldsymbol{R}^N$ if the quantity $\|Dv\|_M$ is finite. Notice that for $v \in W^{1,1}(\boldsymbol{R}^N)$ the left-hand side of (3.1) coincides with $\sum_{i=1}^N \|\partial v/\partial x_i\|_1$. We denote by $BV(\boldsymbol{R}^N)$ the space of all functions of bounded variation in $\boldsymbol{R}^N$. The space $BV(\boldsymbol{R}^N)$ is a Banach space with respect to the norm $\|v\|_{BV} \equiv \|v\|_1 + \|Dv\|_M$. It is easily seen that for $v \in BV(\mathbf{R}^N)$ the distributional derivatives $D_i v$, $i = 1, \ldots, N$, are regarded as elements of the space $M(\mathbf{R}^N)$ of Radon measures on $\mathbf{R}^N$ with finite total variations. We denote by $|Dv|$ the total variation of the vector-valued Radon measure $Dv = (D_1 v, \ldots, D_N v)$. The total variation $|Dv|$ itself defines a measure and for every open set $\Omega \subset \mathbf{R}^N$ the integral of the measure $|Dv|$ over Ω is given by the right-hand side of (3.1) with $\varphi \in C_0^\infty(\mathbf{R}^N; \mathbf{R}^N)$ replaced by $\varphi \in C_0^\infty(\Omega; \mathbf{R}^N)$.

In what follows, we denote by $\int_E f D_i v$ the integral on a Borel set $E \subset \mathbf{R}^N$ of a integrable function $f: E \to \mathbf{R}^1$ with respect to the Radon measure $D_i v$. In this case, we define the vector-valued Radon measure $fDv = (fD_1 v, \ldots, fD_N v)$ by

$$(3.2) \qquad [fD_i v](E) = \int_E f D_i v \qquad \text{for Borel set } E \subset \mathbf{R}^N .$$

We also write $\int_E f Dv = \sum_{i=1}^N \int_E f^i D_i v$ for a vector-valued integrable function $f: E \to \mathbf{R}^N$, $f = (f^1, \ldots, f^N)$.

Let $E \subset \mathbf{R}^N$ be a Lebesgue measurable set. A unit vector n is called a *measure-theoretic normal* to E at a point x ([22], Definition 5.6.4), if

$$\lim_{r \downarrow 0} r^{-N} |B(x, r) \cap \{y; (y - x)\cdot n < 0, y \in E^C\}| = 0$$

and

$$\lim_{r \downarrow 0} r^{-N} |B(x, r) \cap \{y; (y - x)\cdot n > 0, y \in E\}| = 0 .$$

The measure-theoretic normal to E at x is uniquely determined for x and E, and is denoted by $n(x, E)$. We write

$$\partial^* E = \{x \in \mathbf{R}^N; n(x, E) \text{ exists}\} .$$

For $v \in BV(\mathbf{R}^N)$, the Fleming-Rishel coarea formula holds as follows:

$$(3.3) \qquad |Dv|(E) = \int_{-\infty}^{\infty} \mathcal{H}^{N-1}(E \cap \partial^*\{v > t\})\, dt$$

for each Borel set $E \subset \mathbf{R}^N$. As a simple consequence of formula (3.3) we have

$$(3.4) \qquad |Dv|(E) = 0 \qquad \text{for all Borel set } E \subset \mathbf{R}^N \text{ with } \mathcal{H}^{N-1}(E) = 0 .$$

For a Lebesgue measurable set $E \subset \mathbf{R}^N$, the *upper* and *lower density* of E at a point x is defined respectively by

$$\overline{D}(E, x) = \limsup_{r \downarrow 0} |E \cap B(x, r)|/|B(x, r)| ,$$

$$\underline{D}(E, x) = \liminf_{r \downarrow 0} |E \cap B(x, r)|/|B(x, r)| .$$

In the case that the upper and lower densities are equal, we define the *density* of E at a point x, denoted by $D(E, x)$, by their common value.

Let v be a Borel function defined on $\boldsymbol{R}^N$. The *upper* and *lower approximate limit* of v at a point x is defined ([22], Definition 5.9.1) by

$$v^+(x) = \inf\{t \in [-\infty, \infty]; D(A_t, x) = 0\} \qquad \text{where } A_t = \{y \in \boldsymbol{R}^N; v(y) > t\},$$

$$v^-(x) = \sup\{t \in [-\infty, \infty]; D(B_t, x) = 0\} \qquad \text{where } B_t = \{y \in \boldsymbol{R}^N; v(y) < t\},$$

respectively. Notice that v^+ and v^- are both Borel functions, and that $v^- \leqq v^+$.

A point $x \in \boldsymbol{R}^N$ is called a *regular point* of v if $v^+(x)$ and $v^-(x)$ are finite, and the set of all points of jump of v is denoted by Γ_v, that is,

$$\Gamma_v = \{x \in \boldsymbol{R}^N; v^-(x) < v^+(x)\}.$$

The set Γ_v is a Borel set and negligible with respect to the Lebesgue measure.

For a $v \in BV(\boldsymbol{R}^N)$, it is known ([22], Theorem 5.9.6) that

(i) $-\infty < v^-(x) \leqq v^+(x) < \infty$ for $\mathscr{H}^{N-1}$-a.e. $x \in \boldsymbol{R}^N$.

(ii) Γ_v is countably $(N-1)$-rectifiable, i.e., there exists a sequence of compact sets $(K_n)_{n=1}^\infty$ and a set J with $\mathscr{H}^{N-1}(J) = 0$ such that

$$\Gamma_v = (\textstyle\bigcup_{n=1}^\infty K_n) \cup J$$

and each set K_n is contained in a C^1-hypersurface.

(iii) For $\mathscr{H}^{N-1}$-a.e. $x \in \Gamma_v$, the measure-theoretic normal $n(x, A_s)$ to A_s at x exists whenever $v^-(x) < s < v^+(x)$, and $n(x, A_s)$ does not depend on the choice of s. In what follows, $-n(x, A_s)$ is denoted by $\nu_v(x)$ for such $x \in \Gamma_v$. We write ν_v^i for the i^{th} component of ν_v. Therefore $\nu_v = (\nu_v^1, \dots, \nu_v^N)$.

For each $x \in \Gamma_v$ at which the assertion (iii) holds, it is known that

$$\lim_{r\downarrow 0} |B^+(x, r)|^{-1} \int_{B^+(x,r)} |v(y) - v^+(x)|\, dy = 0,$$

$$\lim_{r\downarrow 0} |B^-(x, r)|^{-1} \int_{B^-(x,r)} |v(y) - v^-(x)|\, dy = 0,$$

where

$$B^+(x, r) = \{y \in B(x, r); (y - x)\cdot \nu_v(x) > 0\},$$

$$B^-(x, r) = \{y \in B(x, r); (y - x)\cdot \nu_v(x) < 0\}.$$

As shown in [20] the restrictions of the measure Dv and its total variation to Γ_v admit the following representations: For each Borel set $E \subset \boldsymbol{R}^N$, we have

$$Dv(E \cap \Gamma_v) = \int_{E\cap\Gamma_v} [v^+ - v^-]\nu_v \, d\mathscr{H}^{N-1}, \tag{3.5}$$

(3.6) $$|Dv|(E \cap \Gamma_v) = \int_{E \cap \Gamma_v} [v^+ - v^-] \, d\mathscr{H}^{N-1} .$$

Further, the chain rule for the distributional derivative of the composite function is formulated as follows ([1], Theorem 2.1): If Ψ is a locally Lipschitz continuous function on $\boldsymbol{R}^1$ and $v \in BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, then $D[\Psi(v)]$ is a vector-valued Radon measure on $\boldsymbol{R}^N$ with finite total variation, $\Gamma_{\Psi(v)} \subset \Gamma_v$ and for each Borel set $E \subset \boldsymbol{R}^N$,

(3.7) $$D[\Psi(v)](E \cap \Gamma_{\Psi(v)}) = \int_{E \cap \Gamma_v} [\Psi(v^+) - \Psi(v^-)] \nu_v \, d\mathscr{H}^{N-1} .$$

For a regular point $x \in \boldsymbol{R}^N$ of a Borel function v, the Vol'pert's *averaged superposition* of $f \in C(\boldsymbol{R}^1)$ and v is defined by

(3.8) $$\hat{f}(v)(x) = \int_0^1 f(\theta v^+(x) + (1 - \theta) v^-(x)) \, d\theta .$$

In a similar way, for a regular point $x \in \boldsymbol{R}^N$ of a Borel function v, the *average value* of v at x is defined by

(3.9) $$\bar{v}(x) = 2^{-1}[v^+(x) + v^-(x)] .$$

Notice that $\bar{v}(x) = v^+(x) = v^-(x)$ at any regular point $x \in (\Gamma_v)^C$ and

(3.10) $\bar{v} = v$ a.e. with respect to the Lebesgue measure.

Furthermore it is seen from the definitions that the following relation

(3.11) $$\hat{f}(v) = f(\bar{v}) = \overline{f(v)}$$

holds at the regular points of v outside of Γ_v.

By means of the above-mentioned functions, the following formulas concerning superpositions and products are obtained: For v, $w \in BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$,

(3.12) $D\Psi(v) = \widehat{\Psi'}(v) Dv$ in the sense of measures,

(3.13) $D(vw) = \bar{w} Dv + \bar{v} Dw$ in the sense of measures.

Here, the right-hand sides of (3.12), (3.13) are interpreted in the sense of (3.2).

Finally, we prepare the following two lemmas which will be effectively used in the subsequent discussions.

Let ρ be a Lebesgue measurable function satisfying the following conditions:

$$\rho \in L^\infty(\boldsymbol{R}^N)^+ , \qquad \operatorname{supp} \rho \subset \{x \in \boldsymbol{R}^N; |x| \leqq 1\} , \qquad \int \rho(x) \, dx = 1 .$$

The corresponding *averaging kernel* is defined by $\rho_\delta = \delta^{-N}\rho(\cdot/\delta)$, $\delta > 0$. In the case that the kernel function ρ is radial, the averaging kernel is said to be *symmetric*. Then the average value is characterized in the following way ([20], p. 182 Theorem 1):

LEMMA 3.1. *Let v be a Borel function defined on an open set $\Omega \subset \boldsymbol{R}^N$ and $v \in L^1_{loc}(\Omega)$. If $x \in \Omega$ is a regular point of v, then*

$$[\rho_\delta * v](x) \equiv \int_\Omega \rho_\delta(x-y)v(y)\,dy \to \bar{v}(x) \qquad \text{as } \delta \downarrow 0$$

for any symmetric averaging kernel ρ_δ, $0 < \delta < d(x, \partial\Omega)$.

LEMMA 3.2. *Let $v = (v^1, \dots, v^N) \in BV(\boldsymbol{R}^N; \boldsymbol{R}^N)$ and $\varphi \in C_0^1(\boldsymbol{R}^N)$. Then*

$$\int D(\varphi v)\left(\equiv \int \textstyle\sum_{i=1}^N D_i(\varphi v^i)\right) = 0\,.$$

PROOF. We choose a sequence $(v_k)_{k=1}^\infty \subset C^\infty(\boldsymbol{R}^N; \boldsymbol{R}^N)$ in such a way that

$$\|v_k^i - v^i\|_1 \to 0 \qquad \text{and} \qquad \|Dv_k^i\|_M \to \|Dv^i\|_M \qquad \text{as } k \to \infty$$

by the approximation theorem for BV functions ([22], Theorem 5.3.3). Since $\{Dv_k^i; k \geqq 1\}$ is bounded in $L^1(\boldsymbol{R}^N)^N (\subset M(\boldsymbol{R}^N)^N)$, we may assume that $D_i v_k^i$ converges weakly in $M(\boldsymbol{R}^N)$ to $D_i v^i$ as $k \to \infty$. Hence we have

$$\begin{aligned} \int D(\varphi v) &= \langle v, \nabla\varphi \rangle + \textstyle\sum_i \int \varphi D_i v^i \\ &= \lim_{k\to\infty} (\langle v_k, \nabla\varphi\rangle + \textstyle\sum_i \langle \varphi, D_i v_k^i \rangle) = \lim_{k\to\infty} \int \textstyle\sum_i D_i(\varphi v_k^i)\,dx\,. \end{aligned}$$

Since supp φ is compact, the integral of $\sum_i D_i(\varphi v_k^i)$ over $\boldsymbol{R}^N$ is zero for each k by the classical divergence theorem. Thus we obtain the desired assertion. ■

Using these powerful properties of functions of bounded variation, we can prove the next proposition that is essential to establish our main theorem via the approximation theory for nonlinear semigroups. The idea of its proof is based on the result due to Yin [21].

PROPOSITION 3.3. *The operator B_0 defined by* (1.14) *is dissipative in $L^1(\boldsymbol{R}^N)$; i.e.*

$$\|v - w\|_1 \leqq \|(I - \lambda B_0)v - (I - \lambda B_0)w\|_1 \qquad \text{for } v, w \in D(B_0),\ \lambda > 0\,. \tag{3.14}$$

PROOF. First we assert that $\mathscr{H}^{N-1}(\Gamma_v) = 0$ for $v \in D(B_0)$. Since β is locally Lipschitz continuous and $v \in BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, it follows that $D_i\beta(v) \in$

$M(\boldsymbol{R}^N)$ for $i=1, \ldots, N$ (see [1], Theorem 2.1). Applying formula (3.7) to $D\beta(v)$, the following identity is valid:

$$\int_{E\cap\Gamma_{\beta(v)}} fD_i\beta(v) = \int_{E\cap\Gamma_v} f[\beta(v^+)-\beta(v^-)]\nu_v^i\, d\mathscr{H}^{N-1}$$

for each Borel set $E\subset\boldsymbol{R}^N$ and each Borel function f which is bounded with respect to Hausdorff measure $\mathscr{H}^{N-1}$. Taking $f=\operatorname{sgn}\nu_v^i$ in the above identity and adding up both sides with respect to i from 1 to N, we obtain

$$(3.15)\quad \sum_{i=1}^N \int_{E\cap\Gamma_{\beta(v)}} \operatorname{sgn}\nu_v^i D_i\beta(v) = \int_{E\cap\Gamma_v} [\beta(v^+)-\beta(v^-)]\sum_{i=1}^N |\nu_v^i|\, d\mathscr{H}^{N-1}.$$

Now by the definition of B_0, $v\in D(B_0)\subset BV(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)\subset L^2(\boldsymbol{R}^N)$ and

$$|A^i(v)-A^i(0)| \leqq \sup\{|a^i(s)|;\, |s|\leqq \|v\|_\infty\}|v|.$$

Therefore $A^i(v)-A^i(0)\in L^2(\boldsymbol{R}^N)$ and $D_iA^i(v)\in H^{-1}(\boldsymbol{R}^N)$, where $H^s(\boldsymbol{R}^N)$, $s\in\boldsymbol{R}^1$, denotes the Sobolev spaces defined via the Fourier transform. This together with the relations $B_0v\in BV(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)\subset L^2(\boldsymbol{R}^N)$, we have

$$\triangle\beta(v) = B_0v + \nabla A(v)\in H^{-1}(\boldsymbol{R}^N).$$

Notice that $\beta(v)-\beta(0)\in L^2(\boldsymbol{R}^N)$, we have $D_i\beta(v)\in L^2(\boldsymbol{R}^N)$. From this we see that $D_i\beta(v)\in M(\boldsymbol{R}^N)\cap L^2(\boldsymbol{R}^N)\subset L^1(\boldsymbol{R}^N)$, and hence that the measure $D_i\beta(v)$ dose not have a singular part with respect to the Lebesgue decomposition. This implies that the following identity holds:

$$[D_i\beta(v)](E) = \int_E D_i\beta(v)\,dx \qquad \text{for each Borel set } E\subset\boldsymbol{R}^N.$$

Therefore (3.15) can be rewritten in the following form:

$$(3.16)\quad \sum_{i=1}^N \int_{E\cap\Gamma_{\beta(v)}} \operatorname{sgn}\nu_v^i D_i\beta(v)\,dx = \int_{E\cap\Gamma_v} [\beta(v^+)-\beta(v^-)]\sum_{i=1}^N |\nu_v^i|\, d\mathscr{H}^{N-1}.$$

Since $v^+>v^-$ holds $\mathscr{H}^{N-1}$-a.e. on Γ_v and β is strict increasing, it follows that $\beta(v^+)>\beta(v^-)$ holds $\mathscr{H}^{N-1}$-a.e. on Γ_v. Also, $\sum_{i=1}^N|\nu_v^i(x)|>0$ holds for $\mathscr{H}^{N-1}$-a.e. $x\in\Gamma_v$ from $\nu_v(x)\in S^{N-1}$. Combining these results, we see that the integrand of the right-hand side of (3.16) is strictly positive $\mathscr{H}^{N-1}$-a.e. on Γ_v. On the other hand, since $|\Gamma_{\beta(v)}|=0$, the left-hand side of (3.16) is equal to zero. Therefore it follows that $\mathscr{H}^{N-1}(E\cap\Gamma_v)=0$. Now Borel sets $E\subset\boldsymbol{R}^N$ are arbitrary, the property (ii) of Γ_v yields $\mathscr{H}^{N-1}(\Gamma_v)=0$.

Next we show (3.14). Set $v'=B_0v$, $w'=B_0w$. By definition,

$$(3.17)\quad \langle v'-w', \psi\rangle = -\langle\nabla\beta(v)-\nabla\beta(w), \nabla\psi\rangle - \int\psi D[A(v)-A(w)]$$

for each $\psi \in C_0^\infty(\boldsymbol{R}^N)$. We then introduce a family of functions $\{\Phi_j; j = 1, 2, \dots\} \subset C^2(\boldsymbol{R}^1)$ which is analogous to the family of functions employed in Crandall [9] so as to approximate some discontinuous functions:

$$\Phi_j(s) = \begin{cases} -(j\pi/2)^{-1}\cos[(j\pi/2)s] + j^{-1} & (|s| \leqq j^{-1}), \\ |s| & (|s| \geqq j^{-1}). \end{cases}$$

The following can be easily verified:

(3.18) $\quad \lim_{j\to\infty} \Phi_j(s) = |s|, \qquad \lim_{j\to\infty} \Phi_j'(s) = \operatorname{sgn} s, \qquad \lim_{j\to\infty} s\Phi_j''(s) = 0$

(3.19) $\quad \Phi_j''(s) \geqq 0, \qquad \operatorname{supp} \Phi_j'' = [-j^{-1}, j^{-1}], \qquad |s\Phi_j''(s)| \leqq \pi/2.$

In view of the duality mapping of $L^1(\boldsymbol{R}^N)$, we substitute the test function ψ in (3.17) for $\operatorname{sgn}(v - w)$. To this end, we take any cut-off function $\varphi \in C_0^\infty(\boldsymbol{R}^N)$ with $0 \leqq \varphi \leqq 1$ on $\boldsymbol{R}^N$, and then define the desired test functions by

$$\psi_{j,\delta} = \rho_\delta * [\Phi_j'(\beta(v) - \beta(w))\varphi] \qquad \text{for } \delta > 0,\ j = 1, 2, \dots.$$

Apparently, each function $\psi_{j,\delta}$ satisfies $\psi_{j,\delta} \in C_0^\infty(\boldsymbol{R}^N)$, $\|\psi_{j,\delta}\|_\infty \leqq 1$ and

$$\psi_{j,\delta} \to \Phi_j'(\beta(v) - \beta(w))\varphi \qquad \text{in } H^1(\boldsymbol{R}^N) \text{ as } \delta \downarrow 0,$$

notice that $\Phi_j'(\beta(v) - \beta(w)) \in H^1(\boldsymbol{R}^N)$. Hence

$$\langle v' - w', \psi_{j,\delta} \rangle \to \langle v' - w', \Phi_j'(\beta(v) - \beta(w))\varphi \rangle$$

and

$$\begin{aligned} &-\langle \nabla\beta(v) - \nabla\beta(w), \nabla\psi_{j,\delta} \rangle \\ &\quad \to -\langle \nabla\beta(v) - \nabla\beta(w), \nabla[\Phi_j'(\beta(v) - \beta(w))\varphi] \rangle \\ &\quad = -\int |\nabla\beta(v) - \nabla\beta(w)|^2 \Phi_j''(\beta(v) - \beta(w))\varphi \, dx \\ &\qquad -\langle \nabla\beta(v) - \nabla\beta(w), \Phi_j'(\beta(v) - \beta(w))\nabla\varphi \rangle \\ &\quad \leqq \langle \Phi_j(\beta(v) - \beta(w)), \triangle\varphi \rangle \qquad \text{as } \delta \downarrow 0. \end{aligned}$$

Next, in order to handle the second term on the right-hand side of (3.17), we have to observe the pointwise behavior of $\psi_{j,\delta}$ as $\delta \downarrow 0$. Since $z_j \equiv \Phi_j'(\beta(v) - \beta(w))$ lies in $BV(\boldsymbol{R}^N)$, we have $\mathscr{H}^{N-1}(I) = 0$, where $I = \{x \in \boldsymbol{R}^N;\ x$ is not a regular point of $z_j\varphi\}$ (see [22] p. 254). Applying Lemma 3.1 to $z_j\varphi$, we have

$$\psi_{j,\delta}(x) \to \overline{[z_j\varphi]}(x) \qquad \text{as } \delta \downarrow 0 \text{ for } x \in I^C.$$

Notice that $D[A(v) - A(w)](I) = 0$ by (3.4). It then follows from the dominated convergence theorem that

$$\int \psi_{j,\delta} D[A(v) - A(w)] \to \int \overline{z_j \varphi} D[A(v) - A(w)] \quad \text{as } \delta \downarrow 0 . \tag{3.20}$$

Since $\overline{z_j\varphi} = \Phi_j'(\beta(\bar{v}) - \beta(\bar{w}))\varphi$ holds for $\mathscr{H}^{N-1}$-a.e. on $(\Gamma_v \cup \Gamma_w)^C$ by (3.11), together with $\mathscr{H}^{N-1}(\Gamma_v) = \mathscr{H}^{N-1}(\Gamma_w) = 0$ and (3.4), we obtain

$$\int \overline{z_j \varphi} D[A(v) - A(w)] = \int \Phi_j'(\beta(\bar{v}) - \beta(\bar{w}))\varphi D[A(v) - A(w)] .$$

Therefore, substituting $\psi_{j,\delta}$ for ψ in (3.17) and then passing to the limit as $\delta \downarrow 0$, we have

$$\begin{aligned} \langle v' - w', \Phi_j'(\beta(v) - \beta(w))\varphi \rangle &\leqq \langle \Phi_j(\beta(v) - \beta(w)), \Delta\varphi \rangle \\ &- \int \Phi_j'(\beta(\bar{v}) - \beta(\bar{w}))\varphi D[A(v) - A(w)] . \end{aligned}$$

Using the properties of Φ_j and letting $j \to \infty$ in the above inequality,

$$\begin{aligned} \langle v' - w', \operatorname{sgn}(\beta(v) - \beta(w))\varphi \rangle &\leqq \langle |\beta(v) - \beta(w)|, \Delta\varphi \rangle \\ &- \int \operatorname{sgn}(\beta(\bar{v}) - \beta(\bar{w}))\varphi D[A(v) - A(w)] . \end{aligned}$$

Since β is strictly increasing, $\operatorname{sgn}(\beta(\bar{v}(x)) - \beta(\bar{w}(x))) = \operatorname{sgn}(\bar{v}(x) - \bar{w}(x))$ for $\mathscr{H}^{N-1}$-a.e. x, and so the above inequality can be rewritten as

$$\begin{aligned} &\langle \operatorname{sgn}(v - w)[v' - w'], \varphi \rangle \\ &\quad \leqq \langle |\beta(v) - \beta(w)|, \Delta\varphi \rangle - \int \operatorname{sgn}(\bar{v} - \bar{w})\varphi D[A(v) - A(w)] . \end{aligned}$$

Using the above estimate, we have

$$\begin{aligned} \langle |v - w|, \varphi \rangle &= \langle \operatorname{sgn}(v - w)[(v - \lambda v') - (w - \lambda w')], \varphi \rangle \\ &\quad + \lambda \langle \operatorname{sgn}(v - w)[v' - w'], \varphi \rangle \\ &\leqq \langle |(v - \lambda v') - (w - \lambda w')|, \varphi \rangle + \lambda \langle |\beta(v) - \beta(w)|, \Delta\varphi \rangle \\ &\quad - \lambda \int \operatorname{sgn}(\bar{v} - \bar{w})\varphi D[A(v) - A(w)] , \end{aligned} \tag{3.21}$$

for each $\lambda > 0$. It now remains to show the following identity:

$$\int \operatorname{sgn}(\bar{v} - \bar{w})\psi D[A(v) - A(w)] = -\langle \operatorname{sgn}(v - w)[A(v) - A(w)], \nabla\psi \rangle \tag{3.22}$$

for all $\psi \in C_0^\infty(\boldsymbol{R}^N)$. Indeed, suppose that (3.22) holds for all $\psi \in C_0^\infty(\boldsymbol{R}^N)$. Choose any function $\varphi_0 \in C_0^\infty(\boldsymbol{R}^N)$ satisfying $0 \leqq \varphi_0 \leqq 1$, $\varphi_0(x) = 1$ for $|x| \leqq 1$

and $\varphi_0(x) = 0$ for $|x| \geqq 2$, and define $\varphi_k = \varphi_0(\cdot/k)$. Then

$$\int f\varphi_k\,dx \to \int f\,dx \qquad \text{for all } f \in L^1(\boldsymbol{R}^N), \tag{3.23}$$

$$\|D\varphi_k\|_\infty,\ \|D^2\varphi_k\|_\infty \to 0 \qquad \text{as } k \to \infty. \tag{3.24}$$

Putting $\varphi = \varphi_k$ in (3.21) and then letting k tend to the infinity, we obtain the desired assertion (3.14) from (3.22) through (3.24).

We then demonstrate that (3.22) holds. As seen in the first step of this proof, $\mathscr{H}^{N-1}(\Gamma_v) = \mathscr{H}^{N-1}(\Gamma_w) = 0$ since $v, w \in D(B_0)$. Using this fact, Lemma 3.2, (3.11), (3.13) and by integration by parts, we have

$$\begin{aligned}
&\int \Phi_j'(\bar{v} - \bar{w})\psi D[A(v) - A(w)] \\
&= \int \overline{\Phi_j'(v - w)\psi} D[A(v) - A(w)] \\
&= \int D(\Phi_j'(v - w)\psi[A(v) - A(w)]) - \int \overline{[A(v) - A(w)]} D[\Phi_j'(v - w)\psi] \\
&= -\int [A(\bar{v}) - A(\bar{w})] D[\Phi_j'(v - w)\psi] \\
&= -\int [A(\bar{v}) - A(\bar{w})]\psi D\Phi_j'(v - w) - \langle [A(v) - A(w)]\Phi_j'(v - w), \nabla\psi\rangle
\end{aligned} \tag{3.25}$$

Again, we apply (3.11) and (3.12) to transform the first term of the extreme right-hand side of (3.25) as follows:

$$\begin{aligned}
&-\int [A(\bar{v}) - A(\bar{w})]\psi\widehat{\Phi_j''}(v - w)D[v - w] \\
&= -\int \left(\int_0^1 a(\theta\bar{v} + (1 - \theta)\bar{w})\,d\theta\right)\psi[\bar{v} - \bar{w}]\Phi_j''(\bar{v} - \bar{w})D[v - w].
\end{aligned} \tag{3.26}$$

Here we have used the fact that $\mathscr{H}^{N-1}(\Gamma_v) = \mathscr{H}^{N-1}(\Gamma_w) = 0$. Notice that it is essential in this argument to take the average values $\bar{v}$ and $\bar{w}$. In view of (3.18), the right-hand side of (3.26) tends to zero as $j \to \infty$ by applying the dominated convergence theorem. Therefore letting $j \to \infty$ in (3.25), we obtain the desired assertion (3.22). ■

4. Proof of Theorem

In this section, we give the proof of main theorem. In order to prove the assertion (a), it is sufficient to show the following two propositions by virtue of Proposition 2.1 and Lemma 2.2 (i):

PROPOSITION 4.1. *$R(I-\lambda B)=L^1(\mathbf{R}^N)$ for each $\lambda>0$ and*

$$(I-\lambda B_h)^{-1}v \to (I-\lambda B)^{-1}v$$

as $h\downarrow 0$ in $L^1(\mathbf{R}^N)$ for each $v\in L^1(\mathbf{R}^N)$.

PROPOSITION 4.2. *$D(B)$ is dense in $L^1(\mathbf{R}^N)$.*

The proof of convergence as $h\downarrow 0$ of $(I-\lambda B_h)^{-1}v$ for a general element $v\in L^1(\mathbf{R}^N)$ can be reduced to the case of $v\in BV(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$ since this intersection is dense in $L^1(\mathbf{R}^N)$. In what follows, we set

$$v_h^\lambda=(I-\lambda B_h)^{-1}v \qquad \text{for } v\in L^1(\mathbf{R}^N)\,.$$

Notice that

$$B_h v_h^\lambda=\lambda^{-1}(v_h^\lambda-v)\,.$$

LEMMA 4.3. *Suppose that the positive numbers h and ε satisfy* (1.11). *Let $\delta>0$ be such that $0<\varepsilon<1$ whenever $0<h<\delta$. Then we have:*

(i) *For each $\lambda>0$ and each $v\in L^1(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$, the set $\{v_h^\lambda; h\in(0,\delta)\}$ is precompact in $L^1(\mathbf{R}^N)$.*

(ii) *Let $v\in BV(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$ and let v^λ be any cluster point of the net $\{v_h^\lambda; 0<h<\delta, h\downarrow 0\}$. Then $v^\lambda\in D(B_0)$ and $\lambda^{-1}(v^\lambda-v)=B_0v^\lambda$. Therefore $R(I-\lambda B_0)\supset BV(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$ and*

$$v_h^\lambda\to(I-\lambda B_0)^{-1}v \qquad \text{in } L^1(\mathbf{R}^N) \text{ as } h\downarrow 0\,.$$

PROOF. (i) For λ, $h>0$ and $y\in\mathbf{R}^N$, Lemma 2.3 implies

$$\|v_h^\lambda\|_p\leqq\|v\|_p\,,\qquad p=1,\infty\,,\qquad \|\tau_y v_h^\lambda-v_h^\lambda\|_1\leqq\|\tau_y v-v\|_1\,. \tag{4.1}$$

Therefore the Fréchet–Kolmogorov theorem can be applied to prove the assertion, if it is proved

$$\lim_{\rho\uparrow\infty}\sup_{h\in(0,\delta)}\int_{|x|>\rho}|v_h^\lambda|\,dx=0\,. \tag{4.2}$$

In order to show (4.2), we first note that if $v\in L^1(\mathbf{R}^N)\cap L^\infty(\mathbf{R}^N)$, then estimate (2.6) with $k=0$ is still valid for any bounded continuous function $\varphi\geqq 0$ with bounded continuous derivatives up to the order 2. Take any such φ and fix it, then the application of (2.6) with $v=v_h^\lambda$ and $k=0$ implies

$$\begin{aligned}\lambda^{-1}\langle|v_h^\lambda|,\varphi\rangle-\lambda^{-1}\langle|v|,\varphi\rangle &\leqq\langle \operatorname{sgn}(v_h^\lambda)B_hv_h^\lambda,\varphi\rangle\\ &\leqq h^{-1}\int\langle\operatorname{sgn}(v_h^\lambda)F_\varepsilon(v_h^\lambda,\xi),\varphi(\cdot+h\xi)-\varphi\rangle d\xi\end{aligned} \tag{4.3}$$

$$= \int \langle \operatorname{sgn}(v_h^\lambda) F_\varepsilon(v_h^\lambda, \xi)\xi, \nabla\varphi \rangle \, d\xi$$

$$+ h \sum_{i,j} \int_0^1 d\theta(1-\theta) \int \xi_i \xi_j \langle \operatorname{sgn}(v_h^\lambda) F_\varepsilon(v_h^\lambda, \xi), D_{ij}\varphi(\cdot + \theta h\xi) \rangle \, d\xi$$

$$= I_1 + I_2 .$$

We estimate the extreme right-hand side term by term. The first term I_1 can be written as

$$I_1 = \langle \operatorname{sgn}(v_h^\lambda)[A(v_h^\lambda) - A(0)], \nabla\varphi \rangle = \int_0^1 \langle |v_h^\lambda| a(\sigma v_h^\lambda), \nabla\varphi \rangle \, d\sigma ,$$

and so it follows that

$$|I_1| \leqq \sum_{i=1}^N \sup\{|a^i(s)|; |s| \leqq \|v\|_\infty\} \, \|D\varphi\|_\infty \|v\|_1 .$$

Next, by a change of variables, I_2 is transformed into

$$I_2 = h \sum_{i,j} \int_0^1 d\theta(1-\theta) \iint [\xi_i + a^i(s)][\xi_j + a^j(s)]$$

$$\cdot \langle \operatorname{sgn}(v_h^\lambda) F(v_h^\lambda, s), D_{ij}\varphi(\cdot + \theta h(\xi + a(s))) \rangle \chi_\varepsilon(\xi, s) \, d\xi \, ds$$

$$= h\varepsilon^{-2} \sum_{i,j} \int_0^1 d\theta(1-\theta) \iint [\beta_\varepsilon'(s)^{1/2}\eta_i + \varepsilon a^i(s)][\beta_\varepsilon'(s)^{1/2}\eta_j + \varepsilon a^j(s)]$$

$$\cdot \langle \operatorname{sgn}(v_h^\lambda) F(v_h^\lambda, s), D_{ij}\varphi(\cdot + \theta h\varepsilon^{-1}[\beta_\varepsilon'(s)^{1/2}\eta + \varepsilon a(s)]) \rangle \chi(\eta) \, d\eta \, ds$$

Using the fact that $\operatorname{supp} \chi \subset \{\eta \in \boldsymbol{R}^N; |\eta| \leqq 1\}$, we have

$$|I_2| \leqq h\varepsilon^{-2} \sum_{i,j} \int_0^1 d\theta(1-\theta) \iint \sup\{\beta_\varepsilon'(s)^{1/2} + |a(s)|; |s| \leqq \|v\|_\infty, \varepsilon > 0\}^2$$

$$\cdot \langle |F(v_h^\lambda, s)|, \|D^2\varphi\|_\infty \rangle \chi(\eta) \, d\eta \, ds$$

$$\leqq [N^2 h/(2\varepsilon^2)] \sup\{\beta_\varepsilon'(s)^{1/2} + |a(s)|; |s| \leqq \|v\|_\infty, \varepsilon > 0\}^2 \|D^2\varphi\|_\infty \|v\|_1 .$$

Thus we obtain the following estimate:

$$(4.4) \quad \lambda^{-1}\langle |v_h^\lambda|, \varphi \rangle \leqq \lambda^{-1}\langle |v|, \varphi \rangle + \Big(\sum_{i=1}^N \sup\{|a^i(s)|; |s| \leqq \|v\|_\infty\} \|D\varphi\|_\infty$$
$$+ [N^2 h/(2\varepsilon^2)] \sup\{\beta_\varepsilon'(s)^{1/2} + |a(s)|; |s| \leqq \|v\|_\infty, \varepsilon > 0\}^2 \|D^2\varphi\|_\infty \Big) \|v\|_1 .$$

We now take any function $g \in C^\infty(\boldsymbol{R}^1)$ with the properties

$$g(s) = 1 \quad \text{for } s \geqq 1 , \qquad g(s) = 0 \quad \text{for } s \leqq 0 , \qquad 0 \leqq g \leqq 1 \text{ on } \boldsymbol{R}^1 ,$$

and for any pair ρ, τ with $\rho > \tau > 0$, we define

$$g_{\rho,\tau}(s) \equiv g[(|s| - \tau)(\rho - \tau)^{-1}] \qquad \text{for } s \in \boldsymbol{R}^1 .$$

Then it is easy to check that

$$g_{\rho,\tau}(s) = 1 \text{ for } |s| \geqq \rho , \qquad g_{\rho,\tau}(s) = 0 \text{ for } |s| \leqq \tau$$

$$\|g'_{\rho,\tau}\|_\infty \to 0 , \qquad \|g''_{\rho,\tau}\|_\infty \to 0 \qquad \text{as } \rho \uparrow \infty .$$

If we set $\varphi_{\rho,\tau}(x) = \sum_{i=1}^N g_{\rho,\tau}(x_i)$, then we have

(4.5)
$$0 \leqq \varphi_{\rho,\tau} \leqq N , \quad \varphi_{\rho,\tau}(x) \geqq 1 \quad \text{for} \quad |x| \geqq \rho N^{1/2} , \quad \varphi_{\rho,\tau}(x) = 0 \quad \text{for} \quad |x| \leqq \tau ,$$

(4.6)
$$\|D\varphi_{\rho,\tau}\|_\infty \to 0 , \qquad \|D^2\varphi_{\rho,\tau}\|_\infty \to 0 \qquad \text{as } \rho \uparrow \infty .$$

Thus the function $\varphi_{\rho,\tau}$ is bounded, continuous and has bounded continuous derivatives up to the order 2. Thus we can substitute $\varphi_{\rho,\tau}$ for φ in (4.4) to get

(4.7)
$$\lambda^{-1} \int_{|x| > \rho N^{1/2}} |v_h^\lambda| \, dx \leqq \lambda^{-1} \langle |v_h^\lambda|, \varphi_{\rho,\tau} \rangle$$
$$\leqq \lambda^{-1} \langle |v|, \varphi_{\rho,\tau} \rangle + \big(\textstyle\sum_{i=1}^N \sup\{|a^i(s)|; |s| \leqq \|v\|_\infty\} \|D\varphi_{\rho,\tau}\|_\infty$$
$$+ [N^2 h/(2\varepsilon^2)] \sup \{\beta'_\varepsilon(s)^{1/2} + |a(s)|; |s| \leqq \|v\|_\infty, \varepsilon > 0\}^2 \|D^2\varphi_{\rho,\tau}\|_\infty \big) \|v\|_1$$

by (4.5). Therefore (4.5), (4.6), and the relation $h\varepsilon^{-2}$ = const. together imply

$$\text{limsup}_{\rho \uparrow \infty} \sup_{h \in (0,\delta)} \int_{|x| > \rho N^{1/2}} |v_h^\lambda| \, dx \leqq N \int_{|x| > \tau} |v| \, dx .$$

Since τ is arbitrary, we finally obtain (4.2).

(ii) For simplicity in notation we write $h \downarrow 0$ for the convergence of a null sequence $(h(k))_{k=1}^\infty$ of positive numbers. Thus we write $v_h^\lambda \to v^\lambda$ for the convergence of a subsequence $(v_{h(k)})_{k=1}^\infty$ to a cluster point v^λ. We may assume without loss of generality that $v_h^\lambda \to v^\lambda$ as $h \downarrow 0$ in $L^1(\boldsymbol{R}^N)$ and a.e. in $\boldsymbol{R}^N$. First we will show that the function v^λ satisfies the equation

(4.8)
$$\lambda^{-1}(v^\lambda - v) = \Delta\beta(v^\lambda) - \nabla A(v^\lambda) \qquad \text{in } \mathscr{D}'(\boldsymbol{R}^N) .$$

To this end, for any $\varphi \in C_0^\infty(\boldsymbol{R}^N)$, we see from direct calculations that

(4.9)
$$\lambda^{-1} \langle v_h^\lambda - v, \varphi \rangle = h^{-1} \int \langle F_\varepsilon(v_h^\lambda, \xi), \varphi(\cdot + h\xi) - \varphi \rangle \, d\xi$$
$$= \int \langle F_\varepsilon(v_h^\lambda, \xi)\xi, \nabla\varphi \rangle \, d\xi + (h/2) \textstyle\sum_{i,j} \int \xi_i \xi_j \langle F_\varepsilon(v_h^\lambda, \xi), D_{ij}\varphi \rangle \, d\xi$$

$$+h\sum_{i,j}\int_0^1 d\theta(1-\theta)\int \xi_i\xi_j\langle F_\varepsilon(v_h^\lambda,\xi), D_{ij}\varphi(\cdot+\theta h\xi)-D_{ij}\varphi\rangle\, d\xi$$

$$=J_1+J_2+J_3\,.$$

We wish to check each term on the right-hand side of (4.9). First we see with the aid of (4.1) that

(4.10) $$J_1=\langle A(v_h^\lambda),\nabla\varphi\rangle\to\langle A(v^\lambda),\nabla\varphi\rangle \qquad \text{as } h\downarrow 0\,.$$

Next, J_2 can be transformed into

$$J_2=(h/2)\sum_{i,j}\iint[\xi_i+a^i(s)][\xi_j+a^j(s)]\langle F(v_h^\lambda,s),D_{ij}\varphi\rangle\chi_\varepsilon(\xi,s)\,d\xi\,ds$$

$$=[h/(2\varepsilon^2)]\sum_{i,j}\iint[\beta_\varepsilon'(s)^{1/2}\eta_i+\varepsilon a^i(s)][\beta_\varepsilon'(s)^{1/2}\eta_j+\varepsilon a^j(s)]$$

$$\cdot\langle F(v_h^\lambda,s),D_{ij}\varphi\rangle\chi(\eta)\,d\eta\,ds\,.$$

Since χ is assumed to be a radial function, it is easy to show that

$$\int\eta_i\chi(\eta)\,d\eta=0\,,\qquad \int\eta_i\eta_j\chi(\eta)\,d\eta=N^{-1}\int|\eta|^2\chi(\eta)\,d\eta\,\delta_{ij}\,.$$

Using these facts, one can rewrite J_2 as follows:

$$J_2=[h/(2\varepsilon^2)]\sum_{i,j}\int\beta_\varepsilon'(s)\left(N^{-1}\int|\eta|^2\chi(\eta)\,d\eta\,\delta_{ij}\right)\langle F(v_h^\lambda,s),D_{ij}\varphi\rangle\,ds$$

$$+(h/2)\sum_{i,j}\int a^i(s)a^j(s)\langle F(v_h^\lambda,s),D_{ij}\varphi\rangle\,ds$$

$$=J_{21}+J_{22}\,.$$

Applying the relation (1.11) between h and ε, we see that

(4.11) $$J_{21}=\int\beta_\varepsilon'(s)\langle F(v_h^\lambda,s),\Delta\varphi\rangle\,ds\to\langle\beta(v^\lambda),\Delta\varphi\rangle \qquad \text{as } h\downarrow 0\,,$$

and that

(4.12) $$|J_{22}|\leqq(h/2)\sum_{i,j}\int\sup\{|a(s)|;|s|\leqq\|v\|_\infty\}^2\langle|F(v_h^\lambda,s)|,\|D^2\varphi\|_\infty\rangle\,ds$$

$$\leqq[(N^2h)/2]\sup\{|a(s)|;|s|\leqq\|v\|_\infty\}^2\|v\|_1\|D^2\varphi\|_\infty\to 0 \qquad \text{as } h\downarrow 0\,.$$

The last term J_3 is rewritten in the following way:

$$J_3 = h\varepsilon^{-2}\sum_{i,j}\int_0^1 d\theta(1-\theta)\iint[\beta_\varepsilon'(s)^{1/2}\eta_i + \varepsilon a^i(s)][\beta_\varepsilon'(s)^{1/2}\eta_j + \varepsilon a^j(s)]$$

$$\cdot\langle F(v_h^\lambda, s), D_{ij}\varphi(\cdot + \theta h\varepsilon^{-1}[\beta_\varepsilon'(s)^{1/2}\eta + \varepsilon a(s)]) - D_{ij}\varphi\rangle\chi(\eta)\, d\eta\, ds\,.$$

Since supp $F(v_h^\lambda(x), \cdot)$ is contained in $E \equiv \{s \in \boldsymbol{R}^1; |s| \leqq \|v\|_\infty\}$ and $|F(v_h^\lambda(x), s)| \leqq 1_E(s)$ for a.e. $x \in \boldsymbol{R}^N$ by the definition of F, $|J_3|$ is estimated as

$$(4.13)\quad |J_3| \leqq [(N^2h)/\varepsilon^2]\sup\{\beta_\varepsilon'(s)^{1/2} + |a(s)|; |s| \leqq \|v\|_\infty, \varepsilon > 0\}^2\int_0^1 d\theta(1-\theta)$$

$$\cdot\iint\langle 1_E(s), |D_{ij}\varphi(\cdot + \theta h\varepsilon^{-1}[\beta_\varepsilon'(s)^{1/2}\eta + \varepsilon a(s)]) - D_{ij}\varphi|\rangle\chi(\eta)\, d\eta\, ds$$

$$\leqq [(N^2h)/\varepsilon^2]\sup\{\beta_\varepsilon'(s)^{1/2} + |a(s)|; |s| \leqq \|v\|_\infty, \varepsilon > 0\}^2\|v\|_\infty$$

$$\cdot\sup\{\|D^2\varphi(\cdot + \theta h\varepsilon^{-1}[\beta_\varepsilon'(s)^{1/2}\eta + \varepsilon a(s)]) - D^2\varphi\|_1\,;$$

$$|s| \leqq \|v\|_\infty,\ \ |\eta| \leqq 1\,,\ \ 0 \leqq \theta \leqq 1\} \to 0 \qquad \text{as } h\downarrow 0\,.$$

Therefore, letting $h\downarrow 0$ in (4.9) and using (4.10) through (4.13), we obtain (4.8). Finally, we show that $v^\lambda \in D(B_0)$. It is obvious from (4.1) that $v^\lambda \in L^\infty(\boldsymbol{R}^N)$. On the other hand, it follows from the estimate

$$\|\tau_y v^\lambda - v^\lambda\|_1 \leqq \liminf_{h\downarrow 0}\|\tau_y v_h^\lambda - v_h^\lambda\|_1 \leqq \|\tau_y v - v\|_1$$

and the condition $v \in BV(\boldsymbol{R}^N)$ that $v^\lambda \in BV(\boldsymbol{R}^N)$. Furthermore, by (4.8),

$$\triangle\beta(v^\lambda) - \nabla A(v^\lambda) = \lambda^{-1}(v^\lambda - v) \in BV(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)\,,$$

which means $v^\lambda \in D(B_0)$ and $v^\lambda = (I - \lambda B_0)^{-1}v$. Notice that our argument stated above remains valid even if we start with any subsequence of $(v_{h(k)}^\lambda)_{k=1}^\infty$. This shows that any subsequence of $\{v_h^\lambda; h > 0\}$ contains a subsequence which converges in $L^1(\boldsymbol{R}^N)$ and its limit point is uniquely determined (as $(I - \lambda B_0)^{-1}v$). Thus we can conclude that v_h^λ itself converges to $(I - \lambda B_0)^{-1}v$ in $L^1(\boldsymbol{R}^N)$ as $h\downarrow 0$. ■

PROOF OF PROPOSITION 4.1. Let $(v_k)_{k=1}^\infty$ be a sequence of functions in the class $BV(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)$ such that $v_k \to v$ as $k \to \infty$ in $L^1(\boldsymbol{R}^N)$. If we set

$$v_{h,k}^\lambda = (I - \lambda B_h)^{-1}v_k\,, \qquad v_k^\lambda = (I - \lambda B_0)^{-1}v_k \qquad \text{for } \lambda,\ h > 0\,,$$

then it follows that $v_{h,k}^\lambda \to v_k^\lambda$ as $h\downarrow 0$ in $L^1(\boldsymbol{R}^N)$ for each k by Lemma 4.3 (ii). Hence Lemma 2.3 (i) implies

$$\|v_k^\lambda - v_l^\lambda\|_1 \leqq \liminf_{h\downarrow 0}\|v_{h,k}^\lambda - v_{h,l}^\lambda\|_1 \leqq \|v_k - v_l\|_1\,.$$

This means that the sequence $(v_k^\lambda)_{k=1}^\infty$ in $D(B_0)$ is Cauchy in $L^1(\boldsymbol{R}^N)$, and hence that converges in $L^1(\boldsymbol{R}^N)$ to some element $v^\lambda \in L^1(\boldsymbol{R}^N)$. From this and the

$$\lambda^{-1}[u^\lambda(t) - u^\lambda(t-\lambda)] = \triangle\beta(u^\lambda(t)) - \nabla A(u^\lambda(t))$$

holds in $\mathscr{D}'(\boldsymbol{R}^N)$ for $t \in [s, \infty)$.

Let $\rho_n *$ be the usual Friedrichs mollifier with support $\{x \in \boldsymbol{R}^N; |x| \leqq n^{-1}\}$ defined for positive integer n by a radial function ρ. In what follows, we denote $\rho_n * v$ by v_n for a $v \in L^1_{loc}(\boldsymbol{R}^N)$. Then it is easily seen that

$$(4.17) \qquad \lambda^{-1}[u^\lambda(t) - u^\lambda(t-\lambda)]_n = \triangle\beta(u^\lambda(t))_n - \nabla[A(u^\lambda(t)) - A(0)]_n$$

holds in the classical sense. Multiplying both side of (4.17) by $[\beta(u^\lambda(t)) - \beta(0)]_n$, and then integrating over $\boldsymbol{R}^N$, we have

$$(4.18) \quad \|\nabla\beta(u^\lambda(t))_n\|_2^2 \leqq \lambda^{-1}\|u^\lambda(t) - u^\lambda(t-\lambda)\|_1\|\beta(u^\lambda(t)) - \beta(0)\|_\infty + C(\kappa)\textstyle\sum_i \|A^i(u^\lambda(t)) - A^i(0)\|_2^2 + \kappa\|\nabla\beta(u^\lambda(t))_n\|_2^2$$

where the constant $C(\kappa)$ depends only on $\kappa \in (0, 1)$. Therefore

$$(1-\kappa)\|\nabla\beta(u^\lambda(t))_n\|_2^2 \leqq \lambda^{-1}\|u^\lambda(t) - u^\lambda(t-\lambda)\|_1 \cdot \sup\{\beta'(\sigma); |\sigma| \leqq r\}r + C(\kappa)\|u^\lambda(t)\|_1 r \cdot \textstyle\sum_i \sup\{|a^i(\sigma)|; |\sigma| \leqq r\}^2$$

where $r = \|u_0\|_\infty$. Thus, for each fixed $\lambda > 0$, the family $\{\nabla\beta(u^\lambda)_n; n \geqq 1\}$ is bounded in $L^\infty(s, \infty; L^2(\boldsymbol{R}^N)^N)$; in particular, it is bounded in $L^2(s, T; L^2(\boldsymbol{R}^N)^N)$. On the other hand, the sequence $\nabla\beta(u^\lambda)_n$ itself converges to $\nabla\beta(u^\lambda)$ in $\mathscr{D}'(\boldsymbol{R}^N \times (s, T))^N$ as $n \to \infty$, and so it is easily seen that $\nabla\beta(u^\lambda) \in L^2(s, T; L^2(\boldsymbol{R}^N)^N)$.

Now suppose that u_0 also belongs to $\hat{D}(B)$, hence

$$|Bu_0| \equiv \liminf_{\nu\downarrow 0} \nu^{-1}\|(I - \nu B)^{-1}u_0 - u_0\|_1 < \infty .$$

By virtue of the above observation, we can show that $\nabla\beta(u) \in L^2(0, T; L^2(\boldsymbol{R}^N)^N)$ for each $T > 0$. This will be proved via a compactness argument to the set $\{\nabla\beta(u^\lambda); \lambda \in (0, s)\}$ in the space $L^2(s, T; L^2(\boldsymbol{R}^N)^N)$. Therefore we necessitate showing the following assertion:

$$(4.19) \qquad \lambda^{-1}\|u^\lambda(t) - u^\lambda(t-\lambda)\|_1 \leqq |Bu_0| \qquad \text{for } t \in [s, T] .$$

To this end, for each $\nu > 0$ and integer j, k, with $j \leqq k$

$$\begin{aligned}
&\|(I-\lambda B)^{-k}(I-\nu B)^{-1}u_0 - (I-\lambda B)^{-j}(I-\nu B)^{-1}u_0\|_1 \\
&\quad = \lim_{h\downarrow 0} \|(I-\lambda B_h)^{-k}(I-\nu B_h)^{-1}u_0 - (I-\lambda B_h)^{-j}(I-\nu B_h)^{-1}u_0\|_1 \\
&\quad \leqq \lim_{h\downarrow 0} \textstyle\sum_{i=j+1}^k \|(I-\lambda B_h)^{-i}(I-\nu B_h)^{-1}u_0 - (I-\lambda B_h)^{-i+1}(I-\nu B_h)^{-1}u_0\|_1 \\
&\quad \leqq \lim_{h\downarrow 0} \textstyle\sum_{i=j+1}^k \|(I-\lambda B_h)^{-1}(I-\nu B_h)^{-1}u_0 - (I-\nu B_h)^{-1}u_0\|_1 \\
&\quad = \lim_{h\downarrow 0} (k-j)\|(I-\lambda B_h)^{-1}(I-\nu B_h)^{-1}u_0 - (I-\lambda B_h)^{-1}(I-\lambda B_h)(I-\nu B_h)^{-1}u_0\|_1 \\
&\quad \leqq \lim_{h\downarrow 0} (k-j)\lambda\nu^{-1}\|(I-\nu B_h)^{-1}u_0 - u_0\|_1 . \\
&\quad = (k-j)\lambda\nu^{-1}\|(I-\nu B)^{-1}u_0 - u_0\|_1
\end{aligned}$$

by Lemma 2.3 (i) and Proposition 4.1. Hence, taking limits inferior as $v \downarrow 0$,

$$\lambda^{-1}\|(I-\lambda B)^{-k}u_0-(I-\lambda B)^{-j}u_0\|_1 \leqq (k-j)|Bu_0|\,. \tag{4.20}$$

Putting $k=[t/\lambda]$, $j=[(t-\lambda)/\lambda]$ in the above inequality, we obtain (4.19). Since $\nabla\beta(u^\lambda)_n$ converges weakly in $L^2(s, T; L^2(\boldsymbol{R}^N)^N)$ to $\nabla\beta(u^\lambda)$ as $n\to\infty$,

$$\begin{aligned}(1-\kappa)&\int_s^T \|\nabla\beta(u^\lambda(t))\|_2^2\,dt\\ &\leqq \liminf_{n\to\infty}(1-\kappa)\int_s^T \|\nabla\beta(u^\lambda(t))_n\|_2^2\,dt\\ &\leqq \big(|Bu_0|\cdot\sup\{\beta'(\sigma); |\sigma|\leqq r\}+C(\kappa)\|u_0\|_1\cdot\textstyle\sum_i \sup\{|a^i(\sigma)|; |\sigma|\leqq r\}^2\big)\,rT.\end{aligned}$$

Therefore $\{\nabla\beta(u^\lambda); \lambda\in(0, s)\}$ is bounded in $L^2(s, T; L^2(\boldsymbol{R}^N)^N)$. Since $u^\lambda(t)\to u(t)$ as $\lambda\downarrow 0$ in $L^1(\boldsymbol{R}^N)$ uniformly for bounded t, we see easily that $\nabla\beta(u)\in L^2(s, T; L^2(\boldsymbol{R}^N)^N)$ and

$$\begin{aligned}(1-\kappa)&\int_s^T \|\nabla\beta(u(t))\|_2^2\,dt\\ &\leqq \big(|Bu_0|\cdot\sup\{\beta'(\sigma); |\sigma|\leqq r\}+C(\kappa)\|u_0\|_1\cdot\textstyle\sum_i \sup\{|a^i(\sigma)|; |\sigma|\leqq r\}^2\big)rT.\end{aligned}$$

Since the right-hand side of the above inequality does not depend on s and s can be taken arbitrarily, it follows that $\nabla\beta(u)\in L^2(0, T; L^2(\boldsymbol{R}^N)^N)$.

To conclude that u is a BV solution of the problem (M) provided $u_0\in BV(\boldsymbol{R}^N)\cap L^\infty(\boldsymbol{R}^N)\cap\hat{D}(B)$, it remains to show that $u\in BV(\boldsymbol{R}^N\times(0, T))$ for each $T>0$. But this is obvious from

$$\|Du(t)\|_M \leqq \liminf_{h\downarrow 0}\|Du_h(t)\|_M \leqq \|Du_0\|_M \qquad \text{for } t\geqq 0\,,$$

and the following estimate which is obtained by (4.20) with $k=[t/\lambda]$, $j=[s/\lambda]$:

$$\|u^\lambda(t)-u^\lambda(s)\|_1 \leqq ([t/\lambda]-[s/\lambda])\lambda|Bu_0| \qquad \text{for } 0<s<t\,. \tag{4.21}$$

PROOF OF (ii). We first show that a BV solution z in the sense of Yin satisfies the following identity: For each $T>0$ and $\varphi\in C_0^\infty(\boldsymbol{R}^N)$,

$$\begin{aligned}&\langle\Psi(z(t)), \varphi\rangle-\langle\Psi(z(s)), \varphi\rangle\\ &\qquad-\int_s^t \langle A(z(\tau))-A(0), \nabla([\beta(z(\tau))-\beta(0)]\varphi)\rangle\,d\tau\\ &\qquad=-\int_s^t \langle\nabla\beta(z(\tau)), \nabla([\beta(z(\tau))-\beta(0)]\varphi)\rangle\,d\tau\end{aligned} \tag{4.22}$$

holds for a.e. t, s with $0<s<t<T$ where $\Psi(r)=\displaystyle\int_0^r [\beta(\sigma)-\beta(0)]\,d\sigma$.

fact that $B_0 v_k^\lambda = \lambda^{-1}(v_k^\lambda - v_k)$ converges to $\lambda^{-1}(v^\lambda - v)$ in $L^1(\mathbf{R}^N)$, it follows that $v^\lambda \in D(B)$ and $Bv^\lambda \ni \lambda^{-1}(v^\lambda - v)$; notice that B is the closure of B_0. Hence this shows that $R(I - \lambda B) = L^1(\mathbf{R}^N)$ and $v^\lambda = (I - \lambda B)^{-1} v$. Further, it follows from Lemma 2.3 (i) and Lemma 4.3 (ii) that

$$\begin{aligned}
&\limsup_{h\downarrow 0} \|(I - \lambda B_h)^{-1} v - (I - \lambda B)^{-1} v\|_1 \\
&\leqq \limsup_{h\downarrow 0} [\|(I - \lambda B_h)^{-1} v - (I - \lambda B_h)^{-1} v_k\|_1 + \|v_{h,k}^\lambda - v_k^\lambda\|_1 + \|v_k^\lambda - v^\lambda\|_1] \\
&\leqq \|v - v_k\|_1 + \|v_k^\lambda - v^\lambda\|_1 .
\end{aligned}$$

Passing to the limit as $k \to \infty$, we obtain the desired assertion. ■

PROOF OF PROPOSITION 4.2. Clearly, it suffices to show that $BV(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N) \subset \overline{D(B_0)}$. Let $v \in BV(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$ and $v^\lambda = (I - \lambda B_0)^{-1} v$ for $\lambda > 0$. Then Lemma 4.3 (ii) asserts that $v_h^\lambda \to v^\lambda$ in $L^1(\mathbf{R}^N)$ as $h \downarrow 0$. Hence we have

$$(4.14) \quad \|v^\lambda\|_p \leqq \|v\|_p, \qquad p = 1, \infty, \qquad \|\tau_y v^\lambda - v^\lambda\|_1 \leqq \|\tau_y v - v\|_1 \qquad (y \in \mathbf{R}^N)$$

from (4.1). Moreover, passing to the limit as $h \downarrow 0$ in (4.7), we obtain

$$\begin{aligned}
(4.15) \quad &\lambda^{-1} \int_{|x| > \rho N^{1/2}} |v^\lambda|\, dx \\
&\leqq N\lambda^{-1} \int_{|x| > \tau} |v|\, dx + \big(\textstyle\sum_{i=1}^N \sup\{|a^i(s)|; |s| \leqq \|v\|_\infty\} \|D\varphi_{\rho,\tau}\|_\infty \\
&\qquad + \text{const.} \sup\{\beta_\varepsilon'(s)^{1/2} + |a(s)|; |s| \leqq \|v\|_\infty, \varepsilon > 0\}^2 \|D^2\varphi_{\rho,\tau}\|_\infty\big) \|v\|_1 .
\end{aligned}$$

For any positive number λ_0 (4.6) implies that

$$\limsup_{\rho\uparrow\infty} \sup_{\lambda \in (0,\lambda_0)} \int_{|x| > \rho N^{1/2}} |v^\lambda|\, dx \leqq N \int_{|x| > \tau} |v|\, dx .$$

Therefore the set $\{v^\lambda; \lambda \in (0, \lambda_0)\}$ is precompact in $L^1(\mathbf{R}^N)$. So we can choose a null sequence $(\lambda_n)_{n=1}^\infty$, $v^0 \in L^1(\mathbf{R}^N)$ so that $v^{\lambda_n} \to v^0$ as $n \to \infty$ in $L^1(\mathbf{R}^N)$. On the other hand, we see from the definition of B_0 that

$$(4.16) \qquad v^\lambda - v = \lambda[\Delta\beta(v^\lambda) - \nabla A(v^\lambda)] \qquad \text{in } \mathscr{D}'(\mathbf{R}^N).$$

Substituting $\lambda = \lambda_n$ in (4.16) and letting $n \to \infty$, we see that the right-hand side of (4.16) converges to 0 in $\mathscr{D}'(\mathbf{R}^N)$, since (4.14) and $\lambda_n \downarrow 0$. Hence $v = v^0 \in \overline{D(B_0)}$. ■

The rest of the paper is devoted to show that the limit function $T(\cdot)u_0$ obtained through the convergence (1.16) (with v replaced by the given initial-value u_0) is the desired solution of (M). We first observe that if $u_0 \in L^1(\mathbf{R}^N) \cap L^\infty(\mathbf{R}^N)$, the function $u(\cdot) \equiv T(\cdot)u_0$ lies in $L^\infty(\mathbf{R}^N \times (0, \infty)) \cap C([0, \infty); L^1(\mathbf{R}^N))$.

Indeed, by virtue of (1.16), we see that $u_h(t) \equiv S_h^{[t/h]}u_0 \to u(t)$ as $h \downarrow 0$ in $L^1(\boldsymbol{R}^N)$, uniformly for bounded t. This, together with the fact that $u_h \in C([0, \infty); L^1(\boldsymbol{R}^N))$ and $\|u_h(t)\|_\infty \leqq \|u_0\|_\infty$ implies that $u \in L^\infty(\boldsymbol{R}^N \times (0, \infty)) \cap C([0, \infty); L^1(\boldsymbol{R}^N))$.

Next we will show that u is a distribution solution to the problem (M). Choose any $\varphi \in C_0^\infty(\boldsymbol{R}^N \times (0, \infty))$ and recall that $h^{-1}[u_h(t+h) - u_h(t)] = B_h u_h(t)$ by the definition of u_h. Multiplying both sides by φ, and then integrating the resultant equality with respect to (x, t) over $\boldsymbol{R}^N \times (0, \infty)$, we obtain

$$\int_0^\infty \langle h^{-1}[u_h(t+h) - u_h(t)], \varphi(\cdot, t)\rangle \, dt$$
$$= \int_0^\infty dt \int h^{-1}\langle F_\varepsilon(u_h(t)(\cdot - h\xi), \xi) - F_\varepsilon(u_h(t), \xi), \varphi(\cdot, t)\rangle \, d\xi \, .$$

Recalling that $\operatorname{supp}\varphi$ is compact in $\boldsymbol{R}^N \times (0, \infty)$, we have

$$-\int_h^\infty \langle u_h(t), (-h)^{-1}[\varphi(\cdot, t-h) - \varphi(\cdot, t)]\rangle \, dt$$
$$= \int_0^\infty dt \int \langle F_\varepsilon(u_h(t), \xi), h^{-1}[\varphi(\cdot + h\xi, t) - \varphi(\cdot, t)]\rangle \, d\xi$$

for sufficiently small $h > 0$. Therefore, in a way similar to the derivation of Lemma 4.3 (ii), we obtain

$$-\int_0^\infty \int u\varphi_t \, dx \, dt = \int_0^\infty \int [\beta(u)\,\Delta\varphi + A(u)\nabla\varphi] \, dx \, dt \, .$$

We will prove that (i): u is a BV solution of the problem (M) provided that $u_0 \in BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N) \cap \hat{D}(B)$, and then that (ii): u is a weak solution of the problem (M) provided that $u_0 \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$.

PROOF OF (i). First we define, for $\lambda > 0$, an $L^1(\boldsymbol{R}^N)$-valued function:

$$u^\lambda(t) = (I - \lambda B)^{-[t/\lambda]}u_0 \qquad \text{for } t \geqq 0 \, .$$

Notice that $u^\lambda(t) \to u(t)$ as $\lambda \downarrow 0$ in $L^1(\boldsymbol{R}^N)$ uniformly for bounded t by the exponential formula. Therefore it is seen that if u_0 belongs to $BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, then $\{\nabla\beta(u^\lambda); \lambda \in (0, s)\} \subset L^2(s, T; L^2(\boldsymbol{R}^N)^N)$ for each s, T with $0 < s < T$. Indeed, the restriction of the operator $(I - \lambda B)^{-1}$ to $BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$ coincides with $(I - \lambda B_0)^{-1}$ by Proposition 4.1 and Lemma 4.3 (ii). From this it follows that

$$u^\lambda(t) = (I - \lambda B_0)^{-1}u^\lambda(t - \lambda) \qquad \text{for } t \in [s, \infty)$$

since $u^\lambda(t - \lambda) \in BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$. According to the definition of B_0, We now proceed our argument in a way similar to [18], Proposition 5. By the definition of BV solution in the sense of Yin, it is easily seen that for each $\psi \in C_0^\infty(\mathbf{R}^N \times (0, T))$

$$\int_{\mathbf{R}^N \times (0,T)} \psi(\partial z/\partial \tau) - \int_0^T \langle A(z(\tau)) - A(0), \nabla \psi(\tau) \rangle \, d\tau = -\int_0^T \langle \nabla \beta(z(\tau)), \nabla \psi(\tau) \rangle \, d\tau . \tag{4.23}$$

Now, applying the same argument as in the proof of Proposition 3.3, we will substitute (at least formally) the test function $[\beta(z) - \beta(0)]\varphi 1_{[s,t]}$ for ψ in (4.23), where $1_{[s,t]}$ means the characteristic function of the interval $[s, t]$. For this purpose, let $j \in C_0^\infty(\mathbf{R}^1)$ with $j \geqq 0$, $\operatorname{supp} j \subset [-1, 1]$, $\int_{-\infty}^{\infty} j(\sigma)\, d\sigma = 1$ and define for $\nu > 0$

$$\alpha_\nu(\tau) = \int_{-\infty}^{\tau} j_\nu(\sigma)\, d\sigma \qquad \text{for } \tau \in \mathbf{R}^1 ,$$

where $j_\nu(\sigma) = \nu^{-1} j(\nu^{-1}\sigma)$. Since $\operatorname{supp}[\alpha_\nu(\cdot - s) - \alpha_\nu(\cdot - t)] \subset [s - \nu, t + \nu]$, $\zeta_\nu \equiv \varphi[\alpha_\nu(\cdot - s) - \alpha_\nu(\cdot - t)]$ belongs to $C_0^\infty(\mathbf{R}^N \times (0, T))$, provided $0 < \nu < \min\{s, T - t\}$. We then extend $[\beta(z) - \beta(0)]\zeta_\nu$ as the zero-extention on $\mathbf{R}^N \times \mathbf{R}^1$. Then the desired test functions are defined as follows:

$$\psi_{\nu,\delta} = \rho_\delta * ([\beta(z) - \beta(0)]\zeta_\nu) \qquad \text{for } \delta > 0 ,$$

where $\rho_\delta *$ is the usual $(N + 1)$-dimensional Friedrichs mollifier with support $\{(x, \tau) \in \mathbf{R}^{N+1};\ |(x, \tau)| \leqq \delta\}$ defined for positive number δ by a radial function ρ. Since $\psi_{\nu,\delta} \in C_0^\infty(\mathbf{R}^N \times (0, T))$ for sufficiently small δ, (4.23) is valid for $\psi_{\nu,\delta}$ replaced by ψ. On the other hand, $[\beta(z) - \beta(0)]\zeta_\nu \in BV(\mathbf{R}^N \times (0, T))$ and $\nabla([\beta(z) - \beta(0)]\zeta_\nu) \in L^2(0, T; L^2(\mathbf{R}^N)^N)$, and so we can take the limit as $\delta \downarrow 0$ in (4.23) with $\psi_{\nu,\delta}$ in the same way as in the derivation of (3.20);

(4.24)

$$\int_{\mathbf{R}^N \times (0,T)} \varphi[\alpha_\nu(\cdot - s) - \alpha_\nu(\cdot - t)]\overline{[\beta(z) - \beta(0)]}(\partial z/\partial \tau) - \int_0^T [\alpha_\nu(\tau - s) - \alpha_\nu(\tau - t)]\langle A(z(\tau)) - A(0), \nabla([\beta(z(\tau)) - \beta(0)]\varphi)\rangle \, d\tau$$
$$= -\int_0^T [\alpha_\nu(\tau - s) - \alpha_\nu(\tau - t)]\langle \nabla\beta(z(\tau)), \nabla([\beta(z(\tau)) - \beta(0)]\varphi)\rangle \, d\tau .$$

Owing to the result due to Yin ([21], Lemma 2.3), the set of jump points of z, i.e. $\{(x, \tau) \in \boldsymbol{R}^N \times (0, T); z^-(x, \tau) < z^+(x, \tau)\}$, is negligible with respect to the Hausdorff measure of dimension N. This implies the identities

$$\partial \Psi(z)/\partial \tau = [\hat{\beta}(z) - \beta(0)](\partial z/\partial \tau) = \overline{[\beta(z) - \beta(0)]}(\partial z/\partial \tau)$$

in the sense of measures. Hence the first term on the left-hand side of (4.24) is transformed as follows:

$$\begin{aligned}
&\int_{\boldsymbol{R}^N \times (0,T)} \varphi[\alpha_\nu(\tau - s) - \alpha_\nu(\tau - t)](\partial \Psi(z)/\partial \tau) \\
&= -\int_0^T \langle \Psi(z(\tau)), \varphi\rangle (d/d\tau)[\alpha_\nu(\tau - s) - \alpha_\nu(\tau - t)]\, d\tau \\
&= \int_0^T [j_\nu(\tau - t) - j_\nu(\tau - s)]\langle \Psi(z(\tau)), \varphi\rangle\, d\tau\,.
\end{aligned}$$

Therefore we obtain (4.22) for Lebesgue points t, s $(0 < s < t < T)$ of the function $\langle \Psi(z(\cdot)), \varphi\rangle$ by letting $\nu \downarrow 0$ in (4.24).

We then prove the assertion (ii). For a general element $u_0 \in L^1(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N)$, we can apply Proposition 4.2 to choose a sequence $(u_{0m})_{m=1}^\infty$ contained in $D(B_0)(\subset BV(\boldsymbol{R}^N) \cap L^\infty(\boldsymbol{R}^N) \cap \hat{D}(B))$ so that $u_{0m} \to u_0$ in $L^1(\boldsymbol{R}^N)$ as $m \to \infty$ and $\|u_{0m}\|_\infty \leqq \|u_0\|_\infty$. Set $u_m(\tau) \equiv T(\tau)u_{0m}$ for $\tau \geqq 0$. Then, as shown in (i), u_m is a BV solution (in the sense of Definition in Section 1) with $u_m(0) = u_{0m}$ for $m \geqq 1$. Since each u_m is the semigroup solution, it is obvious that $u_m \in C([0, \infty); L^1(\boldsymbol{R}^N))$. Hence for each $f \in L^\infty(\boldsymbol{R}^N)$, the function $\tau \in [0, \infty) \to \langle \Psi(u_m(\tau)), f\rangle$ is continuous, more precisely Lipschitz continuous by (4.22). Thus, as seen before, each function u_m satisfies (4.22) for all t, s with $0 < s < t < T$.

Therefore we obtain

$$\begin{aligned}
&\int_s^t d\tau \int |\nabla \beta(u_m(\tau))|^2 \varphi\, dx \\
&\leqq \int_s^t \|\nabla \beta(u_m(\tau))\|_2 \|\nabla \varphi\|_\infty \|\beta(u_m(\tau)) - \beta(0)\|_2\, d\tau + \|\Psi(u_m(t)) - \Psi(u_m(s))\|_1 \|\varphi\|_\infty \\
&\quad + \int_s^t \|A(u_m(\tau)) - A(0)\|_2 (\|\nabla \beta(u_m(\tau))\|_2 \|\varphi\|_\infty + \|\beta(u_m(\tau)) - \beta(0)\|_2 \|\nabla \varphi\|_\infty)\, d\tau
\end{aligned}$$

for all t, s with $0 < s < t < T$. Substituting φ_k for φ, where $(\varphi_k)_{k=1}^\infty$ is the sequence of smooth functions as defined in the Section 3, and then letting k tend to infinity, we obtain the following estimate for the energy term by using (3.23) and (3.24):

$$\int_s^t \|\nabla\beta(u_m(\tau))\|_2^2\, d\tau$$

$$\leqq \|\Psi(u_m(t)) - \Psi(u_m(s))\|_1 + \int_s^t \|A(u_m(\tau)) - A(0)\|_2 \|\nabla\beta(u_m(\tau))\|_2\, d\tau$$

$$\leqq \sup\{|\beta(\sigma) - \beta(0)|; |\sigma| \leqq r\} \|u_m(t) - u_m(s)\|_1$$

$$+ \int_s^t (C(\kappa)\|A(u_m(\tau)) - A(0)\|_2^2 + \kappa\|\nabla\beta(u_m(\tau))\|_2^2)\, d\tau\,,$$

where $r = \|u_0\|_\infty$, $0 < \kappa < 1$, and a constant $C(\kappa)$ depends only on κ. Hence

$$(1 - \kappa)\int_s^t \|\nabla\beta(u_m(\tau))\|_2^2\, d\tau$$

$$\leqq 2\sup\{|\beta(\sigma) - \beta(0)|; |\sigma| \leqq r\}\cdot\sup\{\|u_{0m}\|_1; m \geqq 1\}$$

$$+ C(\kappa)r\cdot\sup\{\|u_{0m}\|_1; m \geqq 1\}(t - s)\textstyle\sum_i \sup\{|a^i(\sigma)|; |\sigma| \leqq r\}^2\,.$$

Thus $\{\nabla\beta(u_m); m \geqq 1\}$ is bounded in $L^2(s, t; L^2(\boldsymbol{R}^N)^N)$. Since $u_m(\tau)$ converges in $L^1(\boldsymbol{R}^N)$ to $u(\tau)$ as $m \to \infty$ uniformly for bounded τ, we obtain $\nabla\beta(u) \in L^2(s, t; L^2(\boldsymbol{R}^N)^N)$. Thus u is a weak solution of the problem (M).

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