HIROSHIMA MATH. J.
23 (1993), 565–575

Error bounds for asymptotic expansions of some distributions in a multivariate two-stage procedure

Satoru MUKAIHATA and Yasunori FUJIKOSHI

(Received April 22, 1992)

1. Introduction

In this paper we obtain error bounds for asymptotic expansions of the distributions of test statistics in a multivariate two-stage procedure. These statistics can be expressed as a scale mixture of a chi-square variable χ_p^2 with p degrees of freedom, i.e.

$$X = \sigma\chi_p^2 , \tag{1.1}$$

where σ is a positive-valued random variable and is independent of χ_p^2. The scale factor σ is defined by $\sigma = \frac{1}{p}\mathrm{tr}(S^{-1})$ for the case of a multivariate population, where nS is distributed as a Wishart distribution $\mathrm{W}_p(I_p, n)$. For the case of two multivariate populations, $\sigma = \frac{1}{2}(\sigma_1 + \sigma_2)$, where $\sigma_i = \frac{1}{p}\mathrm{tr}(S_i^{-1})$, $i =$ 1, 2, and n_iS_i are independently distributed as $\mathrm{W}_p(I_p, n_i)$. Hyakutake and Siotani [6] obtained asymptotic expansions of the distribution of X in the two cases. Our purpose of this paper is to obtain explicit bounds for the approximations based on these asymptotic expansions. The method used is based on a general result (see, e.g., Fujikoshi and Shimizu [5]) for scale mixtures of the gamma distribution.

In the use of the result it is necessary that we evaluate the exact moments of $\sigma^{\pm 1}$. But it is generally difficult to obtain the exact ones in these cases. In this paper we derive appropriate approximations for their moments with error bounds. Using these results, we will obtain error bounds for asymptotic expansions of the distribution of X.

2. Scale mixture of a chi-square distribution

Here, we consider a scale mixture of a chi-square variable with p degrees of freedom, i.e.

$$X = \sigma\chi_p^2 . \tag{2.1}$$

Let $G(x; p)$ and $g(x; p)$ be the distribution and the probability density function of χ_p^2, respectively. Then the distribution function $F(x)$ of X can be expressed as

$$F(x) = \mathrm{E}_\sigma[G(x/\sigma; p)] .$$

The following two types of approximations have been proposed (cf. Fujikoshi and Shimizu [5]): For $\delta = -1, 1$,

$$G_{\delta,k}(x; p) = \mathrm{E}_\sigma[G_{\delta,k}(x, \sigma; p)] \tag{2.2}$$

where

$$G_{\delta,k}(x, \sigma; p) = G(x; p) + \sum_{j=1}^{k-1} \frac{1}{j!} b_{\delta,j}(x; p) g(x; p)(\sigma^\delta - 1)^j.$$

Here the coefficients $b_{\delta,j}(x; p)$ are defined by

$$b_{\delta,j}(x; p) = \left(\frac{d}{ds}\right)^j G(xs^{-\delta}; p)\bigg|_{s=1} \bigg/ g(x; p)$$

or equivalently

$$b_{1,j}(x; p) = -xL_{j-1}^{(p/2)}(x/2) ,$$

$$b_{-1,j}(x; p) = (-1)^{j-1} x L_{j-1}^{(p/2-j)}(x/2) ,$$

where $L_j^{(\lambda)}(x)$ is the Laguerre polynomial defined by

$$L_j^{(\lambda)}(x) = (-1)^j \mathrm{e}^x x^{-\lambda} (d/dx)^j (x^{\lambda+j} \mathrm{e}^{-x}) .$$

When the exact moments of σ^δ are not available, we cannot use the approximation $G_{\delta,k}(x; p)$. Therefore, we consider to approximate $(\sigma^\delta - 1)^j$ by a simple variable, $m_{\delta,j}^*(\sigma)$, whose exact moments are available. Namely, suppose that for $j = 1, \ldots, k-1$,

$$(\sigma^\delta - 1)^j = m_{\delta,j}^*(\sigma) + R_{\delta,j}(\sigma) \tag{2.3}$$

and $\mathrm{E}_\sigma[m_{\delta,j}^*(\sigma)]$ can be evaluated. Then, we have the following result:

THEOREM 2.1. *Let $X = \sigma\chi_p^2$ be a scale mixture of a chi-square variable. Suppose that for a given positive integer k, $\mathrm{E}[\sigma^k] < \infty$, $\mathrm{E}[\sigma^{-k}] < \infty$ and $\mathrm{E}_\sigma[m_{\delta,j}^*(\sigma)] < \infty$, $j = 1, \ldots, k-1$. Then, letting*

$$G_{\delta,k}^*(x; p) = G(x; p) + \sum_{j=1}^{k-1} \frac{1}{j!} b_{\delta,j}(x; p) g(x; p) \mathrm{E}_\sigma[m_{\delta,j}^*(\sigma)] , \tag{2.4}$$

it holds that

$$\begin{aligned}\sup_x |F(x) - G^*_{\delta,k}(x; p)| &\le \frac{1}{k!}\bar{b}_{\delta,k}(p)\mathrm{E}_\sigma[(\sigma \vee \sigma^{-1} - 1)^k] \\ &\quad + \sum_{j=1}^{k-1} \frac{1}{j!}\bar{b}_{\delta,j}(p)|\mathrm{E}_\sigma[R_{\delta,j}(\sigma)]| \\ &\le \frac{1}{k!}\bar{b}_{\delta,k}(p)\mathrm{E}_\sigma[|\sigma - 1|^k + |\sigma^{-1} - 1|^k] \\ &\quad + \sum_{j=1}^{k-1} \frac{1}{j!}\bar{b}_{\delta,j}(p)\mathrm{E}_\sigma[|R_{\delta,j}(\sigma)|],\end{aligned}$$

where $\bar{b}_{\delta,j}(p) = \sup_{x>0} |b_{\delta,j}(x; p)g(x; p)|$, $j = 1, \ldots, k$.

PROOF. It is known (cf. Fujikoshi and Shimizu [5]) that

$$\sup_x |F(x) - G_{\delta,k}(x; p)| \le \frac{1}{k!}\bar{b}_{\delta,k}(p)\mathrm{E}_\sigma[(\sigma \vee \sigma^{-1} - 1)^k]. \tag{2.5}$$

Therefore, the result follows from that

$$G_{\delta,k}(x; p) = G^*_{\delta,k}(x; p) + \sum_{j=1}^{k-1} \frac{1}{j!} b_{\delta,j}(x; p)g(x; p)\mathrm{E}_\sigma[R_{\delta,j}(\sigma)].$$

We note that if the first $k - 1$ moments of σ^δ in Theorem 2.1 can be obtained, we can take $R_{\delta,j} = 0$ for $j = 1, \ldots, k - 1$.

3. Test statistics in a multivariate two-stage procedure

Some multivariate two-stage procedures have been proposed (see, e.g., Chattergee [1], Dudewicz and Bishop [2]) for the purpose of overcoming the complexity which arises in statistical analysis of a normal population or several normal populations with different covariance matrices. For a summary, see Siotani, Hayakawa and Fujikoshi [7, Sections 5.6.3 and 6.4.3]. Here we consider test procedures for hypotheses

(i) $H_0: \boldsymbol{\mu} = \mathbf{0}$ in $\mathrm{N}_p(\boldsymbol{\mu}, \Sigma)$

and

(ii) $H_0: \boldsymbol{\mu}_1 = \boldsymbol{\mu}_2$ in $\mathrm{N}_p(\boldsymbol{\mu}_1, \Sigma_1)$ and $N_p(\boldsymbol{\mu}_2, \Sigma_2)$,

under a two-stage sampling scheme. For the test of (i), Chatterjee [1] proposed a statistic whose null distribution can be expressed as

$$X = \sigma\chi^2_p, \tag{3.1}$$

where $\sigma = \frac{1}{p}\operatorname{tr}(S^{-1})$, $nS \sim \mathrm{W}_p(I_p, n)$ and S is independent of χ_p^2. The two-stage procedure for the test of (ii) is based on a statistic whose null distribution can be expressed as

$$\tilde{X} = \tilde{\sigma}\chi_p^2, \tag{3.2}$$

where $\tilde{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_2)$, $\sigma_i = \frac{1}{p}\operatorname{tr}(S_i^{-1})$, $i = 1, 2$, and S_i's are independent of χ_p^2. Here $n_1 S_1$ and $n_2 S_2$ are independently distributed as Wishart distributions $\mathrm{W}_p(I_p, n_1)$ and $\mathrm{W}_p(I_p, n_2)$, respectively, with $n_1 = n_2 = n$.

The exact distributions of X and $\tilde{X}$ are known only when $p = 1$, 2 and $p = 1$, respectively (see, e.g., Hyakutake and Siotani [6]). On the other hand, Hyakutake and Siotani [6] obtained asymptotic expansions of the distributions of X and $\tilde{X}$ with respect to n^{-1}. Our purpose is to obtain error bounds for their asymptotic expansions. For the distribution of $\tilde{X}$, we shall treat the case where n_1 and n_2 may be different.

3.1. The distribution of X

It is known (see, e.g., Watamori [8]) that the first four moments of

$$\sigma = \frac{1}{p}\operatorname{tr}(S^{-1}), \qquad \text{where } nS \sim \mathrm{W}_p(I_p, n) \tag{3.3}$$

are given as follows:

LEMMA 3.1. *Suppose that $nS \sim \mathrm{W}_p(I_p, n)$. Then the k-th moment of $\sigma = \frac{1}{p}\operatorname{tr}(S^{-1})$ exists for $q = n - p \geq 2k$, and*

$$\mathrm{E}(\sigma) = d_1^{-1} n,$$

$$\mathrm{E}(\sigma^2) = d_2^{-1} n^2\{p(q-2) + 2\},$$

$$\mathrm{E}(\sigma^3) = d_3^{-1} n^3\{p^2(q^2 - 5q + 2) + 6p(q-3) + 16\},$$

$$\mathrm{E}(\sigma^4) = d_4^{-1} n^4\{p^3(q^4 - 11q^3 + 28q^2 + 8q - 32) + 12p^2(q^3 - 9q^2 + 18q - 4) + 4p(19q^2 - 117q + 152) + 48(5q - 11)\},$$

where

$$d_1 = q - 1, \qquad d_2 = pq(q-1)(q-3),$$

$$d_3 = p^2 q(q+1)(q-1)(q-3)(q-5)$$

and

$$d_4 = p^3q(q+1)(q+2)(q-1)(q-2)(q-3)(q-5)(q-7).$$

Let

$$m_{\delta,j} = \mathrm{E}_\sigma[(\sigma^\delta - 1)^j] \tag{3.4}$$

for $\delta = 1, -1$. Then, from Lemma 3.1 we obtain that

$$m_{1,1} = d_1^{-1}(p+1), \qquad m_{1,2} = d_2^{-1}[2q^2 + p\{(p^2+2p+7)q - 2p(p-1)\}],$$

$$m_{1,3} = d_3^{-1}[2(3p^2+3p+8)q^3 + p\{(p^4+3p^3+21p^2+p+78)q^2 - p(5p^3-3p^2+45p-123)q + 2p^2(p-1)(p-23)\}],$$

$$\begin{aligned} m_{1,4} = d_4^{-1}[&12pq^6 + 4(3p^4+6p^3+37p^2-5p+60)q^5 + (p^7+4p^6 \\ &+ 42p^5 - 32p^4 + 385p^3 - 588p^2 + 1824p - 528)q^4 \\ &- p\{(11p^6 + 8p^5 + 306p^4 - 504p^3 + 2679p^2 - 5552p \\ &+ 3904)q^3 - 4p(7p^5 - 53p^4 + 153p^3 - 1083p^2 + 2375p \\ &- 2556)q^2 - 4p^2(2p^4 + 170p^3 - 507p^2 + 2388p - 2817)q \\ &+ 16p^3(p-1)(2p^2+19p-250)\}]. \end{aligned} \tag{3.5}$$

On the other hand, it seems that the exact moments of σ^{-1} are very complicated. However, we can obtain the following result:

LEMMA 3.2. *Under the same notation as in Lemma* 3.1, *it holds that*

$$\mathrm{E}_\sigma[|\sigma^{-1} - 1|^j] \le \mathrm{E}_\sigma[|\sigma - 1|^j] + \mathrm{E}[|n^{-1}\chi^2_{n-p+1} - 1|^j]$$

for $j \ge 1$.

PROOF. First for $\sigma \ge 1$, we have

$$0 \le 1 - \sigma^{-1} = \sigma^{-1}(\sigma - 1) \le \sigma - 1,$$

which implies

$$|\sigma^{-1} - 1|^j \le |\sigma - 1|^j, \qquad j = 1, 2, \ldots. \tag{3.6}$$

Next, assume $0 < \sigma \le 1$ and let $H = [\boldsymbol{h}_1, \ldots, \boldsymbol{h}_p]$ be a $p \times p$ orthogonal matrix. Then

$$\begin{aligned} 0 \le \sigma^{-1} - 1 &= \left(\frac{1}{p}\sum_{i=1}^p \boldsymbol{h}_i' S^{-1}\boldsymbol{h}_i\right)^{-1} - 1 \\ &\le \frac{1}{p}\sum_{i=1}^p (\boldsymbol{h}_i' S^{-1}\boldsymbol{h}_i)^{-1} - 1 \\ &= \frac{1}{p}\sum_{i=1}^p \{(\boldsymbol{h}_i' S^{-1}\boldsymbol{h}_i)^{-1} - 1\}. \end{aligned}$$

Here we note that $\{\boldsymbol{h}_i'(nS)^{-1}\boldsymbol{h}_i\}^{-1}$ $(i = 1, \dots, p)$ are identically distributed as χ^2_{n-p+1} (see, e.g., Siotani, Hayakawa and Fujikoshi [7]), but they are not necessarily independent. Hence

$$
\begin{aligned}
(3.7)\quad & \mathrm{E}_S\left[\left|\frac{1}{p}\sum_{i=1}^{p}\{(\boldsymbol{h}_i'S^{-1}\boldsymbol{h}_i)^{-1} - 1\}\right|^j\right] \\
&= \frac{1}{p^j}\mathrm{E}_S\left[\left|\sum_{i_1,\dots,i_j=1}^{p}\{(\boldsymbol{h}_{i_1}'S^{-1}\boldsymbol{h}_{i_1})^{-1} - 1\}\dots\{(\boldsymbol{h}_{i_j}'S^{-1}\boldsymbol{h}_{i_j})^{-1} - 1\}\right|\right] \\
&\le \frac{1}{p^j}\sum_{i_1,\dots,i_j=1}^{p}\mathrm{E}_S[\{|(\boldsymbol{h}_{i_1}'S^{-1}\boldsymbol{h}_{i_1})^{-1} - 1|^j \times \cdots \times |(\boldsymbol{h}_{i_j}'S^{-1}\boldsymbol{h}_{i_j})^{-1} - 1|^j\}^{1/j}] \\
&\le \frac{1}{p^j}\sum_{i_1,\dots,i_j=1}^{p}\mathrm{E}_S\left[\frac{1}{j}\{|(\boldsymbol{h}_{i_1}'S^{-1}\boldsymbol{h}_{i_1})^{-1} - 1|^j + \cdots + |(\boldsymbol{h}_{i_j}'S^{-1}\boldsymbol{h}_{i_j})^{-1} - 1|^j\}\right] \\
&= \mathrm{E}[|n^{-1}\chi^2_{n-p+1} - 1|^j]\,.
\end{aligned}
$$

Using (3.6) and (3.7), we obtain the desired result.

Let

$$
(3.8)\qquad l_{2j} = \mathrm{E}[(n^{-1}\chi^2_{n-p+1} - 1)^{2j}]\,.
$$

Then for $j = 1, 2$,

$$
(3.9)\qquad
\begin{aligned}
l_2 &= \frac{2}{n} + \frac{(p-1)(p-3)}{n^2}\,, \\
l_4 &= \frac{4}{n^2}\left(3 + \frac{3p^2 - 20p + 29}{n}\right) + \frac{(p-1)(p-3)(p-5)(p-7)}{n^4}\,.
\end{aligned}
$$

Using the above results, we obtain the following results:

THEOREM 3.1. *Let $F(x)$ be the distribution function of X defined by* (3.1). *Then for the* (*non-modified*) *approximation $G_{1,k}(x; p)$ defined by* (2.7) *and its error*

$$
\varepsilon_{1,k} = \varepsilon_{1,k}(p) = \sup_{x>0}|F(x) - G_{1,k}(x; p)|\,,
$$

it holds that

$$
\varepsilon_{1,1} \le \bar{b}_{1,1}(p)\{m_{1,2}^{1/2} + (m_{1,2} + l_2)^{1/2}\}\,,
$$

$$
\varepsilon_{1,k} \le \frac{1}{k!}\bar{b}_{1,k}(p)\{m_{1,k} + (m_{1,k} + l_k)\} \qquad (k = 2,\ 4)\,,
$$

where $\bar{b}_{1,j}(p)$'s *are given in Theorem* 2.2, *and* $m_{1,j}$ *and* $l_{2,j}$ *are given by* (3.5) *and* (3.9), *respectively.*

PROOF. These inequalities are immediately obtained from Lemma 3.2 and Theorem 2.2.

Next, we consider the modified approximation $G^*_{\delta,k}(x; p)$ with $\delta = -1$ for $F(x)$. Since

$$\frac{1}{\sigma} - 1 = -(\sigma - 1) + \frac{(\sigma - 1)^2}{\sigma}, \tag{3.10}$$

we can approximate $(\sigma^{-1} - 1)$ by $-(\sigma - 1)$. Further, its error is bounded by

$$\begin{aligned} 0 \le \mathrm{E}\left\{\frac{(\sigma - 1)^2}{\sigma}\right\} &\le /\mathrm{E}\{(1 - \sigma^{-1})^2\}\mathrm{E}\{(\sigma - 1)^2\}]^{1/2} \\ &\le \frac{1}{2}\{E(\sigma - 1)^2 + \mathrm{E}(\sigma^{-1} - 1)^2\}. \end{aligned} \tag{3.11}$$

Using these results, we obtain the following result:

THEOREM 3.2. *Let* $G^*_{-1,2}(x; p)$ *be the modified approximation defined by* (2.9) *with* $m^*_{-1,1}(\sigma) = -(\sigma - 1)$. *Then it holds that*

$$\sup_{x>0} |F(x) - G^*_{-1,2}(x; p)| \le \frac{1}{2}\{\bar{b}_{-1,1}(p) + \bar{b}_{-1,2}(p)\}\{m_{1,2} + (m_{1,2} + l_2)\}.$$

3.2. The distribution of $\tilde{X}$

First, we consider the case when p is arbitrary. The following lemma is useful to obtain error bounds for asymptotic expansions of the distribution of $\tilde{X}$:

LEMMA 3.3. *Let* σ_1 *and* σ_2 *are positive and mutually independent, and let*

$$\tilde{\sigma} = \frac{1}{2}(\sigma_1 + \sigma_2). \tag{3.12}$$

Then for $j = 1, 2, \dots$

(i) $\mathrm{E}_{\tilde{\sigma}}[(\tilde{\sigma} - 1)^j] = 2^{-j} \sum_{i=0}^{j} \binom{j}{i} \mathrm{E}_{\sigma_1}[(\sigma_1 - 1)^{j-i}]\mathrm{E}_{\sigma_2}[(\sigma_2 - 1)^i]$,

(ii) $\mathrm{E}_{\tilde{\sigma}}[|\tilde{\sigma}^{-1} - 1|^j] \le \mathrm{E}_{\sigma_1}[|\sigma_1^{-1} - 1|^j] + \mathrm{E}_{\sigma_2}[|\sigma_2^{-1} - 1|^j]$.

PROOF. (i) is obvious from (3.12). Next, if $0 < \tilde{\sigma} \le 1$,

$$0 \le \tilde{\sigma}^{-1} - 1 \le (\sigma_1 \wedge \sigma_2)^{-1} - 1 = \sigma_1^{-1} \vee \sigma_2^{-1} - 1 .$$

Similarly, if $\tilde{\sigma} > 1$,

$$0 \ge \tilde{\sigma}^{-1} - 1 \ge (\sigma_1 \vee \sigma_2)^{-1} - 1 = \sigma_1^{-1} \wedge \sigma_2^{-1} - 1 .$$

Therefore

$$\begin{aligned} |\tilde{\sigma}^{-1} - 1|^j &\le \max\{|(\sigma_1^{-1} - 1) \vee (\sigma_2^{-1} - 1)|^j, \quad |(\sigma_1^{-1} - 1) \wedge (\sigma_2^{-1} - 1)|^j\} \\ &\le |\sigma_1^{-1} - 1|^j + |\sigma_2^{-1} - 1|^j , \end{aligned}$$

which implies (ii).

Using the above results, we obtain the following results:

THEOREM 3.3. *Let $\tilde{F}(x)$ be the distribution function of $\tilde{X}$ defined by* (3.2). *Then for the approximation $\tilde{G}_{1,k}(x; p)$ defined by* (2.7) *with σ replaced by $\tilde{\sigma}$ and its error*

$$\tilde{\varepsilon}_{1,k} = \sup_{x>0} |\tilde{F}(x) - \tilde{G}_{1,k}(x; p)| ,$$

it holds that

$$\tilde{\varepsilon}_{1,1} \le \bar{b}_{1,1}(p)\left[\tilde{m}_{1,2}^{1/2} + \left\{\sum_{i=1}^{2} (m_{1,2}^{(i)} + l_2^{(i)})\right\}^{1/2}\right],$$

$$\tilde{\varepsilon}_{1,k} \le \frac{1}{k!}\bar{b}_{1,k}(p)\left[\tilde{m}_{1,k} + \sum_{i=1}^{2} \{m_{1,k}^{(i)} + l_k^{(i)}\}\right] \qquad (k = 2,\ 4) ,$$

where

$$\tilde{m}_{1,j} = \mathrm{E}_{\tilde{\sigma}}[(\tilde{\sigma} - 1)^j] , \qquad m_{1,j}^{(i)} = \mathrm{E}_{\sigma_i}[(\sigma_i - 1)^j] \tag{3.13}$$

and

$$l_j^{(i)} = \mathrm{E}[(n_i^{-1}\chi^2_{n_i-p+1} - 1)^j] , \qquad i = 1,\ 2 . \tag{3.14}$$

PROOF. These inequalities are immediately obtained from Lemmas 3.2, 3.3, and Theorem 2.2.

Next, we consider the modified approximation $\tilde{G}^*_{\delta,k}(x; p)$ with $\delta = -1$ for $\tilde{F}(x)$. Using (3.10) and (3.11) with σ replaced by $\tilde{\sigma}$, we obtain the following result:

THEOREM 3.4. *Let $\tilde{G}^*_{-1,2}(x; p)$ be the modified approximation defined by* (2.9) *with $m^*_{-1,1}(\sigma)$ replaced by $-(\tilde{\sigma} - 1)$. Then, it holds that*

$$\sup_{x>0} |\tilde{F}(x) - \tilde{G}^*_{-1,2}(x; p)| \leq \frac{1}{2}\{\bar{b}_{-1,1}(p) + \bar{b}_{-1,2}(p)\}\left[\tilde{m}_{1,2} + \sum_{i=1}^{2} \{m_{1,2}^{(i)} + l_2^{(i)}\}\right].$$

Now, in the following we consider the case $p = 2$. For this special case, we can easily evaluate the exact moments of both $\sigma = \frac{1}{p}\operatorname{tr}(s^{-1})$ and σ^{-1} as follows:

LEMMA 3.4. *Let* $\sigma = \frac{1}{p}\operatorname{tr}(S^{-1})$, *where* $nS \sim \mathrm{W}_p(I_p, n)$. *When* $p = 2$, *it holds that*

(i) *if* $n - 2j - 1 > 0$,

$$\mathrm{E}(\sigma^j) = \frac{n-1}{n-2j-1}\cdot\frac{n^j}{\prod_{i=1}^{j}(n-i)},$$

(ii) *for* $j \geq 1$,

$$\mathrm{E}(\sigma^{-j}) = \frac{n-1}{n+2j-1}\cdot\frac{\prod_{i=0}^{j-1}(n+i)}{n^j}.$$

PROOF. When $p = 2$, we can write nS as

$$nS = \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix}' \begin{pmatrix} t_{11} & t_{12} \\ 0 & t_{22} \end{pmatrix},$$

where $t_{ii}^2 \sim \chi_{n-i+1}^2$, $i = 1, 2$, $t_{12} \sim \mathrm{N}(0, 1)$ and t_{11}, t_{12} and t_{22} are mutually independent. This implies

$$\begin{aligned} \sigma &= \frac{n}{2}(t_{11}^2 + t_{12}^2 + t_{22}^2)/(t_{11}^2 t_{22}^2) \\ &= (4T_1T_2)^{-1}\left(\frac{1}{2n}T_3\right)^{-1}, \end{aligned}$$

where $T_i = t_{ii}^2 T_3^{-1}$, $i = 1, 2$ and $T_3 = t_{11}^2 + t_{12}^2 + t_{22}^2$. It is easily seen that (i) $(T_1, T_2, 1 - T_1 - T_2) \sim \mathrm{D}_3(n/2, (n-1)/2, 1/2)$, a three dimensional Dirichlet distribution with degrees of freedom $n/2$, $(n-1)/2$ and $1/2$, and therefore $4T_1T_2 \sim \mathrm{B}((n-1)/2, 1)$, a beta distribution with degrees of freedom $(n-1)/2$ and 1, (ii) $T_3 \sim \chi_{2n}^2$ and (iii) (T_1, T_2) and T_3 are mutually independent. Using these properties, we can easily obtain the desired results.

From Lemma 3.4 we obtain that for $p = 2$,

$$m_{1,1} = \frac{3}{n-3}, \qquad m_{1,2} = \frac{n^2 + 11n - 30}{(n-2)(n-3)(n-5)},$$

(3.15)
$$m_{1,3} = \frac{13n^2 + 37n - 210}{(n-2)(n-3)(n-5)(n-7)},$$

$$m_{1,4} = \frac{3(n^4 + 38n^3 - 181n^2 - 514n + 2520)}{(n-2)(n-4)(n-3)(n-5)(n-7)(n-9)},$$

$$m_{-1,1} = -\frac{2}{n+1}, \qquad m_{-1,2} = \frac{n^2 + 8n - 1}{n(n+1)(n+3)},$$

(3.16)
$$m_{-1,3} = -\frac{2(2n^3 + 21n^2 - 2n + 3)}{n^2(n+1)(n+3)(n+5)},$$

$$m_{-1,4} = \frac{3n^5 + 62n^4 + 426n^3 + 28n^2 - 45n - 90}{n^3(n+1)(n+3)(n+5)(n+7)}.$$

Using (3.15) and (3.16), we can improve Theorems 3.3 and 3.4 (in the case $p = 2$) as follows:

THEOREM 3.3′. *With the same notations as in Theorem* 3.3 *it holds that for the case* $p = 2$,

$$\tilde{\varepsilon}_{1,1} \le 0.36788[\tilde{m}_{1,2}^{1/2} + \{m_{-1,2}^{(1)} + m_{-1,2}^{(2)}\}^{1/2}],$$

$$\tilde{\varepsilon}_{1,2} \le 0.23058[\tilde{m}_{1,2} + m_{-1,2}^{(1)} + m_{-1,2}^{(2)}],$$

$$\tilde{\varepsilon}_{1,4} \le 0.13242[\tilde{m}_{1,4} + m_{-1,4}^{(1)} + m_{-1,4}^{(2)}],$$

where $m_{\delta,j}^{(i)}$ *are the ones obtained from* $m_{\delta,j}$ *by replacing* n *by* n_i.

THEOREM 3.4′. *With the same notations as in Theorem* 3.4 *it holds that*

$$\sup_{x>0} |\tilde{F}(x) - \tilde{G}^*_{-1,2}(x; 2)| \le 0.31929\{\tilde{m}_{1,2} + m_{-1,2}^{(1)} + m_{-1,2}^{(2)}\}.$$

REMARK. For the numerical values of $\bar{b}_{-1,j}(p)$ and $\bar{b}_{1,j}(p)$, see Fujikoshi [3] and Fujikoshi and Shimizu [4], respectively.

References

[1] S. K. Chattergee, On an extension of Stein's two-sample procedure to multi-normal problem, Culcutta Statistical Association, Bulletin, **8** (1959), 121–148.

[2] E. J. Dudewicz and T. A. Bishop, The heteroscedastic method, Optimizing Methods in Statistics (J. S. Rustagi, ed.), Academic Press, New York, 1979, 183–203.

[3] Y. Fujikoshi, Error bounds for asymptotic expansions of scale mixtures of distributions, Hiroshima Math. J., **17** (1987), 309–324.

[4] Y. Fujikoshi and R. Shimizu, Error bounds for asymptotic expansions of scale mixtures of univariate and multivariate distributions, J. Multivariate Anal., **30** (1989), 279–291.

[5] Y. Fujikoshi and R. Shimizu, Asymptotic expansions of some distributions and their error bounds—the distributions of sums of independent random variables and scale mixtures, Sugaku Expositions, **3** (1990), 75–96.
[6] H. Hyakutake and M. Siotani, The multivariate heteroscedastic method: distributions of statistics and an application, Amer. J. Math. Manage. Sci., **7** (1987), 89–111.
[7] M. Siotani, T. Hayakawa and Y. Fujikoshi, Modern Multivariate Statistical Analysis: A Graduate Course and Handbook, American Science Press, Columbus, Ohio, 1985.
[8] Y. Watamori, On the moments of traces of Wishart and inverted Wishart matrices, South African Statist. J., **24** (1990), 153–176.

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 724, Japan