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Approximations for the quantiles of Student's t and F distributions and their error bounds

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1. Introduction

Let $F_n(x)$ be the distribution function of a random variable X_n depending on some parameter n, not necessary a sample size. A typical form of the asymptotic expansion of $F_n(x)$ around the limiting distribution function $G(x)$ of $F_n(x)$ as $n \to \infty$ is

$$F_n(x) = G(x) + g(x)\left\{\frac{1}{n}a_1(x) + \frac{1}{n^2}a_2(x) + \cdots\right\}, \tag{1.1}$$

or the one with n replaced by $\sqrt{n}$, where $g(x)$ is the density function of $G(x)$, and $a_1(x)$, $a_2(x)$, etc. are suitable polynomials. When $F_n(x)$ is approximated by a function of the form

$$G_{k,n}(x) = G(x) + g(x)\sum_{j=1}^{k-1} a_j(x)n^{-j},$$

it is well known that the error $R_{k,n}(x) = F_n(x) - G_{k,n}(x)$ satisfies

$$R_{k,n}(x) = O(n^{-k})$$

under suitable regularity conditions (see, e.g., Bhattacharya and Ghosh [1]). This means that there exists a positive constant C_k such that for large n

$$|R_{k,n}(x)| \le C_k n^{-k}$$

However, such C_k and n have not been obtained except for some special statistics (see, e.g., Fujikoshi and Shimizu [3]).

On the other hand, it is also important to find out $x_n(u)$ such that

$$F_n(x_n(u)) = G(u). \tag{1.2}$$

Based on the expansion (1.1), we can formally expand $x_n(u)$ as

$$x_n(u) = u + \frac{1}{n}b_1(u) + \frac{1}{n^2}b_2(u) + \cdots. \tag{1.3}$$

This is usually done, first by finding out $u_n(x)$ of the form

$$u_n(x) = x + \frac{1}{n}\tilde{a}_1(x) + \frac{1}{n^2}\tilde{a}_2(x) + \cdots \tag{1.4}$$

which satisfies

$$F_n(x) = G(u_n(x)), \tag{1.5}$$

and then by solving the equation (1.4) of x (see, e.g., Hill and Davis [4]). Expansions (1.3) and (1.4) are called Cornish-Fisher expansions. The quantile $x_n(u)$ of $F_n(x)$ is usually approximated by a function of the form

$$x_{k,n}(u) = u + \sum_{j=1}^{k-1} b_j(u)n^{-j}.$$

However, it is little known about the error estimate of this approximation. An ideal result is to find out an upper bound $\bar{l}_n(u)$ and a lower bound $\underline{l}_n(u)$ of $x_n(u)$, such that

$$\underline{l}_n(u) \le x_n(u) \le \bar{l}_n(u) \tag{1.6}$$

and

$$0 \le \bar{l}_n(u) - \underline{l}_n(u) \le D_k n^{-k}, \tag{1.7}$$

where D_k is a positive constant. In general, it will be difficult to have an error estimate of the form (1.7). So, as a more feasible form we will consider upper and lower bounds such that

$$0 \le \bar{l}_n(u) - \underline{l}_n(u) \le \frac{D}{n}|u|, \tag{1.8}$$

where D is a positive constant. In fact, Wallace [5] obtained upper and lower bounds in the forms (1.6) and (1.7) with $k = \frac{1}{2}$ and (1.8) for $u_n(x)$ of Student's t distribution.

It may be noted that Wallace's results are for $u_n(x)$, not for $x_n(u)$ of Student's t distribution. In this paper, we give an approximation, which has an error estimate in the form (1.8), for $x_n(u)$ of t and F distributions. Our approximations will be proposed with the help of Cornish-Fisher expansions. The proofs are based on the method of Wallace [5].

2. Preliminary results

Let F and G be absolutely continuous distribution functions with density functions f and g, respectively, and let $x(u)$ be the solution of the equation $F(x) = G(u)$ for x in terms of u.

Assume throughout that the density $g(u)$ are positive and continuous for

$c < u < \infty$ and that an approximation $z(u)$ to $x(u)$ is a continuously differentiable, strictly increasing function for $c < u < \infty$. Here c is any appropriately chosen constant which can be $-\infty$. Further, assume that the density $f(x)$ is continuous for $\lim_{u \to c} z(u) < x < \infty$. Let

$$R(u) = \frac{f(z(u))z'(u)}{g(u)}. \tag{2.1}$$

The following theorem and lemma were proved by Wallace [5].

THEOREM 2.1. *If*

(a_1) $\lim_{u \to \infty} z(u) = \infty$
(a_2) $\lim_{u \to c} G(u) = F(\lim_{u \to c} z(u))$
(a_3) $\operatorname{sgn}\{R(u) - 1\}$ *is monotonic function of* u *for* $c < u < \infty$,

then $x(u) \geq z(u)$ *or* $x(u) \leq z(u)$ *for all* $c < u < \infty$ *according as the function in* (a_3) *is increasing or decreasing.*

LEMMA 2.1. *For all* $y > 0$, $h_d(y) = (e^y - 1)/(ye^{dy})$ *is monotone decreasing for* $d \geq 1$, *monotone increasing for* $0 < d \leq \frac{1}{2}$ *and not monotonic for* $\frac{1}{2} < d < 1$.

3. Student's t distribution

Let F_n, f_n be respectively the distribution and the density functions of Student's t with n degrees of freedom, let Φ, ϕ be respectively the standard normal distribution and the density functions, and let $x_n(u)$ be the solution of the equation

$$F_n(x) = G(u) \tag{3.1}$$

for x in terms of u. It is well known that $x_n(u)$ can be formally expanded as

$$x_n(u) = u + \frac{1}{4n}(u^3 + u) + \frac{1}{96n^2}(5u^5 + 16u^3 + 3u) + \cdots.$$

Let $\underline{l}_n(u)$ and $\bar{l}_n(u)$ be two approximations to $x_n(u)$ defined by

$$\begin{aligned} \underline{l}_n(u) &= n^{1/2}(e^{u^2/n} - 1)^{1/2} \\ &= u + \frac{1}{4n}u^3 + \frac{5}{96n^2}u^5 + \cdots. \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} \bar{l}_n(u) &= n^{1/2}\{e^{(1/n)(1-1/(2n))^{-1}u^2} - 1\}^{1/2} \\ &= u + \frac{1}{4n}(u^3 + u) + \frac{1}{96n^2}(5u^5 + 18u^3 + 9u) + \cdots, \end{aligned} \tag{3.3}$$

respectively. We note that these two approximations can be intutively proposed by looking the first two terms in an expansion of $x_n(u)$. Then,

THEOREM 3.1. *For all* $u > 0$,

(i) $x_n(u) \geq \underline{l}_n(u) \qquad (n > 0)$;
(ii) $x_n(u) \leq \bar{l}_n(u) \qquad (n > \frac{1}{2})$.

PROOF. Let $z_n(u) = n^{1/2}(e^{\lambda u^2/n} - 1)^{1/2}$ with a positive constant λ. Then it is easily seen that $z_n(u)$ is continuously differentiable, strictly increasing for $0 < u < \infty$. We can write $R_n(u) = f_n(z_n(u))z_n'(u)/\phi(u)$ as

$$R_n(u) = \frac{\Gamma((n+1)/2)}{\Gamma(n/2)(n/2)^{1/2}}\left(\frac{e^y - 1}{ye^{dy}}\right)^{-1/2},$$

where $y = \lambda u^2/n$ and $d = 1 - n(1 - \lambda^{-1})$. Note that y is a monotone increasing function of u for all $u > 0$. By Lemma 2.1 we have that $R_n(u)$ is monotone increasing for $d = 1$ or $\lambda = 1$ and decreasing for $d = 1/2$ or $\lambda = (1 - 1/(2n))^{-1}$. Hence inequalities (i), (ii) follows from Theorem 2.1. Q.E.D.

Let X_n be a random variable whose distribution is Student's t with n degrees of freedom. We consider a transformed random variable

$$(3.4) \qquad Y_n = \left\{\left(n - \frac{1}{2}\right)\log\left(1 + \frac{1}{n}X_n^2\right)\right\}^{1/2}\mathrm{sgn}(X_n), \qquad \text{for } n > \frac{1}{2}.$$

This variable has a rapid convergence to the standard normal distribution, in a sense of

$$P(Y_n \leq y) = \Phi(y) + O(n^{-2})$$

for all real y. Let $y_n(u)$ be the solution of the equation

$$P(Y_n \leq y) = \Phi(u)$$

for y in terms of u. Then,

THEOREM 3.2. *For all* $u > 0$,

(i) $\left(1 - \frac{1}{2n}\right)^{1/2} u \leq y_n(u) \leq u \qquad \left(n > \frac{1}{2}\right)$;

(ii) $0 \leq u - \left(1 - \frac{1}{2n}\right)^{1/2} u \leq n_0\left\{1 - \left(1 - \frac{1}{2n_0}\right)^{1/2}\right\}\frac{u}{n} \qquad \left(n \geq n_0 > \frac{1}{2}\right)$.

PROOF. We have

$$y_n(u) = \left\{\left(n - \frac{1}{2}\right)\log\left(1 + \frac{1}{n}x_n^2(u)\right)\right\}^{1/2}$$

for all $u > 0$. Therefore, (i) follows immediately from Theorem 3.1. Inequality (ii) follows from

$$1 - \left(1 - \frac{1}{2n}\right)^{1/2} \le \frac{n_0}{n}\left\{1 - \left(1 - \frac{1}{2n_0}\right)^{1/2}\right\}$$

for all $n \ge n_0 > \frac{1}{2}$ which has been proved in Wallace [5]. Q.E.D.

4. *F* distribution

Let F_n and f_n be the distribution and the density functions of a random variable $X_n = \chi_q^2/(\chi_n^2/n)$, respectively, where χ_q^2 and χ_n^2 are mutually independent chi-square variables with q and n degrees of freedom, respectively. Let G and g be the distribution and the density functions of χ_q^2, respectively. It is well known (see, e.g., Fujikoshi [2]) that

$$(4.1)\quad F_n(x) = G(x) - g(x)\left[\frac{1}{n}\left(\frac{1}{2}x^2 - \frac{1}{2}(q-2)x\right) + \frac{1}{n^2}\left\{\frac{1}{16}x^4 - \frac{1}{48}(9q-2)x^3 + \frac{1}{48}(q-2)(9q-4)x^2 - \frac{1}{16}(q-2)(q-4)(3q-2)x\right\} + \cdots\right].$$

First, let $u_n(x)$ be the solution of the equation

$$(4.2)\quad F_n(x) = G(u)$$

for u in terms of x (i.e., $u_n(x)$ is the chi-square deviate corresponding to the argument x of F_n). Then

$$(4.3)\quad u_n(x) = x - \frac{1}{n}\left(\frac{1}{2}x^2 - \frac{1}{2}(q-2)x\right) + \frac{1}{n^2}\left\{\frac{1}{3}x^3 - \frac{7}{24}(q-2)x^2 - \frac{1}{24}(q-2)(q+2)x\right\} - \cdots.$$

Let $w_{1,n}(x)$ and $w_{2,n}(x)$ be two approximations to $u_n(x)$ defined by

$$(4.4)\quad w_{1,n}(x) = n\log\left(1 + \frac{x}{n}\right) = x - \frac{x^2}{2n} + \frac{x^3}{3n^2} - \cdots$$

and

$$(4.5)\quad w_{2,n}(x) = \left\{n + \frac{1}{2}(q-2)\right\}\log\left(1 + \frac{x}{n}\right) = x - \frac{1}{n}\left(\frac{1}{2}x^2 - \frac{1}{2}(q-2)x\right) + \frac{1}{n^2}\left(\frac{1}{3}x^3 - \frac{1}{4}(q-2)x^2\right) - \cdots,$$

respectively. We note that the right-hand sides of (4.4) and (4.5) are closely related to the one of (4.3). Then

THEOREM 4.1. *For all* $x > 0$,

(i) $u_n(x) \le w_{1,n}(x) \quad (0 < q < 2 \text{ and } n > 0)$;
$u_n(x) \ge w_{2,n}(x) \quad (0 < q < 2 \text{ and } n > (2-q)/2)$;
(ii) $u_n(x) = w_{1,n}(x) = w_{2,n}(x) \quad (q = 2 \text{ and } n > 0)$;
(iii) $w_{1,n}(x) \le u_n(x) \le w_{2,n}(x) \quad (q > 2 \text{ and } n > 0)$;
(iv) $|w_{1,n}(x) - w_{2,n}(x)| \le \frac{|q-2|}{2}\frac{x}{n} \quad (q > 0 \text{ and } n > 0)$.

PROOF. Let $z_n(x) = \lambda n \cdot \log(1 + x/n)$, where λ is a positive constant to be chosen. Then consider the function $R_n(x) = g(z_n(x))z_n'(x)/f_n(x)$ which can be written as a function of $y = \log(1 + x/n)$ as follows:

$$R_n(x) = C_n \lambda^{q/2} \frac{y^{q/2-1} \exp\left[\left\{\frac{1}{2}q - 1 + \frac{1}{2}n(1-\lambda)\right\} y\right]}{(e^y - 1)^{q/2-1}}.$$

where

$$C_n = \frac{\Gamma(n/2)(n/2)^{q/2}}{\Gamma((q+n)/2)} \tag{4.6}$$

First, set $\lambda = 1$. Then $z_n(x) = w_{1,n}(x)$ and

$$R_n(x) = C_n \left(\frac{e^y - 1}{ye^y}\right)^{1-q/2}$$

which is monotone decreasing for $0 < q \le 2$ and increasing for $q \ge 2$ from Lemma 2.1.

Next set $\lambda = 1 + (q-2)/(2n)$. Then $z_n(x) = w_{2,n}(x)$ and

$$R_n(x) = C_n \left(1 + \frac{q-2}{2n}\right)^{q/2} \left(\frac{e^y - 1}{ye^{y/2}}\right)^{1-q/2}$$

which is monotone increasing for $0 < q \le 2$ and decreasing for $q \ge 2$.

Hence inequalities (i), (ii) and (iii) follow from Theorem 2.1. Finally, (iv) follows immediately from the definitions of $w_{1,n}(x)$ and $w_{2,n}(x)$. Q.E.D.

Next we consider the quantile $x_n(u)$ of F_n, i.e., the solution of the equation (4.2) for x in terms of u. Then

$$(4.7)\qquad x_n(u) = u + \frac{1}{n}\left(\frac{1}{2}u^2 - \frac{1}{2}(q-2)u\right) + \frac{1}{n^2}\left\{\frac{1}{6}u^3 - \frac{11}{24}(q-2)u^2 + \frac{1}{24}(q-2)(7q-10)u\right\} + \cdots .$$

Let $l_{1,n}(u)$ and $l_{2,n}(u)$ be two approximations to $x_n(u)$ defined by

$$(4.8)\qquad l_{1,n}(u) = n(e^{u/n} - 1) = u + \frac{u^2}{2m} + \frac{u^3}{6n^2} + \cdots$$

and

$$(4.9)\qquad l_{2,n}(u) = n\{e^{1/n(1+(q-2/2n))^{-1}u} - 1\} = u + \frac{1}{n}\left(\frac{1}{2}u^2 - \frac{1}{2}(q-2)u\right) + \frac{1}{n^2}\left\{\frac{1}{6}u^3 - \frac{1}{2}(q-2)u^2 + \frac{1}{4}(q-2)^2 u\right\} + \cdots,$$

respectively. These approximations may be proposed by comparing the expansion (4.7) with the expansions (4.8) and (4.9). Then,

THEOREM 4.2. *For all* $u > 0$,

(i) $x_n(u) \geq l_{1,n}(u)$ $\quad (0 < q < 2$ *and* $n > 0)$;
$x_n(u) \leq l_{2,n}(u)$ $\quad (0 < q < 2$ *and* $n > \frac{1}{2}(2-q))$;
(ii) $x_n(u) = l_{1,n}(u) = l_{2,n}(u)$ $\quad (q = 2$ *and* $n > 0)$;
(iii) $l_{2,n}(u) \leq x_n(u) \leq l_{1,n}(u)$ $\quad (q > 2$ *and* $n > 0)$.

PROOF. Let $z_n(u) = n(e^{\lambda u/n} - 1)$, where λ is a positive constant to be chosen. Then we can write $R_n(u) = f_n(z_n(u))z_n'(u)/g(u)$ as

$$R_n(u) = C_n^{-1}\lambda^{q/2}\frac{(e^y - 1)^{q/2-1}}{y^{q/2-1}\exp\left[\left\{\frac{1}{2}q - 1 + \frac{1}{2}n(1-\lambda^{-1})\right\}y\right]},$$

where $y = \lambda u/n$ and C_n is defined by (4.6). Hence inequalities (i), (ii) and (iii) follow from Theorem 2.1 and Lemma 2.1 Q.E.D.

Consider a transformed random variable

$$Y_n = \left\{n + \frac{1}{2}(b-2)\right\}\log\left(1 + \frac{1}{n}X_n\right),$$

where $n > \max\{0, (2-b)/2\}$. We note that this transformation is based on a Bartllet adjustment for a log-likelihood radio statistic in a linear model. It is known that

$$P(Y_n \le y) = G(y) + O(n^{-2})$$

for all real y. Let $y_n(u)$ be the solution of the equation

$$P(Y_n \le y) = G(u)$$

for y in terms of u. Then,

THEOREM 4.3. *For all* $u > 0$,

(i) $\left(1 + \frac{q-2}{2n}\right)u \le y_n(u) \le u \qquad (0 < q < 2 \text{ and } n > \frac{1}{2}(2 - q))$;

(ii) $y_n(u) = u \qquad (q = 2 \text{ and } n > 0)$;

(iii) $u \le y_n(u) \le \left(1 + \frac{q-2}{2n}\right)u \qquad (q > 2 \text{ and } n > 0)$.

PROOF. It holds that

$$y_n(u) = \left\{n + \frac{1}{2}(q - 2)\right\} \log\left(1 + \frac{1}{n}x_n(u)\right)$$

for all $u > 0$ and $n > \max\left\{\frac{1}{2}(2 - q), 0\right\}$. Hence, inequalities (i), (ii) and (iii) follow immediately from Theorem 4.2. Q.E.D.

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