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Note on selecting the better component of the bivariate exponential distribution

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1. Introduction

Let X and Y be random variables having Marshall and Olkin's [5] bivariate exponential (BVE) distribution whose survival function is given by

$$\bar{F}(x, y) = P(X > x, Y > y)$$
$$= \exp[-\lambda_1 x - \lambda_2 y - \lambda_0 \max(x, y)],$$

where $x > 0$, $y > 0$, $\lambda_1 > 0$, $\lambda_2 > 0$, and $\lambda_0 \geq 0$. This BVE distribution is derived from supposing that failure is caused by three types of Poisson shocks on a system containing two components. The problem is to select the better component with respect to $\lambda_i (i = 1, 2)$ which are viewed as the hazard of the components in lifetime analysis. When $\lambda_1 < \lambda_2$, we define that X-component is better than Y-component and we select X-component. When $\lambda_1 > \lambda_2$, Y-component is selected as the better component. The better component has the larger mean because of $E(X) = 1/(\lambda_1 + \lambda_0)$ and $E(Y) = 1/(\lambda_2 + \lambda_0)$, that is, the better component has larger expected survival time. Throughout this paper, we assume that X-component is better $(\lambda_1 < \lambda_2)$ without loss of generality.

Hyakutake [4] gave selection procedures R_1 and R_2, which are based on a trinomial distribution and the marginal distribution, respectively. By using the central limit theorem, Hyakutake [4] also gave two-stage sampling schemes based on R_1 to satisfy probability requirements. However, the probability of correct selection (CS) of R_1 is less than that of R_2 for small λ_2/λ_1 and small λ_0 by Hyakutake [4]. Hence we give the two-stage sampling scheme based on R_2. The marginal distribution of X (or Y) is exponential with parameter $\lambda_1 + \lambda_0$ (or $\lambda_2 + \lambda_0$). The sample means $\bar{X}$ and $\bar{Y}$ are unbiased estimates of $1/(\lambda_1 + \lambda_0)$ and $1/(\lambda_2 + \lambda_0)$, respectively. By this fact, we select X-component when $\bar{X} > \bar{Y}$. The selection procedure by using sample means, namely R_2, is equivalent to the procedure by using moment type estimates of λ_i $(i = 1, 2, 0)$ derived by Bemis, Bain and Higgins [3].

In Section 2, some known results for selecting the better component of BVE distribution are stated. The probability requirement and the sample size for the requirement are given in Section 3. In Section 4, we give the two-stage sampling scheme to satisfy the requirement by normal approximation.

2. Preliminaries

In this section, the outline of Hyakutake [4] is described. Let $\{(X_r, Y_r), r = 1, 2, \cdots\}$ be a sequence of independent and identical random vectors having BVE distribution. From Arnold [1] or Awad, Azzam and Hamdan [2],

$$P(X_r < Y_r) = p_1, \quad P(X_r > Y_r) = p_2, \quad P(X_r = Y_r) = p_0,$$

where $p_i = \lambda_i/(\lambda_1 + \lambda_2 + \lambda_0)$, $i = 1, 2, 0$. For fixed sample size n, let n_1, n_2 and m denote the number of observations in the regions $\{x < y\}$, $\{x > y\}$ and $\{x = y\}$, respectively, then (n_1, n_2, m) has a trinomial distribution with n and cell probability (p_1, p_2, p_0). The procedure R_1 is based on n_1 and n_2, that is,

$$P(CS \mid R_1) = P(n_1 < n_2) \tag{2.1}$$

by the assumtion $\lambda_1 < \lambda_2$. For the procedure R_1, the sample size satisfying a probability requirement is given in Hyakutake [4]. The procedure R_2 is based on the sample means, that is,

$$P(CS \mid R_2) = P(\bar{X} > \bar{Y}). \tag{2.2}$$

By the central limit theorem, $[(\bar{X} - \bar{Y}) - \mu]/[\sqrt{\sigma^2/n}]$ has the standard normal distribution $N(0, 1)$ as $n \to \infty$, where

$$\mu = \frac{1}{\lambda_1 + \lambda_0} - \frac{1}{\lambda_2 + \lambda_0},$$

$$\sigma^2 = \frac{1}{(\lambda_1 + \lambda_0)^2} + \frac{1}{(\lambda_2 + \lambda_0)^2} - \frac{2\lambda_0}{(\lambda_1 + \lambda_0)(\lambda_2 + \lambda_0)(\lambda_1 + \lambda_2 + \lambda_0)}.$$

For large n, Hyakutake [4] has approximated as follows

$$P(CS \mid R_2) \approx \Phi(c), \tag{2.3}$$

where $\Phi(\cdot)$ is the cumulative distribution function of $N(0, 1)$ and

$$c = \sqrt{n}\,\frac{\lambda_2 - \lambda_1}{\sqrt{(\lambda_1 + \lambda_0)^2 + (\lambda_2 + \lambda_0)^2 - \dfrac{2\lambda_0(\lambda_1 + \lambda_0)(\lambda_2 + \lambda_0)}{\lambda_1 + \lambda_2 + \lambda_0}}}.$$

3. Requirement and sample size

In this section, we consider two indifference zone approaches and give the required sample size based on R_2 by using the normal approximation of (2.3). Since the parameters of the BVE distribution may be viewed as scale parameters, the probability requirement in the indifference zone formulation is generally stated as

$$P(CS) \geq P^* \quad \text{whenever} \quad \lambda_2/\lambda_1 \geq \delta_1^*, \tag{3.1}$$

where P^* $(1/2 < P^* < 1)$ and $\delta_1^* > 1$ are preassigned constants. The problem is to find the required sample size n satisfying (3.1). Let $\lambda_2/\lambda_1 = p_2/p_1 = \delta_1$, then

$$\begin{aligned} c &= \sqrt{n}\,\frac{p_2 - p_1}{\sqrt{(p_1 + p_0)^2 + (p_2 + p_0)^2 - 2p_0(p_1 + p_0)(p_2 + p_0)}} \\ &= \sqrt{n}\,\frac{\delta_1 - 1}{\sqrt{\delta_1^2 + 1 - 2\delta_1 p_0 + 2p_0(1 - p_0)}}, \end{aligned}$$

which is increasing in $\delta_1 > 1$ and increasing in p_0 $(0 \leq p_0 < 1)$. We find that

$$\Phi(c) \geq \Phi\left(\left[n\,\frac{(\delta_1^* - 1)^2}{(\delta_1^*)^2 + 1}\right]^{1/2}\right)$$

by $\delta_1 \geq \delta_1^* > 1$ and $p_0 \geq 0$. Hence (3.1) is satisfied if the sample size n satisfies

$$n \geq \frac{(\delta_1^*)^2 + 1}{(\delta_1^* - 1)^2} z_P^2, \tag{3.2}$$

where z_P is the solution of $\Phi(z_P) = P^*$.

Next we consider the probability requirement

$$P(CS) \geq P^* \quad \text{whenever} \quad \frac{\lambda_2 + \lambda_0}{\lambda_1 + \lambda_0} \geq \delta_2^*, \tag{3.3}$$

for fixed P^* $(1/2 < P^* < 1)$ and $\delta_2^* > 1$. This requirement is based on the parameter of the marginal distribution. Let $(\lambda_2 + \lambda_0)/(\lambda_1 + \lambda_0) = \delta_2$, then

$$c = \sqrt{n}\,\frac{\delta_2 - 1}{\sqrt{\delta_2^2 + 1 - 2\delta_2 p_0}}.$$

By similar argument for (3.1), the required sample size for (3.3) is $n \geq z_P^2((\delta_2^*)^2 + 1)/(\delta_2^* - 1)^2$, which is same as (3.2).

The above result shows that we can obtain the required sample size which is not depend on the parameter λ_0. By (3.2), when $\delta_1^* = 2$ and $P^* = 0.9$, the optimum sample size is $n = 9$, which is not sufficient for normal approximation. If δ_i^* $(i = 1, 2)$ is not small, we need the exact distribution of $\bar{X} - \bar{Y}$, or use the procedure R_1 of Hyakutake [4]. By (3.2), when $\delta_1^* = 1.2$ and $P^* = 0.9$, we must take the observations at least $n = 101$ to satisfy (3.1). This may lead to over sampling except that X and Y are independent, that is, $\lambda_0 = 0$ $(p_0 = 0)$.

4. Two-stage sampling scheme

We propose two-stage sampling scheme to avoid over sampling. In order to satisfy requirement (3.1), the sampling scheme R_{21} based on R_2 is as follows:

We first take a sample of size n_0, which is noderately large (say, $n_0 \geq 30$), from BVE distribution and compute m/n_0. Obviously, m/n_0 is the muximum likelihood estimate of p_0. Define N_1 by

$$(4.1) \qquad N_1 = \max\left\{n_0, \left[z_P^2 \frac{(\delta_1^*)^2 + 1 - 2\delta_1^* m/n_0 + 2m/n_0(1 - m/n_0)}{(\delta_1^* - 1)^2}\right] + 1\right\},$$

where $[a]$ denotes the greatest integer not greater than a. Next we take $N_1 - n_0$ additional observations and compute $\bar{X}$ and $\bar{Y}$ based on N_1 observations. Since $\Phi(c)$ is approximated by

$$\Phi\left(\left[N_1 \frac{(\delta_1 - 1)^2}{\delta_1^2 + 1 - 2\delta_1 m/n_0 + 2m/n_0(1 - m/n_0)}\right]^{1/2}\right),$$

which is greater than $\Phi(z_P)$ by $\delta_1 \geq \delta_1^*$ and (4.1), we have

$$P(CS \mid R_{21}) \geq \Phi(z_P) = P^*.$$

Hence the two-stage sampling scheme R_{21}, namely (4.1), is satisfies (3.1).

For requirement (3.3), the two-stage sampling scheme R_{22} is as follows:

The first stage of the scheme is same as R_{21}. Difine

$$(4.2) \qquad N_2 = \max\left\{n_0, \left[z_P^2 \frac{(\delta_2^*)^2 + 1 - 2\delta_2^* m/n_0}{(\delta_2^* - 1)^2}\right] + 1\right\},$$

where $[a]$ denotes the greatest integer not greater than a. Since $\Phi(c)$ is approximated by

$$\Phi\left(\left[N_2 \frac{(\delta_2^* - 1)^2}{\delta_2^2 + 1 - 2\delta_2 m/n_0}\right]^{1/2}\right),$$

it can be shown that $\Phi(c) \geq P^*$ by $\delta_2 \geq \delta_2^*$ and (4.2). Hence the procedure R_{22}, namely (4.2), satisfies (3.3).

The proposed two-stage sampling schemes are available for small δ_i^*, say $1 < \delta_i^* < 1.5$ $(i = 1, 2)$, because we use the normal approximation for the procedures. For large δ_i^*, the exact distribution, which is difficult to derive, is needed.

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