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On modified singular integrals

Dedicated to Prof. Masanori Kishi on the occasion of his 60th birthday

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1. Introduction

Let R^n be the n-dimensional Euclidean space. E. M. Stein [5] gave a weighted norm inequality for singular integrals on R^n as follows (see also C. Sadosky [4; Theorem 6.1]):

Theorem A. *Let $\Omega(x)$ be a homogeneous function of degree $-n$ on R^n, and suppose that $\Omega(x)$ satisfies the cancellation property*

$$\int_S \Omega(x)\, d\sigma(x) = 0, \tag{1.1}$$

where $d\sigma$ is the induced Euclidean measure on the unit sphere S, and $\Omega(x)$ is bounded on S. Let $Tf(x)$ denote the corresponding singular integral:

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{|x-y| \geq \varepsilon} \Omega(x-y) f(y)\, dy.$$

Then

$$\left(\int |Tf(x)|^p |x|^{-rp} dx \right)^{1/p} \leq C \left(\int |f(y)|^p |y|^{-rp} dy \right)^{1/p}, \tag{1.2}$$

provided that $1 < p < \infty$ and $-n/p' < r < n/p$ where $(1/p) + (1/p') = 1$.

For the ordinary singular integrals the above restriction of r is necessary. Indeed, when $r \geq n/p$, for $f(y) = (1 + \log|y|)^{-1}$, $|y| \geq 1$, we see $\int |f(y)|^p |y|^{-rp} dy < \infty$ and $\int_{|x-y| \geq \varepsilon} |\Omega(x-y) f(y)|\, dy = \infty$, so (1.2) fails. When $r \leq -n/p'$, for $f(y) = (1 - \log|y|)^{-1} |y|^{-\beta}$, $|y| \leq 1$, (1.2) does not hold with $n \leq \beta \leq (n/p) - r$. The purposes of this paper are to introduce modified singular integrals and give integral estimates similar to (1.2) which holds for all $r > -n/p'$ such that $r - (n/p) \neq$ a nonnegative integer.

Let $\Omega(x)$ be a homogeneous function of degree $-n$, and suppose that $\Omega(x)$ satisfies (1.1) and $\Omega(x) \in C^\infty(R^n - \{0\})$. For an integer $k \geq -1$ we set

$$\Omega_k(x, y) = \begin{cases} \Omega(x-y) - \sum_{|\gamma| \le k} (x^\gamma/\gamma!)(D^\gamma \Omega)(-y), & k \ge 0 \\ \Omega(x-y), & k = -1 \end{cases}$$

where γ is a multi-index $(\gamma_1, \dots, \gamma_n)$, $x^\gamma = x_1^{\gamma_1} \cdots x_n^{\gamma_n}$, $D^\gamma = D_1^{\gamma_1} \cdots D_n^{\gamma_n}$, $\gamma! = \gamma_1! \cdots \gamma_n!$ and $|\gamma| = \gamma_1 + \cdots + \gamma_n$. We define modified singular integrals as follows:

$$T_k f(x) = \lim_{\varepsilon \to 0} T_{k,\varepsilon} f(x),$$

$$T_k^* g(y) = \lim_{\varepsilon \to 0} T_{k,\varepsilon}^* g(y)$$

where

$$T_{k,\varepsilon} f(x) = \int_{|x-y| \ge \varepsilon} \Omega_k(x, y) f(y)\, dy,$$

$$T_{k,\varepsilon}^* g(y) = \int_{|x-y| \ge \varepsilon} \Omega_k(x, y) g(x)\, dx$$

for $\varepsilon > 0$.

Throughout this paper we take p as $1 < p < \infty$. For a real number r, we set

$$L^{p,r} = \left\{ f \colon \|f\|_{p,r} = \left(\int |f(x)|^p |x|^{rp} dx \right)^{1/p} < \infty \right\}$$

and simply write $\|f\|_{p,0} = \|f\|_p$. Moreover $[r]$ denotes the integral part of r.

The main results of this paper are the following.

THEOREM 1. *Let $r > -n/p'$, $r - (n/p) \neq$ a nonnegative integer and*

$$k = \begin{cases} [r - (n/p)], & \text{if } r - (n/p) > 0, \\ -1, & \text{if } r - (n/p) < 0. \end{cases}$$

Then

(i) $$\|T_{k,\varepsilon} f\|_{p,-r} \le C \|f\|_{p,-r},$$

(ii) $$\|T_{k,\varepsilon}^* g\|_{p',r} \le C \|g\|_{p',r},$$

where C is a constant independent of f, g and ε.

THEOREM 2. *Let r and k be as in Theorem 1. Then*
(i) *for $f \in L^{p,-r}$*

$$\|T_{k,\varepsilon_1} f - T_{k,\varepsilon_2} f\|_{p,-r} \longrightarrow 0 \qquad (\varepsilon_1, \varepsilon_2 \longrightarrow 0),$$

(ii) *for* $g \in L^{p',r}$

$$\| T^*_{k,\varepsilon_1} g - T^*_{k,\varepsilon_2} g \|_{p',r} \longrightarrow 0 \qquad (\varepsilon_1, \varepsilon_2 \longrightarrow 0).$$

On account of Theorems 1 and 2 we have

COROLLARY. *Let r and k be as in Theorem* 1.

(i) *If* $f \in L^{p,-r}$, *then* $\lim_{\varepsilon \to 0} T_{k,\varepsilon} f = T_k f$ *exists in the* $L^{p,-r}$*-norm and*

$$\| T_k f \|_{p,-r} \le C \| f \|_{p,-r}.$$

(ii) *If* $g \in L^{p',r}$, *then* $\lim_{\varepsilon \to 0} T^*_{k,\varepsilon} g = T^*_k g$ *exists in the* $L^{p',r}$*-norm and*

$$\| T^*_k g \|_{p',r} \le C \| g \|_{p',r}.$$

REMARK. *For the kernel functions*

$$K_{m,\lambda,k}(x, y) = \begin{cases} D^\lambda k_m(x - y) - \sum_{|\nu| \le k} (x^\nu/\nu!) D^{\nu+\lambda} k_m(-y), & |y| \ge 1 \\ D^\lambda k_m(x - y), & |y| < 1 \end{cases}$$

where $k_m(x)$ *is the Riesz kernel of order* $2m$ *and* $|\lambda| = 2m$, *Y. Mizuta* [2] *gave the following weighted* L^p*-estimates*:

$$\int \left| \int K_{m,\lambda,k}(x, y) f(y)\, dy \right|^p \omega^*(|x|)\, dx \le C \int |f(y)|^p \omega(|x|)\, dx.$$

In our case, $\omega(t) = \omega^*(t) = t^{-rp}$.

2. Lemmas

In this section we prepare several lemmas which are necessary for the proofs of Theorems 1 and 2. Hereafter the letter C is used for a generic positive constant whose value may be different at each occurrence. First, by Taylor's theorem we have

LEMMA 2.1. (cf. [1; Lemma 3.1]) *For* $|x - y| \ge 3|x|/2 > 0$ *and an integer* $k \ge -1$, *it holds*

$$|\Omega_k(x, y)| \le C |x|^{k+1} |y|^{-k-1-n}.$$

The following Lemmas 2.2 and 2.3 follow from Hardy's inequalities [6; p. 272].

LEMMA 2.2. *If* $\alpha > n/p$, *then*

$$\left(\int \left| |x|^{-\alpha} \int_{|y| \le 5|x|/2} |y|^{\alpha-n} f(y)\, dy \right|^p dx \right)^{1/p} \le C \| f \|_p.$$

LEMMA 2.3. *If* $\alpha < n/p$, *then*

$$\left(\int \left| |x|^{-\alpha} \int_{|y| \geq 2|x|/5} |y|^{\alpha-n} f(y)\, dy \right|^p dx\right)^{1/p} \leq C \| f \|_p.$$

To prove the Lemmas 2.5 and 2.6 below, we use Okikiolu's result.

LEMMA 2.4. ([3]) *Let* $K(x, y)$ *be a nonnegative measurable function on* $R^n \times R^n$. *Suppose that there are measurable functions* $\phi_1 > 0, \phi_2 > 0$ *and constants* $M_1 > 0, M_2 > 0$ *such that*

$$\int \phi_2(y)^{p'} K(x, y)\, dy \leq M_1^{p'} \phi_1(x)^{p'}, \tag{2.1}$$

$$\int \phi_1(x)^p K(x, y)\, dx \leq M_2^p \phi_2(y)^p \tag{2.2}$$

for all $x, y \in R^n$. *Then the operator* Kf *defined by*

$$Kf(x) = \int K(x, y) f(y)\, dy$$

satifies

$$\| Kf \|_p \leq M_1 M_2 \| f \|_p.$$

LEMMA 2.5. *If* $\alpha > 0$, *then*

$$\left(\int \left| |x|^{-\alpha} \int_{|x-y| < 3|x|/2} |x - y|^{\alpha-n} f(y)\, dy \right|^p dx\right)^{1/p} \leq C \| f \|_p.$$

PROOF: For $\phi_1(x) = \phi_2(x) = |x|^{-b} (0 < b < n/p')$ and

$$K(x, y) = \begin{cases} |x|^{-\alpha} |x - y|^{\alpha-n}, & \text{if } |x - y| < 3|x|/2 \\ 0, & \text{if } |x - y| \geq 3|x|/2 \end{cases}$$

we shall verify (2.1) and (2.2). From the conditions $\alpha > 0$ and $b < n/p'$ it follows that

$$\begin{aligned} \int \phi_2(y)^{p'} K(x, y)\, dy &= \int_{|x-y| < 3|x|/2} |y|^{-bp'} |x|^{-\alpha} |x - y|^{\alpha-n}\, dy \\ &= |x|^{-bp'} \int_{|x'-z| < 3/2} |z|^{-bp'} |x' - z|^{\alpha-n}\, dz \\ &= C|x|^{-bp'} = C\phi_1(x)^{p'} \end{aligned}$$

with $x' = x/|x| (x \neq 0)$. Moreover by $\alpha > 0$ and $b > 0$ we have

$$\int \phi_1(x)^p K(x, y)\,dx = \int_{|x-y|<3|x|/2} |x|^{-bp}|x|^{-\alpha}|x-y|^{\alpha-n}\,dx$$
$$= |y|^{-bp}\int_{|z-y'|<3|z|/2} |z|^{-bp-\alpha}|z-y'|^{\alpha-n}\,dz$$
$$= C|y|^{-bp} = C\phi_2(y)^p.$$

Thus we obtain (2.1) and (2.2) and so the lemma follows from Lemma 2.4.

LEMMA 2.6. *If* $\alpha < n/p$, *then*

$$\left(\int \left| |x|^{-\alpha}\int_{|x-y|\geq|x|/2} |x-y|^{\alpha-n} f(y)dy\right|^p dx\right)^{1/p} \leq C\|f\|_p.$$

PROOF: For $\phi_1(x) = \phi_2(x) = |x|^{-n/(pp')}$ and

$$K(x, y) = \begin{cases} |x|^{-\alpha}|x-y|^{\alpha-n}, & \text{if } |x-y| \geq |x|/2 \\ 0, & \text{if } |x-y| < |x|/2 \end{cases}$$

we shall show (2.1) and (2.2). By the condition $\alpha < n/p$ we have

$$\int \phi_2(y)^{p'} K(x, y)\,dy = \int_{|x-y|\geq|x|/2} |y|^{-n/p}|x|^{-\alpha}|x-y|^{\alpha-n}\,dy$$
$$= |x|^{-n/p}\int_{|x'-z|\geq 1/2} |z|^{-n/p}|x'-z|^{\alpha-n}\,dz$$
$$= C|x|^{-n/p} = C\phi_1(x)^{p'}$$

and

$$\int \phi_1(x)^p K(x, y)\,dx = \int_{|x-y|\geq|x|/2} |x|^{-n/p'}|x|^{-\alpha}|x-y|^{\alpha-n}\,dx$$
$$= |y|^{-n/p'}\int_{|z-y'|\geq|z|/2} |z|^{-(n/p')-\alpha}|z-y'|^{\alpha-n}\,dz$$
$$= C|y|^{-n/p'} = C\phi_2(y)^p.$$

Hence by Lemma 2.4 we obtain the present lemma.

As a consequence of Lemmas 2.2, 2.3 and 2.5 we obtain the following lemma.

LEMMA 2.7. *Let* $L(x, y) = ||x|^{-r} - |y|^{-r}||y|^r|x-y|^{-n}$.

(i) *If* $r < n/p$, *then*

$$\left(\int \left|\int_{|x-y|\geq 3|x|/2} L(x, y) f(y)\,dy\right|^p dx\right)^{1/p} \leq C\|f\|_p.$$

(ii) *If* $r > -n/p'$, *then*

$$\left(\int \left| \int_{|x|/2 \le |x-y| < 3|x|/2} L(x, y) f(y)\, dy \right|^p dx \right)^{1/p} \le C \|f\|_p.$$

(iii) *For all* r,

$$\left(\int \left| \int_{|x-y| < |x|/2} L(x, y) f(y)\, dy \right|^p dx \right)^{1/p} \le C \|f\|_p.$$

PROOF: (i) We note that $|x-y| \ge 3|x|/2$ implies $|x-y| \ge 3|y|/5$. Hence for $|x-y| \ge 3|x|/2$,

$$\begin{aligned} L(x, y) &\le \left(\frac{5}{3}\right)^n (|x|^{-r} + |y|^{-r}|)|y|^r |y|^{-n} \\ &= \left(\frac{5}{3}\right)^n |x|^{-r}|y|^{r-n} + \left(\frac{5}{3}\right)^n |y|^{-n}. \end{aligned}$$

Since $|x-y| \ge 3|x|/2$ implies $|y| \ge |x|/2$, (i) follows from Lemma 2.3.

(ii) For $|x|/2 \le |x-y|$,

$$\begin{aligned} L(x, y) &\le 2^n(|x|^{-r} + |y|^{-r})|y|^r |x|^{-n} \\ &= 2^n |x|^{-r-n}|y|^r + 2^n |x|^{-n}. \end{aligned}$$

Since $|x-y| \le 3|x|/2$ implies $|y| \le 5|x|/2$, (ii) follows from Lemma 2.2.

(iii) By the mean value theorem,

$$||x|^{-r} - |y|^{-r}| \le |r|\,|x-y|\,|x + \theta(y-x)|^{-r-1}, \qquad 0 < \theta < 1.$$

If $|x-y| < |x|/2$, then $|x|/2 < |x + \theta(y-x)| < 3|x|/2$. Therefore for $|x-y| < |x|/2$,

$$||x|^{-r} - |y|^{-r}| \le C|x|^{-r-1}|x-y|$$

and hence

$$\left| \int_{|x-y| < |x|/2} L(x, y) f(y)\, dy \right| \le C|x|^{-1} \int_{|x-y| < |x|/2} |x-y|^{1-n}|f(y)|\, dy.$$

Consequently (iii) follows from Lemma 2.5.

The following lemma is proved in [6] (Theorem 2 in Chap. II).

LEMMA 2.8.

$$\left(\int \left| \int_{|x-y| \ge 1} \Omega(x-y) f(y)\, dy \right|^p dx \right)^{1/p} \le C \|f\|_p.$$

Let $G(x, y)$ be a measurable function on $R^n \times R^n$ such that

$$G(\delta x, \delta y) = \delta^{-n} G(x, y) \qquad \text{for } \delta > 0 \tag{2.3}$$

and let $D(x, \varepsilon)$ be measurable subsets of R^n such that

$$D(\delta x, \delta\varepsilon) = \delta D(x, \varepsilon) \qquad \text{for } \delta > 0,\ x \in R^n,\ \varepsilon > 0. \tag{2.4}$$

We set

$$G_\varepsilon^f(x) = \int_{D(x,\varepsilon)} G(x, y) f(y)\, dy.$$

By the change of variables, we see

$$G_\varepsilon^f(x) = G_1^{f_\varepsilon}(x/\varepsilon),$$

where $f_\varepsilon(x) = f(\varepsilon x)$. Therefore we obtain

LEMMA 2.9. *If*

$$\| G_1^f \|_{p,-r} \le M \| f \|_{p,-r}$$

for all $f \in L^{p,-r}$, *then*

$$\| G_\varepsilon^f \|_{p,-r} \le M \| f \|_{p,-r}$$

for all $f \in L^{p,-r}$ *with the same constant* M.

In the proof of Theorem 2, we use the fact that a certain class of C^1-functions is dense in $L^{p,-r}$, namely the following lemma.

LEMMA 2.10. *If* $f \in L^{p,-r}$, *then there exists a sequence* $\{\phi_j\} \subset C^1 \cap L^{p,-r}$ *such that*

$$\int |\nabla \phi_j(x)|^p |x|^{-rp}\, dx < \infty, \tag{i}$$

$$\| \phi_j - f \|_{p,-r} \longrightarrow 0 \qquad (j \longrightarrow \infty), \tag{ii}$$

where $\nabla \phi$ *denotes the gradient of* ϕ.

PROOF: Since $f(x)|x|^{-r} \in L^p$, there exists a sequence $\{\psi_j\} \subset C^1 \cap L^p$ such that

$$\operatorname{supp} \psi_j \subset \{x;\ a_j \le |x| \le b_j\}, \quad 0 < a_j < b_j < \infty$$

and $\psi_j \to f(x)|x|^{-r}$ in L^p as $j \to \infty$. The sequence $\{\psi_j(x)|x|^r\} \subset C^1 \cap L^{p,-r}$ satisfies the conditions (i) and (ii).

The final lemma is easily seen.

LEMMA 2.11. *If* α *is a real number and* f *is a nonnegative locally integrable* *function such that*

$$\int |y|^{\alpha} f(y)\,dy < \infty,$$

then

$$\int_{|x-y| \geq 2|x|/3} |x-y|^{\alpha} f(y)\,dy$$

is bounded on $\{|x| \leq 1\}$.

3. Proof of Theorem 1

Since $G(x, y) = \Omega_k(x, y)$ and $D(x, \varepsilon) = \{y;\, |x-y| \geq \varepsilon\}$ satisfy the conditions (2.3) and (2.4), respectively, by Lemma 2.9 it suffices to show the theorem for $\varepsilon = 1$.

(i) We decompose $T_{k,1}f$ as follows:

$$\begin{aligned} T_{k,1}f(x) &= \int_{|x-y| \geq 1,\, |x-y| < 3|x|/2} \Omega(x-y) f(y)\,dy \\ &\quad - \sum_{|r| \leq k} \int_{|x-y| \geq 1,\, |x-y| < 3|x|/2} \frac{x^{\gamma}}{\gamma!} (D^{\gamma}\Omega)(-y) f(y)\,dy \\ &\quad + \int_{|x-y| \geq 1,\, |x-y| \geq 3|x|/2} \Omega_k(x, y) f(y)\,dy \\ &= I_1(x) - \sum_{|\gamma| \leq k} I_2^{\gamma}(x) + I_3(x). \end{aligned}$$

For $I_3(x)$, since $r - (n/p) < k+1$, by Lemmas 2.1 and 2.3 we have

$$\begin{aligned} &\left(\int |I_3(x)|^p |x|^{-rp}\,dx \right)^{1/p} \\ (3.1) \qquad &\leq C \left(\int \left(\int_{|y| \geq |x|/2} |x|^{k+1-r} |y|^{-k-1+r-n} |f(y)|\, |y|^{-r}\,dy \right)^p dx \right)^{1/p} \\ &\leq C \left(\int |f(y)|^p |y|^{-rp}\,dy \right)^{1/p}. \end{aligned}$$

Since $r - (n/p)$ is not a nonnegative integer and $|\gamma| \leq k$, we see that $|\gamma| < r - (n/p)$. Hence by Lemma 2.2,

$$\left(\int |I_2^\gamma(x)|^p |x|^{-rp}\,dx\right)^{1/p}$$

(3.2)
$$\le C\left(\int\left(\int_{|y|<5|x|/2} |x|^{|\gamma|-r}|y|^{r-|\gamma|-n}|f(y)||y|^{-r}\,dy\right)^p dx\right)^{1/p}$$
$$\le C\left(\int |f(y)|^p|y|^{-rp}\,dy\right)^{1/p}.$$

For $I_1(x)$, we have

$$\left(\int |I_1(x)|^p|x|^{-rp}\,dx\right)^{1/p}$$
$$= \left(\int \Big| \int_{|x-y|\ge 1,|x-y|<3|x|/2} |x|^{-r}\Omega(x-y)f(y)\,dy \right.$$
$$- \int_{|x-y|\ge 1,|x-y|<3|x|/2} \Omega(x-y)f(y)|y|^{-r}\,dy$$
$$\left. + \int_{|x-y|\ge 1,|x-y|<3|x|/2} \Omega(x-y)f(y)|y|^{-r}\,dy\Big|^p dx\right)^{1/p}$$
$$\le C\left(\int\left(\int_{|x-y|<3|x|/2} ||x|^{-r}-|y|^{-r}||y|^r|x-y|^{-n}|f(y)||y|^{-r}\,dy\right)^p dx\right)^{1/p}$$
$$+ \left(\int\left|\int_{|x-y|\ge 1,|x-y|<3|x|/2} \Omega(x-y)f(y)|y|^{-r}\,dy\right|^p dx\right)^{1/p}$$
$$= I_{11} + I_{12}.$$

Since $r > -n/p'$, it follows from Lemma 2.7 (ii) and (iii) that

$$I_{11} \le C\left(\int |f(y)|^p|y|^{-rp}\,dy\right)^{1/p}.$$

Moreover by Lemmas 2.6 and 2.8 we have

$$I_{12} \le \left(\int\left|\int_{|x-y|\ge 1} \Omega(x-y)f(y)|y|^{-r}\,dy\right|^p dx\right)^{1/p}$$
$$+ C\left(\int\left(\int_{|x-y|\ge 3|x|/2} |x-y|^{-n}|f(y)||y|^{-r}\,dy\right)^p dx\right)^{1/p}$$
$$\le C\left(\int |f(y)|^p|y|^{-rp}\,dy\right)^{1/p}.$$

Thus we obtain (i).

(ii) When $-n/p' < r < n/p$, $T^*_{k,1} = T^{\Omega_1}_{k,1}$ with $\Omega_1(x) = \Omega(-x)$; hence (ii) follows from (i). Let $r > n/p$ and $r - (n/p) \neq a$ nonnegative integer. We shall prove that for $g \in L^{p',r}$ and $f \in L^{p,-r} \cap L^{1,-r}$,

$$\int\left(\int_{|x-y|\geq 1} \Omega_k(x, y) f(y)\, dy\right) g(x)\, dx = \int\left(\int_{|x-y|\geq 1} \Omega_k(x, y) g(x)\, dx\right) f(y)\, dy. \tag{3.3}$$

From this equality, (ii) readily follows from (i), since $L^{p,-r} \cap L^{1,-r}$ is dense in $L^{p,-r}$. We have

$$\begin{aligned}
&\int\left(\int_{|x-y|\geq 1} |\Omega_k(x, y) f(y)|\, dy\right) |g(x)|\, dx \\
&\leq \int\left(\int_{|x-y|\geq 1, |x-y|<3|x|/2} |\Omega(x-y) f(y)|\, dy\right) |g(x)|\, dx \\
&\quad + \sum_{|\gamma|\leq k} \int\left(\int_{|x-y|\geq 1, |x-y|<3|x|/2} \left|\frac{x^\gamma}{\gamma!}(D^\gamma \Omega)(-y) f(y)\right| dy\right) |g(x)|\, dx \\
&\quad + \int\left(\int_{|x-y|\geq 1, |x-y|\geq 3|x|/2} |\Omega_k(x, y) f(y)|\, dy\right) |g(x)|\, dx \\
&= A_1 + \sum_{|\gamma|\leq k} A_2^\gamma + A_3.
\end{aligned}$$

For A_3, since $r - (n/p) < k + 1$, by a calculation similar to (3.1) and Hölder's inequality we have

$$\begin{aligned}
A_3 &\leq C \int\left(\int_{|y|\geq |x|/2} |x|^{k+1-r} |y|^{r-k-1-n} |f(y)|\, |y|^{-r}\, dy\right) |g(x)|\, |x|^r\, dx \\
&\leq C \|f\|_{p,-r} \|g\|_{p',r} < \infty.
\end{aligned}$$

For A_2^γ, since $|\gamma| < r - (n/p)$, by (3.2) and Hölder's inequality we have

$$\begin{aligned}
A_2^\gamma &\leq C \int\left(\int_{|y|<5|x|/2} |x|^{|\gamma|-r} |y|^{r-|\gamma|-n} |f(y)|\, |y|^{-r}\, dy\right) |g(x)|\, |x|^r\, dx \\
&\leq C \|f\|_{p,-r} \|g\|_{p',r} < \infty.
\end{aligned}$$

For A_1, since $r > n/p > 0$, and $|x - y| < 3|x|/2$ implies $|x|/|y| > 2/5$, we have

$$A_1 \leq C \int\left(\int_{|x-y|\geq 1} |x-y|^{-n} |f(y)|\, |y|^{-r}\, dy\right) |g(x)|\, |x|^r\, dx.$$

For s with $1 < s < p$, we put $1/t = (1/s') + (1/p)$. Since $t > 1$ and $f \in L^{s,-r}$, by Young's inequality we see that

$$A_1 \leq C\left(\int_{|x| \geq 1} |x|^{-tn}\, dx\right)^{1/t} \|f\|_{s,-r} \|g\|_{p',r} < \infty.$$

Thus by Fubini's theorem we obtain (3.3), and hence (ii) is proved.

REMARK. *By Theorem* 1 (i) *and* (3.3), *for* $f \in L^{p,-r}$ and $g \in L^{p',r}$ *we have*

$$\int T_{k,\varepsilon} f(x) g(x)\, dx = \int T_{k,\varepsilon}^{*} g(y) f(y)\, dy.$$

4. Proof of Theorem 2

(i) By Lemma 2.10 and Theorem 1 it is sufficient to show (i) for $f \in C^1 \cap L^{p,-r}$ such that

$$\int |\nabla f(x)|^p |x|^{-rp}\, dx < \infty. \tag{4.1}$$

Let $0 < \varepsilon_1 < \varepsilon_2 < 1$. We decompose $T_{k,\varepsilon_1} f - T_{k,\varepsilon_2} f$ as follows:

$$\begin{aligned} &T_{k,\varepsilon_1} f(x) - T_{k,\varepsilon_2} f(x) \\ &\quad = \int_{\varepsilon_1 \leq |x-y| < \varepsilon_2,\, |x-y| < 3|x|/2} \Omega(x-y) f(y)\, dy \\ &\quad\quad - \sum_{|\gamma| \leq k} \int_{\varepsilon_1 \leq |x-y| < \varepsilon_2,\, |x-y| < 3|x|/2} (x^\gamma/\gamma!) D^\gamma \Omega(-y) f(y)\, dy \\ &\quad\quad + \int_{\varepsilon_1 \leq |x-y| < \varepsilon_2,\, |x-y| \geq 3|x|/2} \Omega_k(x, y) f(y)\, dy \\ &\quad = J_1(x) - \sum_{|\gamma| \leq k} J_2^\gamma(x) + J_3(x). \end{aligned}$$

By Lemma 2.1 and (3.1) we have

$$\begin{aligned} &\left(\int |J_3(x)|^p |x|^{-rp}\, dx\right)^{1/p} \\ &\quad \leq C\left(\int_{|x| < 2\varepsilon_2/3} \left(\int_{|y| \geq |x|/2} |x|^{k+1-r} |y|^{r-k-1-n} |f(y)|\, |y|^{-r}\, dy\right)^p dx\right)^{1/p} \\ &\quad \longrightarrow 0 \qquad (\varepsilon_2 \longrightarrow 0). \end{aligned}$$

Since $|\gamma| < r - (n/p)$, by (3.2) and Lebesgue's dominated convergence theorem we have

$$\left(\int |J_2^y(x)|^p |x|^{-rp} dx\right)^{1/p}$$
$$\leq C\left(\int\left(|x|^{|\gamma|-r}\int_{|x-y|<\varepsilon_2,|y|<5|x|/2} |y|^{r-|\gamma|-n}|f(y)||y|^{-r}dy\right)^p dx\right)^{1/p}$$
$$\longrightarrow 0 \qquad (\varepsilon_2 \longrightarrow 0).$$

For J_1, by the cancellation property of Ω we see that

$$B=\left(\int |J_1(x)|^p |x|^{-rp} dx\right)^{1/p}$$
$$=\left(\int\left|\int_{\varepsilon_1\leq|y|<\varepsilon_2,|y|<3|x|/2} \Omega(y)(f(x-y)-f(x))\,dy\right|^p |x|^{-rp}dx\right)^{1/p}$$
$$=\left(\int\left|\int_{\varepsilon_1\leq|y|<\varepsilon_2,|y|<3|x|/2} \Omega(y)\left(\int_0^{|y|} \nabla f(x-sy')\cdot(-y')\,ds\right)dy\right|^p |x|^{-rp}dx\right)^{1/p}$$
$$\leq \int_{\varepsilon_1\leq|y|<\varepsilon_2} |\Omega(y)|\left(\int_0^{|y|}\left(\int_{|x|>2|y|/3} |\nabla f(x-sy')|^p|x|^{-rp}dx\right)^{1/p} ds\right)dy,$$

where $y'=y/|y|$ and $x\cdot y=\sum_{i=1}^n x_i y_i$. Therefore it follows from (4.1) and Lemma 2.11 that

$$B\leq C\int_{\varepsilon_1\leq|y|<\varepsilon_2} |y|^{1-n}dy \longrightarrow 0 \qquad (\varepsilon_2\longrightarrow 0).$$

We have completed the proof of (i).

(ii) By Theorem 1 (ii) and Lemma 2.10, We may assume that $g\in C^1\cap L^{p',r}$ and

$$\int |\nabla g(x)|^{p'}|x|^{rp'}dx<\infty.$$

For $0<\varepsilon_1<\varepsilon_2<1$, we decompose $T^*_{k,\varepsilon_1}g - T^*_{k,\varepsilon_2}g$ as follows:

$$T^*_{k,\varepsilon_1}g(y) - T^*_{k,\varepsilon_2}g(y)$$
$$=\int_{\varepsilon_1\leq|x-y|<\varepsilon_2,|x-y|<3|y|/5} \Omega(x-y)g(x)\,dx$$
$$+\int_{\varepsilon_1\leq|x-y|<\varepsilon_2,|x-y|\geq 3|y|/5,|x-y|<3|x|/2} \Omega(x-y)g(x)\,dx$$

$$- \sum_{|\gamma| \le k} \int_{\varepsilon_1 \le |x-y| < \varepsilon_2, |x-y| < 3|x|/2} (x^\gamma/\gamma!)(D^\gamma \Omega)(-y) g(x)\, dx$$

$$+ \int_{\varepsilon_1 \le |x-y| < \varepsilon_2, |x-y| \ge 3|x|/2} \Omega_k(x, y) g(x)\, dx$$

$$= H_1(y) + H_2(y) - \sum_{|\gamma| \le k} H_3^\gamma(y) + H_4(y).$$

By the same discussion as for $J_1(x)$ of (i), we see that

$$\left(\int |H_1(y)|^{p'} |y|^{rp'} dy \right)^{1/p'} \longrightarrow 0 \qquad (\varepsilon_2 \longrightarrow 0).$$

Since $|x-y| \ge 3|y|/5$ and $|x-y| < 3|x|/2$ imply $|x-y| \ge 3|x|/8$ and $|x| \ge 2|y|/5$, by $r > -n/p'$ and Lemma 2.3 we have

$$\left(\int \left(\int_{|x-y| \ge 3|y|/5, |x-y| < 3|x|/2} |\Omega(x-y) g(x)|\, dx \right)^{p'} |y|^{rp'} dy \right)^{1/p'}$$

$$\le C \left(\int \left(|y|^r \int_{|x| \ge 2|y|/5} |x|^{-r-n} |g(x)|\, |x|^r dx \right)^{p'} dy \right)^{1/p'}$$

$$\le C \left(\int |g(x)|^{p'} |x|^{rp'} dx \right)^{1/p'} < \infty.$$

Hence

$$\left(\int |H_2(y)|^{p'} |y|^{rp'} dy \right)^{1/p'}$$

$$\le C \left(\int_{|y| < 5\varepsilon_2/3} \left(\int_{|x-y| \ge 3|y|/5, |x-y| < 3|x|/2} |\Omega(x-y) g(x)| dx \right)^{p'} |y|^{rp'} dy \right)^{1/p'}$$

$$\longrightarrow 0 \qquad (\varepsilon_2 \longrightarrow 0).$$

We note that $\int_{|x-y| < \varepsilon_2} |x|^{|\gamma| - r} |g(x)|\, |x|^r dx \to 0$ $(\varepsilon_2 \to 0)$ since $|\gamma| < r - (n/p)$. Hence by Lemma 2.3 and Lebesgue's dominated convergence theorem

$$\left(\int |H_3^\gamma(y)|^{p'} |y|^{rp'} dy \right)^{1/p'}$$

$$\le C \left(\int \left(|y|^{r - |\gamma| - n} \int_{|x-y| < \varepsilon_2, |x| > 2|y|/5} |x|^{|\gamma| - r} |g(x)|\, |x|^r dx \right)^{p'} dy \right)^{1/p'}$$

$$\longrightarrow 0 \qquad (\varepsilon_2 \longrightarrow 0).$$

Finally by Lemmas 2.1, 2.2 and $r - (n/p) < k + 1$,

$$\left(\int |H_4(y)|^{p'} |y|^{rp'} dy\right)^{1/p'}$$

$$\leq C\left(\int_{|y| \leq 5\varepsilon_2/3}\left(\int_{|x| \leq 2|y|} |x|^{k+1-r}|y|^{r-k-1-n}|g(x)||x|^r dx\right)^{p'} dy\right)^{1/p'}$$

$$\longrightarrow 0 \qquad (\varepsilon_2 \longrightarrow 0).$$

This completes the proof of (ii).

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