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Stability of singularly perturbed solutions to nonlinear diffusion systems arising in population dynamics

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1. Introduction

One of the important subjects in mathematical biology is to understand theoretically the interaction between the distribution of organisms and their environment. In particular, the study of coexistence of multi-competing species is a very important problem in this field. For the theoretical study, many mathematical models have been proposed so far in the framework of reaction-diffusion equations. Among them, Shigesada et al. [17] proposed the following system for two competing species:

$$(1.1)\qquad \begin{cases} u_t = \Delta[(d_1 + \alpha_{12}v)u] + (a_1 - b_1 u - c_1 v)u, \\ v_t = \Delta[(d_2 + \alpha_{21}u)v] + (a_2 - b_2 u - c_2 v)v, \qquad x \in \Omega,\ t > 0, \end{cases}$$

$$(1.2)\qquad \frac{\partial}{\partial \boldsymbol{n}}[(d_1 + \alpha_{12}v)u] = 0 = \frac{\partial}{\partial \boldsymbol{n}}[(d_2 + \alpha_{21}u)v], \qquad x \in \partial\Omega,\ t > 0,$$

$$(1.3)\qquad u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0, \quad x \in \bar{\Omega},$$

where Δ is the Laplace operator in $\mathbf{R}^n$. Ω is a bounded domain in $\mathbf{R}^n$ with smooth boundary $\partial\Omega$. $\boldsymbol{n}$ is the outer unit normal vector on $\partial\Omega$. u and v are the population densities of two competing species. a_i is the intrinsic growth rate $(i = 1, 2)$. b_1, c_2 and c_1, b_2 are the coefficients of intraspecific and interspecific competitions, respectively. d_1, d_2 and α_{12}, α_{21} are the self-diffusion rates and the cross-diffusion rates, respectively. All parameters are non-negative constants. We should briefly explain the meaning of the differential operator in (1.1) from a biological aspect. We rewrite it as the following two terms:

$$\Delta[(d_1 + \alpha_{12}v)u] = \operatorname{div}[(d_1 + \alpha_{12}v)\nabla u] + \alpha_{12}\operatorname{div}[u\nabla v].$$

The first term is a nonlinear diffusion with Fickian type, and the second one is the advection term such that u migrates to the direction of lower density of v with the speed $\alpha_{12}u|\nabla v|$, which means the intensity of the escape from the population pressure of the other species. For the precise interpretation, we refer to the excellent book by Okubo [14].

By a suitable normalization on u and v, we conveniently rewrite (1.1)–(1.3) as

$$(1.4)\qquad \begin{cases} u_t = \varepsilon^2 \Delta[(1+\alpha v)u] + \bar{f}(u, v), \\ v_t = D\Delta[(1+\beta u)v] + \bar{g}(u, v), \qquad x \in \Omega,\ t > 0, \end{cases}$$

$$(1.5)\qquad \frac{\partial}{\partial \boldsymbol{n}}[(1+\alpha v)u] = 0 = \frac{\partial}{\partial \boldsymbol{n}}[(1+\beta u)v], \qquad x \in \partial\Omega,\ t > 0,$$

$$(1.6)\qquad u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0, \quad x \in \bar{\Omega},$$

where $\bar{f}(u, v) = (1 - u - cv)u$ and $\bar{g}(u, v) = (a - bu - v)v$ with positive constants a, b and c.

First of all, we review the known results on a special case of (1.4)–(1.6) when α and β are absent. The resulting system is a usual reaction-diffusion system.

$$(1.7)\qquad \begin{cases} u_t = \varepsilon^2 \Delta u + \bar{f}(u, v), \\ v_t = D\Delta v + \bar{g}(u, v), \qquad x \in \Omega,\ t > 0, \end{cases}$$

$$(1.8)\qquad \frac{\partial}{\partial \boldsymbol{n}} u = 0 = \frac{\partial}{\partial \boldsymbol{n}} v, \qquad x \in \partial\Omega,\ t > 0,$$

$$(1.9)\qquad u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0, \quad x \in \bar{\Omega}.$$

The asymptotic behaviour of solution to (1.7)–(1.9) is classified into four cases (for example, see de Mottoni [11]):

(I) If $a < b, 1/c$, then $\lim_{t\to\infty}(u(t, x), v(t, x)) = (1, 0)$.

(II) If $b < a < 1/c$, then $\lim_{t\to\infty}(u(t, x), v(t, x)) = ((1 - ac)/(1 - bc), (a - b)/(1 - bc))$.

(III) If $1/c < a < b$, then $(1, 0)$ and $(0, a)$ are locally stable. Which species can survive in competition depends on the initial data.

(IV) If $b, 1/c < a$, then $\lim_{t\to\infty}(u(t, x), v(t, x)) = (0, a)$.

Case (III) is further investigated. Kishimoto and Weinberger [4] showed that even if any non-constant stationary solutions of (1.7)–(1.9) exist, these are unstable when Ω is convex. On the other hand, Matano and Mimura [7] proved that there exist stable non-constant stationary solutions for a suitable dumbbell shaped (non-convex) domain Ω. These results imply that the coexistence of two competing species crucially depends on the shape of the domain.

Motivated by the above, we study the problem as to whether or not (1.4)–(1.6) possibly has *stable* non-constant stationary solutions when Ω is convex. In ecological terms, this problem means whether or not the coexistence of two competing species is possible by introducing cross-diffusion effects in the convex habitat.

In this paper, we consider this problem by studying a simple case of (1.4)–(1.6) when $\alpha = 0$ and $\beta > 0$ in one dimensional space $\Omega = I = (0, 1)$. Then (1.4)–(1.6) is written as

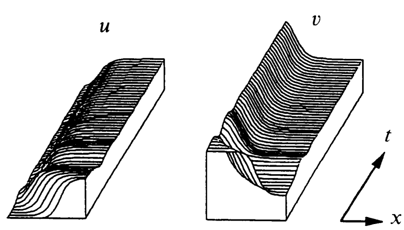
$$\begin{cases} u_t = \varepsilon^2 u_{xx} + \bar{f}(u, v), \\ v_t = D[(1+\beta u)v]_{xx} + \bar{g}(u, v), \qquad x \in I, \ t > 0, \end{cases} \tag{1.10}$$

$$u_x = 0 = [(1+\beta u)v]_x, \qquad x = 0, 1, \ t > 0, \tag{1.11}$$

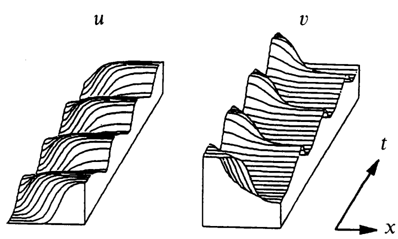
$$u(0, x) = u_0(x) \geq 0, \quad v(0, x) = v_0(x) \geq 0, \quad x \in [0, 1]. \tag{1.12}$$

For existence problems, there are many studies of the local existence of solutions of more general evolution equations including (1.10)–(1.12) in suitable function spaces (for example, see Amann [1]). Masuda and Mimura [6] proved the global existence of solutions of (1.10)–(1.12) for any initial data in $H_N^2(I) \times H_N^2(I)$, where $H_N^2(I) = \{u \in H^2(I) | u_x = 0 \text{ at } x = 0, 1\}$. Along this line, we also refer to Pozio and Tesei [16] and Yagi [18].

For stationary problems of (1.10)–(1.12), Mimura [8] and Mimura et al. [9] proved the existence of non-constant stationary solutions under some conditions when $\varepsilon > 0$ is sufficiently small. However, the stability of these solutions has not yet been proved. The difficulty is that β is present in (1.10)



(i) $a = 1.0$



(ii) $a = 0.992$

Figure 1. Spatially inhomogeneous solution of (1.10)–(1.12) where $\varepsilon^2 = 0.001$, $D = 10.0$, $\beta = 3.0$ and $b = c = 1.1$.

so that the system possesses a truly nonlinear diffusion, and that these solutions are not necessarily stable as suggested by numerical simulations. Figure 1 clearly shows that the stability depends on the values of parameters included in the system.

The purpose of this paper is to give the criterion of the stability of the stationary solutions of (1.10)–(1.12) constructed in [9].

In section 2, we introduce the stationary solutions constructed in [9]. In section 3, we study the distribution of eigenvalues of the linearized eigenvalue problems associated with these solutions by using the *SLEP method* proposed by Nishiura and Fujii [13]. In section 4, we show that the stability of these stationary solutions changes by Hopf bifurcation when some parameter is varied.

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2. Preliminary

In this section, we consider the stationary problem of (1.10)–(1.12) which is described by

$$\text{(2.1)} \qquad \begin{cases} 0 = \varepsilon^2 u_{xx} + \bar{f}(u, v), \\ 0 = D[(1 + \beta u)v]_{xx} + \bar{g}(u, v), \qquad x \in I, \end{cases}$$

$$\text{(2.2)} \qquad u_x = 0 = [(1 + \beta u)v]_x, \qquad x = 0, 1.$$

We impose the following (A.1)–(A.3) on (2.1)–(2.2).

(A.1) a, b and c satisfy the condition of Case (III) or Case (IV).

(A.2) $\beta > 1$.

Using the change of variables, $w = u$ and $z = (1 + \beta u)v$, we conveniently rewrite (2.1)–(2.2) as

$$\text{(2.3)} \qquad \begin{cases} 0 = \varepsilon^2 w_{xx} + f(w, z), \\ 0 = Dz_{xx} + g(w, z), \qquad x \in I, \end{cases}$$

$$\text{(2.4)} \qquad w_x = 0 = z_x, \quad x = 0, 1,$$

where

$$f(w, z) = \left(1 - w - \frac{cz}{1 + \beta w}\right) w,$$

$$g(w, z) = \left(a - bw - \frac{z}{1 + \beta w}\right)\frac{z}{1 + \beta w}.$$

We first consider the reduced problem ($\varepsilon = 0$) of (2.3)–(2.4):

$$\text{(2.5)} \qquad \begin{cases} 0 = f(w, z), \\ 0 = Dz_{xx} + g(w, z), \qquad x \in I, \end{cases}$$

$$\text{(2.6)} \qquad z_x = 0, \quad x = 0, 1.$$

From the first equation of (2.5), we obtain three different solutions:

$$w = h_-(z) = 0,$$

$$w = h_0(z) = \frac{\beta - 1 - \sqrt{(\beta + 1)^2 - 4c\beta z}}{2\beta},$$

$$w = h_+(z) = \frac{\beta - 1 + \sqrt{(\beta + 1)^2 - 4c\beta z}}{2\beta}.$$

Define z^* by the zero of $\int_{h_-(z)}^{h_+(z)} f(s, z)\,ds$ which is uniquely determined by the assumption (A.2).

(A.3) $a < z^*$ and $h_+(z^*) < (1 - ac)/(1 - bc)$ (Figure 2).

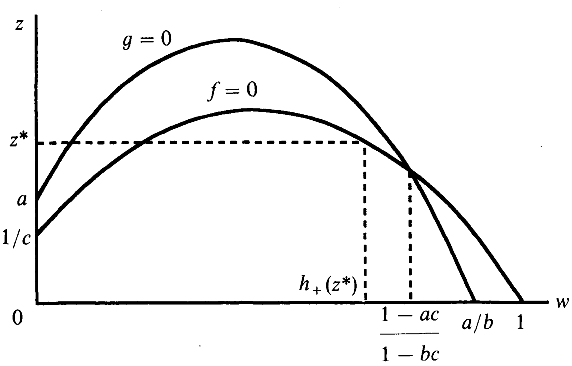

Figure 2. Functional form of f and g.

For any fixed $0 < \xi < (1 + \beta)^2/(4c\beta)$, we define $H(z; \xi)$ by

$$H(z; \xi) = \begin{cases} h_-(z) & 0 < z < \xi \\ h_+(z) & z > \xi. \end{cases}$$

Then, substituting $w = H(z; \xi)$ into the second equation of (2.5), we obtain the scalar equation with respect to z:

$$(2.7)\qquad \begin{cases} 0 = Dz_{xx} + g(H(z;\xi), z), & x \in I, \\ z_x = 0, \quad x = 0, 1. \end{cases}$$

Since $H(z;\xi)$ is discontinuous at $z = \xi$, we note that $g(H(z;\xi), z)$ is also discontinuous at $z = \xi$ (Figure 3). It is proved in Mimura et al. [10] that there is $D_N > 0$ such that, for $D_N < D$, there exist N non-constant solutions $z_i(x;\xi)$ $(i = 1, 2,\dots, N)$ of (2.7) which have the following properties:

(i) For each $i (i = 1, 2,\dots, N)$, z_i is a C^1-class function satisfying $1/c < z_i(x;\xi) < (\beta + 1)^2/(4c\beta)$ for all $x \in [0, 1]$.

(ii) For each $i (i = 1, 2,\dots, N)$, there exists a strictly increasing sequence $\{x_j\}_{j=1}^i$ such that $z_i(x_j;\xi) = \xi$ for any $j (j = 1, 2,\dots, i)$.

(iii) For each i and $j (i = 1, 2,\dots, N; j = 0, 1,\dots, i - 1)$, z_i satisfies $(z_i(x;\xi) - \xi)(z_i(y;\xi) - \xi) < 0$ for all $x \in (x_j, x_{j+1})$ and $y \in (x_{j+1}, x_{j+2})$, where $x_0 = 0$ and $x_{i+1} = 1$.

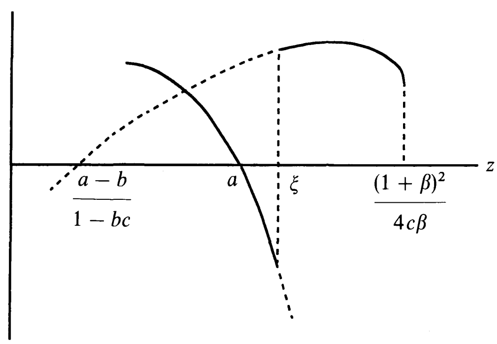

Figure 3. Functional form of $g(H(z;\xi), z)$.

Thus, putting $z = z_i(x;\xi)$ into $H(z;\xi)$, we obtain the solution $(w(x;\xi), z(x;\xi)) = (H(z_i(x;\xi);\xi), z_i(x;\xi))$ of (2.5)–(2.6).

For the sake of brevity, we consider the simple case when $N = 1$ only. So, it turns out that $z(x;\xi)$ is monotone increasing or decreasing in I, and that $z(x^*;\xi) = \xi$ at the only one point $x = x^*$. We assume here the case when $z(x;\xi)$ is monotone increasing in I. Then $w(x;\xi)$ is given by

$$w(x;\xi) = \begin{cases} h_-(z(x;\xi))(\equiv 0) & 0 < x < x^* \\ h_+(z(x;\xi)) & x^* < x < 1 \end{cases}$$

(Figure 4).

THEOREM 2.1. (Mimura et al. [9]). *Suppose a, b, c, β and z^* satisfy* (A.1)–(A.3). *Then there exist positive constants ε_0 and D_0 such that non-negative*

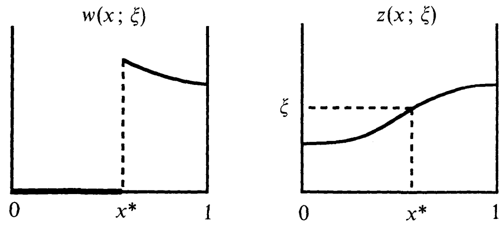

Figure 4. Reduced solution $(w(x;\xi), z(x;\xi))$.

solution $(w^{\varepsilon,D}(x), z^{\varepsilon,D}(x))$ *of* (2.3)–(2.4) *exists for all* $(\varepsilon, D) \in \Omega_0 = \{(\varepsilon, D) \in \mathbf{R}^2 \mid 0 < \varepsilon < \varepsilon_0, D > D_0\}$, *and satisfies*

$$\lim_{\varepsilon \downarrow 0} z^{\varepsilon,D}(x) = z(x; z^*),$$

$$\lim_{\varepsilon \downarrow 0} w^{\varepsilon,D}(x) = H(z(x; z^*); z^*) \qquad \textit{compact uniformly in } x \in \bar{I} \setminus \{x^*\}.$$

We call $(w^{\varepsilon,D}(x), z^{\varepsilon,D}(x))$ a *singularly perturbed solution* of (2.3)–(2.4) since $w^{\varepsilon,D}$ possesses an internal layer near $x = x^*$ when $\varepsilon > 0$ is sufficiently small (Figure 5).

By the inverse transformations, $u = w$ and $v = z/(1 + \beta w)$, it turns out that a solution $(u^{\varepsilon,D}(x), v^{\varepsilon,D}(x))$ of (1.10)–(1.12) is given by the form $(w^{\varepsilon,D}(x), z^{\varepsilon,D}(x)/(1 + \beta w^{\varepsilon,D}(x)))$, where

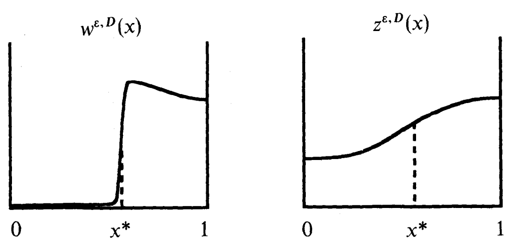

Figure 5. Singularly perturbed solution $(w^{\varepsilon,D}(x), z^{\varepsilon,D}(x))$.

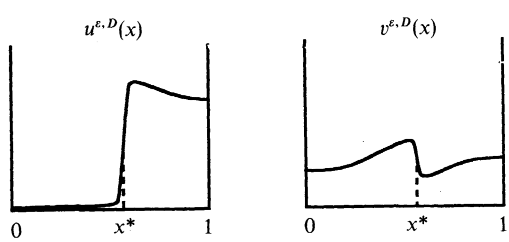

Figure 6. Singularly perturbed solution $(u^{\varepsilon,D}(x), v^{\varepsilon,D}(x))$.

$$\lim_{\varepsilon\downarrow 0}(u^{\varepsilon,D}(x), v^{\varepsilon,D}(x)) = \begin{cases} (0, z(x; z^*)) & 0 \le x < x^* \\ (h_+(z(x; z^*)), z(x; z^*)/(1+\beta h_+(z(x; z^*)))) & x^* < x \le 1. \end{cases}$$

We note that both u and v exhibit an internal layer near $x = x^* \in I$ (Figure 6).

3. Stability of singularly perturbed solutions

In this section, we study the stability of singularly perturbed solutions given in the previous section. Let $\varphi(u, v)$ and $\psi(u, v)$ be $\varphi(u, v) = u$ and $\psi(u, v) = (1 + \beta u)v$, respectively. Since $\det \dfrac{\partial(\varphi, \psi)}{\partial(u, v)} = 1 + \beta u > 0$ for all $(u, v) \in \mathcal{O} \equiv [0, \infty) \times [0, \infty)$, there exists a smooth inverse map (Φ, Ψ) of (φ, ψ) in $\mathcal{O}$. By the change of the variables used in the last section, $w = \varphi(u, v)$ and $z = \psi(u, v)$, (1.10)–(1.12) becomes

$$\begin{cases} \Phi_w w_t + \Phi_z z_t = \varepsilon^2 w_{xx} + f(w, z), \\ \Psi_w w_t + \Psi_z z_t = D z_{xx} + g(w, z), & x \in I, \ t > 0, \end{cases} \tag{3.1}$$

$$w_x = 0 = z_x, \quad x = 0, 1, \quad t > 0, \tag{3.2}$$

$$\begin{cases} w(0, x) = \varphi(u_0(x), v_0(x)) (\ge 0), \\ z(0, x) = \psi(u_0(x), v_0(x)) (\ge 0), & x \in [0, 1]. \end{cases} \tag{3.3}$$

Let $\boldsymbol{w}^{\varepsilon,D} = (w^{\varepsilon,D}, z^{\varepsilon,D})$ be a non-negative singularly perturbed stationary solution of (3.1)–(3.3) given in Theorem 2.1 for $(\varepsilon, D) \in \Omega_0$. Then $\boldsymbol{u}^{\varepsilon,D} = (u^{\varepsilon,D}, v^{\varepsilon,D}) = (\Phi(\boldsymbol{w}^{\varepsilon,D}), \Psi(\boldsymbol{w}^{\varepsilon,D}))$ is a solution of (1.10)–(1.12). By $H^2(I) \subset C^1(\bar{I})$ and the regularity of (Φ, Ψ), we find that the stability of $\boldsymbol{u}^{\varepsilon,D}$ coincides with that of $\boldsymbol{w}^{\varepsilon,D}$. So, we may only consider the stability of $\boldsymbol{w}^{\varepsilon,D}$.

We define $J(z)$ by $J(z) = \int_{h_-(z)}^{h_+(z)} f(s, z)\,ds$, and also define R_-, R_+ by

$$R_- = \{(w, z) \mid w = h_-(z) \text{ for } 1/c < z \le z^*\},$$
$$R_+ = \{(w, z) \mid w = h_+(z) \text{ for } z^* \le z < (\beta + 1)^2/(4c\beta)\},$$

respectively. Then we obtain

$$\frac{dJ}{dz}(z^*) = -\int_{h_-(z^*)}^{h_+(z^*)} \frac{cs}{1 + \beta s}\,ds (< 0),$$

$$f_w(w, z) = \begin{cases} 1 - cz (< 0) & (w, z) \in R_- \\ -h_+(z)\dfrac{2\beta h_+(z) - \beta + 1}{1 + \beta h_+(z)} (< 0) & (w, z) \in R_+. \end{cases}$$

By Theorem 2.1, we find that $\lim_{\varepsilon\downarrow 0} w^{\varepsilon,D}(x) \in R_- \cup R_+$ for all $x \in [0, 1] \setminus \{x^*\}$, i.e., $\lim_{\varepsilon\downarrow 0} f_w(w^{\varepsilon,D}) < 0$ in $[0, 1] \setminus \{x^*\}$.

The linearized eigenvalue problem associated with $w^{\varepsilon,D}$ is given by

$$(3.4)\qquad \begin{cases} \lambda(\Phi_w^{\varepsilon,D} W + \Phi_z^{\varepsilon,D} Z) = \varepsilon^2 W_{xx} + f_w^{\varepsilon,D} W + f_z^{\varepsilon,D} Z, \\ \lambda(\Psi_w^{\varepsilon,D} W + \Psi_z^{\varepsilon,D} Z) = D Z_{xx} + g_w^{\varepsilon,D} W + g_z^{\varepsilon,D} Z, \qquad x \in I, \\ W_x = 0 = Z_x, \quad x = 0, 1, \end{cases}$$

where $f_w^{\varepsilon,D}(x) = f_w(w^{\varepsilon,D}(x))$ and other partial derivatives are defined similarly. In order to study the distribution of eigenvalues of (3.4), we apply the algorithm of the SLEP method, which was proposed by Nishiura and Fujii [13], to (3.4).

Let us first consider the following Sturm-Liouville problem:

$$(3.5)\qquad \begin{cases} 0 = L^{\varepsilon,D}(\lambda) W \equiv \varepsilon^2 W_{xx} + (f_w^{\varepsilon,D} - \lambda \Phi_w^{\varepsilon,D}) W, \qquad x \in I, \\ W_x = 0, \quad x = 0, 1. \end{cases}$$

By the definition of (Φ, Ψ), we have

$$\Phi_w^{\varepsilon,D} = \psi_v^{\varepsilon,D} / \det \frac{\partial(\varphi, \psi)^{\varepsilon,D}}{\partial(u, v)} > 0.$$

By virtue of the above inequality, it turns out that the sequence of eigenvalues and the corresponding eigenfunctions $\{(\lambda_n^{\varepsilon,D}, \phi_n^{\varepsilon,D})\}_{n\geq 0}$ of (3.5) satisfies the following properties:

(i) For each $n \geq 0$, $\lambda_n^{\varepsilon,D}$ is real and $\lambda_n^{\varepsilon,D} \geq \lambda_{n+1}^{\varepsilon,D}$. Moreover, $\lambda_n^{\varepsilon,D} \to -\infty$ as $n \to \infty$.

(ii) $\phi_n^{\varepsilon,D}$ has n zeros for each $n \geq 0$.

(iii) $\{\phi_n^{\varepsilon,D}\}_{n\geq 0}$ is a complete orthonormal set (CONS) in $L^2(I)$ with respect to the weight $\Phi_w^{\varepsilon,D}$.

We normalize $\phi_n^{\varepsilon,D}$ as $\langle |\phi_n^{\varepsilon,D}|^2, \Phi_w^{\varepsilon,D} \rangle = 1$ for all $n \geq 0$, where $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(I)$. Throughout this paper, we denote $h^{0,D}$, $h^{\varepsilon,\infty}$ and $h^{0,\infty}$ by

$$h^{0,D} = \lim_{\varepsilon\downarrow 0} h^{\varepsilon,D}, \ h^{\varepsilon,\infty} = \lim_{D\uparrow\infty} h^{\varepsilon,D} \quad \text{and} \quad h^{0,\infty} = \lim_{\varepsilon\downarrow 0} \lim_{D\uparrow\infty} h^{\varepsilon,D},$$

respectively. We also denote the stretched function of $h \in C^2(I)$ by $\tilde{h} \in C^2(\tilde{I})$ where $\tilde{I} = (-x^*/\varepsilon, (1 - x^*)/\varepsilon)$, i.e., $\tilde{h}(y) = h(x^* + \varepsilon y)$, and put $\hat{\phi}_0^{\varepsilon,D} = \sqrt{\varepsilon}\, \tilde{\phi}_0^{\varepsilon,D}$.

LEMMA 3.1. (Lemmas 1.2 and 1.4 in [13]). *$\phi_0^{\varepsilon,D}$ satisfies the following properties:*

(i) $\hat{\phi}_0^{\varepsilon,D} \to \hat{\phi}_0^{0,D} = \kappa^*(d\tilde{w}/dy)^{0,D}$ *as* $\varepsilon \to 0$, *where* $(\kappa^*)^{-1} = \|(d\tilde{w}/dy)^{0,D}\|_{L^2(\mathbf{R})}$. *The convergence is uniform on any compact subset of* $\mathbf{R}$.

(ii) $\int_I \phi_0^{\varepsilon,D} dx = \sqrt{\varepsilon} L(\varepsilon, D)$ *as* $\varepsilon \to 0$, *where* $L(\varepsilon, D)$ *is a positive continuous function of* $(\varepsilon, D) \in \Omega_0$. *Moreover,* $L^* = \lim_{\varepsilon \downarrow 0} L(\varepsilon, D) = \kappa^* \{h_+(z^*) - h_-(z^*)\}$.

LEMMA 3.2. (Corollary 1.3 in [13]). *There exist positive constants* Δ^*, C_1 *and* γ_1 *independently of* $(\varepsilon, D) \in \Omega_0$ *such that*

$$\lambda_0^{\varepsilon,D} = \varepsilon \hat{\lambda}_0(\varepsilon, D) + Exp(\varepsilon, D) \quad \text{and} \quad \lambda_1^{\varepsilon,D} \leq -\Delta^* (< 0)$$

hold for all $(\varepsilon, D) \in \Omega_0$ *where*

$$\hat{\lambda}_0^* = \lim_{\varepsilon \downarrow 0} \hat{\lambda}_0(\varepsilon, D) = \frac{(\kappa^*)^2}{D} \frac{dJ}{dz}(z^*) \int_0^{x^*} g(w^{0,D}(x))\, dx,$$

$$|Exp(\varepsilon, D)| \leq C_1 \exp(-\gamma_1/\varepsilon).$$

Since $w^{\varepsilon,D}$ is uniformly L^∞-bounded for $(\varepsilon, D) \in \Omega_0$, there exists $\mu_0 > 0$ such that

$$\sup_{D > D_0} \sup_{x \in [0,1] \setminus \{x^*\}} \frac{f_w^{0,D}(x)}{\Phi_w^{0,D}(x)} \leq -\mu_0 < 0.$$

Let ω and μ be constants satisfying $0 < \omega < \pi/2$ and $0 < \mu < \min(\Delta^*, \mu_0)$, respectively, and define a subset $\Sigma(\omega, \mu)$ in the complex plane $\mathbf{C}$ by

$$\Sigma(\omega, \mu) = \{\lambda \in \mathbf{C} \mid \operatorname{Re}\lambda \geq -\mu \text{ or } |\arg \lambda| \leq \pi/2 + \omega\}.$$

Set $W = \sum_{n=0}^\infty W_n \phi_n^{\varepsilon,D} = L^{\varepsilon,D}(\lambda)^{-1} h$ where $h \in L^2(I)$ and $\lambda \notin \{\lambda_n^{\varepsilon,D}\}_{n \geq 0}$. Since $\{\phi_n^{\varepsilon,D}\}_{n \geq 0}$ is a CONS in $L^2(I)$ with respect to the weight $\Phi_w^{\varepsilon,D}$, we have

$$W = \sum_{n=0}^\infty \frac{\langle h, \phi_n^{\varepsilon,D} \rangle}{\lambda_n^{\varepsilon,D} - \lambda} \phi_n^{\varepsilon,D}.$$

Define $L^{\varepsilon,D}(\lambda)^\dagger$ by

$$L^{\varepsilon,D}(\lambda)^\dagger h = \sum_{n=1}^\infty \frac{\langle h, \phi_n^{\varepsilon,D} \rangle}{\lambda_n^{\varepsilon,D} - \lambda} \phi_n^{\varepsilon,D}$$

for all $h \in L^2(I)$. Then the following lemma is easily obtained.

LEMMA 3.3.

$$\| L^{\varepsilon,D}(\lambda)^\dagger \|_{L^2(I)} \leq \frac{C_2}{\operatorname{dist}(\{\lambda_n^{\varepsilon,D}\}_{n \geq 1}, \lambda)}$$

for all $\lambda \in \Sigma(\omega, \mu)$ *where*

$$C_2^{-1} = \inf_{(\varepsilon, D, x) \in \Omega_0 \times [0,1]} \Phi_w^{\varepsilon,D} (> 0).$$

By the first equation of (3.4), we have

$$W = \frac{\langle -(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D})Z, \phi_0^{\varepsilon,D}\rangle}{\lambda_0^{\varepsilon,D} - \lambda} \phi_0^{\varepsilon,D} \tag{3.6}$$
$$+ L^{\varepsilon,D}(\lambda)^{\dagger}[-(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D})Z].$$

LEMMA 3.4. (Lemma 2.2 in [13]). *Let $F(w, z)$ be a smooth function of w and z. Then*

$$L^{\varepsilon,D}(\lambda)^{\dagger}[F^{\varepsilon,D}h] \xrightarrow{\varepsilon \downarrow 0} \frac{F^{0,D}h}{f_w^{0,D} - \lambda \Phi_w^{0,D}} \qquad \textit{strongly in } L^2(I)$$

for any $h \in L^2(I) \cap L^\infty(I)$ and $\lambda \in \Sigma(\omega, \mu)$, where $F^{\varepsilon,D} = F(w^{\varepsilon,D})$ and $F^{0,D} = F(w^{0,D})$. The convergence is uniform for $D > D_0$ and $\lambda \in \Sigma(\omega, \mu)$.

LEMMA 3.5. (Lemma 2.3 in [13]).

$$-\frac{1}{\sqrt{\varepsilon}}(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D})\phi_0^{\varepsilon,D} \xrightarrow{\varepsilon \downarrow 0} \left\{c_1^* + \kappa^* \lambda \int_{\mathbf{R}} \tilde{\Phi}_z^{0,D} \frac{d\tilde{w}^{0,D}}{dy} dy\right\} \delta^*$$

in $H^{-1}(I)$-sense and

$$\frac{1}{\sqrt{\varepsilon}}(g_w^{\varepsilon,D} - \lambda \Psi_w^{\varepsilon,D})\phi_0^{\varepsilon,D} \xrightarrow{\varepsilon \downarrow 0} \left\{c_2^* - \kappa^* \lambda \int_{\mathbf{R}} \tilde{\Psi}_w^{0,D} \frac{d\tilde{w}^{0,D}}{dy} dy\right\} \delta^*$$

in $H^{-1}(I)$-sense hold uniformly for $D > D_0$ where δ^* is the Dirac's δ-function at $x = x^*$, and

$$c_1^* = -\kappa^* \frac{dJ}{dz}(z^*)(>0), \quad c_2^* = \kappa^* \{g(h_+(z^*), z^*) - g(h_-(z^*), z^*)\}.$$

LEMMA 3.6. *Let h be a C^2-class function. Then*

$$\lambda L^{\varepsilon,D}(\lambda)^{\dagger}[h] \longrightarrow -\frac{h}{\Phi_w^{\varepsilon,D}} + \langle h, \phi_0^{\varepsilon,D}\rangle \phi_0^{\varepsilon,D} \qquad \textit{strongly in } L^2(I)$$

as $|\lambda| \to \infty$ and $\lambda \in \Sigma(\omega, \mu)$. The convergence is uniform for $(\varepsilon, D) \in \Omega_0$.

PROOF. Set $\lambda L^{\varepsilon,D}(\lambda)^{\dagger}[h] = H^{\varepsilon,D,\lambda} - (h/\Phi_w^{\varepsilon,D}) + \langle h, \phi_0^{\varepsilon,D}\rangle \phi_0^{\varepsilon,D}$. By multiplying $L^{\varepsilon,D}(\lambda)$ into the both hands, we have $L^{\varepsilon,D}(\lambda)H^{\varepsilon,D,\lambda} = R^{\varepsilon,D,\lambda}$ where $R^{\varepsilon,D,\lambda} = L^{\varepsilon,D}(0)[h/\Phi_w^{\varepsilon,D}] - \lambda_0^{\varepsilon,D}\langle h, \phi_0^{\varepsilon,D}\rangle \Phi_w^{\varepsilon,D}\phi_0^{\varepsilon,D}$. From the regularity of $w^{\varepsilon,D}$ for any fixed $(\varepsilon, D) \in \Omega_0$, it follows that $\| L^{\varepsilon,D}(0)[h/\Phi_w^{\varepsilon,D}] \|_{L^2(I)}$ and $\| \Phi_w^{\varepsilon,D}\phi_0^{\varepsilon,D} \|_{L^2(I)}$ are bounded. Therefore, $\| R^{\varepsilon,D,\lambda} \|_{L^2(I)}$ is uniformly bounded with respect to λ. Since $\langle R^{\varepsilon,D,\lambda}, \phi_0^{\varepsilon,D}\rangle = 0$, we obtain $H^{\varepsilon,D,\lambda} = L^{\varepsilon,D}(\lambda)^{\dagger} R^{\varepsilon,D,\lambda}$. By Lemma 3.3, we find

$$\begin{aligned}\| H^{\varepsilon,D,\lambda} \|_{L^2(I)} &\le \| L^{\varepsilon,D}(\lambda)^{\dagger} \|_{L^2(I)} \| R^{\varepsilon,D,\lambda} \|_{L^2(I)} \\ &\le C_2 \frac{\| R^{\varepsilon,D,\lambda} \|_{L^2(I)}}{\operatorname{dist}(\{\lambda_n^{\varepsilon,D}\}_{n\ge 1}, \lambda)} \\ &\longrightarrow 0 \quad \text{as} \quad |\lambda| \longrightarrow \infty \quad \text{and} \quad \lambda \in \Sigma(\omega, \mu).\end{aligned}$$

Substituting (3.6) into the second equation of (3.4), we have

$$\begin{aligned}0 = DZ_{xx} &+ \frac{\langle -(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D})Z, \phi_0^{\varepsilon,D} \rangle}{\lambda_0^{\varepsilon,D} - \lambda} (g_w^{\varepsilon,D} - \lambda \Psi_w^{\varepsilon,D}) \phi_0^{\varepsilon,D} \\ &+ (g_w^{\varepsilon,D} - \lambda \Psi_w^{\varepsilon,D}) L^{\varepsilon,D}(\lambda)^{\dagger}[-(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D})Z] \\ &+ (g_z^{\varepsilon,D} - \lambda \Psi_z^{\varepsilon,D})Z. \end{aligned} \tag{3.7}$$

We introduce the following bilinear form associated with the above equation:

$$\begin{aligned}B^{\varepsilon,D,\lambda}(Z^1, Z^2) &= D\langle Z_x^1, Z_x^2 \rangle \\ &- \frac{\langle -(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D})Z^1, \phi_0^{\varepsilon,D} \rangle}{\lambda_0^{\varepsilon,D} - \lambda} \langle (g_w^{\varepsilon,D} - \lambda \Psi_w^{\varepsilon,D}) \phi_0^{\varepsilon,D}, Z^2 \rangle \\ &- \langle (g_w^{\varepsilon,D} - \lambda \Psi_w^{\varepsilon,D}) L^{\varepsilon,D}(\lambda)^{\dagger}[-(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D})Z^1], Z^2 \rangle \\ &- \langle (g_z^{\varepsilon,D} - \lambda \Psi_z^{\varepsilon,D})Z^1, Z^2 \rangle \qquad \text{for all } Z^1, Z^2 \in H_N^1(I). \end{aligned}$$

Let λ and Z be an eigenvalue and its eigenfunction which is normalized as

$$\left\langle \frac{1}{\Phi_w^{\varepsilon,D}} \det \frac{\partial(\Phi, \Psi)^{\varepsilon,D}}{\partial(w, z)}, |Z|^2 \right\rangle = 1.$$

Then Z satisfies $C_3 \| Z \|_{L^2(I)}^2 \le 1 \le C_4 \| Z \|_{L^2(I)}^2$, where

$$C_3 = \inf_{(\varepsilon, D, x) \in \Omega_0 \times [0,1]} \left\{ \frac{1}{\Phi_w^{\varepsilon,D}} \det \frac{\partial(\Phi, \Psi)^{\varepsilon,D}}{\partial(w, z)} \right\} (>0),$$

$$C_4 = \sup_{(\varepsilon, D, x) \in \Omega_0 \times [0,1]} \left\{ \frac{1}{\Phi_w^{\varepsilon,D}} \det \frac{\partial(\Phi, \Psi)^{\varepsilon,D}}{\partial(w, z)} \right\} (>0).$$

LEMMA 3.7. *There exists $M_1 > 0$ such that $|\lambda| \le M_1$ for all $(\varepsilon, D) \in \Omega_0$ and $\lambda \in \Sigma(\omega, \mu) \cap \sigma(3.4)$, where $\sigma(3.4)$ represents a set of all eigenvalues of* (3.4).

PROOF. Let Z be an eigenfunction associated with the eigenvalue λ. It follows from Lemma 3.6 that

$$0 = \frac{B^{\varepsilon,D,\lambda}(Z, Z)}{\lambda} = \frac{D \| Z_x \|_{L^2(I)}^2}{\lambda} + 1 + R^{\varepsilon,D}(\lambda, Z)$$

where $R^{\varepsilon,D}(\lambda, Z) = O(\| Z \|_{L^2(I)}/\lambda)$ as $|\lambda| \to \infty$ and $\lambda \in \Sigma(\omega, \mu)$. Since C_3 is independent of ε, D and λ, we know that

$$R^{\varepsilon,D}(\lambda, Z) \longrightarrow 0 \quad \text{as} \quad |\lambda| \longrightarrow \infty \quad \text{and} \quad \lambda \in \Sigma(\omega, \mu).$$

This implies that the eigenvalue λ with $\lambda \in \Sigma(\omega, \mu)$ is uniformly bounded for all $(\varepsilon, D) \in \Omega_0$.

LEMMA 3.8. *Let Z be an eigenfunction associated with $\lambda \in \Sigma(\omega, \mu) \cap \sigma(3.4)$. Then there exists $M_2 > 0$ such that*

$$\sqrt{\frac{1}{C_4}} \leq \|Z\|_{H^1(I)} \leq \sqrt{\frac{1 + M_2}{C_3}}.$$

PROOF. Since λ and Z satisfy $B^{\varepsilon,D,\lambda}(Z, Z) = 0$ and $w^{\varepsilon,D}$ is L^∞-bounded, we find that $\|Z_x\|^2_{L^2(I)} \leq M_2 \|Z\|^2_{L^2(I)}$ where $M_2 > 0$ is independent of $(\varepsilon, D) \in \Omega_0$ and $\lambda \in \Sigma(\omega, \mu) \cap \{\lambda \in \mathbf{C} \mid |\lambda| \leq M_1\}$.

Let B_δ be a closed ball with center at the origin and radius δ in the complex plane $\mathbf{C}$. We shall say that $\lambda = \lambda(\varepsilon, D)$ is a *non-critical eigenvalue* of (3.4) if there exists $\delta > 0$ such that $\lambda \notin B_\delta$ for small $\varepsilon > 0$, and that $\lambda = \lambda(\varepsilon, D)$ is a *critical eigenvalue* of (3.4) if λ is not a non-critical eigenvalue of (3.4).

Let $\delta > 0$ be an arbitrary fixed constant. Since $\lim_{\varepsilon \downarrow 0} \lambda_0^{\varepsilon,D} = 0$ and $\int_I \phi_0^{\varepsilon,D} dx = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$, it holds that

$$|\text{the second term of } B^{\varepsilon,D,\lambda}(Z^1, Z^2)| \leq \frac{C_5(\lambda)\varepsilon}{\delta} \|Z^1\|_{H^1(I)} \|Z^2\|_{H^1(I)}$$

for $(\varepsilon, D) \in \Omega_0$ and $\lambda \in (\Sigma(\omega, \mu) \cap B_{M_1}) \setminus B_\delta$, where $C_5(\lambda) > 0$ depends on λ only. Therefore, we obtain

$$\begin{aligned}
|B^{\varepsilon,D,\lambda}(Z^1, Z^2)| \leq\ & D \|Z_x^1\|_{L^2(I)} \|Z_x^2\|_{L^2(I)} \\
& + \frac{C_5(\lambda)\varepsilon}{\delta} \|Z^1\|_{H^1(I)} \|Z^2\|_{H^1(I)} \\
& + C_6(\lambda) \|Z^1\|_{L^2(I)} \|Z^2\|_{L^2(I)},
\end{aligned}$$

where $C_6(\lambda)$ depends on λ only. The above inequality shows the boundedness of $B^{\varepsilon,D,\lambda}$. By Lemma 3.4, we have

$$\begin{aligned}
&\langle (g_w^{\varepsilon,D} - \lambda \Psi_w^{\varepsilon,D}) L^{\varepsilon,D}(\lambda)^\dagger [-(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D}) Z], Z \rangle \\
&\qquad \xrightarrow{\varepsilon \downarrow 0} \left\langle -\frac{(f_z^{0,D} - \lambda \Phi_z^{0,D})(g_w^{0,D} - \lambda \Psi_w^{0,D})}{f_w^{0,D} - \lambda \Phi_w^{0,D}}, |Z|^2 \right\rangle,
\end{aligned}$$

$$\langle (g_z^{\varepsilon,D} - \lambda \Psi_z^{\varepsilon,D}) Z, Z \rangle \xrightarrow{\varepsilon \downarrow 0} \langle g_z^{0,D} - \lambda \Psi_z^{0,D}, |Z|^2 \rangle$$

uniformly for $\|Z\|_{H^1(I)} \leq \sqrt{(1+M_2)/C_3}$ and $\lambda \in \Sigma(\omega, \mu) \cap B_{M_1}$. We define $B^{0,D,\lambda}$ by

$$B^{0,D,\lambda}(Z^1, Z^2) = D\langle Z_x^1, Z_x^2 \rangle + \left\langle \frac{f_z^{0,D} - \lambda \Phi_z^{0,D}}{f_w^{0,D} - \lambda \Phi_w^{0,D}} Z^1, (g_w^{0,D} - \lambda \Psi_w^{0,D}) Z^2 \right\rangle - \langle (g_z^{0,D} - \lambda \Psi_z^{0,D}) Z^1, Z^2 \rangle.$$

If $|B^{0,D,\lambda}(Z, Z)| \geq C_7 \|Z\|_{H^1(I)}$ holds for some positive constant C_7, then the similar inequality holds for $B^{\varepsilon,D,\lambda}(Z, Z)$ with the appropriate change of constant. Rewrite $B^{0,D,\lambda}$ as

$$B^{0,D,\lambda}(Z, Z) = D\langle Z_x, Z_x \rangle + \langle Q^{0,D}(\lambda), |Z|^2 \rangle,$$

where

$$Q^{\varepsilon,D}(\lambda) = Q_1^{\varepsilon,D}\lambda + Q_2^{\varepsilon,D} + \frac{Q_3^{\varepsilon,D}}{\Phi_w^{\varepsilon,D}\lambda - f_w^{\varepsilon,D}},$$

$$Q_1^{\varepsilon,D} = \frac{1}{\Phi_w^{\varepsilon,D}} \det \frac{\partial(\Phi, \Psi)^{\varepsilon,D}}{\partial(w, z)} \quad (> 0),$$

$$Q_2^{\varepsilon,D} = -\frac{1}{(\Phi_w^{\varepsilon,D})^2} \left(\det \frac{\partial(\Phi, \Psi)^{\varepsilon,D}}{\partial(w, z)} \right)^2 [\bar{f}_v^{\varepsilon,D} \psi_u^{\varepsilon,D} + \bar{g}_v^{\varepsilon,D} \psi_v^{\varepsilon,D}],$$

$$Q_3^{\varepsilon,D} = -\frac{1}{(\Phi_w^{\varepsilon,D})^2} (f_w^{\varepsilon,D} \Psi_w^{\varepsilon,D} - g_w^{\varepsilon,D} \Phi_w^{\varepsilon,D})(f_w^{\varepsilon,D} \Phi_z^{\varepsilon,D} - f_z^{\varepsilon,D} \Phi_w^{\varepsilon,D})$$

$$= f_w^{\varepsilon,D} Q_2^{\varepsilon,D} + \det \frac{\partial(f, g)^{\varepsilon,D}}{\partial(w, z)}.$$

We assume

(A.4)
$$[\bar{f}_v \psi_u + \bar{g}_v \psi_v]|_{(\Phi, \Psi)(R_+ \cup R_-)} < 0,$$

(A.5)
$$\det \left. \frac{\partial(f, g)}{\partial(w, z)} \right|_{R_+ \cup R_-} > 0.$$

$\operatorname{Re} Q^{0,D}(\lambda)$ satisfies

$$\operatorname{Re} Q^{0,D}(\lambda) = Q_1^{0,D} \operatorname{Re}\lambda + \frac{1}{|\Phi_w^{0,D}\lambda - f_w^{0,D}|^2} \times \left\{ Q_2^{0,D} \Phi_w^{0,D} (\Phi_w^{0,D} |\lambda|^2 - f_w^{0,D} \operatorname{Re}\lambda) + \det \frac{\partial(f, g)^{0,D}}{\partial(w, z)} (\Phi_w^{0,D} \operatorname{Re}\lambda - f_w^{0,D}) \right\}.$$

By (A.4), (A.5) and $f_w|_{R_+ \cup R_-} < 0$, we find that there exist positive constants $\mu_1(\leq \mu_0)$ and C_8 independently of $\delta > 0$ such that $\operatorname{Re} Q^{0,D}(\lambda) > C_8$ for all $x \in [0, 1] \setminus \{x^*\}$ and $\lambda \in \{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda > -\mu_1, |\lambda| \leq M_1\}$. Therefore we find that

$$|B^{0,D,\lambda}(Z, Z)| \geq |\operatorname{Re} B^{0,D,\lambda}(Z, Z)| \geq C_9 \|Z\|_{H^1(I)},$$

where $C_9 = \min\{D, C_8\}$.

We put $\omega^* = \tan^{-1}(\mu_1/M_1) \in (0, \pi/2)$. From Lemma 3.7 and the Lax-Milgram theorem, we have the following lemma:

LEMMA 3.9. *Suppose λ is an arbitrary non-critical eigenvalue of* (3.4). *If the assumptions* (A.4) *and* (A.5) *are satisfied, then* $\lambda \in \mathbf{C} \setminus \Sigma(\omega^*, \mu_1)$.

Now we consider the distribution of critical eigenvalues. To do it, we take the asymptotic form of $\lambda(\varepsilon, D)$ as $\lambda(\varepsilon, D) = \varepsilon\tau^{\varepsilon,D}$ where $\tau^{\varepsilon,D}$ is a continuous function of ε and D. Let $Z^{\varepsilon,D}$ be an eigenfunction associated with $\lambda(\varepsilon, D)$. By using Lemma 3.5, we have

$$\frac{\langle -(f_z^{\varepsilon,D} - \lambda \Phi_z^{\varepsilon,D}) Z^{\varepsilon,D}, \phi_0^{\varepsilon,D} \rangle}{\lambda_0^{\varepsilon,D} - \lambda(\varepsilon, D)} (g_w^{\varepsilon,D} - \lambda \Psi_w^{\varepsilon,D}) \phi_0^{\varepsilon,D}$$

$$\longrightarrow \frac{c_1^* c_2^* \langle Z^{0,D}, \delta^* \rangle \delta^*}{\hat{\lambda}_0^* - \tau^{0,D}} \qquad \text{as } \varepsilon \longrightarrow 0$$

in $H^{-1}(I)$-sense where $\tau^{0,D} = \lim_{\varepsilon \downarrow 0} \tau^{\varepsilon,D}$. Normalizing $Z^{0,D}$ as $\langle Z^{0,D}, \delta^* \rangle = 1$, we obtain the following equation:

$$\begin{cases} 0 = D Z_{xx}^{0,D} + \dfrac{1}{f_w^{0,D}} \det \dfrac{\partial(f, g)^{0,D}}{\partial(w, z)} Z^{0,D}, & x \in I \setminus \{x^*\}, \\ Z_x^{0,D} = 0, \quad x = 0, 1, \\ D[Z_x^{0,D}] = -\dfrac{c_1^* c_2^*}{\hat{\lambda}_0^* - \tau^{0,D}}, \end{cases}$$

where $[Z_x^{0,D}] = \lim_{\delta \downarrow 0} \{Z_x^{0,D}(x^* + \delta) - Z_x^{0,D}(x^* - \delta)\}$. As $D \to \infty$, we have

$$\tau^{0,\infty} = (c_1^* c_2^*) \Big/ \left\langle \frac{1}{f_w^{0,\infty}} \det \frac{\partial(f, g)^{0,\infty}}{\partial(w, z)}, 1 \right\rangle. \tag{3.8}$$

By (A.5) and the definition of c_2^*, if $g(h_+(z^*), z^*) > g(h_-(z^*), z^*)$, we find $\tau^{0,\infty} < 0$. Putting $\mu^*(\varepsilon) = \min\{\mu_1/\varepsilon, -\tau^{0,\infty}/2\} (> 0)$, we know that there exist positive constants ε^* and D^* such that $\tau^{\varepsilon,D} < -\mu^*(\varepsilon)$ for all $(\varepsilon, D) \in \Omega^* = \{(\varepsilon, D) \mid 0 < \varepsilon < \varepsilon^*, D > D^*\}$. We now arrive at the following theorem.

THEOREM 3.10. *If the assumptions* (A.4), (A.5) *and*

(A.6) $$g|_{R_-} < 0 < g|_{R_+}$$

are satisfied, then $\Sigma(\omega^*, \varepsilon\mu^*(\varepsilon)) \subset \rho(3.4)$ *for all* $(\varepsilon, D) \in \Omega^*$, *where* $\rho(3.4)$ *represents a resolvent set of* (3.4).

The problem which we should do is to show that the stability of stationary solutions of (3.1)–(3.3) is given by the argument on the corresponding linearized eigenvalue problem.

Let us introduce the following theorem which justifies the linearized stability principle for the quasilinear parabolic equations including (3.1)–(3.3). The interpolation spaces in the sense of Lions and Peetre [5] are denoted by $[Y, X]_{\theta,p}$ and their norms by $\|\cdot\|_{\theta,p}$.

THEOREM 3.11. (Potier-Ferry [15]). *Let X and Y be Banach spaces with Y dense in X. Let the norm of X be denoted by $\|\cdot\|$. Let $0 < \theta \leq 1$, $0 \leq \theta' < 1$, $1 \leq p \leq \infty$. For each u in a neighborhood of $u = 0$ in $[Y, X]_{\theta,p}$, let $T(u)$: $Y \to X$ be a closed linear operator. Let f be a nonlinear map from a neighborhood of $u = 0$ in Y into $[Y, X]_{\theta',p}$. Suppose that*

(i) *there exist positive constants ω_1, μ_2 and C_{10} such that $\Sigma(\omega_1, \mu_2) \subset \rho(T(0))$ and*

$$\|[T(0) + \lambda]^{-1}\| \leq \frac{C_{10}}{1 + |\lambda|}$$

for any $\lambda \in \Sigma(\omega_1, \mu_2)$,

(ii) *for any $x \in Y$, the map $u \to T(u)x$ from a neighborhood of $u = 0$ in $[Y, X]_{\theta,p}$ into X is differentiable and there exist positive constants η and C_{11} such that*

$$\|[T'(u_1)v - T'(u_2)v]x\| \leq C_{11}(\|u_2 - u_1\|_{\theta,p})^{\eta} \|v\|_{\theta,p} \|x\|_Y,$$

(iii) *there exists $C_{12} > 0$ such that*

$$\|f(u_1) - f(u_2)\|_{\theta',p} \leq C_{12}\|u_1 - u_2\|_Y, \qquad \text{and}$$

(iv) *there exists $C_{13} > 0$ such that $\|f(u)\|_{\theta' p} \leq C_{13}\|u\|_Y^2$.*

Then

$$\frac{du}{dt} + T(u)u = f(u), \quad u(0) = u_0 \in Y$$

has a global solution u on the time interval $[0, \infty)$, and for sufficiently small $\|u_0\|_Y$ there exist positive constants C_{14} and μ_3 such that

$$\|u(t)\|_Y \le C_{14} \|u_0\|_Y \exp(-\mu_3 t) \qquad \text{for } t \ge 0.$$

To apply the above theorem to (3.1)–(3.3), we substitute $\boldsymbol{w} = (w, z) = \boldsymbol{w}^{\varepsilon, D} + \boldsymbol{W}$ into (3.1)–(3.3) where $\boldsymbol{W} = (W, Z)$, and obtain

$$\frac{d\boldsymbol{W}}{dt} + \boldsymbol{T}(\boldsymbol{W})\boldsymbol{W} = \boldsymbol{f}(\boldsymbol{W}) \tag{3.9}$$

where

$$\boldsymbol{T}(\boldsymbol{W})\boldsymbol{W} = -\left[\frac{\partial(\Phi, \Psi)}{\partial(w, z)}(\boldsymbol{w}^{\varepsilon, D} + \boldsymbol{W})\right]^{-1} \boldsymbol{L}^{\varepsilon, D}\boldsymbol{W},$$

$$\boldsymbol{L}^{\varepsilon, D}\boldsymbol{W} = \begin{pmatrix} \varepsilon^2 W_{xx} + f_w(\boldsymbol{w}^{\varepsilon, D})W + f_z(\boldsymbol{w}^{\varepsilon, D})Z \\ DZ_{xx} + g_w(\boldsymbol{w}^{\varepsilon, D})W + g_z(\boldsymbol{w}^{\varepsilon, D})Z \end{pmatrix},$$

$$\boldsymbol{f}(\boldsymbol{W}) = \left[\frac{\partial(\Phi, \Psi)}{\partial(w, z)}(\boldsymbol{w}^{\varepsilon, D} + \boldsymbol{W})\right]^{-1}$$
$$\times \begin{pmatrix} f(\boldsymbol{w}^{\varepsilon, D} + \boldsymbol{W}) - f(\boldsymbol{w}^{\varepsilon, D}) - f_w(\boldsymbol{w}^{\varepsilon, D})W - f_z(\boldsymbol{w}^{\varepsilon, D})Z \\ g(\boldsymbol{w}^{\varepsilon, D} + \boldsymbol{W}) - g(\boldsymbol{w}^{\varepsilon, D}) - g_w(\boldsymbol{w}^{\varepsilon, D})W - g_z(\boldsymbol{w}^{\varepsilon, D})Z \end{pmatrix}.$$

Let $\boldsymbol{X} = L^2(I) \times L^2(I)$ and $\boldsymbol{Y} = H_N^2(I) \times H_N^2(I)$. Theorem 3.10 implies that the assumption (i) in Theorem 3.11 holds for (3.9). In order to check that the assumptions (ii), (iii) and (iv) in Theorem 3.11 hold for (3.9), we put $\theta \in (0, 1/4)$ and $\theta' \in (3/4, 1)$. By the embedding theorem and the interpolation theorem (see Grisvard [3]), we know

$$H_N^2(I) \subset [H^2(I), L^2(I)]_{\theta,2} = W^{2(1-\theta),2}(I) \subset C^{1/2}(\bar{I}),$$
$$H_N^2(I) \subset C^{1/2}(\bar{I}) \subset W^{2(1-\theta'),2}(I) = [H_N^2(I), L^2(I)]_{\theta',2}.$$

Therefore, it turns out that

$$\boldsymbol{Y} \subset [\boldsymbol{Y}, \boldsymbol{X}]_{\theta,2} \subset C^{1/2}(\bar{I}) \times C^{1/2}(\bar{I}), \quad \boldsymbol{Y} \subset C^{1/2}(\bar{I}) \times C^{1/2}(\bar{I}) \subset [\boldsymbol{Y}, \boldsymbol{X}]_{\theta',2}.$$

By using the above inclusion, the regularities for $\boldsymbol{T}$ and $\boldsymbol{f}$ and the boundedness of $\boldsymbol{w}^{\varepsilon, D}$, we thus find that the assumptions (ii), (iii) and (iv) in Theorem 3.11 hold. Consequently, the following main theorem is obtained:

THEOREM 3.12. *Suppose φ, ψ, $\bar{f}$ and $\bar{g}$ satisfy the assumptions* (A.4), (A.5) *and* (A.6). *Then, for any fixed $\varepsilon \in (0, \varepsilon^*)$, $\boldsymbol{w}^{\varepsilon, D}$ is exponentially stable for $D > D^*$.*

4. Stability of singularly perturbed solutions when $D > 0$ is sufficiently large

The final problem is to find the parameter regions with respect to

$(a, b, c, \beta, \varepsilon, D)$ in which $\boldsymbol{w}^{\varepsilon,D}$ is stable by using the results shown in the previous section. It is not easy to do this, because the stability of $\boldsymbol{w}^{\varepsilon,D}$ depends on its spatial profiles. To do it, we study a special case when $D > 0$ is sufficiently large. Nishiura [12] proved that, for any fixed small $\varepsilon > 0$, $\boldsymbol{w}^{\varepsilon,D}$ converges to a solution, say $\boldsymbol{w}^{\varepsilon,\infty}$, of

$$\text{(4.1)} \qquad \begin{cases} 0 = \varepsilon^2 w_{xx} + f(w, z), & x \in I, \\ w_x = 0, \quad x = 0, 1, \\ 0 = \displaystyle\int_I g(w, z)\, dx \end{cases}$$

in the $C^2(\bar{I}) \times C^2(\bar{I})$-topology as $D \to \infty$. By this result and the construction of $\boldsymbol{w}^{\varepsilon,D}$, we find that $\boldsymbol{w}^{0,\infty} = (w^{0,\infty}, z^{0,\infty}) = \lim_{\varepsilon \downarrow 0} \boldsymbol{w}^{\varepsilon,\infty}$ satisfies $z^{0,\infty}(x) \equiv z^*$ and

$$w^{0,\infty}(x) = \begin{cases} h_-(z^*)(=0) & 0 \le x < x^* \\ h_+(z^*) & x^* < x \le 1. \end{cases}$$

By using this information, let us examine whether or not $\boldsymbol{w}^{\varepsilon,D}$ is stable when $\varepsilon > 0$ is sufficiently small and $D > 0$ is sufficiently large.

We put

$$\boldsymbol{w}_\pm = (w_\pm, z_\pm) = (h_\pm(z^*), z^*), \quad \boldsymbol{u}_\pm = (u_\pm, v_\pm) = (\Phi(\boldsymbol{w}_\pm), \Psi(\boldsymbol{w}_\pm)),$$

$$a_\pm = bu_\pm + v_\pm, \quad G_\pm = g(\boldsymbol{w}_\pm), \quad Q_\pm = \left[\frac{1}{f_w} \det \frac{\partial(f, g)}{\partial(w, z)}\right](\boldsymbol{w}_\pm),$$

$$Q^* = \left\langle \frac{1}{f_w^{0,\infty}} \det \frac{\partial(f, g)^{0,\infty}}{\partial(w, z)}, 1 \right\rangle (= -\langle Q^{0,\infty}(0), 1\rangle),$$

$$a_{min} = \min(a_+, a_-), \quad a_{max} = \max(a_+, a_-),$$

respectively. By some computations, we have

$$\bar{f}_u^{0,\infty} = \begin{cases} 1 - cz^*(<0) & 0 \le x < x^* \\ -h_+(z^*)(<0) & x^* < x \le 1, \end{cases}$$

$$\bar{f}_v^{0,\infty} = -cu^{0,\infty}(\le 0), \quad \bar{g}_u^{0,\infty} = -bv^{0,\infty}(<0),$$

$$\frac{\partial}{\partial a} G_\pm = v_\pm (>0),$$

$$\frac{\partial}{\partial a} Q_\pm = \frac{\bar{f}_u(\boldsymbol{u}_\pm)}{f_w(\boldsymbol{w}_\pm)} \det \frac{\partial(\Phi, \Psi)}{\partial(w, z)} (\boldsymbol{w}_\pm)(>0),$$

respectively. Since z^* is independent of a, we find that G_+, G_-, Q_+ and Q_- are increasing linear functions with respect to a. Thus, the following lemma is obtained.

LEMMA 4.1.

(i) $a_- < a_+$ (*resp.* $a_+ < a_-$) *if and only if* $\beta h_+(z^*) + bc - \beta > 0$ (*resp.* < 0). *Moreover, if* $bc < 1$, *then* $\beta h_+(z^*) + bc - \beta < 0$.

(ii) *If* $a_- < a$ (*resp.* $a < a_-$), *then* $0 < G_-$ (*resp.* $G_- < 0$).

(iii) $\operatorname{sign} Q_- = \operatorname{sign}(a - 2a_-)$ *and* $\operatorname{sign}(Q_+|_{a=a_+}) = \operatorname{sign}(bc - 1)$, *where* $\operatorname{sign} z$ *means the sign of* z.

(iv) *If* $G_+ \geq 0$ *and* $bc > 1$, *then* $Q_+ > 0$.

(v) *If* $\det \dfrac{\partial(f, g)}{\partial(w, z)}^{0,\infty} > 0$, *then* $\bar{f}_v^{0,\infty} \psi_u^{0,\infty} + \bar{g}_v^{0,\infty} \psi_v^{0,\infty} < 0$.

(vi) *If* $Q^* > 0$, *then there exists a positive eigenvalue of* (3.4) *for sufficiently small* $\varepsilon > 0$ *and large* $D > 0$.

PROOF. By using $a_- = z^*$ and $a_+ = bh_+(z^*) + z^*/(1 + \beta h_+(z^*))$, we have

$$a_+ - a_- = \frac{h_+(z^*)}{c}\{\beta h_+(z^*) + bc - \beta\}, \quad G_- = (a - a_-)z^*,$$

$$\det \frac{\partial(\bar{f}, \bar{g})}{\partial(u, v)}(\boldsymbol{u}_-) = (1 - cz^*)(a - 2a_-),$$

$$\det \frac{\partial(\bar{f}, \bar{g})}{\partial(u, v)}(\boldsymbol{u}_+) = -\frac{u_+}{v_+}G_+ + (1 - bc)u_+v_+.$$

Thus (i), (ii), (iii) and (iv) are obviously obtained.

By using $\det \dfrac{\partial(\bar{f}, \bar{g})}{\partial(u, v)} = \bar{f}_u\bar{g}_v - \bar{f}_v\bar{g}_u = \det \dfrac{\partial(f, g)}{\partial(w, z)} \det \dfrac{\partial(\varphi, \psi)}{\partial(u, v)}$, we have

$$\begin{aligned}&\bar{f}_v^{0,\infty}\psi_u^{0,\infty} + \bar{g}_v^{0,\infty}\psi_v^{0,\infty}\\ &= \bar{f}_v^{0,\infty}\psi_u^{0,\infty} + \frac{\psi_v^{0,\infty}}{\bar{f}_u^{0,\infty}}\left\{\bar{f}_v^{0,\infty}\bar{g}_u^{0,\infty} + \det \frac{\partial(f, g)}{\partial(w, z)}^{0,\infty} \det \frac{\partial(\varphi, \psi)}{\partial(u, v)}^{0,\infty}\right\} < 0.\end{aligned}$$

Substitute $Z^1 = 1 = Z^2$ into $B^{\varepsilon,D,\lambda}(Z^1, Z^2)$. Then if $\lambda \neq 0$, we obtain

$$B^{\varepsilon,D,\lambda}(1, 1) \longrightarrow \langle Q^{0,\infty}(\lambda), 1\rangle \quad \text{as } \varepsilon \longrightarrow 0 \text{ and } D \longrightarrow \infty.$$

Since

$$\langle Q^{0,\infty}(0), 1\rangle = -Q^*(< 0), \ \lim_{\lambda \to +\infty} \frac{1}{\lambda}\langle Q^{0,\infty}(\lambda), 1\rangle = \langle Q_1^{0,\infty}, 1\rangle (> 0)$$

and $\langle Q^{0,\infty}(\lambda), 1\rangle$ is continuous for all $\lambda \in [0, +\infty)$, we find that $\langle Q^{0,\infty}(\lambda), 1\rangle = 0$ has a positive solution. Therefore, for sufficiently small $\varepsilon > 0$ and large $D > 0$, (3.4) has a positive eigenvalue and its eigenfunction $Z \equiv 1$.

The above lemma shows that (A.4) is obtained by (A.5).

Since $G_+|_{a=a_+} = 0$ and $G_-|_{a=a_-} = 0$, and since G_+ and G_- are increasing with respect to a, it is shown that, for $a \in (a_{min}, a_{max})$ and $\beta > 1$, there exists a solution of (4.1) for sufficiently small $\varepsilon > 0$. From $\int_I g(w^{0,\infty})dx = 0$, we obtain $x^* = G_+/(G_+ - G_-)$ and $Q^* = (G_+Q_- - G_-Q_+)/(G_+ - G_-)$. From $x^* \in (0, 1)$, it follows that G_+ and G_- satisfy either $G_- < 0 < G_+$ for all $a \in (a_{min}, a_{max})$ or $G_+ < 0 < G_-$ for all $a \in (a_{min}, a_{max})$. Since G_+, G_-, Q_+ and Q_- are linear with respect to a, we know that $G_+Q_- - G_-Q_+$ is a quadratic function with respect to a, and that

$$\begin{aligned} &\operatorname{sign}(Q^*|_{a=a_-} Q^*|_{a=a_+}) \\ &= \operatorname{sign}([G_+Q_- - G_-Q_+]|_{a=a_-}[G_+Q_- - G_-Q_+]|_{a=a_+}) \\ &= \operatorname{sign}(Q_-|_{a=a_-} Q_+|_{a=a_+}) = \operatorname{sign}(1 - bc) \end{aligned}$$

holds. If $bc > 1$, then $G_+Q_- - G_-Q_+$ has unique zero a_c in (a_{min}, a_{max}), i.e., Q^* also has unique zero a_c in (a_{min}, a_{max}).

4.1. Case of $bc < 1$

In this case, we refer to [8] and [9] for the construction of $w^{\varepsilon,D}$ as shown in Theorem 2.1, and find that the results shown in the previous section are still valid.

By Lemma 4.1, we have $a_{min} = a_+ < a_- = a_{max}$, $Q_+|_{a=a_+} < 0$, $Q_- < 0$ for $a \in [a_{min}, a_{max}]$ and $G_- < 0 < G_+$ for $a \in (a_{min}, a_{max})$. This implies that $w^{\varepsilon,D}$ satisfies (A.6) for sufficiently small $\varepsilon > 0$ and large $D > 0$. Since Q_+ is increasing with respect to a, it follows that there exists $a_0 \in (a_{min}, a_{max}]$ such that $Q_+ < 0$ for $a \in [a_{min}, a_0)$. From the definition of Q_+ and Q_-, we find that, for $a \in [a_{min}, a_0)$, $w^{\varepsilon,D}$ satisfies (A.5) for sufficiently small $\varepsilon > 0$ and large $D > 0$.

THEOREM 4.2. *Suppose $bc < 1$ and $\beta > 1$. For any $a \in (a_{min}, a_0)$, there exist positive constants ε_a and D_a such that $w^{\varepsilon,D}$ is stable for all $0 < \varepsilon < \varepsilon_a$ and $D > D_\alpha$.*

4.2. Case of $bc > 1$

We first assume $\beta h_+(z^*) + bc - \beta > 0$. From Lemma 4.1, we have $a_{min} = a_- < a_+ = a_{max}$, $G_+ < 0 < G_-$ for $a \in (a_{min}, a_{max})$ and $Q_+|_{a=a_+} > 0$. Then Q^* satisfies

$$Q^* \begin{cases} < 0 & a \in (a_{min}, a_c) \\ = 0 & a = a_c \\ > 0 & a \in (a_c, a_{max}). \end{cases}$$

For $a \in (a_c, a_{max})$, from Lemma 4.1, it turns out that there exists an eigenvalue with positive real part of (3.4) for sufficiently small $\varepsilon > 0$ and large $D > 0$. For $a \in (a_{min}, a_c)$, we have $c_2^* < 0$. By (3.8), we find that the critical eigenvalue of (3.4) is positive for sufficiently small $\varepsilon > 0$ and large $D > 0$.

THEOREM 4.3. *Suppose* $bc > 1$, $\beta > 1$ *and* $\beta h_+(z^*) + bc - \beta > 0$. *For any* $a \in (a_{min}, a_{max}) \setminus \{a_c\}$, *there exist positive constants* ε_a *and* D_a *such that* $w^{\varepsilon,D}$ *is unstable for all* $0 < \varepsilon < \varepsilon_a$ *and* $D > D_a$.

Next we assume $\beta h_+(z^*) + bc - \beta < 0$. From Lemma 4.1, we have $a_{min} = a_+ < a_- = a_{max}$, $G_- < 0 < G_+$ for $a \in (a_{min}, a_{max})$, $Q_- < 0$ for $a \in [a_{min}, a_{max}]$ and $Q_+ > 0$ for $a \in [a_{min}, a_{max}]$. Then it turns out that Q^* satisfies

$$Q^* \begin{cases} > 0 & a \in (a_{min}, a_c) \\ = 0 & a = a_c \\ < 0 & a \in (a_c, a_{max}). \end{cases}$$

For $a \in (a_{min}, a_c)$, it follows from Lemma 4.1 that there exists an eigenvalue with positive real part of (3.4) for sufficiently small $\varepsilon > 0$ and large $D > 0$.

For $a \in (a_c, a_{max})$, it follows from (3.8) that the critical eigenvalue of (3.4) is negative for sufficiently small $\varepsilon > 0$ and large $D > 0$. Let $\lambda^{\varepsilon,D}$ be a non-critical eigenvalue of (3.4) which is uniformly bounded for all $(\varepsilon, D) \in \Omega_0$, and let $(W^{\varepsilon,D}, Z^{\varepsilon,D})$ be its eigenfunction. For any fixed $\varepsilon \in (0, \varepsilon_0)$, normalizing $(W^{\varepsilon,D}, Z^{\varepsilon,D})$ as $\max(\| W^{\varepsilon,D} \|_{C^2(\bar{I})}, \| Z^{\varepsilon,D} \|_{C^2(\bar{I})}) = 1$ for all $D > D_0$, it follows from (3.7) that $Z^{\varepsilon,D}$ converges to a constant function as $D \to \infty$. Then we find that $Z^{\varepsilon,\infty} = \lim_{D \uparrow \infty} Z^{\varepsilon,D} = 0$ or

$$\text{(4.2)} \qquad \frac{1}{\lambda^{\varepsilon,\infty}} \langle f_z^{\varepsilon,\infty} - \lambda^{\varepsilon,\infty} \Phi_z^{\varepsilon,\infty}, \phi_0^{\varepsilon,\infty} \rangle \langle g_w^{\varepsilon,\infty} - \lambda^{\varepsilon,\infty} \Psi_w^{\varepsilon,\infty}, \phi_0^{\varepsilon,\infty} \rangle - \langle Q^{\varepsilon,\infty}(\lambda^{\varepsilon,\infty}), 1 \rangle = 0$$

holds. Since $\lambda^{\varepsilon,D}$ is a non-critical eigenvalue, $Z^{\varepsilon,\infty} = 0$ shows that $\lambda^{\varepsilon,\infty}$ need to satisfy $\lambda^{\varepsilon,\infty} = \lambda_n^{\varepsilon,\infty} (< 0)$ for some integer $n \geq 1$. By (4.2) and $\int_I \phi_0^{\varepsilon,D} dx = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$, we obtain $\langle Q^{0,\infty}(\lambda^{0,\infty}), 1 \rangle = 0$. In consideration of Lemma 3.7, we may study the solutions of $\langle Q^{0,\infty}(\lambda), 1 \rangle = 0$.

By simple computations, we obtain $Q_2^{0,\infty}|_{a=a_-} = a_- > 0$ and $Q_3^{0,\infty}|_{a=a_-} = 0$ for $x \in [0, 1)$. By taking the real part of $\langle Q^{0,\infty}(\lambda)|_{a=a_-}, 1 \rangle$, if $\mathrm{Re}\,\lambda \geq 0$, then

$$\mathrm{Re} \langle Q^{0,\infty}(\lambda)|_{a=a_-}, 1 \rangle = \langle Q_1^{0,\infty}|_{a=a_-}, 1 \rangle \mathrm{Re}\,\lambda + \langle Q_2^{0,\infty}|_{a=a_-}, 1 \rangle > 0$$

shows that $\langle Q^{0,\infty}(\lambda)|_{a=a_-}, 1 \rangle = 0$ has no solution with non-negative real part. Since $\langle Q^{0,\infty}(\lambda), 1 \rangle$ is a continuous function with respect to $a \in [a_{min}, a_{max}]$, there exists $a_1 \in [a_c, a_{max}]$ such that $\langle Q^{0,\infty}(\lambda), 1 \rangle = 0$ has no solution with non-negative real part for all $a \in (a_1, a_{max}]$, and that $\langle Q^{0,\infty}(\lambda), 1\rangle = 0$ has a solution with non-negative real part at $a = a_1$. Since $\langle Q^{0,\infty}(0), 1\rangle = -Q^* > 0$ for $a \in (a_c, a_{max})$, if $a_c < a_1$, we find that $\langle Q^{0,\infty}(\lambda), 1\rangle = 0$ has a pair of solutions with pure imaginary number at $a = a_1$.

THEOREM 4.4. *Suppose $bc > 1$, $\beta > 1$ and $\beta h_+(z^*) + bc - \beta < 0$.*

(i) *For any $a \in (a_1, a_{max})$, there exist positive constants ε_a and D_a such that $w^{\varepsilon,D}$ is stable for all $0 < \varepsilon < \varepsilon_a$ and $D > D_a$. Moreover, if $a_c < a_1$, then there exists a periodic solution of* (1.10)–(1.12) *for sufficiently small $\varepsilon > 0$ and large $D > 0$ when a is varied in the neighborhood of $a = a_1$.*

(ii) *For any $a \in (a_{min}, a_c)$, there exist positive constants ε_a and D_a such that $w^{\varepsilon,D}$ is unstable for all $0 < \varepsilon < \varepsilon_a$ and $D > D_a$.*

When the assumptions in Theorem 4.4 and $a_1 = a_c$ are satisfied and a takes the value near a_c, we shall show that a periodic solution appears. Since $w^{\varepsilon,D}$ is uniformly bounded for all $(\varepsilon, D) \in \Omega_0$, there exists $M_3 > 0$ such that $w^{\varepsilon,D}(x) \in [0, M_3] \times [0, M_3]$ $(\equiv \mathscr{C})$ for all $x \in [0, 1]$ and $(\varepsilon, D) \in \Omega_0$. Since $w^{\varepsilon,D}$ converges to $w^{\varepsilon,\infty}$ in $C^2([0, 1]) \times C^2([0, 1])$ as $D \to \infty$, if $B^{\varepsilon,\infty,\lambda}(1, 1) = 0$ has a solution $\lambda(\varepsilon)$ without the multiplicity, then $B^{\varepsilon,D,\lambda}(1, 1) = 0$ also has a solution near $\lambda(\varepsilon)$ for sufficiently large $D > 0$. In consideration of the proof of Theorem 4.4, we may only show that $B^{\varepsilon,\infty,\lambda}(1, 1) = 0$ has a pair of zeros which converge to 0 as $\varepsilon \to 0$, and which are pure imaginary numbers at $a = a_c$.

First of all, we put $\lambda = \lambda(\varepsilon) = \varepsilon^{\eta_1}\hat{\lambda}(\varepsilon)$ where $0 < \eta_1 < 1$ and $\lim_{\varepsilon \downarrow 0}\hat{\lambda}(\varepsilon) = \hat{\lambda}^* \neq 0$, and study the order estimate of $\langle g_w^{\varepsilon,\infty}, L^{\varepsilon,\infty}(\lambda)^\dagger f_z^{\varepsilon,\infty}\rangle$ as $\varepsilon \to 0$. After some computations, we find that $L^{\varepsilon,\infty}(\lambda)^\dagger f_z^{\varepsilon,\infty}$ is rewritten as

$$L^{\varepsilon,\infty}(\lambda)^\dagger f_z^{\varepsilon,\infty} = \frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}} - \left\langle \frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}}, \Phi_w^{\varepsilon,\infty}\phi_0^{\varepsilon,\infty} \right\rangle \phi_0^{\varepsilon,\infty} + L^{\varepsilon,\infty}(\lambda)^\dagger R^{\varepsilon,\infty},$$

where

$$R^{\varepsilon,\infty} = -\varepsilon^2\left[\frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}}\right]_{xx} + \lambda\frac{f_z^{\varepsilon,\infty}\Phi_w^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}} - \langle f_z^{\varepsilon,\infty}, \phi_0^{\varepsilon,\infty}\rangle \Phi_w^{\varepsilon,\infty}\phi_0^{\varepsilon,\infty}$$
$$+ (\lambda_0^{\varepsilon,\infty} - \lambda)\left\langle \frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}}, \Phi_w^{\varepsilon,\infty}\phi_0^{\varepsilon,\infty} \right\rangle \Phi_w^{\varepsilon,\infty}\phi_0^{\varepsilon,\infty}.$$

From the definition of $L^{\varepsilon,D}(\lambda)^\dagger$, it is shown that $\langle L^{\varepsilon,D}(\lambda)^\dagger h_1, h_2\rangle = \langle h_1, L^{\varepsilon,D}(\lambda)^\dagger h_2\rangle$ for all $h_1, h_2 \in L^2(I)$. Then we have

$$\langle g_w^{\varepsilon,\infty}, L^{\varepsilon,\infty}(\lambda)^\dagger f_z^{\varepsilon,\infty}\rangle = \left\langle g_w^{\varepsilon,\infty}, \frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}} \right\rangle + \langle L^{\varepsilon,\infty}(\lambda)^\dagger g_w^{\varepsilon,\infty}, R^{\varepsilon,\infty}\rangle$$

$$- \left\langle \frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}}, \Phi_w^{\varepsilon,\infty} \phi_0^{\varepsilon,\infty} \right\rangle \langle g_w^{\varepsilon,\infty}, \phi_0^{\varepsilon,\infty} \rangle.$$

Since $w^{\varepsilon,\infty}$ satisfies

$$\varepsilon^2 w_{xx} + f(w, z) = 0 \quad \text{and} \quad \frac{\varepsilon^2}{2}(w_x)^2 + \int_{w(0)}^{w(x)} f(s, z)\, ds = 0,$$

we have

$$- \varepsilon^2 \left[\frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}} \right]_{xx} = - \varepsilon^2 \{ h_1^{\varepsilon,\infty} w_{xx}^{\varepsilon,\infty} + h_2^{\varepsilon,\infty} (w_x^{\varepsilon,\infty})^2 \}$$

$$= h_1^{\varepsilon,\infty} f^{\varepsilon,\infty} + 2 h_2^{\varepsilon,\infty} \int_{w^{\varepsilon,\infty}(0)}^{w^{\varepsilon,\infty}(x)} f(s, z^{\varepsilon,\infty})\, ds,$$

where

$$h_1^{\varepsilon,\infty} = \frac{\partial}{\partial w} \left[\frac{f_z}{f_w} \right]^{\varepsilon,\infty} \quad \text{and} \quad h_2^{\varepsilon,\infty} = \frac{\partial^2}{\partial w^2} \left[\frac{f_z}{f_w} \right]^{\varepsilon,\infty}.$$

From $f(w^{0,\infty}(x)) = 0$ and $\int_{w^{0,\infty}(0)}^{w^{0,\infty}(x)} f(s, z^{0,\infty})\, ds = 0$ for all $x \in [0, 1] \setminus \{x^*\}$, we also have

$$|f(w^{\varepsilon,\infty}(x))| = |f(w^{\varepsilon,\infty}(x)) - f(w^{0,\infty}(x))|$$

$$\leq \|\nabla f\|_{L^\infty(\mathscr{C})} \{ |w^{\varepsilon,\infty}(x) - w^{0,\infty}(x)| + |z^{\varepsilon,\infty} - z^{0,\infty}| \},$$

$$\left| \int_{w^{\varepsilon,\infty}(0)}^{w^{\varepsilon,\infty}(x)} f(s, z^{\varepsilon,\infty})\, ds \right|$$

$$= \left| \int_{w^{\varepsilon,\infty}(0)}^{w^{0,\infty}(0)} f(s, z^{\varepsilon,\infty})\, ds + \int_{w^{0,\infty}(x)}^{w^{\varepsilon,\infty}(x)} f(s, z^{\varepsilon,\infty})\, ds + \int_{w^{0,\infty}(0)}^{w^{0,\infty}(x)} \{ f(s, z^{\varepsilon,\infty}) - f(s, z^{0,\infty}) \}\, ds \right|$$

$$\leq \|f\|_{L^\infty(\mathscr{C})} \{ |w^{\varepsilon,\infty}(0) - w^{0,\infty}(0)| + |w^{\varepsilon,\infty}(x) - w^{0,\infty}(x)| \} + \|\nabla f\|_{L^\infty(\mathscr{C})} |w^{0,\infty}(x) - w^{0,\infty}(0)|\, |z^{\varepsilon,\infty} - z^{0,\infty}|$$

for all $x \in [0, 1] \setminus \{x^*\}$.

For the stretched function $\tilde{w}^{\varepsilon,D}(y)$ of $w^{\varepsilon,D}(x)$, we know the following result:

LEMMA 4.5. (Fife [2]). *There exist positive constants C_{15} and γ_2 independent of $(\varepsilon, D) \in \Omega_0$ such that*

$$|\tilde{w}^{\varepsilon,D}(y) - h_\pm(z^*)| \leq C_{15} \exp(-\gamma_2 |y|) \qquad \text{as } y \longrightarrow \pm\infty.$$

From the above lemma and the construction of the solution of the singular perturbation method, we obtain

$$|w^{0,\infty}(0) - w^{0,\infty}(0)| = O(e^{-\gamma_2/\varepsilon}), \quad \int_I |w^{\varepsilon,\infty}(x) - w^{0,\infty}(x)| dx = O(\varepsilon),$$

$$|z^{\varepsilon,\infty} - z^{0,\infty}| = O(\varepsilon)$$

as $\varepsilon \to 0$ where γ_2 is the constant given in Lemma 4.5, that is, $\int_I |f(w^{\varepsilon,\infty}(x))| dx = O(\varepsilon)$ and $\int_I |\int_{w^{\varepsilon,\infty}(0)}^{w^{\varepsilon,\infty}(x)} f(s, z^{\varepsilon,\infty}) ds| dx = O(\varepsilon)$ as $\varepsilon \to 0$. Since $h_1^{\varepsilon,\infty}$, $h_2^{\varepsilon,\infty}$ and $L^{\varepsilon,\infty}(\lambda)^\dagger g_w^{\varepsilon,\infty}$ are bounded in $L^\infty(I)$, we have

$$\left| \left\langle L^{\varepsilon,\infty}(\lambda)^\dagger g_w^{\varepsilon,\infty},\ \varepsilon^2 \left[\frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}} \right]_{xx} \right\rangle \right| = O(\varepsilon)$$

as $\varepsilon \to 0$. From $\int_I \phi_0^{\varepsilon,\infty} dx = O(\sqrt{\varepsilon})$ as $\varepsilon \to 0$, we obtain

$$\langle L^{\varepsilon,\infty}(\lambda)^\dagger g_w^{\varepsilon,\infty}, R^{\varepsilon,\infty} \rangle = \varepsilon^{\eta_1} \hat{\lambda}^* \left\langle \frac{f_z^{0,\infty} g_w^{0,\infty} \Phi_w^{0,\infty}}{\{f_w^{0,\infty}\}^2}, 1 \right\rangle + o(\varepsilon^{\eta_1}),$$

$$\left\langle \frac{f_z^{\varepsilon,\infty}}{f_w^{\varepsilon,\infty}}, \Phi_w^{\varepsilon,\infty} \phi_0^{\varepsilon,\infty} \right\rangle \langle g_w^{\varepsilon,g}, \phi_0^{\varepsilon,\infty} \rangle = O(\varepsilon)$$

as $\varepsilon \to 0$.

Put $a = a_c + \varepsilon^{\eta_2} \hat{a} + o(\varepsilon^{\eta_2})$ where $0 < \eta_2 < 1$. For any function $h^{\varepsilon,D}$ which is dependent of (ε, D, a), we denote $h^{\varepsilon,D,*}$ by $h^{\varepsilon,D,*} = h^{\varepsilon,D}|_{a=a_c}$. Then we have

$$\langle L^{\varepsilon,\infty}(\lambda)^\dagger g_w^{\varepsilon,\infty}, R^{\varepsilon,\infty} \rangle = \varepsilon^{\eta_1} \hat{\lambda}^* \left\langle \frac{f_z^{0,\infty,*} g_w^{0,\infty,*} \Phi_w^{0,\infty,*}}{\{f_w^{0,\infty,*}\}^2}, 1 \right\rangle + o(\varepsilon^{\eta_1}),$$

$$Q^* = \varepsilon^{\eta_2} \hat{a} Q_4^* + o(\varepsilon^{\eta_2})$$

as $\varepsilon \to 0$ where $Q_4^* = \frac{\partial}{\partial a} Q^*|_{a=a_c} (< 0)$. By $B^{\varepsilon,\infty,\lambda}(1, 1) = 0$, we have

$$0 = \frac{c_1^* c_2^*}{\hat{\lambda}^*} \varepsilon^{1-\eta_1} - \hat{a} Q_4^* \varepsilon^{\eta_2} + \hat{\lambda}^* Q_5^* \varepsilon^{\eta_1} + o(\varepsilon^{\eta_1}, \varepsilon^{1-\eta_1}, \varepsilon^{\eta_2}) \tag{4.3}$$

as $\varepsilon \to 0$ where

$$Q_5^* = \left\langle \frac{g_w^{0,\infty,*} \bar{f}_v^{0,\infty,*} + f_w^{0,\infty,*} \bar{f}_u^{0,\infty,*}}{\{f_w^{0,\infty,*}\}^2} \det \frac{\partial(\Phi, \Psi)^{0,\infty,*}}{\partial(w, z)}, 1 \right\rangle.$$

When $\eta_1 = \eta_2 = 1/2$, the both sides of (4.3) are well-balanced and $\hat{\lambda}^*$ satisfies

$$(\hat{\lambda}^*)^2 Q_5^* - \hat{a} Q_4^* \hat{\lambda}^* + c_1^* c_2^* = 0.$$

By simple computations, we have $g_w^{0,\infty,*} \bar{f}_v^{0,\infty,*} + f_w^{0,\infty,*} \bar{f}_u^{0,\infty,*} > 0$ in $[0, x^*)$.

To show $g_w^{0,\infty,*}\bar{f}_v^{0,\infty,*}+f_w^{0,\infty,*}\bar{f}_u^{0,\infty,*}>0$ in $(x^*, 1]$, we derive a contradiction by assuming $g_w^{0,\infty,*}\geq 0$ for some $x\in(x^*, 1]$. From $Q_+>0$, $\Phi_z\equiv 0$, $f_w^{0,\infty,*}<0$ and $f_z^{0,\infty,*}\leq 0$, we have

$$g_z^{0,\infty,*}=\bar{g}_v^{0,\infty,*}\Psi_z^{0,\infty,*}=Q_+ + \frac{f_z^{0,\infty,*}g_w^{0,\infty,*}}{f_w^{0,\infty,*}}>0,$$

i.e., $\bar{g}_v^{0,\infty,*}>0$ because of $\Psi_z^{0,\infty,*}>0$. From $\Phi_w\equiv 1$, $\Psi_w^{0,\infty,*}<0$ and $\bar{g}_u^{0,\infty,*}<0$, we have

$$0\leq g_w^{0,\infty,*}=\bar{g}_u^{0,\infty,*}\Phi_w^{0,\infty,*}+\bar{g}_v^{0,\infty,*}\Psi_w^{0,\infty,*}<0.$$

This implies a contradiction, i.e., $g_w^{0,\infty,*}<0$ in $(x^*, 1]$. From $\bar{f}_u^{0,\infty,*}\leq 0$ and $\bar{f}_v^{0,\infty,*}\leq 0$, we obtain $Q_5^*>0$. From $c_1^*>0$ and $c_2^*>0$, we have the following.

THEOREM 4.6. *Fix b, c and β to satisfy the assumptions of Theorem* 4.4, *and suppose $a_1=a_c$. Then there exists a periodic solution of* (1.10)–(1.12) *for sufficiently small $\varepsilon>0$ and large $D>0$ when a is varied in the neighborhood of $a=a_c$.*

5. Concluding Remarks

We have shown the criterion of the stability of the singularly perturbed solution $w^{\varepsilon,D}$ when ε is small and D is large. In this section, by numerical computations, we show the bifurcation diagram of this solution when a is globally varied.

First of all, we consider the case when $bc\leq 1$. When a is increasing, the situation varies from Case (I) to Case (IV) through Case (II). Figure 7 shows the bifurcation diagram. E_{+0}, E_{0+} and E_{++} denote the branches of the equilibrium points (1, 0), (0, a) and $((1-ac)/(1-bc), (a-b)/(1-bc))$, respectively. A non-constant stationary solution branch bifurcates from the E_{++}-branch at $a=\underline{a}$, then turns back at $a=a_t$ and connects with the

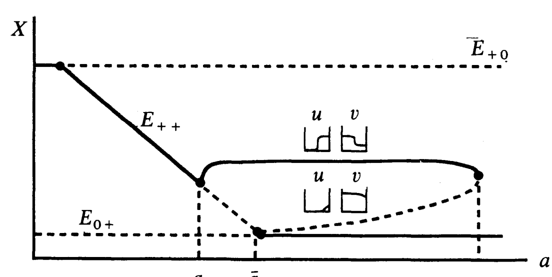

Figure 7. Bifurcation diagram where $\varepsilon^2=0.001$, $D=10.0$, $\beta=3.0$ and $b=c=0.9$. The solid curves are stable and the broken ones are unstable.

E_{++}-branch at $a = \bar{a}$. Numerical computations show that the upper branch corresponds to stable singularly perturbed solutions shown in the previous sections, and that the lower one is unstable. We have not touched with the lower branch. But for its construction, we refer to [9].

Next we consider the case when $bc > 1$. As a is increasing, the situation varies from Case(I) to Case(IV) through Case(III) (Figure 8). The great difference from the case $bc \leq 1$ is that the secondary bifurcation of Hopf type occurs at $a = a_c$. This bifurcation is already shown by the stability analysis in the last section. By numerical computations, as a is decreasing, the periodic

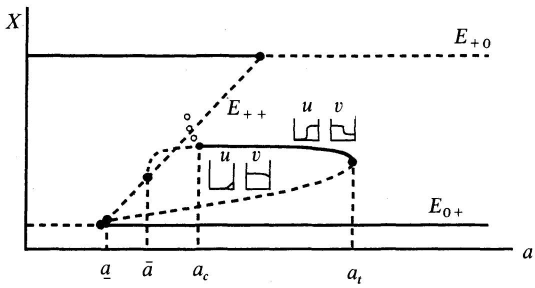

Figure 8. Bifurcation diagram where $\varepsilon^2 = 0.001$, $D = 10.0$, $\beta = 3.0$ and $b = c = 1.1$.
—: stable stationary solution.
---: unstable stationary solution.
∘∘∘: stable periodic solution.

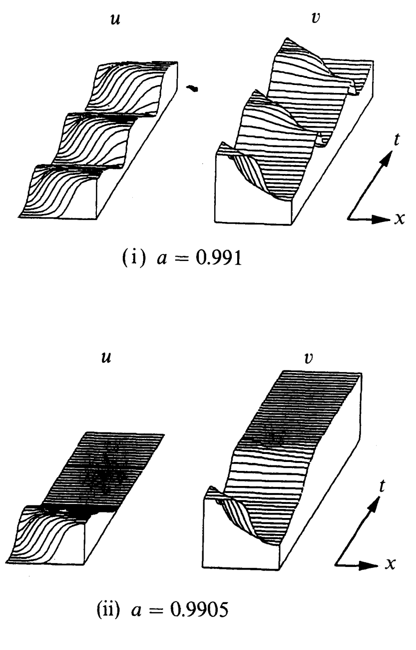

Figure 9. Spatially inhomogeneous solution of (1.10)–(1.12) where ε, D, β, b and c are the same as in Figure 1.

solution disappear suddenly (Figure 9). In this paper we do not touch with periodic solutions. The complete understanding of these global bifurcation diagrams is a future work for us.

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