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Inertial manifold theory for a class of reaction-diffusion equations on thin tubular domains

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§1. Introduction

During the past several decades, reaction-diffusion equations have been proposed to understand spatio-temporal structures of nonlinear phenomena arising in population ecology (Cantrell and Cosner [2], Kan-on and Yanagida [25], Levin [29], Mimura et al. [33]–[37]), neurobiology (Henry [19]), fission reactors (Leung [28]), chemical reactions (Fife [15], Smoller [42]), combustions (Ei et al. [11]–[13]) and other applied sciences. For the qualitative study of solutions of these equations, fruitful mathematical methods have been extensively developed in the field of applied analysis (Ei [9], Ei and Mimura [10], Mimura et al. [34], for instance).

Most of reaction-diffusion equations are described by the following semilinear parabolic system of equations:

$$u_t = div(D(x, u)\nabla u) + f(x, u), \qquad (t, x) \in (0, \infty) \times \Omega, \tag{1.1}$$

where Ω is a bounded domain in $\mathbf{R}^n$, $u = (u_1, \cdots, u_m) \in \mathbf{R}^m$, $D(x, u)$ is a nonnegative definite matrix and f is a kinetic function from $\mathbf{R}^n \times \mathbf{R}^m$ into $\mathbf{R}^m$. In most applications, D is a constant diagonal matrix and f is independent of x. The resulting system is simply

$$u_t = D\Delta u + f(u), \qquad (t, x) \in (0, \infty) \times \Omega. \tag{1.2}$$

The variables u usually denote the quantities such as densities of biological populations in ecology, concentrations of substances in chemical reaction, for instance. In qualitatively understanding the behavior of solutions to (1.1) or (1.2), the studies of existence and stability of equilibrium and periodic solutions to (1.1) and (1.2) are very important. In fact, it is known that stable spatially inhomogeneous equilibrium solutions play, among other things, an important role in the formation of patterns arising in reacting and diffusing medium.

As for stable spatially inhomogeneous equilibrium solutions, we have the following problem: How is the relation between the stability of spatially inhomogeneous equilibrium solutions and the shape of domain Ω. Along this line, Matano [31] and Casten and Holland [3] showed that any stable equilibrium solution of the scalar equation of (1.2) with $m=1$ is *constant* under the zero-flux boundary condition if Ω is convex, that is, there exist no stable spatially inhomogeneous equilibrium solutions. (We refer to Chafee [4] in one dimension case). Under the same boundary conditions, Kishimoto and Weinberger [27] extends this result to the system of (1.2) where f is restricted to be $(\partial/\partial u_j)f_i > 0$ for $i \neq j$, under which (1.2) is called the cooperation-diffusion system. The conclusion is also valid for (1.2) with $m=2$ where $(\partial/\partial u_j)f_i < 0$ for $i \neq j$ $(i, j = 1, 2)$, under which (1.2) is called the competition-diffusion system. On the other hand, Matano and Mimura [32] and Jimbo [23] showed that for some appropriate f, stable spatially inhomogeneous equilibrium solutions of the competition-diffusion system with $m=2$ exist in a suitable dumbbell-shaped nonconvex domain under the zero-flux boundary conditions. Since then, the dependency of spatial domains on equilibrium solutions of (1.2) has been intensively investigated under the zero-flux or Dirichlet boundary conditions (Dancer [7] and [8], Ei et al. [13], Keyfitz and Kuiper [26], Vegas [44] and others). Hale and Vegas [16] and Vegas [43] first parametrized a family of dumbbell-shaped domains which are introduced in [31] and [32]. For this special type of domain, Hale and Vegas [16] and Jimbo [22] studied the structure of equilibrium solutions of the scalar equation of (1.2) when f takes εf with a small parameter ε, and Morita [39] showed that the origional system (1.2) can be reduced to a finite dimensional ordinary differential equations on a Lipschitz continuous invariant manifold. Fang [14] and Mimura, Ei and Fang [38] studied in detail the dependency of the domain Ω as well as the diffusion coefficients D on solutions of (1.2) and, as an application, considered the bifurcation problem for the competition-diffusion system with $m=2$ when the shape of domain varies. Recently, Jimbo and Morita [24] and Morita and Jimbo [40] have considered the system (1.2) on a domain Ω which consists of many dumbbells connected by narrow handles.

From the aspect of the dependency of domain-shape on solutions, the effect of tubular domains on solutions has been recently investigated. Let $p(x)$, $x \in [0, L]$ satisfying $|p_x(x)| = 1$ be a smooth curve which does not intersect itself in $\mathbf{R}^{N+1}$, N_x an N-dimensional normal plane at $p(x)$ and let $q^i(x)$ satisfying $|q^i(x)| = 1$ $(i = 1, \cdots, N)$ be an orthonormal basis of N_x. Also let $D_x \subset \mathbf{R}^N$ be a simply connected bounded domain with smooth boundary. N_x and D_x are assumed to depend on x smoothly. With a small parameter $\varepsilon > 0$, define $\Omega_\varepsilon \subset \mathbf{R}^{N+1}$ by

$$\Omega_\varepsilon = \{p(x) + \varepsilon \textstyle\sum_{i=1}^N y_i q^i(x) | y = (y_1, \cdots, y_N) \in D_x, \quad x \in (0, L)\}.$$

Yanagida [45] considered the following scalar equation in Ω_ε:

$$(1.3)\qquad \begin{cases} u_t = \Delta u + f(u), & (t, x) \in (0, \infty) \times \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0, & (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon. \end{cases}$$

Using the upper- and lower-solution methods, he showed that if $\psi(x)$ is an asymptotically stable equilibrium solution of the following equation:

$$(1.4)\qquad \begin{cases} v_t = \dfrac{1}{a(x)}(a(x)v_x)_x + f(v) & \text{for } x \in (0, L), \\ \dfrac{\partial v}{\partial n} = 0 & \text{for } x = 0, L, \end{cases}$$

where $a(x)$ is the N-dimensional volume of D_x, then there exists a stable equilibrium solution $\phi(x, y)$ of (1.3) such that $\phi(x, y) \to \psi(x)$ as $\varepsilon \downarrow 0$ uniformly in Ω_ε. This result suggests that when ε is sufficiently small, (1.4) is a nice approximating equation to (1.3), but it was not justified.

However, from the view point of dynamical theory, Hale and Raugel [17] quite recently have justified it for the scalar reaction-diffusion equation (1.3) on Ω_ε in $\mathbf{R}^{N+1}$, $N \leq 2$. But this problem remains to be unsolved for the system version of (1.2).

The purpose of this paper is to study this problem for the following reaction-diffusion system with two components:

$$(1.5)\qquad \begin{cases} u_t = d_1 \Delta u + f(u, v, x, y), \\ v_t = d_2 \Delta v + g(u, v, x, y), \end{cases} \qquad (t; x, y) \in (0, \infty) \times \Omega_\varepsilon$$

on the symmetric thin tubular domain $\Omega_\varepsilon \subset \mathbf{R}^{N+1}$, where d_1, d_2 are positive constants and $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y_1^2 + \cdots + \partial^2/\partial y_N^2$. The result of Yanagida [45] suggests us that in the limit $\varepsilon \downarrow 0$, (1.5) is reduced to the following system in one dimensional space:

$$(1.6)\qquad \begin{cases} \bar{u}_t = \dfrac{d_1}{a(x)}(a(x)\bar{u}_x)_x + f(\bar{u}, \bar{v}, x, 0), \\ \bar{v}_t = \dfrac{d_2}{a(x)}(a(x)\bar{v}_x)_x + g(\bar{u}, \bar{v}, x, 0). \end{cases} \qquad (t, x) \in (0, \infty) \times (0, L)$$

What we want to do here is to verify the validity that the dynamics of solutions to the system (1.5) is approximated by those of the system (1.6) when ε is sufficiently small. Furthermore, we want to show that if (1.6) has an equilibrium solution $(\bar{u}_0, \bar{v}_0)$ which is nondegenerate, then (1.5) has the corresponding equilibrium solution $(u_\varepsilon, v_\varepsilon)$ when $\varepsilon > 0$ is sufficiently small and moreover, the stability of $(u_\varepsilon, v_\varepsilon)$ in (1.5) is inherited to the stability of $(\bar{u}_0, \bar{v}_0)$

in (1.6). The method which we use here is the inertial manifold approach developed by Mallet-Paret and Sell [30] and Hale and Raugel [17].

Throughout the paper, we assume that (1.5) possesses a positively invariant region, that is, there exists a region $\Sigma \subset \mathbf{R}^m$ such that the solution of (1.5) stays in Σ for its existing time interval if the initial and boundary values are in Σ. If it is bounded, then it is well known that the global existence and other dynamical properties such as the existence of compact attractors are obtained [19]. For the construction of such invariant regions, we refer to Chueh, Conley and Smoller [5], Smoller [42] and the references therein. In practical applications, there is the situation where, with the initial and boundary values in Σ, the solution globally exists and eventually enters into Σ although it may be temporarily out of Σ. We call such Σ an asymptotically invariant region. In Section 2, we give the formulation of the problem and show some examples of systems which possess bounded invariant regions or bounded asymptotically invariant regions.

In Section 3, we formulate the systems (1.5) and (1.6) in an abstract setting and give some preliminaries for these systems.

In Section 4, we consider the eigenvalue problems corresponding to the systems (1.5) and (1.6). Let $\{\mu_{\varepsilon,i}\}$ $(i = 1, 2, \cdots)$ satisfying $\mu_{\varepsilon,1} \leq \mu_{\varepsilon,2} \leq \cdots$ be the eigenvalues of the operator $\boldsymbol{B}_\varepsilon$ in Ω_ε, where $\boldsymbol{B}_\varepsilon(u, v) = (-d_1 \Delta u + \alpha_1 u, -d_2 \Delta v + \alpha_2 v)$ with $\alpha_1, \alpha_2 > 0$ such that $\alpha_1/d_1 = \alpha_2/d_2$, and $\{\mu_i\}$ $(i = 1, 2, \cdots)$ satisfying $\mu_1 \leq \mu_2 \leq \cdots$ be the eigenvalues of $\boldsymbol{B}_0$ in $(0, L)$, where $\boldsymbol{B}_0(u, v) = (-(d_1/a(x))(a(x)u_x)_x + \alpha_1 u, -(d_2/a(x))(a(x)v_x)_x + \alpha_2 v)$. It is shown that

$$(1.7) \qquad \limsup_{n \to \infty} (\mu_{n+1} - \mu_n) = \infty \quad \text{and} \quad \mu_{\varepsilon,m} \longrightarrow \mu_m \ \text{ as } \ \varepsilon \downarrow 0$$

for any fixed $m \in \mathbf{N}$. This fact, toghther with other conditions, allows us to use the existence theorem of inertial manifolds in Mallet-Paret and Sell [30]. In Section 5, inertial manifolds for (1.5) and (1.6), say, $\mathfrak{M}^\varepsilon$ and $\mathfrak{M}^0$, are constructed as graphes Φ^ε and Φ^0 over the finite dimensional linear space, which is determined by the span of the eigenfunctions corresponding to the first N_1 eigenvalues of the operators $\boldsymbol{B}_\varepsilon$ and $\boldsymbol{B}_0$, respectively. By using (1.7), the inertial manifolds $\mathfrak{M}^\varepsilon$ for (1.5) with sufficiently small $\varepsilon > 0$ and $\mathfrak{M}^0$ for (1.6) can be constructed in the same finite dimensions. In Section 6, using the techniques developed by Hale and Raugel [17], we show that $\Phi^\varepsilon \to \Phi^0$ as $\varepsilon \downarrow 0$ in C^1-topology. As a result, one can find that the dynamics of the original system (1.5) with sufficiently small $\varepsilon > 0$ on the inertial manifold $\mathfrak{M}^\varepsilon$ and the reduced system of (1.6) on the inertial manifold $\mathfrak{M}^0$ are governed by the same finite dimensional ordinary differential equations. Unfortunately, the dimensions of $\mathfrak{M}^\varepsilon$ and $\mathfrak{M}^0$ can not be determined easily.

Finally, in Section 7, we discuss the relation between equilibrium solutions of (1.5) and those of (1.6). Our discussion may also be extended to periodic solutions.

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§ 2. Formulation of the problem and examples

We are concerned with the following reaction-diffusion system for two components $u, v \in \mathbf{R}$:

$$(2.1) \qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + f(u, v; x), \\ \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + g(u, v; x), \end{cases} \qquad (t; x) \in (0, \infty) \times \Omega_\varepsilon,$$

subject to one of the following three boundary conditions:

$$(2.2) \qquad \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \qquad (t; x) \in (0, \infty) \times \partial\Omega_\varepsilon$$

and

$$(2.3) \qquad \begin{cases} u = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_1 \quad \text{and} \quad \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_2, \\ \\ \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon \end{cases}$$

and

$$(2.4) \qquad \begin{cases} u = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_1 \quad \text{and} \quad \dfrac{\partial u}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_2, \\ \\ v = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_1 \quad \text{and} \quad \dfrac{\partial v}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_2. \end{cases}$$

Here, Ω_ε is a bounded domain with piecewise smooth boundary which is defined as follows: Let $D \subset \mathbf{R}^N$ be a bounded simply connected convex domain with $0 \in D$. $r(s): [0, L] \to \mathbf{R}_+$ belongs to C^2-class and there exist positive constants r_1, r_2 and r_3 such that

$$r_1 \le r(s) \le r_2, \ |r'(s)| \le r_3, \ |r''(s)| \le r_3 \qquad \text{for } s \in [0, L].$$

Let $D_s = \{(z_1, \cdots, z_N) \in \mathbf{R}^N | (z_1, \cdots, z_N)/r(s) \in D\}$ for $s \in [0, L]$. Define Ω_ε by

(2.5) $$\Omega_\varepsilon = \{x = (s, y) = (s, y_1, \cdots, y_N) \in \mathbf{R}^{N+1} | s \in (0, L), \ y/\varepsilon \in D_s\}$$

with a small parameter ε satisfying $0 < \varepsilon < \varepsilon_0$ (see Figure 1). The boundaries $(\partial\Omega_\varepsilon)_1$ and $(\partial\Omega_\varepsilon)_2$ are

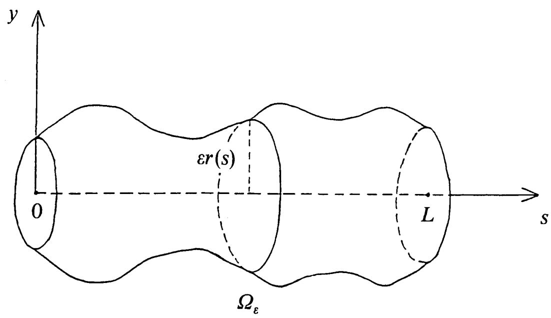

Figure 1. The domain shape of Ω_ε.

$$(\partial\Omega_\varepsilon)_1 = \{x = (s, y) \in \mathbf{R}^{N+1} | s = 0, \ L, \ y/\varepsilon \in D_s\}$$

and

$$(\partial\Omega_\varepsilon)_2 = \{x = (s, y) \in \mathbf{R}^{N+1} | 0 < s < L, \ y/\varepsilon \in \partial D_s\},$$

respectively. $\Delta = \partial^2/\partial s^2 + \partial^2/\partial y_1^2 + \cdots + \partial^2/\partial y_N^2$. The functions $f, g: \mathbf{R} \times \mathbf{R} \times Q \to \mathbf{R}$ belong to C^2 with respect to u, v and to $W^{1,\infty}(Q)$ with respect to x, where Q is a bounded domain in $\mathbf{R}^{N+1}$ such that $Q \supset cl.(\Omega_\varepsilon)$ for $0 < \varepsilon \le \varepsilon_0$. If the boundary condition takes either (2.3) or (2.4), we also assume the compatibility condition

(2.6) $$\begin{cases} f(0, 0; x) = 0 \\ g(0, 0; x) = 0 \end{cases} \qquad \text{for } x \in (\partial\Omega_\varepsilon)_1.$$

We make the following assumption to the system (2.1):

[A] $[0, K_1] \times [0, K_2]$ is an invariant region (or asymptotically invariant region) of the system (2.1) with (2.2) or (2.3) or (2.4) for any suitable large $K_1, K_2 > 0$.

That is, if the initial values $u(0)$, $v(0)$ satisfy $0 \le u(0) \le K_1$ and $0 \le v(0) \le K_2$, then there exists $t_0 \ge 0$ ($t_0 = 0$ in the case when invariant region holds) such that the solutions $u(t)$, $v(t)$ of (2.1) satisfy $0 \le u(t) \le K_1$ and $0 \le v(t) \le K_2$ for $t \ge t_0$.

Let us show some examples.

EXAMPLE 1. (competition-diffusion system)

The first system is

$$(2.7)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + (R_1(x) - a_1(x)u - b_1(x)v)u, \\ \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + (R_2(x) - b_2(x)u - a_2(x)v)v, \end{cases} \qquad (t, x) \in (0, \infty) \times \Omega_\varepsilon,$$

which is called the competition-diffusion system with Gause-Lotka-Volterra dynamics describing the dynamics of two competing species that move by diffusion, where $u(t, x)$, $v(t, x)$ are the population densities of two competing species at position x and time t, and d_1, d_2 are the diffusion rates of two species, respectively. $a_i(x)$ $(i = 1, 2)$ are the intraspecific competition rates and $b_i(x)$ $(i = 1, 2)$ do the interspecific competition rates at position x. $R_i(x)$ $(i = 1, 2)$ are the intrinsic growth rates. All of the coefficients are positive and bounded functions in Ω_ε but if the environment is homogeneous, these are constants (Ahmad and Lazer [1], Pao [41], for instance).

It is easily found that $[0, K_1] \times [0, K_2]$ is an invariant region of (2.7) under the boundary conditions (2.2) for any large $K_1, K_2 > 0$ (Smoller [42] or Leung [28]).

EXAMPLE 2. (prey-predator system)

The second system is

$$(2.8)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + r\left(1 - \dfrac{u}{K(x)}\right)u - \dfrac{mvu}{a+u}, \\ \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + \left(-R + \dfrac{c}{v}u\right)v, \end{cases} \qquad (t, x) \in (0, \infty) \times \Omega_\varepsilon,$$

where u, v are the densities of a prey species and its predator and d_1, d_2 are the diffusion rates. r is the intrinsic growth rate, $K(x)$ is the carrying capacity, and m is the maximum predation rate with intensity a. R is the death rate and c is the growth rate by predation. All of the parameters are positive. Then under the boundary conditions (2.2), it is obvious that $[0, K_1] \times [0, K_2]$ is an invariant region of (2.8) for any large $K_1, K_2 > 0$ satisfying $K_2 > cK_1/R$.

EXAMPLE 3. (liquids superconductivity system)

The third system is

$$
(2.9)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d_1 \Delta u + (1 - u^2 - v^2)u, \\ \\ \dfrac{\partial v}{\partial t} = d_2 \Delta v + (1 - u^2 - v^2)v, \end{cases} \qquad (t, x) \in (0, \infty) \times \Omega_\varepsilon
$$

which is proposed in the theory of superconductivity of liquids (Chueh, Conley and Smoller [5]). Under the boundary conditions (2.2), it is known that $[0, K_1] \times [0, K_2]$ is an invariant region for any large $K_1, K_2 > 0$.

EXAMPLE 4. (combustion system)

The forth system is

$$
(2.10)\qquad \begin{cases} \dfrac{\partial \theta}{\partial t} = \Delta\theta + cf(\theta), \\ \\ \dfrac{\partial c}{\partial t} = d\Delta c - \delta c f(\theta) + I(x), \end{cases} \qquad (t, x) \in (0, \infty) \times \Omega_\varepsilon
$$

which arises in the theory of combustion (Ei and Mimura [11] and the references therein). Here, $\theta(t, x)$ and $c(t, x)$ denote the nondimensionalized temperature and concentration of fuel at position x and time t, respectively. The reaction term $f(\theta)$ takes $f(\theta) = exp(\theta/(1 + \beta\theta))$ with β some positive constant. d and δ denote the diffusion rate of fuel and the thermal effect of the reaction, respectively. $I(x)$ (≥ 0) denotes the supply of fuel which satisfies $I \in W^{1,\infty}(Q)$. The boundary conditions for (2.10) often take the following forms

$$
(2.11)\qquad \begin{cases} \theta = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_1 \quad \text{and} \quad \dfrac{\partial \theta}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_2, \\ \\ \dfrac{\partial c}{\partial n} = 0 \quad \text{on} \quad \partial\Omega_\varepsilon. \end{cases}
$$

It is not so trivial to check that the problem (2.10), (2.11) satisfies the assumption [A]. Therefore, we will show that (2.10), (2.11) possesses an asymptotically invariant region. We take the initial conditions for (2.10) as

$$
(2.12)\qquad \theta(0, x) = \theta_0(x) \geq 0, \quad c(0, x) = c_0(x) \geq 0, \quad x \in \Omega_\varepsilon,
$$

where θ_0 and c_0 are bounded on Ω_ε. Noting that $1 \leq f(\theta) \leq e^{1/\beta}$ for $\theta \geq 0$, we can obtain the global existence of the solution of (2.10) $\sim$ (2.12) in a standard manner. Let $I_1 = sup_{x \in Q} I(x) \geq 0$ and $M_1 = I_1/\delta$ and consider the following auxiliary equations for $(\tilde{c}, \tilde{\theta})$:

$$
(2.13)\qquad \begin{cases} \dfrac{\partial \tilde{c}}{\partial t} = d\Delta \tilde{c} - \delta \tilde{c} + I_1, & (t, x) \in (0, \infty) \times \Omega_\varepsilon, \\[2ex] \dfrac{\partial \tilde{c}}{\partial n} = 0, & (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon, \\[2ex] \tilde{c}(0, x) = c_0(x) \geq 0 \end{cases}
$$

and

$$
(2.14)\qquad \begin{cases} \dfrac{\partial \tilde{\theta}}{\partial t} = \Delta \tilde{\theta} + exp(1/\beta)\tilde{c}, \qquad (t, x) \in (0, \infty) \times \Omega_\varepsilon, \\[2ex] \tilde{\theta} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_1 \quad \text{and} \quad \dfrac{\partial \tilde{\theta}}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_2, \\[2ex] \tilde{\theta}(0, x) = \theta_0(x) \geq 0. \end{cases}
$$

LEMMA 2.1. *$0 \leq \theta(t, x) \leq \tilde{\theta}(t, x)$ and $0 \leq c(t, x) \leq \tilde{c}(t, x)$ for all $t > 0$ and $x \in \Omega_\varepsilon$.*

The proof is easily given by the minimum and maximum principles, so we omit it.

LEMMA 2.2. *If $0 \leq c_0(x) \leq K_2$ for some $K_2 > 0$, then for any $c^* > 0$, there exists $t_0 = t_0(K_2, c^*) \geq 0$ such that $0 \leq c(t, x) \leq c^*$ for $t \geq t_0$ and $x \in \Omega_\varepsilon$.*

PROOF. By Lemma 2.1, it is sufficient to show $\tilde{c}(t, x) \leq c^*$ for $t \geq t_0$ for some t_0. Let $u(t, x) = c^* - \tilde{c}(t, x)$. Then u satisfies the following equation:

$$
(2.15)\qquad \begin{cases} \dfrac{\partial u}{\partial t} = d\Delta u - \delta u + \delta c^* + I_1, & (t, x) \in (0, \infty) \times \Omega_\varepsilon, \\[2ex] \dfrac{\partial u}{\partial n} = 0, & (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon, \\[2ex] u(0, x) = c^* - c_0(x), & x \in \Omega_\varepsilon. \end{cases}
$$

If $u(0, x) \geq 0$, then it follows from the minimum principle that $u(t, x) \geq 0$ and so $\tilde{c}(t, x) \leq c^*$ for $t \geq 0$ and $x \in \Omega_\varepsilon$. On the other hand, if $inf_{x \in \Omega_\varepsilon} u(0, x) < 0$, we may consider the following equation:

$$
(2.16)\qquad \begin{cases} \dfrac{d\bar{u}}{dt} = -\delta \bar{u} + \delta c^* + I_1, \\[2ex] \bar{u}(0) = inf_{x \in \Omega_\varepsilon} u(0, x). \end{cases}
$$

Using the fact that $\bar{u}(0) < 0$ and $\bar{u} = 0$ is not an equilibrium of (2.16), we know that there exists a finite $t_0 = t_0(K_2, c^*) > 0$ such that $\bar{u}(t)$ increases in $[0, t_0]$ and $\bar{u}(t_0) = 0$. Putting $v(t, x) = \bar{u}(t) - u(t, x)$, we find that v satisfies the following equation:

$$\text{(2.17)}\qquad \begin{cases} \dfrac{\partial v}{\partial t} = d\Delta v - \delta v, & (t, x) \in (0, \infty) \times \Omega_\varepsilon, \\ \dfrac{\partial v}{\partial n} = 0, & (t, x) \in (0, \infty) \times \partial\Omega_\varepsilon, \\ v(0, x) = \bar{u}(0) - u(0, x) \le 0, & x \in \Omega_\varepsilon. \end{cases}$$

The maximum principle indicates that $v(t, x) \le 0$ for $t \ge 0$ and $x \in \Omega_\varepsilon$. Therefore $\bar{u}(t) \le u(t, x)$ for $t \ge 0$ and $x \in \Omega_\varepsilon$, and $u(t_0, x) \ge \bar{u}(t_0) = 0$. Applying the minimum principle again to the following differential inequality:

$$\begin{cases} \dfrac{\partial u}{\partial t} \ge d\Delta u - \delta u, & (t, x) \in (t_0, \infty) \times \Omega_\varepsilon, \\ \dfrac{\partial u}{\partial n} = 0, & (t, x) \in (t_0, \infty) \times \partial\Omega_\varepsilon, \\ u(t_0, x) \ge 0, & x \in \Omega_\varepsilon, \end{cases}$$

we know that $u(t, x) \ge 0$ for $t \ge t_0$ and $x \in \Omega_\varepsilon$. That is $\tilde{c}(t, x) \le c^*$ for $t \ge t_0$ and $x \in \Omega_\varepsilon$.

LEMMA 2.3. *Let* $c_0, \theta_0 \in L^p(\Omega_\varepsilon)$ *with* $p > N + 1$, *then there are* $K_1^0 > 0$ *independent of* ε, θ_0 *and* t, *and* $\bar{t}_0 = \bar{t}_0(\varepsilon, \theta_0) > 0$ *such that* $0 \le \theta(t, x) \le K_1^0$ *for* $t \ge \bar{t}_0$ *and* $x \in \Omega_\varepsilon$.

PROOF. By Lemmas 2.1 and 2.2, we know that there is a finite $t_0 > 0$ such that $0 \le \theta(t, x) \le \tilde{\theta}(t, x)$ and $0 \le \tilde{c}(t, x) \le c^*$ for $t \ge t_0$ and $x \in \Omega_\varepsilon$. Let $\bar{\theta}(t, x)$ be the solution of the following equation:

$$\text{(2.18)}\qquad \begin{cases} \dfrac{\partial \bar{\theta}}{\partial t} = \Delta\bar{\theta} + exp(1/\beta)c^*, & (t, x) \in (t_0, \infty) \times \Omega_\varepsilon, \\ \bar{\theta} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_1 \quad \text{and} \quad \dfrac{\partial\bar{\theta}}{\partial n} = 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_2, \\ \bar{\theta}(t_0, x) = \tilde{\theta}(t_0, x), & x \in \Omega_\varepsilon \end{cases}$$

and let $\bar{\theta}^*$ be the equilibrium solution of (2.18), which is unique. Then $0 \le \tilde{\theta}(t, x) \le \bar{\theta}(t, x)$ follows for $t \ge t_0$ and $x \in \Omega_\varepsilon$. Therefore, it is sufficient to estimate $\bar{\theta}(t, x)$.

Define the operator $\mathscr{B}_\varepsilon$ by $\mathscr{B}_\varepsilon(u_1, u_2) = (-\Delta u_1, -d\Delta u_2)$ with $u_1 = 0$ on

$(\partial\Omega_\varepsilon)_1$, $\dfrac{\partial u_1}{\partial n} = 0$ on $(\partial\Omega_\varepsilon)_2$ and $\dfrac{\partial u_2}{\partial n} = 0$ on $\partial\Omega_\varepsilon$. Then it turns out that $-\mathscr{B}_\varepsilon$ generates a semigroup $exp(-\mathscr{B}_\varepsilon t)$ in $(L^p(\Omega_\varepsilon))^2$ (Henry [19]). Let $\mathscr{D}(\mathscr{B}_\varepsilon)$ be $\mathscr{D}(\mathscr{B}_\varepsilon) = \left\{(u_1, u_2) \in (W^{2,p}(\Omega_\varepsilon))^2 \,|\, u_1 = 0 \text{ on } (\partial\Omega_\varepsilon)_1, \dfrac{\partial u_1}{\partial n} = 0 \text{ on } (\partial\Omega_\varepsilon)_2 \text{ and } \dfrac{\partial u_2}{\partial n} = 0 \text{ on } \partial\Omega_\varepsilon\right\}$. Then if $(\theta_0, c_0) \in (L^p(\Omega_\varepsilon))^2$, we know that $(\theta(t), c(t))$, $(\tilde{\theta}(t), \tilde{c}(t)) \in \mathscr{D}(\mathscr{B}_\varepsilon)$. Define the operator $\mathscr{A}_\varepsilon$ by $\mathscr{A}_\varepsilon u = -\Delta u$ with $u = 0$ on $(\partial\Omega_\varepsilon)_1$ and $\dfrac{\partial u}{\partial n} = 0$ on $(\partial\Omega_\varepsilon)_2$ in $L^p(\Omega_\varepsilon)$. Letting the eigenvalues of $\mathscr{A}_\varepsilon$ be $\{\lambda_i^\varepsilon\}$ $(i = 1, 2, \cdots)$ satisfying $0 < \lambda_1^\varepsilon \le \lambda_2^\varepsilon \le \cdots$ and $exp(-\mathscr{A}_\varepsilon t)$ be the semigroup generated by $-\mathscr{A}_\varepsilon$, we write (2.18) into an integral form:

$$\bar{\theta}(t) = exp(-\mathscr{A}_\varepsilon(t - t_0))\bar{\theta}(t_0) + \int_{t_0}^{t} exp(-\mathscr{A}_\varepsilon(t - s))\, exp(1/\beta)c^* ds.$$

Letting $\mathscr{X}_\varepsilon^\alpha = \mathscr{D}(\mathscr{A}_\varepsilon^\alpha)$ with $\alpha \in ((N+1)/2p, 1)$ (Henry [19]), we have

$$\|\bar{\theta}(t)\|_{L^\infty} \le \|\bar{\theta}(t)\|_{\mathscr{X}_\varepsilon^\alpha} \le c_1\, exp(-\lambda_1^\varepsilon(t - t_0))\, \|\bar{\theta}(t_0)\|_{\mathscr{X}_\varepsilon^\alpha} + c_1 \int_{t_0}^{t} \frac{exp(-\lambda_1^\varepsilon(t - s))}{(t - s)^\alpha} \|exp(1/\beta)c^*\|_{L^p} ds \le c_2$$

for some constants $c_1 > 0$ and $c_2 > 0$. Since (2.18) has the Lyapunov functional

$$V_\varepsilon(u) = \int_{\Omega_\varepsilon} (|\nabla u|^2 - exp(1/\beta)c^* u)\, dx,$$

the dynamical theory (for example, see Matano [31]) shows that $\bar{\theta}(t)$ converges to an equilibrium solution $\bar{\theta}_\infty$ of (2.18) as $t \to \infty$. On the other hand, as (2.18) has the unique equilibrium solution $\bar{\theta}^*$, we know that $\|\bar{\theta}(t) - \bar{\theta}^*\|_{L^\infty} \to 0$ as $t \to \infty$ for any initial value $\theta_0 \in L^p(\Omega_\varepsilon)$ of (2.12). Therefore, if we can obtain the upper-bound of $\bar{\theta}^*$ which is independent of ε, the proof of this lemma is complete. If we can construct a bounded function $U(x)$ independent of ε, which satisfies

$$\text{(2.19)} \qquad \begin{aligned} &\Delta U + exp(1/\beta)c^* \le 0 \quad \text{in } \Omega_\varepsilon, \\ &U \ge 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_1 \quad \text{and} \quad \frac{\partial U}{\partial n} \ge 0 \quad \text{on} \quad (\partial\Omega_\varepsilon)_2, \end{aligned}$$

it is a super-solution of (2.10), (2.11). The construction of $U(x)$ is done in a similar manner to the one in Yanagida [45]. Let $\kappa^i(\xi)$ $(i = 1, \cdots, N-1)$ be the orthogonal unit tangent vectors of $\partial(\varepsilon D_s)$ in N-dimensional y-space at

$\xi \in \partial(\varepsilon D_s)$ $(s \in (0, L))$. Choose $\xi_1 \in \partial(\varepsilon D_{s_1})$ for s_1 very close to s such that $\xi_1 - \xi$ is orthogonal to $\kappa^i(\xi)$ $(i = 1, \cdots, N-1)$. Let

$$\kappa^N(\xi) = lim_{s_1 \to s} \frac{\xi_1 - \xi}{s_1 - s}$$

and n_Q be the outward normal unit vector in N-dimensional y-space at $\partial(\varepsilon D_s)$. Then it is known that $\kappa^N(\xi)$ is a tangent vector on $\partial(\varepsilon D_s)$, which is represented as

$$\kappa^N(\xi) = \gamma_1(\xi)(1, 0, \cdots, 0) + \gamma_2(\xi)(0, n_Q)$$

with $\gamma_1(\xi) = 1 + O(\varepsilon^2)$ and $\gamma_2 = O(\varepsilon)$. Also let n_D be the outward normal unit vector in N-dimensional y-space at ∂D_s, a and $b(s)$ the N-dimensional volume of D and the N-1-dimensional volume of ∂D_s, respectively, $\Delta_N = \partial^2/\partial y_1^2 + \cdots + \partial^2/\partial y_N^2$. We choose $V(s)$ as a bounded function satisfying

$$\begin{cases} \dfrac{1}{r^N(s)} \dfrac{d}{ds}\left(r^N(s)\dfrac{dV}{ds}\right) + exp(1/\beta)c^* < -1, & s \in (0, L), \\ V(0) > 0, \quad V(L) > 0. \end{cases} \tag{2.20}$$

For example, by taking $V(s) = C - e^{\nu s}$ for $s \in (0, L)$ where C and ν are sufficiently large positive constants satisfying $C > e^{\nu L}$, it turns out that $V(s)$ satisfies (2.20).

Consider the following boundary value problem for any constant ζ:

$$\begin{cases} \Delta_N W = \dfrac{Nr'(s)}{r(s)} V'(s) + \zeta b(s), & y \in D_s, \\ \dfrac{\partial W}{\partial n_D} = \dfrac{\gamma_2(\xi)}{\varepsilon} V'(s) + \zeta a r^N(s), & y \in \partial D_s. \end{cases}$$

Then it is shown in [45] that (2.21) has the solution

$$W(y) = -\int_{D_s} K(y, z)\left\{\frac{Nr'(s)}{r(s)} V'(s) + \zeta b(s)\right\} dz + \int_{\partial D_s} K(y, \eta)\left\{\frac{\gamma_2(\xi)}{\varepsilon} V'(s) + \zeta a r^N(s)\right\} d\eta + c$$

for the Neumann function $K(y, z)$ and any constant c. We denote $W(y)$ by $W(s, y, \varepsilon)$ and define $U(s, y)$ by $U(s, y) = V(s) + \varepsilon^2 W(s, y/\varepsilon, \varepsilon)$ in Ω_ε. Then, in a similar manner to the one in [45], we know that $U(s, y) = V(s) + O(\varepsilon^2)$ and

$$\Delta U + exp(1/\beta)c^* < -1 + \zeta b(s) + O(\varepsilon) \qquad \text{in } \Omega_\varepsilon,$$

$$\frac{\partial U}{\partial n}(\xi) = \varepsilon\zeta a r^N(s) + O(\varepsilon^2) \qquad \text{on } (\partial\Omega_\varepsilon)_2.$$

Therefore, if $\varepsilon > 0$ and $\zeta > 0$ are chosen to be sufficiently small, we find that $U(s, y)$ satisfies (2.19).

By using the above Lemmas, we arrive at the following result:

THEOREM 2.4. *There exist positive constants* K_1^0 *and* K_2^0 *such that for any* $K_1 \geq K_1^0$ *and* $K_2 \geq K_2^0$, *if* $0 \leq \theta(0) \leq K_1$ *and* $0 \leq c(0) \leq K_2$, *then the solutions* $\theta(t)$ *and* $c(t)$ *of* (2.10) ~ (2.12) *satisfy*

$$0 \leq \theta(t) \leq K_1, \quad 0 \leq c(t) \leq K_2 \qquad \textit{for } \ t \geq t_1 > 0,$$

where $t_1 = t_1(\varepsilon, K_2, \theta_0) < \infty$.

Theorem 2.4 immediately shows that (2.10), (2.11) has an asymptotically invariant region.

§ 3. Abstract formulation

In this section, we treat the system (2.1) ~ (2.4) in an abstract setting. Taking the transformations

$$\text{(3.1)} \qquad \begin{cases} s = s, \\ y_1 = \varepsilon r(s) z_1, \\ \cdots\cdots\cdots\cdots\cdots \\ y_N = \varepsilon r(s) z_N, \end{cases}$$

and letting $\nabla = {}^t(\partial/\partial s, \partial/\partial y_1, \cdots, \partial/\partial y_N)$ and $\nabla_z = {}^t(\partial/\partial s, \partial/\partial z_1, \cdots, \partial/\partial z_N)$, we know

$$\nabla_z = J^t \nabla,$$

where

$$J = \frac{\partial(s, y_1, \cdots, y_N)}{\partial(s, z_1, \cdots, z_N)} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ \varepsilon r'(s) z_1 & \varepsilon r(s) & 0 & \cdots & 0 \\ \varepsilon r'(s) z_2 & 0 & \varepsilon r(s) & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ \varepsilon r'(s) z_N & 0 & 0 & \cdots & \varepsilon r(s) \end{pmatrix}.$$

Under the transformations (3.1), the system (2.1) becomes

$$\text{(3.2)} \qquad \begin{cases} \dfrac{\partial u}{\partial t} = -\mathscr{L}^\varepsilon(d_1)u - \alpha_1 u + f_\varepsilon(u, v; s, z), \\[2ex] \dfrac{\partial v}{\partial t} = -\mathscr{L}^\varepsilon(d_2)v - \alpha_2 v + g_\varepsilon(u, v; s, z), \end{cases} \qquad (t; s, z) \in (0, \infty) \times \Omega$$

with $\Omega = (0, L) \times D$, where

$$\mathscr{L}^{\varepsilon}(d)u = -(d/r^N(s))\nabla_z \cdot \Gamma_\varepsilon u \qquad \text{for } d > 0,\ \varepsilon > 0$$

and

$$\Gamma_\varepsilon u = \begin{pmatrix} r^N \dfrac{\partial u}{\partial s} - r^{N-1} r' \left(z_1 \dfrac{\partial u}{\partial z_1} + \cdots + z_N \dfrac{\partial u}{\partial z_N} \right) \\ - r^{N-1} r' z_1 \dfrac{\partial u}{\partial s} + r^{N-2} r'^2 z_2 \left(z_1 \dfrac{\partial u}{\partial z_1} + \cdots + z_N \dfrac{\partial u}{\partial z_N} \right) + \dfrac{r^{N-2}}{\varepsilon^2} \dfrac{\partial u}{\partial z_1} \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ - r^{N-1} r' z_N \dfrac{\partial u}{\partial s} + r^{N-2} r'^2 z_N \left(z_1 \dfrac{\partial u}{\partial z_1} + \cdots + z_N \dfrac{\partial u}{\partial z_N} \right) + \dfrac{r^{N-2}}{\varepsilon^2} \dfrac{\partial u}{\partial z_N} \end{pmatrix},$$

$$f_\varepsilon(u, v; s, z) = f(u, v; s, \varepsilon r(s) z) + \alpha_1 u,$$

$$g_\varepsilon(u, v; s, z) = g(u, v; s, \varepsilon r(s) z) + \alpha_2 v$$

with $\alpha_1, \alpha_2 > 0$ satisfying $\alpha_1/d_1 = \alpha_2/d_2$. The boundary conditions (2.2), (2.3) and (2.4) become

$$\text{(3.3)} \qquad \frac{\partial u}{\partial n_\varepsilon} = \Gamma_\varepsilon u \cdot n = 0, \quad \frac{\partial v}{\partial n_\varepsilon} = 0, \qquad (t; s, z) \in (0, \infty) \times \partial\Omega,$$

$$\text{(3.4)} \qquad \begin{cases} u = 0 \quad \text{on} \quad \{0, L\} \times D, \quad \dfrac{\partial u}{\partial n_\varepsilon} = 0 \quad \text{on} \quad (0, L) \times \partial D, \\ \dfrac{\partial v}{\partial n_\varepsilon} = 0 \quad \text{on} \quad \partial\Omega \end{cases}$$

and

$$\text{(3.5)} \qquad \begin{cases} u = 0 \quad \text{on} \quad \{0, L\} \times D, \quad \dfrac{\partial u}{\partial n_\varepsilon} = 0 \quad \text{on} \quad (0, L) \times \partial D, \\ v = 0 \quad \text{on} \quad \{0, L\} \times D, \quad \dfrac{\partial v}{\partial n_\varepsilon} = 0 \quad \text{on} \quad (0, L) \times \partial D, \end{cases}$$

respectively. Here n is the outward normal unit vector on $\partial\Omega$.

The terms $\partial u/\partial z_i$ $(i = 1, 2, \cdots, N)$ should be very small formally as $\varepsilon \downarrow 0$. Because the diffusion in each direction of $z_1, \cdots, z_N$ is very large when ε is sufficiently small, u and v depend very small on $z_1, \cdots, z_N$ and the reduced problem of (3.2) $\sim$ (3.5) as $\varepsilon \downarrow 0$ should be

(3.6) $$\begin{cases} \dfrac{\partial u}{\partial t} = -\mathscr{L}^0(d_1)u - \alpha_1 u + f_0(u, v; s), \\ \dfrac{\partial v}{\partial t} = -\mathscr{L}^0(d_2)v - \alpha_2 v + g_0(u, v; s), \end{cases} \qquad (t; s) \in (0, \infty) \times (0, L),$$

(3.7) $$\frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, \qquad (t, s) \in (0, \infty) \times \{0, L\},$$

(3.8) $$u = 0, \quad \frac{\partial v}{\partial n} = 0, \qquad (t, s) \in (0, \infty) \times \{0, L\},$$

(3.9) $$u = 0, \quad v = 0, \qquad (t, s) \in (0, \infty) \times \{0, L\},$$

where

$$\mathscr{L}^0(d)u = -(d/r^N(s))\frac{d}{ds}\left(r^N(s)\frac{du}{ds}\right) \qquad \text{for } d > 0,$$

$$f_0(u, v; s) = f(u, v; s, 0) + \alpha_1 u,$$

$$g_0(u, v; s) = g(u, v; s, 0) + \alpha_2 v$$

and n is the outward normal unit vector on $\partial(0, L)$.

In order to know the relation between (3.2) ~ (3.5) and (3.6) ~ (3.9), it will be convenient to rewrite the systems in an abstract form. We first consider the system (3.2), (3.3). Let $\|\cdot\|_{0,\Omega}$ be the usual norm in $L^2(\Omega)$. Let $\mathbf{H}_\varepsilon(\Omega)$ be $L^2(\Omega)$ equipped with the norm $\|u\|_L = \left(\int_\Omega r^N(s)u^2\,dsdz\right)^{1/2}$ and the inner product $\langle u, v\rangle_L = \int_\Omega r^N(s)\,uvdsdz$. Also let $(\mathbf{H}_\varepsilon(\Omega))^2$ be $(L^2(\Omega))^2$ equipped with the norm $\|\Phi\|_{\mathbf{H}} = (\|u\|_L^2 + \|v\|_L^2)^{1/2}$ for $\Phi = (u, v)$ and the inner product $\langle \Phi_1, \Phi_2\rangle_{\mathbf{H}} = \langle u_1, u_2\rangle_L + \langle v_1, v_2\rangle_L$ for $\Phi_1 = (u_1, v_1)$ and $\Phi_2 = (u_2, v_2)$. Let $\mathbf{H}_\varepsilon^1(\Omega)$ be $H^1(\Omega)$ equipped with the norm $\|u\|_{\varepsilon,L} = \left(\|u\|_{0,\Omega}^2 + \frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial z_1}\right\|_{0,\Omega}^2 + \cdots + \frac{1}{\varepsilon^2}\left\|\frac{\partial u}{\partial z_N}\right\|_{0,\Omega}^2\right)^{1/2}$ and let $(\mathbf{H}_\varepsilon^1(\Omega))^2$ be $(H^1(\Omega))^2$ equipped with the norm $\|\Phi\|_{\varepsilon,\mathbf{H}} = (\|u\|_{\varepsilon,L}^2 + \|v\|_{\varepsilon,L}^2)^{1/2}$ for $\Phi = (u, v)$. We define the gradient operator $\mathscr{L}_\varepsilon^{1/2}(d)$ in $H^1(\Omega)$ by

$$\mathscr{L}_\varepsilon^{1/2}(d)u = \left(\sqrt{d}\frac{\partial u}{\partial s} - \sqrt{d}\frac{r'(s)}{r(s)}\left(z_1\frac{\partial u}{\partial z_1} + \cdots + z_N\frac{\partial u}{\partial z_N}\right), \frac{\sqrt{d}\,\partial u}{\varepsilon r(s)\partial z_1}, \ldots, \frac{\sqrt{d}\,\partial u}{\varepsilon r(s)\partial z_N}\right)$$

for $d > 0$ and $\varepsilon > 0$. Here $\mathscr{L}_\varepsilon^{1/2}\Phi$ means

$$\mathscr{L}_\varepsilon^{1/2}\Phi = (\mathscr{L}_\varepsilon^{1/2}(d_1)u, \mathscr{L}_\varepsilon^{1/2}(d_2)v) \qquad \text{for } \Phi = (u, v) \in (H^1(\Omega))^2.$$

The bilinear forms $\mathbf{a}_\varepsilon^i(\cdot,\cdot)$ $(i=1,2)$ in $(H^1(\Omega))^2$ and $\mathbf{b}_\varepsilon(\cdot,\cdot)$ in $(H^1(\Omega))^2 \times (H^1(\Omega))^2$ are respectively defined by

$$\mathbf{a}_\varepsilon^i(u,v) = \langle \mathscr{L}_\varepsilon^{1/2}(d_i)u, \quad \mathscr{L}_\varepsilon^{1/2}(d_i)v\rangle_{\varepsilon,\mathbf{L}} + \alpha_i\langle u, v\rangle_L \qquad (i=1,2)$$

for $u, v\in H^1(\Omega)$ and

$$\mathbf{b}_\varepsilon(\Phi_1, \Phi_2) = \mathbf{a}_\varepsilon^1(u_1, u_2) + \mathbf{a}_\varepsilon^2(v_1, v_2)$$

for $\Phi_1 = (u_1, v_1)$, $\Phi_2 = (u_2, v_2)\in(H^1(\Omega))^2$, where

$$\langle \mathscr{L}_\varepsilon^{1/2}(d_i)u, \mathscr{L}_\varepsilon^{1/2}(d_i)v\rangle_{\varepsilon,\mathbf{L}} = \int_\Omega r^N(s)\mathscr{L}_\varepsilon^{1/2}(d_i)u\cdot\mathscr{L}_\varepsilon^{1/2}(d_i)v ds dz.$$

It turns out that $\mathbf{a}_\varepsilon^i(\cdot,\cdot)$ $(i=1,2)$ and $\mathbf{b}_\varepsilon(\cdot,\cdot)$ are elliptic forms and, from the assumptions on $r(s)$, it follows that there exist $\varepsilon_1>0$ $(\varepsilon_1\le\varepsilon_0)$ and $\bar{c}_1>0$ and $\bar{c}_2>0$ such that for $0<\varepsilon\le\varepsilon_1$

$$\bar{c}_1^2\|u\|_{\varepsilon,L}^2 \le \mathbf{a}_\varepsilon^i(u,u) \le \bar{c}_2^2\|u\|_{\varepsilon,L}^2 \qquad \text{for } u\in H^1(\Omega)\ (i=1,2) \tag{3.10}$$

and

$$\bar{c}_1^2\|\Phi\|_{\varepsilon,\mathbf{H}}^2 \le \mathbf{b}_\varepsilon(\Phi,\Phi) \le \bar{c}_2^2\|\Phi\|_{\varepsilon,\mathbf{H}}^2 \qquad \text{for } \Phi\in(H^1(\Omega))^2. \tag{3.11}$$

By using the above notations, operators $\mathbf{S}_\varepsilon^i$ $(i=1,2)$ in $H^1(\Omega)$ can be defined by $\{\mathbf{H}_\varepsilon(\Omega), H^1(\Omega), \mathbf{a}_\varepsilon^i(\cdot,\cdot)\}$, that is, $u\in\mathscr{D}(\mathbf{S}_\varepsilon^i)$ if and only if (iff) the functional $\mathbf{a}_\varepsilon^i(u,\cdot)$ is continuous in $H^1(\Omega)$ with respect to the topology of $\mathbf{H}_\varepsilon(\Omega)$ and $\langle\mathbf{S}_\varepsilon^i u, w\rangle_L = \mathbf{a}_\varepsilon^i(u,w)$. Similarly, an operator $\mathbf{B}_\varepsilon$ in $(H^1(\Omega))^2$ can be defined by $\{(\mathbf{H}_\varepsilon(\Omega))^2, (H^1(\Omega))^2, \mathbf{b}_\varepsilon(\cdot,\cdot)\}$, that is, $\Phi\in\mathscr{D}(\mathbf{B}_\varepsilon)$ iff the functional $\mathbf{b}_\varepsilon(\Phi,\cdot)$ is continuous in $(H^1(\Omega))^2$ with respect to the topology of $(\mathbf{H}_\varepsilon(\Omega))^2$ and $\langle\mathbf{B}_\varepsilon\Phi, \Psi\rangle_\mathbf{H} = \mathbf{b}_\varepsilon(\Phi,\Psi)$. By the Riesz Representation Theorem, the Green Formula and regularity properties (see Hale and Raugel [17] and [18]), it turns out that $\mathbf{S}_\varepsilon^i$ and $\mathbf{B}_\varepsilon$ are well defined with $\mathscr{D}(\mathbf{S}_\varepsilon^i) = \left\{u\in H^2(\Omega) \mid \gamma\dfrac{\partial u}{\partial n} = 0 \text{ on } \partial\Omega\right\}$, $\mathbf{S}_\varepsilon^i u = \mathscr{L}^\varepsilon(d_i)u + \alpha_i u$ and $\mathscr{D}(\mathbf{B}_\varepsilon) = \left\{\Phi = (u,v)\in(H^2(\Omega))^2 \mid \gamma\dfrac{\partial u}{\partial n} = 0,\ \gamma\dfrac{\partial v}{\partial n} = 0 \text{ on } \partial\Omega\right\}$ and $\mathbf{B}_\varepsilon\Phi = (\mathscr{L}^\varepsilon(d_1)u + \alpha_1 u, \mathscr{L}^\varepsilon(d_2)v + \alpha_2 v)$, respectively, where γ is the trace operator on $\partial\Omega$. Moreover, it follows from (3.10) and (3.11) that there is a constant $\bar{c}>0$ such that

$$\|u\|_{\varepsilon,L} \le \bar{c}\|\mathbf{S}_\varepsilon^i u\|_L \qquad \text{for } u\in\mathscr{D}(\mathbf{S}_\varepsilon^i) \quad (i=1,2)$$

and

$$\|\Phi\|_{\varepsilon,\mathbf{H}} \le \bar{c}\|\mathbf{B}_\varepsilon\Phi\|_\mathbf{H} \qquad \text{for } \Phi\in\mathscr{D}(\mathbf{B}_\varepsilon).$$

We note that $\mathbf{S}_\varepsilon^i$ $(i = 1, 2)$ and $\mathbf{B}_\varepsilon$ are self-adjoint, positive and sectorial operators in $H^1(\Omega)$ and $(H^1(\Omega))^2$, respectively (See Henry [19]). The fractional powers of $\mathbf{S}_\varepsilon^i$ and $\mathbf{B}_\varepsilon$ can be defined and $\mathbf{Y}_\varepsilon^i \equiv \mathscr{D}((\mathbf{S}_\varepsilon^i)^{1/2})$ is $H^1(\Omega)$ equipped with the norm

$$\| u \|_{\varepsilon, \mathbf{Y}^i} = \| (\mathbf{S}_\varepsilon^i)^{1/2} u \|_L = (\mathbf{a}_\varepsilon^i(u, u))^{1/2} (= (\| \mathscr{L}_\varepsilon^{1/2}(d_i) u \|_L^2 + \alpha_i \| u \|_L^2)^{1/2}),$$

and $\mathbf{X}_\varepsilon \equiv \mathscr{D}(\mathbf{B}_\varepsilon^{1/2})$ is $(H^1(\Omega))^2$ equipped with the norm

$$\| \Phi \|_{\varepsilon, \mathbf{X}} = \| \mathbf{B}_\varepsilon^{1/2} \Phi \|_{\mathbf{H}} = (\mathbf{b}_\varepsilon(\Phi, \Phi))^{1/2} (= (\mathbf{a}_\varepsilon^1(u, u) + \mathbf{a}_\varepsilon^2(v, v))^{1/2})$$

for $\Phi = (u, v)$. The following inequalities follow from (3.10) and (3.11):

$$\bar{c}_1 \| u \|_{\varepsilon, L} \leq \| u \|_{\varepsilon, \mathbf{Y}^i} \leq \bar{c}_2 \| u \|_{\varepsilon, L} \qquad \text{for } u \in \mathbf{Y}_\varepsilon^i \quad (i = 1, 2) \tag{3.12}$$

and

$$\bar{c}_1 \| \Phi \|_{\varepsilon, \mathbf{H}} \leq \| \Phi \|_{\varepsilon, \mathbf{X}} \leq \bar{c}_2 \| \Phi \|_{\varepsilon, \mathbf{H}} \qquad \text{for } \Phi \in \mathbf{X}_\varepsilon. \tag{3.13}$$

We next consider the system (3.6), (3.7). Let $\| \cdot \|_{1,I}$ and $\| \cdot \|_{1,\mathbf{H}_0}$ be the usual norm in $H^1(0, L)$ and the usual product norm in $(H^1(0, L))^2$, respectively. Let $\mathbf{H}_0(0, L)$ be $L^2(0, L)$ equipped with the norm $\| u \|_{0,I} = \left(\int_0^L r^N(s) u^2 \, ds \right)^{1/2}$ and the inner product $\langle u, v \rangle_{0,I} = \int_0^L r^N(s) uv ds$. Also let $(\mathbf{H}_0(0, L))^2$ be $(L^2(0, L))^2$ equipped with the norm $\| \Phi \|_{0,\mathbf{H}_0} = (\| u \|_{0,I}^2 + \| v \|_{0,I}^2)^{1/2}$ for $\Phi = (u, v)$ and the inner product $\langle \Phi_1, \Phi_2 \rangle_{0,\mathbf{H}_0} = \langle u_1, u_2 \rangle_{0,I} + \langle v_1, v_2 \rangle_{0,I}$ for $\Phi_1 = (u_1, v_1)$, $\Phi_2 = (u_2, v_2)$. We define the bilinear forms $\mathbf{a}_0^i(\cdot, \cdot)$ $(i = 1, 2)$ in $(H^1(0, L))^2$ and $\mathbf{b}_0(\cdot, \cdot)$ in $(H^1(0, L))^2 \times (H^1(0, L))^2$, respectively, by

$$\mathbf{a}_0^i(u, v) = d_i \left\langle \frac{du}{ds}, \frac{dv}{ds} \right\rangle_{0,I} + \alpha_i \langle u, v \rangle_{0,I} \qquad (i = 1, 2)$$

for $u, v \in H^1(0, L)$ and

$$\mathbf{b}_0(\Phi_1, \Phi_2) = \mathbf{a}_0^1(u_1, u_2) + \mathbf{a}_0^2(v_1, v_2)$$

for $\Phi_1 = (u_1, v_1)$, $\Phi_2 = (u_2, v_2) \in (H^1(0, L))^2$.

It is known that $\mathbf{a}_0^i(\cdot, \cdot)$ $(i = 1, 2)$ and $\mathbf{b}_0(\cdot, \cdot)$ are elliptic forms and there are positive constants $\bar{c}_3$ and $\bar{c}_4$ such that

$$\bar{c}_3^2 \| u \|_{1,I}^2 \leq \mathbf{a}_0^i(u, u) \leq \bar{c}_4^2 \| u \|_{1,I}^2 \qquad \text{for } u \in H^1(0, L) \quad (i = 1, 2) \tag{3.14}$$

and

$$\bar{c}_3^2 \| \Phi \|_{1,\mathbf{H}_0}^2 \leq \mathbf{b}_0(\Phi, \Phi) \leq \bar{c}_4^2 \| \Phi \|_{1,\mathbf{H}_0}^2 \qquad \text{for } \Phi \in (H^1(0, L))^2. \tag{3.15}$$

Operators $\mathbf{S}_0^i$ $(i = 1, 2)$ in $H^1(0, L)$ can be defined by $\{\mathbf{H}_0(0, L), H^1(0, L), \mathbf{a}_0^i(\cdot, \cdot)\}$, that is, $u \in \mathscr{D}(\mathbf{S}_0^i)$ iff the functional $\mathbf{a}_0^i(u, \cdot)$ is continuous in $H^1(0, L)$

with respect to the topology of $\mathbf{H}_0(0, L)$ and $\langle \mathbf{S}_0^i u, w\rangle_{0,I} = \mathbf{a}_0^i(u, w)$. Similarly, an operator $\mathbf{B}_0$ in $(H^1(0, L))^2$ can be defined by $\{(\mathbf{H}_0(0, L))^2, (H^1(0, L))^2, \mathbf{b}_0(\cdot, \cdot)\}$, that is, $\Phi \in \mathscr{D}(\mathbf{B}_0)$ iff the functional $\mathbf{b}_0(\Phi, \cdot)$ is continuous in $(H^1(0, L))^2$ with respect to the topology of $(\mathbf{H}_0(0, L))^2$ and $\langle \mathbf{B}_0 \Phi, \Psi\rangle_{0,\mathbf{H}_0} = \mathbf{b}_0(\Phi, \Psi)$. By the Riesz Representation Theorem, the Green Formula and regularity properties (see Hale and Raugel [17] and [18]), we find that $\mathbf{S}_0^i$ and $\mathbf{B}_0$ are well defined with $\mathscr{D}(\mathbf{S}_0^i) = \left\{u \in H^2(0, L) | \gamma \dfrac{\partial u}{\partial n} = 0 \text{ when } s = 0, L\right\}$, $\mathbf{S}_0^i u = \mathscr{L}^0(d_i)u + \alpha_i u$ and $\mathscr{D}(\mathbf{B}_0) = \left\{\Phi = (u, v) \in (H^2(0, L))^2 | \gamma \dfrac{\partial u}{\partial n} = 0,\ \gamma \dfrac{\partial v}{\partial n} = 0 \right.$ when $\left. s = 0, L\right\}$ and $\mathbf{B}_0 \Phi = (\mathscr{L}^0(d_1)u + \alpha_1 u, \mathscr{L}^0(d_2)v + \alpha_2 v)$, respectively, where γ is the trace operator on $\partial(0, L)$. Moreover, there is a constant $\bar{c} > 0$ such that

$$\|u\|_{1,I} \le \bar{c}\|\mathbf{S}_0^i u\|_{0,I} \qquad \text{for } u \in \mathscr{D}(\mathbf{S}_0^i) \quad (i = 1, 2)$$

and

$$\|\Phi\|_{1,\mathbf{H}_0} \le \bar{c}\|\mathbf{B}_0 \Phi\|_{0,\mathbf{H}_0} \qquad \text{for } \Phi \in \mathscr{D}(\mathbf{B}_0).$$

It is known that $\mathbf{S}_0^i$ $(i = 1, 2)$ and $\mathbf{B}_0$ are self-adjoint, positive and sectorial operators in $H^1(0, L)$ and $(H^1(0, L))^2$, respectively. $\mathbf{Y}_0^i \equiv \mathscr{D}((\mathbf{S}_0^i)^{1/2})$ is $H^1(0, L)$ equipped with the norm

$$\|u\|_{0,\mathbf{Y}^i} = \|(\mathbf{S}_0^i)^{1/2} u\|_{0,I} = (\mathbf{a}_0^i(u, u))^{1/2}$$
$$\left(= \left(d_i \left\| \frac{du}{ds} \right\|_{0,I}^2 + \alpha_i \|u\|_{0,I}^2 \right)^{1/2} \right)$$

and $\mathbf{X}_0 \equiv \mathscr{D}(\mathbf{B}_0^{1/2})$ is $(H^1(0, L))^2$ equipped with the norm

$$\|\Phi\|_{0,\mathbf{X}} = \|\mathbf{B}_0^{1/2}\Phi\|_{0,\mathbf{H}_0} = (\mathbf{b}_0(\Phi, \Phi))^{1/2} (= (\mathbf{a}_0^1(u, u) + \mathbf{a}_0^2(v, v))^{1/2})$$

for $\Phi = (u, v)$. The following inequalities follow from (3.14) and (3.15):

$$\bar{c}_3\|u\|_{1,I} \le \|u\|_{0,\mathbf{Y}^i} \le \bar{c}_4\|u\|_{1,I} \qquad \text{for } u \in \mathbf{Y}_0^i \quad (i = 1, 2) \tag{3.16}$$

and

$$\bar{c}_3\|\Phi\|_{1,\mathbf{H}_0} \le \|\Phi\|_{0,\mathbf{X}} \le \bar{c}_4\|\Phi\|_{1,\mathbf{H}_0} \qquad \text{for } \Phi \in \mathbf{X}_0. \tag{3.17}$$

Similar abstract formulations can be given to (3.2), (3.4) and (3.6), (3.8) as well as to (3.2), (3.5) and (3.6), (3.9). We only note that for the former case, $H^1(\Omega)$ for $i = 1$ and $(H^1(\Omega))^2$ are replaced by $V_0(\Omega)$ and $V_0(\Omega) \times H^1(\Omega)$, respectively, where $V_0(\Omega) = \{u \in H^1(\Omega) | u = 0 \text{ on } \{0, L\} \times D\}$, and $H^1(0, L)$ for $i = 1$ and $(H^1(0, L))^2$ are replaced by $V_0(0, L)$ and $V_0(0, L) \times H^1(0, L)$, respectively, where $V_0(0, L) = \{u \in H^1(0, L) | u = 0 \text{ on } \{0, L\}\}$. For the latter case, $H^1(\Omega)$ for $i = 1, 2$ and $(H^1(\Omega))^2$ are replaced by $V_0(\Omega)$ and $(V_0(\Omega))^2$, and $H^1(0, L)$ for $i = 1, 2$ and $(H^1(0, L))^2$ are replaced by $V_0(0, L)$ and $(V_0(\Omega))^2$, respectively. All the notations are needed not to be changed without confusions. We only note, for example, that $\mathbf{B}_\varepsilon$ and $\mathbf{B}_0$ are defined by $\{(\mathbf{H}_\varepsilon(\Omega))^2, V_0(\Omega) \times H^1(\Omega), \mathbf{b}_\varepsilon(\cdot, \cdot)\}$ and $\{(\mathbf{H}_0(0, L))^2, V_0(0, L) \times H^1(0, L), \mathbf{b}_0(\cdot, \cdot)\}$, respectively for the former case and defined by $\{(\mathbf{H}_\varepsilon(\Omega))^2, (V_0(\Omega))^2, \mathbf{b}_\varepsilon(\cdot, \cdot)\}$ and $\{(\mathbf{H}_0(0, L))^2, (V_0(0, L))^2, \mathbf{b}_0(\cdot, \cdot)\}$, respectively for the latter.

§4. Eigenvalue problems

Let $\{\mu_n\}$ and $\{\omega_n\}$ $(n = 1, 2, \cdots)$ be the eigenvalues and the corresponding eigenfunctions of $\mathbf{B}_0$ normalized in $(\mathbf{H}_0(0, L))^2$. It is known that $0 < \mu_1 \le \mu_2 \le \cdots$ and $\mu_n \to \infty$ as $n \to \infty$. Consider the following eigenvalue problem associated with the operator $\mathbf{S}_0^1$:

$$(4.1) \qquad -\frac{d_1}{r^N(s)}(r^N(s)\phi_s)_s + \alpha_1 \phi = \lambda\phi \qquad \text{in } (0, L)$$

with the homogeneous Neumann boundary conditions

$$(4.2) \qquad \phi_s(0) = \phi_s(L) = 0$$

or the Dirichlet boundary conditions

$$(4.3) \qquad \phi(0) = \phi(L) = 0.$$

Let $\{\lambda_n\}$ and $\{\phi_n\}$ $(n = 1, 2, \cdots)$ be the eigenvalues and the corresponding eigenfunctions of (4.1) normalized in $\mathbf{H}_0(0, L)$. It is known that λ_i is simple and $0 < \lambda_1 < \lambda_2 < \cdots$ with $\lambda_n \to \infty$ as $n \to \infty$.

By the transformation $\varphi = (r^N(s))^{1/2}\phi$, (4.1) $\sim$ (4.3) becomes

$$(4.4) \qquad \varphi_{ss} - h(s)\varphi + \frac{\lambda}{d_1}\varphi = 0 \qquad \text{in } (0, L),$$

$$(4.5) \qquad \varphi_s(0) - \beta_0\varphi(0) = 0, \quad \varphi_s(L) - \beta_1\varphi(L) = 0,$$

$$(4.6) \qquad \varphi(0) = 0, \quad \varphi(L) = 0,$$

where $h(s) = (N/2)(N/2 - 1)(r'/r)^2 + (N/2)r''/r + \alpha$, $\beta_0 = (N/2)(r'/r)(0)$, $\beta_1 = (N/2)(r'/r)(L)$. It follows from Hochstadt [21] that there are functions $P_1(\lambda)$ and $P_2(\lambda)$ bounded for all λ such that

$$sin(L\sqrt{\lambda/d_1}) = P_1(\lambda)/\sqrt{\lambda} \qquad \text{for (4.4), (4.5)}$$

and

$$sin(L\sqrt{\lambda/d_1}) = P_2(\lambda)/\sqrt{\lambda} \qquad \text{for (4.4), (4.6).}$$

Letting $L\sqrt{\lambda/d_1} = \pi(n+\nu)$, we have

$$sin(\pi\nu) = (-1)^n \frac{P_i(d_1\pi^2(n+\nu)^2/L^2)}{\sqrt{d_1}\,\pi(n+\nu)/L},$$

which has a unique solution ν_n in $(-1/2, 1/2)$ satisfying $lim_{n\to\infty} \nu_n = 0$ when n is sufficiently large. We thus know that for any $\delta > 0$, there exists $N_0 > 0$ such that

$$\frac{d_1\pi^2}{L^2}(n-\delta)^2 \le \lambda_n \le \frac{d_1\pi^2}{L^2}(n+\delta)^2 \qquad \text{for } n \ge N_0. \tag{4.7}$$

Let $\{\lambda_n^E\}$ and $\{\phi_n^E\}$ $(n = 1, 2, \cdots)$ be the eigenvalues and the corresponding eigenfunctions of (4.1) normalized in $\mathbf{H}_0(0, L)$ with the boundary conditions (4.2) (or (4.3)) associated with the operator $\mathbf{S}_0^2$. The inequality (4.7) also holds for λ_n^E for large n. Note that the sequence of eigenvalues $\{\mu_1, \mu_2, \cdots\}$ of $\mathbf{B}_0$ are rearrangement of $\{\lambda_1, \lambda_2, \cdots\} \cup \{\bar{d}\lambda_1^E, \bar{d}\lambda_2^E, \cdots\}$, where $\bar{d} = d_2/d_1$. Thus we know that the multiplicity of μ_i is at most two and the corresponding normalized eigenfunction is $(\Phi_n, 0)$ or $(0, \Phi_k^E)$.

For sufficiently large k, $n (\ge N_0)$, let $\lambda_{n-1} \le \bar{d}\lambda_k^E < \lambda_n$.

(i) If $\lambda_{n-1} \le \bar{d}\lambda_k^E \le (\lambda_{n-1} + \lambda_n)/2$, we have

$$\lambda_n - \bar{d}\lambda_k^E \ge (\lambda_n - \lambda_{n-1})/2 > d_1 n/L^2$$

and

$$\bar{d}\lambda_{k+1}^E - \bar{d}\lambda_k^E > 2\bar{d}d_1 k/L^2 \ge \sqrt{d_1 d_2}\, n/L^2,$$

by using the estimate (4.7).

(ii) If $(\lambda_{n-1} + \lambda_n)/2 \le \bar{d}\lambda_k^E < \lambda_n$, we have

$$\bar{d}\lambda_k^E - \lambda_{n-1} \ge (\lambda_n - \lambda_{n-1})/2 > d_1 n/L^2$$

and

$$\bar{d}\lambda_k^E - \bar{d}\lambda_{k-1}^E > 2\bar{d}d_1 k/L^2 \ge \sqrt{d_1 d_2}\, n/L^2$$

also by (4.7).

Thus we obtain the following result:

LEMMA 4.1. *The multiplicity of the eigenvalue* μ_n *of* $\mathbf{B}_0$ *is at most two* *and* $limsup_{n\to\infty}(\mu_{n+1}-\mu_n)=\infty$.

We will discuss the relation between the eigenvalues of $\mathbf{B}_\varepsilon$ and $\mathbf{B}_0$. Let $\{\lambda_{\varepsilon,n}\}$ and $\{\phi_{\varepsilon,n}\}$ $(n=1,2,\cdots)$ be the eigenvalues and the corresponding eigenfunctions of the following eigenvalue problem associated with the operator $\mathbf{S}^1_\varepsilon$:

$$\mathscr{L}^\varepsilon(d_1)\phi+\alpha_1\phi=\lambda\phi \qquad \text{in } \Omega \tag{4.8}$$

with the homogeneous Neumann boundary condition

$$\frac{\partial\phi}{\partial n_\varepsilon}=0 \qquad \text{on } \partial\Omega \tag{4.9}$$

or the mixed boundary conditions

$$\phi=0 \quad \text{on} \quad \{0,L\}\times D, \quad \frac{\partial\phi}{\partial n_\varepsilon}=0 \quad \text{on} \quad (0,L)\times\partial D, \tag{4.10}$$

where ϕ is normalized in $\mathbf{H}_\varepsilon(\Omega)$. It is known that $0<\lambda_{\varepsilon,1}\le\lambda_{\varepsilon,2}\le\cdots$ and $\lambda_{\varepsilon,n}\to\infty$ as $n\to\infty$. We will show the following result:

LEMMA 4.2. *For any positive integer* N_1, *there exists* $\varepsilon_2=\varepsilon_2(N_1)>0$ *such that for* $0<\varepsilon\le\varepsilon_2$,

$$max_{1\le j\le N_1}\left\{|\lambda_{\varepsilon,j}-\lambda_j|+\left\|\phi_{\varepsilon,j}-\frac{1}{\sqrt{|D|}}\phi_j\right\|_{\varepsilon,\mathbf{Y}^1}\right\}\le c\lambda_{N_1}\varepsilon \tag{4.11}$$

holds for some positive constant c, where $|D|$ *is the N-dimensional volume of D.*

Thus, for sufficiently small ε, we know that $\lambda_{\varepsilon,j}$ is simple and that for any $K>0$ there are a positive integer N_2 and $\varepsilon_3=\varepsilon_3(N_2)>0$ such that

$$inf_{0\le\varepsilon\le\varepsilon_3}(\lambda_{\varepsilon,N_2+1}-\lambda_{\varepsilon,N_2})\ge K \tag{4.12}$$

holds.

The gap property of eigenvalues of $\mathbf{B}_\varepsilon$ follows from Lemma 4.2. Let $\{\mu_{\varepsilon,n}\}$ and $\{\omega_{\varepsilon,n}\}$ $(n=1,2,\cdots)$ be the eigenvalues and the corresponding eigenfunctions of $\mathbf{B}_\varepsilon$ which is normalized in $(\mathbf{H}_\varepsilon(\Omega))^2$. Similarly, let $\{\lambda^E_{\varepsilon,n}\}$ and $\{\phi^E_{\varepsilon,n}\}$ $(n=1,2,\cdots)$ be the eigenvalues and the corresponding eigenfunctions of (4.8), which is normalized in $\mathbf{H}_\varepsilon(\Omega)$, with the boundary condition (4.9) or (4.10) associated with the operator $\mathbf{S}^2_\varepsilon$. Note that $\{\mu_{\varepsilon,1},\mu_{\varepsilon,2},\cdots\}$ are rearrangement of $\{\lambda_{\varepsilon,1},\lambda_{\varepsilon,2},\cdots\}\cup\{\bar{d}\lambda^E_{\varepsilon,1},\bar{d}\lambda^E_{\varepsilon,2},\cdots\}$ and that the multiplicity of $\mu_{\varepsilon,n}$ is at most two and $\omega_{\varepsilon,n}$ is $(\phi_{\varepsilon,m},0)$ or $(0,\phi^E_{\varepsilon,k})$. The following result is obtained by Lemma 4.2:

THEOREM 4.3. *For any positive integer N_1, there exists $\varepsilon_4 = \varepsilon_4(N_1) > 0$ such that for $0 < \varepsilon \le \varepsilon_4$,*

$$\max_{1 \le j \le N_1} \left\{ |\mu_{\varepsilon,j} - \mu_j| + \left\| \omega_{\varepsilon,j} - \frac{1}{\sqrt{|D|}} \omega_j \right\|_{\varepsilon,\mathbf{X}} \right\} \le c\mu_{N_1}\varepsilon \tag{4.13}$$

holds for some positive constant c. Also for any $K > 0$ there are a positive integer N_3 and $\varepsilon_5 = \varepsilon_5(N_3) > 0$ such that

$$\inf_{0 \le \varepsilon \le \varepsilon_5} (\mu_{\varepsilon, N_3+1} - \mu_{\varepsilon, N_3}) \ge K \tag{4.14}$$

holds.

Next we will prove Lemma 4.2. To do it, the following three Lemmas are needed.

LEMMA 4.4. *For any $w \in H^1(\Omega)$:*

$$\| w - \mathscr{M} w \|_L \le c\varepsilon \| w \|_{\varepsilon, \mathbf{Y}} \tag{4.15}$$

holds for some positive constant c, where $(\mathscr{M}w)(s) = (1/|D|) \int_D w(s, z)\,dz$ and $\mathbf{Y}$ *means* $\mathbf{Y}^1$ or $\mathbf{Y}^2$.

PROOF. For any $w \in C^\infty(\bar{\Omega})$, we have

$$\begin{aligned} w(s, z_1, z_2, \cdots, z_N) &= w(s, \tau_1, z_2, \cdots, z_N) + \int_{\tau_1}^{z_1} \frac{\partial w}{\partial z_1}(s, \xi_1, z_2, \cdots, z_N)\,d\xi_1 \\ &= w(s, \tau_1, \tau_2, z_3, \cdots, z_N) + \int_{\tau_1}^{z_1} \frac{\partial w}{\partial z_1}(s, \xi_1, z_2, \cdots, z_N)\,d\xi_1 \\ &\qquad + \int_{\tau_2}^{z_2} \frac{\partial w}{\partial z_2}(s, \tau_1, \xi_2, z_3, \cdots, z_N)\,d\xi_2 \\ &= w(s, \tau_1, \tau_2, \cdots, \tau_N) + \bar{I}, \end{aligned}$$

where $\bar{I} = \int_{\tau_1}^{z_1} \frac{\partial w}{\partial z_1}(s, \xi_1, z_2, \cdots, z_N)d\xi_1 + \int_{\tau_2}^{z_2} \frac{\partial w}{\partial z_2}(s, \tau_1, \xi_2, z_3, \cdots, z_N)d\xi_2 + \cdots + \int_{\tau_N}^{z_N} \frac{\partial w}{\partial z_N}(s, \tau_1, \tau_2, \cdots, \tau_{N-1}, \xi_N)d\xi_N$. Taking the spatial average of the above equality on D with respect to $z_1, \cdots, z_N$, we have

$$(\mathscr{M}w)(s) = w(s, \tau_1, \tau_2, \cdots, \tau_N) + \frac{1}{|D|} \int_D \bar{I}\,dz_1\,dz_2 \cdots dz_N$$

and

$$w(s, z_1, \cdots, z_N) - (\mathscr{M}w)(s) = \bar{I} - \frac{1}{|D|}\int_D \bar{I}\, dz_1 \cdots dz_N. \tag{4.16}$$

Then let us take the spatial average of (4.16) on D with respect to $\tau_1, \cdots, \tau_N$. It follows that

$$w(s, z_1, \ldots, z_N) - (\mathscr{M}w)(s) = \frac{1}{|D|}\int_D \bar{I}\, d\tau_1 \cdots d\tau_N$$
$$- \frac{1}{|D|^2}\iint_{D \times D} \bar{I}\, dz_1 \cdots dz_N d\tau_1 \cdots d\tau_N.$$

Also by taking the integration on Ω of the above equality with respect to $s, z_1, \ldots, z_N$, we have

$$\| w - \mathscr{M}w \|_{0,\Omega} \le c_1\left(\left\| \frac{\partial w}{\partial z_1} \right\|_{0,\Omega} + \cdots + \left\| \frac{\partial w}{\partial z_N} \right\|_{0,\Omega} \right)$$

for some positive constant c_1. Therefore we know

$$\| w - \mathscr{M}w \|_L \le c_2 \| w - \mathscr{M}w \|_{0,\Omega} \le c_1 c_2 \left(\left\| \frac{\partial w}{\partial z_1} \right\|_{0,\Omega} + \cdots + \left\| \frac{\partial w}{\partial z_N} \right\|_{0,\Omega} \right)$$
$$\le c\varepsilon \| w \|_{\varepsilon, \mathbf{Y}}$$

for some positive constants c_2 and c, by using (3.12) and the definition of $\| \cdot \|_{\varepsilon,L}$. Since $C^\infty(\bar{\Omega})$ is dense in $H^1(\Omega)$, (4.15) holds for any $w \in H^1(\Omega)$.

We define the inverse of $\mathbf{S}_\varepsilon^i$ and $\mathbf{S}_0^i$ ($i = 1, 2$) by

$$(\mathbf{S}_\varepsilon^i)^{-1} u = w \text{ iff } \mathbf{a}_\varepsilon(w, v) = \langle u, v \rangle_L \qquad \text{for any } v \in \mathbf{Y}_\varepsilon^i$$

and

$$(\mathbf{S}_0^i)^{-1} u = w \text{ iff } \mathbf{a}_0(w, v) = \langle u, v \rangle_{0,I} \qquad \text{for any } v \in \mathbf{Y}_0^i,$$

respectively. The Lax-Milgram Theorem shows that $(\mathbf{S}_\varepsilon^i)^{-1}$ and $(\mathbf{S}_0^i)^{-1}$ are well defined. By using regularity properties (see Hale and Raugel [17] and [18]), we know that $(\mathbf{S}_\varepsilon^i)^{-1} \in \mathscr{L}(\mathbf{H}_\varepsilon(\Omega); H^2(\Omega))$ and $(\mathbf{S}_0^i)^{-1} \in \mathscr{L}(\mathbf{H}_0(0, L); H^2(0, L))$. Therefore, it turns out that $(\mathbf{S}_\varepsilon^i)^{-1}$ and $(\mathbf{S}_0^i)^{-1}$ are both compact operators from $\mathbf{Y}_\varepsilon^i$ and $\mathbf{Y}_0^i$ into $\mathbf{Y}_\varepsilon^i$ and $\mathbf{Y}_0^i$, respectively.

LEMMA 4.5. (Hale and Raugel [17]) *There are* $\varepsilon_6 > 0$ *and some positive constant* $\bar{c}_5$ *such that*

$$\| (\mathbf{S}_\varepsilon^i)^{-1} \|_{\mathscr{L}(\mathbf{H}_\varepsilon(\Omega); H^2(\Omega))} + \| (\mathbf{S}_0^i)^{-1} \|_{\mathscr{L}(\mathbf{H}_0(0,L); H^2(0,L))} \le \bar{c}_5 \qquad (i = 1, 2)$$

holds for $0 < \varepsilon \le \varepsilon_6$.

LEMMA 4.6. *There exists some positive constant* $\bar{c}_6$ *such that for* $0 < \varepsilon \le \varepsilon_6$ *and any* $h \in \mathbf{H}_0(0, L)$,

$$\|((\mathbf{S}_\varepsilon^i)^{-1} - (\mathbf{S}_0^i)^{-1})h\|_{\varepsilon, \mathbf{Y}^i} \le \bar{c}_6 \varepsilon \|h\|_{0,I} \qquad (i = 1, 2).$$

PROOF. We note that $h \in \mathbf{H}_0(0, L)$ is considered as an element in $\mathbf{H}_\varepsilon(\Omega)$ and

$$\mathbf{a}_0((\mathbf{S}_0^i)^{-1}h, u) = \langle h, u\rangle_{0,I} \qquad \text{for } u \in H^1(0, L), \tag{4.17}$$

$$\mathbf{a}_\varepsilon((\mathbf{S}_\varepsilon^i)^{-1}h, w) = \langle h, w\rangle_L \qquad \text{for } w \in H^1(\Omega). \tag{4.18}$$

It follows from (4.17) that

$$\int_0^L d_1 r^N(s) \frac{d}{ds}((\mathbf{S}_0^i)^{-1}h) \frac{du}{ds} ds + \alpha_1 \int_0^L r^N(s)((\mathbf{S}_0^i)^{-1}h) u ds = \int_0^L r^N(s) h u ds.$$

Substituting $u = \int_D w(s, z) dz$ into the above, we have

$$\int_\Omega d_1 r^N(s) \frac{d}{ds}((\mathbf{S}_0^i)^{-1}h) \frac{\partial w}{\partial s} dsdz + \alpha_1 \int_\Omega r^N(s)((\mathbf{S}_0^i)^{-1}h) w dsdz = \int_\Omega r^N(s) h w dsdz$$

and then

$$\mathbf{a}_\varepsilon((\mathbf{S}_0^i)^{-1}h, w) = \langle h, w\rangle_L - \int_\Omega d_1 r^{N-1} r' \frac{d}{ds}((\mathbf{S}_0^i)^{-1}h) \textstyle\sum_{j=1}^N \left(z_j \dfrac{\partial w}{\partial z_j}\right) dsdz.$$

By using (4.18) and letting $w = ((\mathbf{S}_\varepsilon^i)^{-1} - (\mathbf{S}_0^i)^{-1})h$ in the above, we obtain

$$\begin{aligned}\|((\mathbf{S}_\varepsilon^i)^{-1} - (\mathbf{S}_0^i)^{-1})h\|_{\varepsilon, \mathbf{Y}^i}^2 &= \mathbf{a}_\varepsilon(((\mathbf{S}_\varepsilon^i)^{-1} - (\mathbf{S}_0^i)^{-1})h, ((\mathbf{S}_\varepsilon^i)^{-1} - (\mathbf{S}_0^i)^{-1})h) \\ &\le c\varepsilon \|h\|_{0,I} \|((\mathbf{S}_\varepsilon^i)^{-1} - (\mathbf{S}_0^i)^{-1})h\|_{\varepsilon, L} \le c'\varepsilon \|h\|_{0,I} \|((\mathbf{S}_\varepsilon^i)^{-1} - (\mathbf{S}_0^i)^{-1})h\|_{\varepsilon, \mathbf{Y}^i}\end{aligned}$$

for some positive constants c and c'. Here we used (3.12) and Lemma 4.5. Thus, Lemma 4.6 immediately follows.

PROOF OF LEMMA 4.2. The proof consists of three steps.

Step I: Let λ_j and ϕ_j be one of the eigenvalues and the corresponding normalized eigenfunctions of $\mathbf{S}_0^i$ $(i = 1, 2)$. We will prove that there is only one simple eigenvalue $\lambda_{\varepsilon,j}$ of $\mathbf{S}_\varepsilon^i$ in a small neighborhood of λ_j for sufficiently small ε and the corresponding normalized eigenfunction $\phi_{\varepsilon,j}$ with the property that $\lambda_{\varepsilon,j} \to \lambda_j$ and $\phi_{\varepsilon,j} \to (1/\sqrt{|D|})\phi_j$ as $\varepsilon \downarrow 0$. We now define the operators $\mathscr{g}_0$ and $\mathscr{g}_\varepsilon$ by

$$\mathscr{g}_0 : \mathbb{R} \times \mathbf{Y}_0^i \longrightarrow \mathbb{R} \times \mathbf{Y}_0^i \quad \text{with} \quad \mathscr{g}_0(\tau, u) = (\zeta_0(u - \phi_j), u - \tau(\mathbf{S}_0^i)^{-1}u)$$

and

$$\mathscr{g}_\varepsilon : \mathbb{R} \times \mathbf{Y}^i_\varepsilon \longrightarrow \mathbb{R} \times \mathbf{Y}^i_\varepsilon \quad \text{with} \quad \mathscr{g}_\varepsilon(\tau, w) = (\zeta_\varepsilon(w - \phi_j),\ w - \tau(\mathbf{S}^i_\varepsilon)^{-1} w),$$

where $\zeta_0(u) = \lambda_j^{-1}\mathbf{a}_0(u, \phi_j)$ and $\zeta_\varepsilon(w) = \lambda_j^{-1}\mathbf{a}_\varepsilon(w, \phi_j)$. The norms in $\mathbb{R} \times \mathbf{Y}^i_0$ and $\mathbb{R} \times \mathbf{Y}^i_\varepsilon$ are defined by $\|(\tau, u)\|_0 = |\tau| + \|u\|_{0,\mathbf{Y}^i}$ and $\|(\tau, w)\|_\varepsilon = |\tau| + \|w\|_{\varepsilon,\mathbf{Y}^i}$, respectively. It is known that $\mathscr{g}_0(\lambda_j, \phi_j) = 0$ and that if $\mathscr{g}_\varepsilon(\tau, w) = 0$, then τ is an eigenvalue of $\mathbf{S}^i_\varepsilon$ with the corresponding eigenfunction w. From Lemmas 4.4, 4.5 and 4.6, the following result holds:

LEMMA 4.7. (Hale and Raugel [17]) *$D\mathscr{g}_0(\lambda_j, \phi_j)$ is an isomorphism from $\mathbb{R} \times \mathbf{Y}^i_0$ into itself and there are $\varepsilon_{0,j} > 0$ and $\gamma_0 > 0$ such that for $0 < \varepsilon \le \varepsilon_{0,j}$, $D\mathscr{g}_\varepsilon(\lambda_j, \phi_j)$ is invertible and*

$$\gamma_\varepsilon = \|(D\mathscr{g}_\varepsilon(\lambda_j, \phi_j))^{-1}\|_{\mathscr{L}(\mathbb{R}\times\mathbf{Y}^i_\varepsilon, \mathbb{R}\times\mathbf{Y}^i_\varepsilon)} \le \gamma_0.$$

We introduce the following notations: $\delta_\varepsilon = \|\mathscr{g}_\varepsilon(\lambda_j, \phi_j)\|_\varepsilon$, $\beta_\varepsilon(\eta) = sup\{\|D\mathscr{g}_\varepsilon(\tau, w) - D\mathscr{g}_\varepsilon(\lambda_j, \phi_j)\|_{\mathscr{L}(\mathbb{R}\times\mathbf{Y}^i_\varepsilon, \mathbb{R}\times\mathbf{Y}^i_\varepsilon)} | (\tau, w) \in \mathscr{B}(\varepsilon, \lambda_j, \phi_j; \eta)\}$, where $\mathscr{B}(\varepsilon, \lambda_j, \phi_j; \eta) = \{(\tau, w) \in \mathbb{R} \times \mathbf{Y}^i_\varepsilon | \|(\tau, w) - (\lambda_j, \phi_j)\|_\varepsilon \le \eta\}$.

LEMMA 4.8. (Crouzeix [6]) *If $2\gamma_\varepsilon\beta_\varepsilon(2\gamma_\varepsilon\delta_\varepsilon) < 1$, then for any $\eta > 0$ satisfying $\gamma_\varepsilon\beta_\varepsilon(\eta) < 1$ and $\eta \ge 2\gamma_\varepsilon\delta_\varepsilon$, $\mathscr{g}_\varepsilon(\tau, w) = 0$ has only one solution $(\tau_\varepsilon, w_\varepsilon)$ in $\mathscr{B}(\varepsilon, \lambda_j, \phi_j; \eta)$. Moreover, τ_ε is a simple eigenvalue of $\mathbf{S}_\varepsilon$ and*

$$\|(\tau_\varepsilon, w_\varepsilon) - (\lambda_j, \phi_j)\|_\varepsilon \le \gamma_\varepsilon\delta_\varepsilon.$$

Since the proof is directly obtained by using the contraction mapping theorem, we omit it.

Since $\mathscr{g}_\varepsilon(\lambda_j, \phi_j) = \mathscr{g}_0(\lambda_j, \phi_j) + (0, \lambda_j((\mathbf{S}^i_0)^{-1} - (\mathbf{S}^i_\varepsilon)^{-1})\phi_j)$, it follows from Lemma 4.6 that $\delta_\varepsilon \le c\varepsilon\lambda_j$ for some positive constant c and $0 < \varepsilon \le \varepsilon_6$. Also since

$$D\mathscr{g}_\varepsilon(\tau, w)(v, \upsilon) = (\zeta_\varepsilon(\upsilon),\ \upsilon - \tau(\mathbf{S}^i_\varepsilon)^{-1}\upsilon - v(\mathbf{S}^i_\varepsilon)^{-1} w),$$

we have

$$(D\mathscr{g}_\varepsilon(\lambda_j, \phi_j) - D\mathscr{g}_\varepsilon(\tau, w))(v, \upsilon) = (0, (\tau - \lambda_j)(\mathbf{S}^i_\varepsilon)^{-1}\upsilon - v(\mathbf{S}^i_\varepsilon)^{-1}(\phi_j - w)).$$

Therefore, by Lemma 4.5, we have $\beta_\varepsilon(\eta) \le c'\eta$ for some positive constant c' and $0 < \varepsilon \le \varepsilon_6$. Then we can use Lemma 4.8 by choosing $\eta = \eta_{1,j} = 1/(2\gamma_0 c')$, $\varepsilon_{1,j} > 0$ and $4\gamma_0^2 cc'\varepsilon_{1,j}\lambda_j < 1$ and therefore we know that $\mathscr{g}_\varepsilon(\tau, w) = 0$ has only one solution $(\lambda_{\varepsilon,j}, w_{\varepsilon,j})$ in $\mathscr{B}(\varepsilon, \lambda_j, \phi_j; \eta_{1,j})$ for $0 < \varepsilon \le \varepsilon_{1,j}$, and that $\lambda_{\varepsilon,j}$ is simple and

$$|\lambda_{\varepsilon,j} - \lambda_j| + \|w_{\varepsilon,j} - \phi_j\|_{\varepsilon,\mathbf{Y}^i} \le c\gamma_0\lambda_j\varepsilon, \qquad 0 < \varepsilon \le \varepsilon_{1,j}. \tag{4.19}$$

We normalize $w_{\varepsilon,j}$ by letting $\phi_{\varepsilon,j} = w_{\varepsilon,j}/\|w_{\varepsilon,j}\|_L$ so that $\|\phi_{\varepsilon,j}\|_L = 1$. It follows from (4.19) that

$$| \|w_{\varepsilon,j}\|_L - \|\phi_j\|_L | \le c\gamma_0 \lambda_j \varepsilon \qquad \text{for } 0 < \varepsilon \le \varepsilon_{1,j}.$$

Using $\|\phi_j\|_L = \sqrt{|D|}\,\|\phi_j\|_{0,I} = \sqrt{|D|}$, we have

$$\left\| \phi_{\varepsilon,j} - \frac{1}{\sqrt{|D|}}\phi_j \right\|_{\varepsilon,\mathbf{Y}^i} \le \tilde{c}\lambda_j \varepsilon \qquad \text{for } 0 < \varepsilon \le \varepsilon_{2,j}$$

for some positive constants $\tilde{c}$ and $\varepsilon_{2,j}$.

Step II: For any positive integer N_1, let $\varepsilon_2 = min\{\varepsilon_{2,1}, \varepsilon_{2,2}, \dots, \varepsilon_{2,N_1}\}$. It follows from the results in Step I that for $0 < \varepsilon \le \varepsilon_2$, $\mathbf{S}_\varepsilon^i$ has simple eigenvalues $\lambda_{\varepsilon,1}, \dots, \lambda_{\varepsilon,N_1}$ in $\mathscr{R}_{N_1}^1 = \bigcup_{j=1}^{N_1} [\lambda_j - \eta_{1,j}, \lambda_j + \eta_{1,j}]$ and

$$max_{1\le j\le N_1} \left\{ |\lambda_{\varepsilon,j} - \lambda_j| + \left\| \phi_{\varepsilon,j} - \frac{1}{\sqrt{|D|}}\phi_j \right\|_{\varepsilon,\mathbf{Y}^i} \right\} \le c\lambda_{N_1}\varepsilon$$

for some positive constant c.

Step III. Letting $\mathscr{R}_{N_1}^2 = (0, \lambda_1 - \eta_{1,1}) \cup (\bigcup_{j=2}^{N_1} (\lambda_{j-1} + \eta_{1,j-1}, \lambda_j - \eta_{1,j}))$, we will prove that $\mathbf{S}_\varepsilon^i$ has no eigenvalue in $\mathscr{R}_{N_1}^2$ when ε is sufficiently small. Suppose that this is not true, then threre exists a sequence $\{\varepsilon_n\}$ ($\varepsilon_n \downarrow 0$ *as* $n \to \infty$) such that $\lambda_{\varepsilon_n} \in \mathscr{R}_{N_1}^2$ is the eigenvalue of $\mathbf{S}_{\varepsilon_n}^i$. Letting ϕ_{ε_n} be the corresponding eigenfunction of $\mathbf{S}_{\varepsilon_n}^i$ normalized in $\mathbf{H}_{\varepsilon_n}(\Omega)$, then we have

$$\bar{c}_1^2 \|\phi_{\varepsilon_n}\|_{\varepsilon_n,L}^2 \le \mathbf{a}_{\varepsilon_n}^i(\phi_{\varepsilon_n}, \phi_{\varepsilon_n}) = \langle \mathbf{S}_{\varepsilon_n}^i \phi_{\varepsilon_n}, \phi_{\varepsilon_n} \rangle_L = \lambda_{\varepsilon_n} \le c$$

for some positive constant c. Therefore $\|\phi_{\varepsilon_n}\|_{\varepsilon_n,L}$ are uniformly bounded and by the compactness of $(\mathbf{S}_\varepsilon^i)^{-1}$ there exists a subsequence of $\{\varepsilon_n\}$, also denoted by $\{\varepsilon_n\}$, such that $\lambda_{\varepsilon_n} \to \lambda_0$ and $\phi_{\varepsilon_n} \to \phi_0$ strongly in $H^1(\Omega)$ for some $\lambda_0 \in \mathscr{R}_{N_1}^2$ and $\phi_0 \in H^1(\Omega)$ with $\|\phi_0\|_{0,\Omega} \ne 0$ from the normality of ϕ_{ε_n}. By the definition of $\|\cdot\|_{\varepsilon,L}$, we know that $\phi_0(s, z) = \phi_0(s)$ does not depend on $z_1, \dots, z_N$. Thus it follows from Lemmas 4.5 and 4.6 that

$$\begin{aligned} \|(\mathbf{S}_{\varepsilon_n}^i)^{-1}\phi_{\varepsilon_n} - (\mathbf{S}_0^i)^{-1}\phi_0\|_{0,\Omega} &\le \|(\mathbf{S}_{\varepsilon_n}^i)^{-1}(\phi_{\varepsilon_n} - \phi_0)\|_{0,\Omega} \\ &+ \|((\mathbf{S}_{\varepsilon_n}^i)^{-1} - (\mathbf{S}_0^i)^{-1})\phi_0\|_{0,\Omega} \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty. \end{aligned}$$

Together with $\phi_{\varepsilon_n} = \lambda_{\varepsilon_n}(\mathbf{S}_{\varepsilon_n}^i)^{-1}\phi_{\varepsilon_n}$, we have $\phi_0 = \lambda_0(\mathbf{S}_0^i)^{-1}\phi_0$. That is, $\mathbf{S}_0^i\phi_0 = \lambda_0\phi_0$ which implies that λ_0 is an eigenvalue of $\mathbf{S}_0^i$ with the corresponding eigenfunction ϕ_0. This is a contradiction so that we find that the conclusion is true.

§5. Existence of inertial manifolds

We rewrite the systems (3.2) ~ (3.5) and (3.6) ~ (3.9) as the following abstract forms:

$$(5.1)\qquad \begin{cases} U_t = -\mathbf{B}_\varepsilon U + F_\varepsilon(U), & t > 0, \\ U(0) = U_0 \in \mathbf{K}_\mathrm{I} \end{cases}$$

and

$$(5.2)\qquad \begin{cases} U_t = -\mathbf{B}_0 U + F_0(U), & t > 0, \\ U(0) = U_0 \in \mathbf{K}_\mathrm{I}, \end{cases}$$

respectively, where $U = (u, v)$, $F_\varepsilon(U)(s, z) = (f_\varepsilon(u, v; s, z), g_\varepsilon(u, v; s, z))$, $F_0(U)(s) = (f_0(u, v; s), g_0(u, v; s))$ and $\mathbf{K}_\mathrm{I} = \{(u, v) | 0 \le u \le K_1,\ 0 \le v \le K_2\}$. Because [A] holds for (5.1), we can modify the nonlinear terms F_ε outside $\mathbf{K}_\mathrm{I}$ as follows: for $0 < \varepsilon \le \varepsilon_4$, F_ε satisfies

$$(5.3)\qquad F_\varepsilon(U) = 0 \qquad \text{if } U \bar{\in} \mathbf{K}_\mathrm{II} = \{(u, v) | -1 \le u \le K_1 + 1,\ -1 \le v \le K_2 + 1\},$$

$$(5.4)\qquad sup\{|F_\varepsilon(U)|, |DF_\varepsilon(U)|, |D^2 F_\varepsilon(U)|\} \le M_1 \qquad \text{for } U \in \mathbf{R}^2,$$

$$(5.5)\qquad |F_\varepsilon(U_1) - F_\varepsilon(U_2)| \le M_1 |U_1 - U_2|, |F_\varepsilon(U)| \le M_1 |U| + M_2 \quad \text{for } U, U_1, U_2 \in \mathbf{R}^2$$

for some positive constants M_1 and M_2 uniformly with respect to (s, z), since we only want to discuss the dynamics of bounded solutions of (5.1). Since f and g belong to $W^{1,\infty}(\Omega)$, the following inequality holds for some positive constants $\bar{c}_6$ and $\bar{c}_7$:

$$(5.6)\qquad \| F_\varepsilon(U) - F_0(U) \|_\mathbf{H} \le \bar{c}_6 \varepsilon \| U \|_\mathbf{H} + \bar{c}_7 \varepsilon \qquad \text{for } U \in (L^2(\Omega))^2.$$

We will discuss (5.1) only, because (5.2) can be considered similarly. We want to construct inertial manifold for the following initial value problem of (5.1):

$$(5.7)\qquad \begin{cases} U_t = -\mathbf{B}_\varepsilon U + F_\varepsilon(U), & t > 0, \\ U(0) = U_0 \in \mathbf{K}_\mathrm{I} \cap \mathbf{K}_\mathrm{III}, \end{cases}$$

where $\mathbf{K}_\mathrm{III} = \{U \in \mathbf{X}^\varepsilon_{p,\beta} | \| U \|_{\mathbf{X}^\varepsilon_{p,\beta}} \le K_3\}$. Here, $\mathbf{X}^\varepsilon_{p,\beta} = \mathscr{D}(\mathbf{B}^\beta_\varepsilon)$ when $\mathbf{B}_\varepsilon$ is considered as an operator in $(L^p(\Omega))^2$ with $\| U \|_{\mathbf{X}^\varepsilon_{p,\beta}} = \| \mathbf{B}^\beta_\varepsilon U \|_{(L^p(\Omega))^2}$, $p > N + 1$ and $\beta \in ((p + N + 1)/2p, 1)$. Fix K_1, K_2 and K_3 sufficiently large. Then we know that there exists a unique global solution $U(t)$ of (5.7) such that $U(t) \in C^1([0, \infty), \mathbf{X}_\varepsilon) \cap \mathscr{D}(\mathbf{B}_\varepsilon)$ (Henry [19]) and that $U(t) \in \mathbf{K}_\mathrm{I}(\mathbf{K}_\mathrm{II})$ if the system (5.7) has the (asymptotically) invariant region assumed in [A]. Moreover, the following result holds (for example, Ei and Mimura [11]):

LEMMA 5.1. *There exists a positive constant* $\bar{c}$ *such that any solution* $U(t)$ *of* (5.7) *satisfies*

$$\| U(t) \|_{(C^1(\Omega))^2} \le \bar{c} \qquad \textit{for } t > 0.$$

By Theorem 4.3, we know that for any large $\mu > 0$ there exist some positive integer N_1 and $\varepsilon_4(N_1) > 0$ such that $\mu_{N_1} < \mu < \mu_{N_1+1}$ and $\mu_{\varepsilon,N_1} < \mu < \mu_{\varepsilon,N_1+1}$ for $0 < \varepsilon \le \varepsilon_4$. Let $\mathbf{V}_\mu^\varepsilon = span\{\omega_{\varepsilon,1},\cdots,\omega_{\varepsilon,N_1}\}$ and $\mathbf{W}_\mu^\varepsilon = (\mathbf{V}_\mu^\varepsilon)^\perp = clspan\{\omega_{\varepsilon,j} | j > N_1\}$. We define $\mathbf{P}_\mu^\varepsilon$ and $\mathbf{Q}_\mu^\varepsilon$ be the natural projection from $(\mathbf{H}_\varepsilon(\Omega))^2$ into $\mathbf{V}_\mu^\varepsilon$ and $\mathbf{Q}_\mu^\varepsilon = \mathbf{I} - \mathbf{P}_\mu^\varepsilon$, respectively. Let $p = \mathbf{P}_\mu^\varepsilon U$ and $q = \mathbf{Q}_\mu^\varepsilon U$. Then (5.7) can be written as

$$\begin{cases} p_t = -\mathbf{B}_\varepsilon p + \mathbf{P}_\mu^\varepsilon(F_\varepsilon(p, q)), \\ q_t = -\mathbf{B}_\varepsilon q + \mathbf{Q}_\mu^\varepsilon(F_\varepsilon(p, q)), \end{cases} \tag{5.8}$$

where $F_\varepsilon(p, q)$ means $F_\varepsilon(p + q)$. Let θ: $[0, \infty) \to [0, 1]$ be a smooth function such that $\theta(\xi) = 1$ if $0 \le \xi \le K_4(max\{K_1 + 1, K_2 + 1\})^2$ and $\theta(\xi) = 0$ if $\xi > 4K_4(max\{K_1 + 1, K_2 + 1\})^2$, where $K_4 > 1$ is a sufficiently large fixed constant. By following Mallet-Paret and Sell [30], (5.8) can be rewritten as

$$\begin{cases} p_t = -B_\varepsilon p + \theta(\|p\|_{\mathbf{H}}^2)\mathbf{P}_\mu^\varepsilon(F_\varepsilon(p, q)) \equiv \mathscr{F}_\varepsilon(p, q), \\ q_t = -\mathbf{B}_\varepsilon q + \theta(\|p\|_{\mathbf{H}}^2)\mathbf{Q}_\mu^\varepsilon(F_\varepsilon(p, q)) \equiv -\mathbf{B}_\varepsilon q + \mathscr{G}_\varepsilon(p, q). \end{cases} \tag{5.9}$$

LEMMA 5.2. *Let $U(t)$ be a solution of* (5.9). *For $0 < \varepsilon \le \varepsilon_4$ there exists $t_1 > 0$ such that*

$$\|U(t)\|_{\mathbf{H}} < \frac{1}{2}K_4(max\{K_1 + 1, K_2 + 1\})^2 \qquad \textit{for } t \ge t_1.$$

PROOF. Rewrite (5.9) as

$$U_t = -\mathbf{B}_\varepsilon U + \theta(\|p\|_{\mathbf{H}}^2)F_\varepsilon(U).$$

Then we have

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_{\mathbf{H}}^2 + \langle \mathbf{B}_\varepsilon U, U\rangle_{\mathbf{H}} = \langle \theta(\|p\|_{\mathbf{H}}^2)F_\varepsilon(U), U\rangle_{\mathbf{H}}.$$

It follows from the modification of F_ε that there exists some positive constant c such that

$$\frac{1}{2}\frac{d}{dt}\|U(t)\|_{\mathbf{H}}^2 + \bar{\alpha}\|U(t)\|_{\mathbf{H}}^2 \le \bar{\alpha}/2\|U(t)\|_{\mathbf{H}}^2 + c,$$

where $\bar{\alpha} = min\{\alpha_1, \alpha_2\}$. Therefore, by using the Grownwall inequality, we have

$$\|U(t)\|_{\mathbf{H}} \le \|U_0\|_{\mathbf{H}}e^{-\bar{\alpha}t/2} + c' < \frac{1}{2}K_4(max\{K_1 + 1, K_2 + 1\})^2 \qquad \text{for } t \ge t_1$$

for some positive constants c' and t_1 and sufficiently large $K_4(> 1)$.

Lemma 5.2 implies that the attractors of (5.8) and (5.9) coincide with each other. In order to use the existence theorem of inertial manifolds in Mallet-Paret and Sell [30], we introduce some notations. Let

$$\mathrm{A}^{\varepsilon,\mu} = \{p \in \mathbf{V}_\mu^\varepsilon \mid \|p\|_{\mathbf{H}} \le R_1\},$$
$$\mathrm{C}^{\varepsilon,\mu} = \{q \in \mathscr{D}(\mathbf{B}_\varepsilon) \cap \mathbf{W}_\mu^\varepsilon \mid \|\mathbf{B}_\varepsilon q\|_{\mathbf{H}} \le R_2\},$$
$$\bar{\mathbf{K}}_{\mathrm{II}} = \mathbf{K}_{\mathrm{II}} \cap \{U \mid \|U\|_{(C^1(\Omega))^2} \le R_3\}$$

for fixed large positive constants R_1, R_2 and R_3. Note that for U, V and $U + V \in (\mathbf{A}^{\varepsilon,\mu} \times \mathbf{C}^{\varepsilon,\mu}) \cap \bar{\mathbf{K}}_{\mathrm{II}}$,

$$F_3(U+V)(x) - F_\varepsilon(U)(x) - (\mathrm{D}F_\varepsilon(U))(x)V(x)$$
$$= \int_0^1 [\mathrm{D}F_\varepsilon(U+\tau V)(x) - \mathrm{D}F_\varepsilon(U)(x)]\,V(x)\,d\tau$$

and that $\|U\|_{\mathbf{H}} \to 0$ implies $|U(x)| \to 0$ in Ω. Then it follows from the modification of F_ε that for any $\eta > 0$, there is $\delta > 0$ such that

$$\|F_\varepsilon(U+V) - F_\varepsilon(U) - \mathrm{D}F_\varepsilon(U)V\|_{\mathbf{H}} \le \eta \|V\|_{\mathbf{H}}$$

if $\|V\|_{\mathbf{H}} \le \delta$ and that

$$\|\mathrm{D}F_\varepsilon(U) - \mathrm{D}F_\varepsilon(V)\|_{\mathscr{L}((\mathbf{H}_\varepsilon(\Omega))^2, (\mathbf{H}_\varepsilon(\Omega))^2)} \le \eta$$

if $\|U - V\|_{\mathbf{H}} \le \delta$. Therefore, we obtain the following two lemmas.

LEMMA 5.3. (Regularity Condition) *$\mathscr{F}_\varepsilon$ and $\mathscr{G}_\varepsilon$ are C^1 in* $(\mathrm{A}^{\varepsilon,\mu} \times \mathrm{C}^{\varepsilon,\mu}) \cap \bar{\mathbf{K}}_{\mathrm{II}}$ *for $0 < \varepsilon \le \varepsilon_4$.*

LEMMA 5.4. (Dissipative Condition) *If $R_1 > 2\sqrt{K_4}\, max\{K_1 + 1, K_2 + 1\}$, then $\mathscr{G}_\varepsilon(p, 0) = 0$ and $\langle p, \mathscr{F}_\varepsilon(p, 0)\rangle_{\mathbf{H}} < 0$ for $p \in cl(\mathbf{V}_\mu^\varepsilon \setminus \mathrm{A}^{\varepsilon,\mu})$.*

LEMMA 5.5. (Sobolev Condition) *For any $R_1 > 0$ there exists R_2^0 such that if $p_0 \in \mathrm{A}^{\varepsilon,\mu}$ and $p(t; p_0, 0) \in \mathrm{A}^{\varepsilon,\mu}$ in $[0, t_0]$ for $t_0 \ge 0$, then $q(t; p_0, 0) \in \mathrm{C}^{\varepsilon,\mu}$ in $[0, t_0]$ for $R_2 \ge R_2^0$ and $0 < \varepsilon \le \varepsilon_4$.*

PROOF. Letting $exp(-\mathbf{B}_\varepsilon t)$ be the semigroup generated by $-\mathbf{B}_\varepsilon$, we know that for $U \in (\mathbf{H}_\varepsilon(\Omega))^2$,

$$\|exp(-\mathbf{B}_\varepsilon t)\mathbf{Q}_\mu^\varepsilon U\|_{\mathbf{H}} \le e^{-\mu t}\|U\|_{\mathbf{H}}, \qquad t \ge 0, \tag{5.10}$$

and

$$\|\mathbf{B}_\varepsilon^{1/2} exp(-\mathbf{B}_\varepsilon t)\mathbf{Q}_\mu^\varepsilon U\|_{\mathbf{H}} \le \mu^{1/2} b(\mu t)\|U\|_{\mathbf{H}}, \qquad t \ge 0, \tag{5.11}$$

where

$$b(\tau) = \begin{cases} (2e\tau)^{-1/2} & \text{in } (0, 1/2], \\ e^{-\tau} & \text{in } [1/2, \infty) \end{cases}$$

and $\int_0^\infty c^{1/2} b(ct)\,dt = 2(ce)^{-1/2}$. By using arguments similar to the one in Mallet-Paret and Sell [30] or Hale and Raugel [17], we know that when $\tilde{F}_\varepsilon(U)$ takes $F_\varepsilon(U)$ for the Neumann boundary conditions or $F_\varepsilon(U) - F_\varepsilon(0)$ for the mixed boundary conditions, $\tilde{F}_\varepsilon(U) \in \mathscr{D}(\mathbf{B}_\varepsilon^{1/2})$ holds if $U \in \mathscr{D}(\mathbf{B}_\varepsilon^{1/2})$. Moreover, there are positive constants M_3 and M_4 such that

$$\| \mathbf{B}_\varepsilon^{1/2} \tilde{F}_\varepsilon(U) \|_{\mathbf{H}} \leq M_3 \| \mathbf{B}_\varepsilon^{1/2} U \|_{\mathbf{H}} + M_4 \qquad \text{for } U \in \mathscr{D}(\mathbf{B}_\varepsilon^{1/2}) \tag{5.12}$$

for both boundary conditions and especially

$$\| \mathbf{B}_\varepsilon^{1/2} F_\varepsilon(0) \|_{\mathbf{H}} \leq M_4 \qquad \text{for the mixed boundary conditions.} \tag{5.13}$$

Let $\mu > max\{1 + M_1, 1 + 2M_3\}$. Then $q(t; p_0, 0)$ satifies the following integral equation:

$$q(t) = \int_0^t exp(-\mathbf{B}_\varepsilon(t-s))\mathbf{Q}_\mu^\varepsilon(\theta(\| p \|_{\mathbf{H}}^2)\mathbf{Q}_\mu^\varepsilon(F_\varepsilon(p(s), q(s))))\,ds. \tag{5.14}$$

We thus have

$$\| q(t) \|_{\mathbf{H}} \leq \int_0^t e^{-\mu(t-s)}(M_1 R_1 + M_1 \| q(s) \|_{\mathbf{H}} + |\Omega|^{1/2} M_2)\,ds, \qquad t \in [0, t_0]$$

and then by using the Grownwall inequality,

$$\| q(t) \|_{\mathbf{H}} \leq \frac{1}{\mu - M_1}(M_1 R_1 + |\Omega|^{1/2} M_2), \qquad t \in [0, t_0]. \tag{5.15}$$

Applying $\mathbf{B}_\varepsilon^{1/2}$ to the formula (5.14), we obtain

$$\| \mathbf{B}_\varepsilon^{1/2} q(t) \|_{\mathbf{H}} \leq \int_0^t \mu^{1/2} b(\mu(t-s))(M_1 R_1 + |\Omega|^{1/2} M_2 + M_1 \| q(s) \|_{\mathbf{H}})\,ds.$$

Substituting (5.15) into it, we have

$$\| \mathbf{B}_\varepsilon^{1/2} q(t) \|_{\mathbf{H}} \leq M_5 \mu^{-1/2}, \qquad t \in [0, t_0] \tag{5.16}$$

for some positive constant M_5. On the other hand, we know

$$\| \mathbf{B}_\varepsilon^{1/2} p(t) \|_{\mathbf{H}} \leq \mu^{1/2} R_1, \qquad t \in [0, t_0]. \tag{5.17}$$

When the Neumann boundary conditions are imposed, applying $\mathbf{B}_\varepsilon$ to the formula (5.14), we have

$$\mathbf{B}_\varepsilon q(t) = \int_0^t \mathbf{B}_\varepsilon^{1/2} exp(-\mathbf{B}_\varepsilon(t-s))\mathbf{B}_\varepsilon^{1/2}(\theta(\|p(s)\|_{\mathbf{H}}^2)\mathbf{Q}_\mu^\varepsilon(\tilde{F}_\varepsilon(p(s), q(s))))\, ds$$

and then by using (5.16) and (5.17)

$$\begin{aligned}\|\mathbf{B}_\varepsilon q(t)\|_{\mathbf{H}} &\le \int_0^t \mu^{1/2} b(\mu(t-s))[M_3\|\mathbf{B}_\varepsilon^{1/2}p(s)\|_{\mathbf{H}} + M_3\|\mathbf{B}_\varepsilon^{1/2}q(s)\|_{\mathbf{H}} + M_4]\, ds \\ &\le M_6 + M_7\mu^{-1/2}, \qquad t\in[0, t_0]\end{aligned}$$

for some positive constants M_6 and M_7. When the mixed boundary conditions are imposed, the above estimate can be similarly obtained by showing the additional term

$$\int_0^t \mathbf{B}_\varepsilon^{1/2} exp(-\mathbf{B}_\varepsilon(t-s))\mathbf{B}_\varepsilon^{1/2}(\theta(\|p(s)\|_{\mathbf{H}}^2)\mathbf{Q}_\mu^\varepsilon F_\varepsilon(0, 0))\, ds$$

is bounded by some constant indepedently of μ and t. Therefore, by choosing R_2 sufficiently large, we have

$$\|\mathbf{B}_\varepsilon q(t)\|_{\mathbf{H}} \le R_2, \qquad t\in[0, t_0].$$

LEMMA 5.6. (Linear Stability Condition) *For any* $R_1, R_2 > 0$ *and* $0<\varepsilon\le\varepsilon_4$, *if the constant* l *satisfies* $l \ge sup\{\|D\mathscr{G}_\varepsilon(p, q)\|_{\mathscr{L}((\mathbf{H}_\varepsilon(\Omega))^2,(\mathbf{H}_\varepsilon(\Omega))^2)}\,|(p, q)\in(\mathrm{A}^{\varepsilon,\mu}\times \mathrm{C}^{\varepsilon,\mu})\cap\bar{\mathbf{K}}_{\mathrm{II}}\}$, *then there exists* $\Lambda\ge 2l$ *such that* $\langle q, \mathbf{B}_\varepsilon q\rangle_{\mathbf{H}} \ge \Lambda\|q\|_{\mathbf{H}}^2$.

PROOF. Since

$$D\mathscr{G}_\varepsilon(p, q)(\rho, \sigma) = 2D\theta(\|p\|_{\mathbf{H}}^2)\langle p, \rho\rangle_{\mathbf{H}}\mathbf{Q}_\mu^\varepsilon F_\varepsilon(p, q) - \theta(\|p\|_{\mathbf{H}}^2)\mathbf{Q}_\mu^\varepsilon DF_\varepsilon(p, q)(\rho, \sigma),$$

we know that

$$\|D\mathscr{G}_\varepsilon(p, q)\|_{\mathscr{L}((\mathbf{H}_\varepsilon(\Omega))^2,(\mathbf{H}_\varepsilon(\Omega))^2)} \le 2sup(|\theta'|)R_1M_1(\bar{c}_1|\Omega|)^{1/2} + M_1c$$

for some positive constant c. On the other hand, we know

$$\langle q, \mathbf{B}_\varepsilon q\rangle_{\mathbf{H}} \ge \mu\|q\|_{\mathbf{H}}^2 \quad \text{for} \quad q\in\mathbf{W}_\mu^\varepsilon\cap\mathscr{D}(\mathbf{B}_\varepsilon) \quad \text{and} \quad 0<\varepsilon\le\varepsilon_4.$$

Thus, the proof can be shown by choosing μ to be sufficiently large.

Let ρ and σ satisfy the following equations:

$$(5.18)\qquad \begin{cases}\rho_t = D\mathscr{F}_\varepsilon(p, q)(\rho, \sigma), \\ \sigma_t = -\mathbf{B}_\varepsilon\sigma + D\mathscr{G}_\varepsilon(p, q)(\rho, \sigma)\end{cases}$$

and let $\mathbf{I} = \frac{1}{2}(\|\sigma\|_{\mathbf{H}}^2 - \|\rho\|_{\mathbf{H}}^2)$. It is known that $\mathbf{I}' = \langle\sigma, -\mathbf{B}_\varepsilon\sigma + D\mathscr{G}_\varepsilon(p, q)(\rho, \sigma)\rangle_{\mathbf{H}} - \langle\rho, D\mathscr{F}_\varepsilon(p, q)(\rho,\sigma)\rangle_{\mathbf{H}}$.

LEMMA 5.7. (Uniform Cone Condition) *For any* $R_2 > 0$ *and* $R_1 \geq 2\sqrt{K_4}\, max\{K_1 + 1, K_2 + 1\}$, *there exists a positive integer* N_1 *such that for* $0 < \varepsilon \leq \varepsilon_4(N_1)$, $(p, q) \in ((\mathrm{A}^{\varepsilon,\mu} \times \mathrm{C}^{\varepsilon,\mu}) \cup (\mathbf{V}_\mu^\varepsilon \times \{0\})) \cap \bar{\mathbf{K}}_{\mathrm{II}}$, $\rho \in \mathbf{V}_\mu^\varepsilon$ *and* $\sigma \in \mathbf{W}_\mu^\varepsilon \cap \mathscr{D}(\mathbf{B}_\varepsilon)$ satisfying (5.18), *if* $\|\rho\|_{\mathbf{H}} = \|\sigma\|_{\mathbf{H}} = 1$, *then we have* $\mathbf{I}' \leq -1$.

PROOF. We rewrite $\mathbf{I}'$ as

$$\mathbf{I}' = -\langle \sigma, \mathbf{B}_\varepsilon \sigma \rangle_{\mathbf{H}} + \langle \rho, \mathbf{B}_\varepsilon \rho \rangle_{\mathbf{H}} + H,$$

where

$$H = 2\theta'(\|p\|_{\mathbf{H}}^2)\langle p, \rho \rangle_{\mathbf{H}} \langle F_\varepsilon(p, q), \sigma - \rho \rangle_{\mathbf{H}} + \theta(\|p\|_{\mathbf{H}}^2)\langle \mathrm{D}F_\varepsilon(p, q)(\rho, \sigma), \sigma - \rho \rangle_{\mathbf{H}}.$$

By using the modification of $F_\varepsilon(p, q)$, we find $|H| \leq M_8$ for some constant $M_8 > 0$ independently of ε and μ. Thus we know that

$$\mathbf{I}' \leq -\mu_{\varepsilon,N_1+1} + \mu_{\varepsilon,N_1} + M_8.$$

Since Theorem 4.3 shows that there is a positive integer N_1 such that

$$\mu_{\varepsilon,N_1+1} - \mu_{\varepsilon,N_1} \geq M_8 + 1 \text{ for } 0 < \varepsilon \leq \varepsilon_4(N_1),$$

we immediately find that $\mathbf{I}' \leq -1$.

REMARK 5.8. More precisely speaking, it can be shown by Theorem 4.3 that there is a positive integer N_1 such that $\mathbf{I}' \leq -2M_8$ for $0 < \varepsilon \leq \varepsilon_4(N_1)$.

A similar argument can be done to the case when $\varepsilon = 0$. We only note that $\mathbf{V}_\mu^0$ and $\mathbf{W}_\mu^0$ are defined as $\mathbf{V}_\mu^0 = span\{\omega_1/\sqrt{|D|}, \cdots, \omega_{N_1}/\sqrt{|D|}\}$, $\mathbf{W}_\mu^0 = clspan\{\omega_j/\sqrt{|D|}\,|\, j > N_1\}$, respectively, and that (5.8) is modified by multiplying $\theta(\|\sqrt{|D|}p\|_{0,\mathbf{H}_0}^2)$ in place of $\theta(\|p\|_{\mathbf{H}}^2)$. That is, we may consider the following equation in place of (5.9):

$$(5.9)' \qquad \begin{cases} p_t = -\mathbf{B}_0 p + \theta(\|\sqrt{|D|}p\|_{0,\mathbf{H}_0}^2)\mathbf{P}_\mu^0(F_0(p, q)) \equiv \mathscr{F}_0(p, q), \\ q_t = -\mathbf{B}_0 q + \theta(\|\sqrt{|D|}p\|_{0,\mathbf{H}_0}^2)\mathbf{Q}_\mu^0(F_0(p, q)) \equiv -\mathbf{B}_0 q + \mathscr{G}_0(p, q). \end{cases}$$

The results similar to Lemmas 5.2 ~ 5.7 hold for (5.9)′.

By applying the result in Mallet-Paret and Sell [30] to the above, we find that the following existence theorem of inertial manifolds holds, where $\|\cdot\|$ means $\|\cdot\|_{\mathbf{H}}$ or $\|\cdot\|_{0,\mathbf{H}_0}$.

THEOREM 5.9. *There is a positive integer* N_1 *such that for* $0 \leq \varepsilon \leq \varepsilon_4(N_1)$, *there exists a* C^1*-function* $\Phi^\varepsilon \colon \mathbf{V}_\mu^\varepsilon \to \mathbf{W}_\mu^\varepsilon$ *with* $\|\mathrm{D}\Phi^\varepsilon\|_{\infty,\varepsilon} = sup\{\|\mathrm{D}\Phi^\varepsilon(p)\rho\|\,|\,p, \rho \in \mathbf{V}_\mu^\varepsilon, \|\rho\| \leq 1\} \leq 1$ *satisfying* $\Phi^\varepsilon(p) \in \mathrm{C}^{\varepsilon,\mu}$ *for* $p \in \mathbf{V}_\mu^\varepsilon$, $supp\, \Phi^\varepsilon \in \mathrm{A}^{\varepsilon,\mu}$, *and* $\mathfrak{M}^\varepsilon = graph\, \Phi^\varepsilon$ *is an invariant manifold of* (5.9) ((5.9)′ *if* $\varepsilon = 0$), *which is locally attracting in the sense that if* $U(t)$ *is the solution of* (5.9) ((5.9)′ *if* $\varepsilon = 0$) *and*

$U(t) \in E^{\varepsilon,\mu} = \{(p, q) \in \mathrm{A}^{\varepsilon,\mu} \times \mathrm{C}^{\varepsilon,\mu} \mid \|q\| \le dist(p, bodry\, \mathrm{A}^{\varepsilon,\mu})\}$, *then*

$$dist(U(t), \mathfrak{M}^{\varepsilon}) \le 2e^{-\kappa t} diam\, \mathrm{C}^{\varepsilon,\mu} \le cR_2 e^{-\kappa t} \qquad for \ \ t > 0,$$

where κ *and* c *are positive constants independent of* ε, μ *and* t.

REMARK 5.10. Let $(p(t), q(t))$ $(-\infty < t < \infty)$ be the solution of (5.9) or (5.9)′ for $0 \le \varepsilon \le \varepsilon_4(N_1)$. If $(p(t), q(t)) \in \mathrm{A}^{\varepsilon,\mu} \times \mathrm{C}^{\varepsilon,\mu}$ for all t, then $(p(t), q(t)) \in \mathfrak{M}^{\varepsilon}$ holds for all t. Especially, if $(\bar{p}, \bar{q})$ is a bounded equilibrium solution of (5.9) or (5.9)′, then $(\bar{p}, \bar{q}) \in \mathfrak{M}^{\varepsilon}$ for some appropriate constants μ, R_1 and R_2.

Let $\mathfrak{M}_0^{\varepsilon}$ and $\mathfrak{M}_t^{\varepsilon}$ denote respectively $\mathbf{V}_{\mu}^{\varepsilon}$ and the image of $\mathfrak{M}_0^{\varepsilon}$ under the flow of (5.9) (or (5.9)′) at time t. It is shown by Mallet-Paret and Sell [30] that there exists a Lipschitz continuous function $\Phi_t^{\varepsilon}: \mathbf{V}_{\mu}^{\varepsilon} \to \mathbf{W}_{\mu}^{\varepsilon}$ with the Lipschitz constant less than 1 such that $\mathfrak{M}_t^{\varepsilon} = graph\, \Phi_t^{\varepsilon}$. The following results which are obtained in [30] will be useful in the next section.

PROPOSITION 5.11.

(i) $\|\Phi^{\varepsilon}(p) - \Phi_t^{\varepsilon}(p)\|_{\mathbf{H}} \le \bar{M} R_2 e^{-\kappa t}$ *holds for* $t \ge 0$, $0 \le \varepsilon \le \varepsilon_4$ *and* $p \in \mathbf{V}_{\mu}^{\varepsilon}$, *where* $\bar{M}$ *is some positive constant independent of* ε *and* μ.

(ii) *Let* $\tau > 0$ *be a given small constant. Then for any* $\eta > 0$ *there is a positive integer* N_2 *such that for any integer* m, *it holds that* $\Phi_{(m+N_2)\tau}^{\varepsilon} \in C^1$ *and* $\|\mathrm{D}\Phi_{(m+N_2)\tau}^{\varepsilon}(p)\rho - \mathrm{D}\Phi^{\varepsilon}(p)\rho\|_{\mathbf{H}} \le \eta \|\rho\|_{\mathbf{H}}$ *for* $0 \le \varepsilon \le \varepsilon_4$ *and* $p, \rho \in \mathbf{V}_{\mu}^{\varepsilon}$.

§6. Reduced forms

We know in the previous section that (5.9) and (5.9)′ on the inertial manifolds $\mathfrak{M}^{\varepsilon}$ and $\mathfrak{M}^{0}$ are

$$\frac{dp}{dt} = \mathscr{F}_{\varepsilon}(p, \Phi^{\varepsilon}(p)) \qquad \text{for } p \in \mathbf{V}_{\mu}^{\varepsilon} \tag{6.1}$$

and

$$\frac{dp}{dt} = \mathscr{F}_{0}(p, \Phi^{0}(p)) \qquad \text{for } p \in \mathbf{V}_{\mu}^{0}, \tag{6.2}$$

respectively. Define $\pi_{\varepsilon}: \mathbf{V}_{\mu}^{\varepsilon} \to \mathbf{R}^{N_1}$ for $0 < \varepsilon \le \varepsilon_4(N_1)$ by $\pi_{\varepsilon}(p) = \beta = (\beta_1, \cdots, \beta_{N_1}) \in \mathbf{R}^{N_1}$ for $p = \sum_{j=1}^{N_1} \beta_j \omega_{\varepsilon,j}$, and $\pi_0: \mathbf{V}_{\mu}^{0} \to \mathbf{R}^{N_1}$ by $\pi_0(p) = \beta \in \mathbf{R}^{N_1}$ for $p = \sum_{j=1}^{N_1} \beta_j \frac{1}{\sqrt{|D|}} \omega_j$. By letting $\Phi^{*\varepsilon}(\beta) = \Phi^{\varepsilon}(\pi_{\varepsilon}^{-1}\beta)$ and $\Phi^{*0}(\beta) = \Phi^{0}(\pi_0^{-1}\beta)$, (6.1) and (6.2) can be written as

$$\frac{d\beta}{dt} = \pi_{\varepsilon}\mathscr{F}_{\varepsilon}(\pi_{\varepsilon}^{-1}\beta, \Phi^{*\varepsilon}(\beta)) \equiv \mathscr{F}_{\varepsilon}^{*}(\beta) \tag{6.3}$$

and

(6.4)
$$\frac{d\beta}{dt} = \pi_0 \mathscr{F}_0(\pi_0^{-1}\beta,\ \Phi^{*0}(\beta)) \equiv \mathscr{F}_0^*(\beta),$$

respectively. Let $\mathbf{B}_{r_0} = \{\beta \in \mathbf{R}^{N_1} \mid |\beta| \le r_0\}$, where $|\cdot|$ is the usual norm in $\mathbf{R}^{N_1}$. By using the similar argument to the one in Hale and Raugel [17], we will show the following result:

THEOREM 6.1. *There exists $r_0^* > 0$ such that for $r_0 \ge r_0^*$, $\Phi^{*\varepsilon} \to \Phi^{*0}$ in $C^1(\mathbf{B}_{r_0}; (L^2(\Omega))^2)$ as $\varepsilon \downarrow 0$.*

The above result implies that $\mathscr{F}_\varepsilon^* \to \mathscr{F}_0^*$ holds in $C^1(\mathbf{B}_{r_0}; \mathbf{R}^{N_1})$ as $\varepsilon \downarrow 0$.

PROOF OF THEOREM 6.1. The following two lemmas are obtained from Proposition 5.11 and Theorem 4.3:

LEMMA 6.2. *For any $\eta > 0$, there is $t_0 = t_0(\eta) > 0$ such that for $\beta, \tilde{\beta} \in \mathbf{R}^{N_1}$ and $0 < \varepsilon \le \varepsilon_4$,*

$$\|\Phi^{*\varepsilon}(\beta) - \Phi^{*0}(\beta)\|_{\mathbf{H}} \le \frac{\eta}{4} + \|\Phi_{t_0}^{*\varepsilon}(\beta) - \Phi_{t_0}^{*0}(\beta)\|_{\mathbf{H}}$$

and

$$\|\mathrm{D}\Phi^{*\varepsilon}(\beta)\tilde{\beta} - \mathrm{D}\Phi^{*0}(\beta)\tilde{\beta}\|_{\mathbf{H}} \le \frac{\eta}{8}|\tilde{\beta}| + \|\mathrm{D}\Phi_{t_0}^{*\varepsilon}(\beta)\tilde{\beta} - \mathrm{D}\Phi_{t_0}^{*0}(\beta)\tilde{\beta}\|_{\mathbf{H}}.$$

LEMMA 6.3. *There exists a constant $\bar{c}_8 > 0$ such that for $0 < \varepsilon \le \varepsilon_4$ and $U \in (\mathbf{H}_0(0, L))^2$,*

$$\|\mathbf{P}_\mu^\varepsilon U - \mathbf{P}_\mu^0 U\|_{\mathbf{H}} \le \bar{c}_8 \varepsilon\mu \|U\|_{0,\mathbf{H}_0}.$$

From now on, we will omit μ from superscripts and subscripts for simplicity only. Let us introduce the following notations: For $\beta \in \mathbf{R}^{N_1}$, let $U^\varepsilon(t; \beta) = (p^\varepsilon(t; \beta), q^\varepsilon(t; \beta))$ be the solution of (5.9) or (5.9)′ with the initial value $U^\varepsilon(0; \beta) = (\pi_\varepsilon^{-1}\beta, 0)$ for $0 \le \varepsilon \le \varepsilon_4$. For $t_0 > 0$, which was stated in Lemma 6.2, there exists $\delta^\varepsilon(\beta) \in \mathbf{R}^{N_1}$ such that $U^\varepsilon(t_0; \delta^\varepsilon(\beta)) = (\pi_\varepsilon^{-1}\beta, \Phi_{t_0}^{*\varepsilon}(\beta))$ $(=(\pi_\varepsilon^{-1}\beta, \Phi_{t_0}^\varepsilon(\pi_\varepsilon^{-1}\beta)))$. For simplicity, we write $U^\varepsilon(t; \delta^\varepsilon(\beta))$, $p^\varepsilon(t; \delta^\varepsilon(\beta))$, $q^\varepsilon(t; \delta^\varepsilon(\beta))$ as $U^\varepsilon(t)$, $p^\varepsilon(t)$, $q^\varepsilon(t)$ and $U^0(t; \delta^\varepsilon(\beta))$, $p^0(t; \delta^\varepsilon(\beta))$, $q^0(t; \delta^\varepsilon(\beta))$ as $U^0(t)$, $p^0(t)$, $q^0(t)$, respectively.

LEMMA 6.4. *Let $0 < \varepsilon \le \varepsilon_4$. If $\beta \in \mathbf{B}_{r_0}$, then*

$$\max\{\|\mathbf{B}_\varepsilon U^\varepsilon(t)\|_{\mathbf{H}}, \|\mathbf{B}_0 U^0(t)\|_{0,\mathbf{H}_0}\} \le \max\{2\mu e^{\mu t_0} r_0, \mu R_1 + R_2\} \quad \text{for } t \ge 0.$$

PROOF. It there is $t_1 \ge 0$ such that

$$\| \mathbf{B}_\varepsilon U^\varepsilon(t_1) \|_{\mathbf{H}} > max\{2\mu e^{\mu t_0} r_0,\ \mu R_1 + R_2\} > \mu R_1 + R_2,$$

it follows from

$$\| \mathbf{B}_\varepsilon U^\varepsilon(t_1) \|_{\mathbf{H}} = \| \mathbf{B}_\varepsilon p^\varepsilon(t_1) + \mathbf{B}_\varepsilon q^\varepsilon(t_1) \|_{\mathbf{H}} \leq \mu \| p^\varepsilon(t_1) \|_{\mathbf{H}} + \| \mathbf{B}_\varepsilon q^\varepsilon(t_1) \|_{\mathbf{H}}$$

that

$$\| p^\varepsilon(t_1) \|_{\mathbf{H}} > R_1 \quad \text{or} \quad \| \mathbf{B}_\varepsilon q^\varepsilon(t_1) \|_{\mathbf{H}} > R_2.$$

In the former case, since $R_1 \geq 2\sqrt{K_4}\, max\{K_1 + 1,\ K_2 + 1\}$, U^ε satisfies

$$U_t = -\mathbf{B}_\varepsilon U$$

and then

$$(p^\varepsilon(t))_t = -\mathbf{B}_\varepsilon p^\varepsilon(t),$$

if t is close to t_1 but $t \leq t_1$. It implies that

$$\| p^\varepsilon(t) \|_{\mathbf{H}} \geq e^{-\mu_0(t - t_1)} \| p^\varepsilon(t_1) \|_{\mathbf{H}} \geq \| p^\varepsilon(t_1) \|_{\mathbf{H}} > R_1$$

for some positive constant $\mu_0 > 0$ as long as $U^\varepsilon_t = -\mathbf{B}_\varepsilon U^\varepsilon$ holds. On the other hand, $\| p^\varepsilon(\tilde{t}) \|_{\mathbf{H}} > R_1$ implies $U^\varepsilon_t = -\mathbf{B}_\varepsilon U^\varepsilon$, if t closes to $\tilde{t}$ but $t \leq \tilde{t}$. Therefore, we have

$$U^\varepsilon_t = -\mathbf{B}_\varepsilon U^\varepsilon \qquad \text{for } 0 \leq t \leq t_1.$$

Thus it follows from the initial value $q^\varepsilon(0, \delta^\varepsilon(\beta)) = 0$ that

$$q^\varepsilon(t) = 0 \qquad \text{for } 0 \leq t \leq t_1$$

and

$$\| \pi_\varepsilon^{-1} \delta^\varepsilon(\beta) \|_{\mathbf{H}} = \| p^\varepsilon(0) \|_{\mathbf{H}} \geq \| p^\varepsilon(t_1) \|_{\mathbf{H}} \geq \frac{1}{\mu} \| \mathbf{B}_\varepsilon p^\varepsilon(t_1) \|_{\mathbf{H}}$$

$$= \frac{1}{\mu} \| \mathbf{B}_\varepsilon U^\varepsilon(t_1) \|_{\mathbf{H}} > 2 e^{\mu t_0} r_0.$$

That is, if r_0 is chosen sufficiently large, we find

$$\| \pi_\varepsilon^{-1} \delta^\varepsilon(\beta) \|_{\mathbf{H}} > 2 e^{\mu t_0} r_0 \geq max\{R_1, |\beta|\} 2 e^{\mu t_0}. \tag{6.5}$$

We note that since $R_1 > 2\sqrt{K_4}\, max\{K_1 + 1,\ K_2 + 1\}$, $U^\varepsilon(t)$ satisfies

$$U_t = -\mathbf{B}_\varepsilon U$$

as long as $\| p^\varepsilon(t) \|_{\mathbf{H}} > R_1$, and when it holds, we find that $q^\varepsilon(t) = 0$ and

$$\| p^\varepsilon(t) \|_{\mathbf{H}} \geq e^{-\mu t} \| \pi_\varepsilon^{-1} \delta^\varepsilon(\beta) \|_{\mathbf{H}}.$$

Since (6.5) leads to $\|p^\varepsilon(t_0)\|_{\mathbf{H}} \geq e^{-\mu t_0}\|\pi_\varepsilon^{-1}\delta^\varepsilon(\beta)\|_{\mathbf{H}} \geq 2R_1$, we have

$$q^\varepsilon(t)=0 \quad \text{and} \quad \|p^\varepsilon(t)\|_{\mathbf{H}} \geq e^{-\mu t}\|\pi_\varepsilon^{-1}\delta^\varepsilon(\beta)\|_{\mathbf{H}} \qquad \text{for } 0\leq t\leq t_0.$$

Especially, when $t=t_0$, we know by the above inequality and (6.5) that

$$\|p^\varepsilon(t_0)\|_{\mathbf{H}} \geq e^{-\mu t_0}\|\pi_\varepsilon^{-1}\delta^\varepsilon(\beta)\|_{\mathbf{H}} \geq 2|\beta|.$$

This is a contradiction to $\|p^\varepsilon(t_0)\|_{\mathbf{H}} = \|\pi_\varepsilon^{-1}\beta\|_{\mathbf{H}} = |\beta|$.

The latter case does not occur because $\|\mathbf{B}_\varepsilon q^\varepsilon(t_1)\|_{\mathbf{H}} \leq R_2$ holds for $\|p^\varepsilon(t_1)\|_{\mathbf{H}} \leq R_1$ by Lemma 5.5.

As the case when $\varepsilon=0$ can be studied similarly, we omit its discussion.

REMARK 6.5. $\mathbf{B}_\varepsilon^{-1}$ and $\mathbf{B}_0^{-1}$ can be defined in a similar way to $(\mathbf{S}_\varepsilon^i)^{-1}$ and $(\mathbf{S}_0^i)^{-1}$ $(i=1,2)$ and by Lemma 4.5, $\mathbf{B}_\varepsilon^{-1}$ and $\mathbf{B}_0^{-1}$ are respectively uniformly bounded in $\mathscr{L}((\mathbf{H}_\varepsilon(\Omega))^2;(H^2(\Omega))^2)$ and $\mathscr{L}((\mathbf{H}_0(0,L))^2;(H^2(0,L))^2)$ for $0<\varepsilon\leq\varepsilon_6$. It is thus shown that they are compact operators from $\mathbf{X}_\varepsilon$ and $\mathbf{X}_0$ into $\mathbf{X}_\varepsilon$ and $\mathbf{X}_0$, respectively. Together with Lemma 6.4, we have

$$\max\{\|U^\varepsilon(t)\|_{\mathbf{H}}, \|U^0(t)\|_{0,\mathbf{H}_0}\} \leq \bar{c}_9 \qquad \text{for } t\geq 0 \tag{6.6}$$

for some positive constant $\bar{c}_9$ which may depend on t_0, μ and r_0.

LEMMA 6.6. *Let* $0<\varepsilon\leq\varepsilon_4$. *If* $\beta\in\mathbf{B}_{r_0}$, *then*

$$\|U^\varepsilon(t)-U^0(t)\|_{\mathbf{H}} \leq \bar{c}_{10}\sqrt{\varepsilon}e^{\bar{c}_{11}t} \qquad \textit{holds for } 0\leq t\leq t_0,$$

where $\bar{c}_{10}$ *and* $\bar{c}_{11}$ *are positive constants which may depend on* t_0, μ *and* r_0.

PROOF. Note that $U^0(t)$ satisfies

$$\frac{dU^0}{dt} = -\mathbf{B}_0U^0 + \theta(\|\sqrt{|D|}p^0\|_{0,\mathbf{H}_0}^2)F_0(U^0).$$

Taking the inner product with $h\in(H^1(0,L))^2$, we have

$$\left\langle\frac{dU^0}{dt}, h\right\rangle_{0,\mathbf{H}_0} = -\mathbf{b}_0(U^0,h) + \langle\theta(\|\sqrt{|D|}p^0\|_{0,\mathbf{H}_0}^2)F_0(U^0), h\rangle_{0,\mathbf{H}_0}. \tag{6.7}$$

Letting $h=\int_D w(s,z)\,ds$ for $w\in(H^1(\Omega))^2$ and substituting it into (6.7), we have

$$\left\langle\frac{dU^0}{dt}, w\right\rangle_{\mathbf{H}} + \mathbf{b}_\varepsilon(U^0,w) = \langle\theta(\|\sqrt{|D|}p^0\|_{0,\mathbf{H}_0}^2)F_0(U^0), w\rangle_{\mathbf{H}}$$

$$-d_1\sum_{j=1}^{N_1}\left\langle\frac{r'(s)\partial}{r(s)\partial s}(U^0)_1, z_j\frac{\partial}{\partial z_j}(w)_1\right\rangle_L - d_2\sum_{j=1}^{N_1}\left\langle\frac{r'(s)\partial}{r(s)\partial s}(U^0)_2, z_j\frac{\partial}{\partial z_j}(w)_2\right\rangle_L,$$

where $U^0 = ((U^0)_1, (U^0)_2)$ and $w = ((w)_1, (w)_2)$. On the other hand, using

$$\left\langle \frac{dU^\varepsilon}{dt}, w \right\rangle_{\mathbf{H}} + \mathbf{b}_\varepsilon(U^\varepsilon, w) = \langle \theta(\| p^\varepsilon \|_{\mathbf{H}}^2) F_\varepsilon(U^\varepsilon), w \rangle_{\mathbf{H}},$$

we have

$$\begin{aligned}
&\left\langle \frac{d}{dt}(U^\varepsilon - U^0), w \right\rangle_{\mathbf{H}} + \mathbf{b}_\varepsilon(U^\varepsilon - U^0, w) \\
&= \langle \theta(\| p^\varepsilon \|_{\mathbf{H}}^2) F_\varepsilon(U^\varepsilon) - \theta(\| \sqrt{|D|} p^0 \|_{0,\mathbf{H}_0}^2) F_0(U^0), w \rangle_{\mathbf{H}} \\
&+ d_1 \textstyle\sum_{j=1}^{N_1} \left\langle \dfrac{r'(s)\partial}{r(s)\partial s}(U^0)_1,\ z_j \dfrac{\partial}{\partial z_j}(w)_1 \right\rangle_L \\
&+ d_2 \textstyle\sum_{j=1}^{N_1} \left\langle \dfrac{r'(s)\partial}{r(s)\partial s}(U^0)_2,\ z_j \dfrac{\partial}{\partial z_j}(w)_2 \right\rangle_L .
\end{aligned}$$

Putting $w = U^\varepsilon - U^0$ in the above formula, we obtain

$$\begin{aligned}
(6.8) \qquad \frac{d}{dt} \| U^\varepsilon - U^0 \|_{\mathbf{H}}^2 \le c_1 \{ &\| U^\varepsilon - U^0 \|_{\mathbf{H}}^2 \\
&+ \| U^\varepsilon - U^0 \|_{\mathbf{H}} | \| p^\varepsilon \|_{\mathbf{H}}^2 - \| \sqrt{|D|} p^0 \|_{0,\mathbf{H}_0}^2 | + c_2 \varepsilon \}
\end{aligned}$$

for some positive constants c_1 and c_2. Here we know

$$| \| p^\varepsilon \|_{\mathbf{H}}^2 - \| \sqrt{|D|} p^0 \|_{0,\mathbf{H}_0}^2 | \le \| p^\varepsilon - p^0 \|_{\mathbf{H}} (\| p^\varepsilon \|_{\mathbf{H}} + \| \sqrt{|D|} p^0 \|_{0,\mathbf{H}_0})$$

and by Lemma 6.3

$$\begin{aligned}
\| p^\varepsilon(t; \delta^\varepsilon(\beta)) - p^0(t; \delta^\varepsilon(\beta)) \|_{\mathbf{H}} &\le \| \mathbf{P}^\varepsilon U^0(t; \delta^\varepsilon(\beta)) - \mathbf{P}^0 U^0(t; \delta^\varepsilon(\beta)) \|_{\mathbf{H}} \\
&+ \| \mathbf{P}^\varepsilon U^\varepsilon(t; \delta^\varepsilon(\beta)) - \mathbf{P}^\varepsilon U^0(t; \delta^\varepsilon(\beta)) \|_{\mathbf{H}} \\
&\le c_3 \varepsilon\mu \| U^0(t; \delta^\varepsilon(\beta)) \|_{0,\mathbf{H}_0} + \| U^\varepsilon(t) - U^0(t) \|_{\mathbf{H}}
\end{aligned}$$

for some positive constant c_3. By using Remark 6.5, we thus obtain

$$| \| p^\varepsilon \|_{\mathbf{H}}^2 - \| \sqrt{|D|} p^0 \|_{0,\mathbf{H}_0}^2 | \le c_4 (\varepsilon\mu + \| U^\varepsilon - U^0 \|_{\mathbf{H}})$$

for some positive constant c_4. Substituting it into (6.8), we have

$$(6.9) \qquad \frac{d}{dt} \| U^\varepsilon - U^0 \|_{\mathbf{H}}^2 \le c_5 \varepsilon + c_6 \| U^\varepsilon - U^0 \|_{\mathbf{H}}^2$$

for some positive constants c_5 and c_6. Since Theorem 4.3 indicates that $U^\varepsilon(0; \delta^\varepsilon(\beta)) = \pi_\varepsilon^{-1} \delta^\varepsilon(\beta)$, $U^0(0; \delta^\varepsilon(\beta)) = \pi_0^{-1} \delta^\varepsilon(\beta)$ and $\| U^\varepsilon(0) - U^0(0) \|_{\mathbf{H}} \le c_7 \varepsilon$ for some positive constant c_7, it follows from (6.9) and the Grownwall inequality that there are $\bar{c}_{10}, \bar{c}_{11} > 0$ such that Lemma 6.6 holds.

LEMMA 6.7. *$\Phi^{*\varepsilon} \to \Phi^{*0}$ in $C(\mathbf{B}_{r_0}, (L^2(\Omega))^2)$ as $\varepsilon \downarrow 0$.*

PROOF. Since $\Phi_{t_0}^0$ has a Lipschitz constant less than 1, we know

$$\begin{aligned}\|\Phi_{t_0}^{*\varepsilon}(\beta) - \Phi_{t_0}^{*0}(\beta)\|_{\mathbf{H}} &\le \|\Phi_{t_0}^{*\varepsilon}(\pi_\varepsilon p^\varepsilon(t_0; \delta^\varepsilon(\beta))) - \Phi_{t_0}^{*0}(\pi_0 p^0(t_0; \delta^\varepsilon(\beta)))\|_{\mathbf{H}} \\ &+ \|\Phi_{t_0}^{*0}(\beta) - \Phi_{t_0}^{*0}(\pi_0 p^0(t_0; \delta^\varepsilon(\beta)))\|_{\mathbf{H}} \\ &\le \|q^\varepsilon(t_0; \delta^\varepsilon(\beta)) - q^0(t_0; \delta^\varepsilon(\beta))\|_{\mathbf{H}} + \|\pi_0^{-1}\beta - p^0(t_0; \delta^\varepsilon(\beta))\|_{\mathbf{H}} \\ &\le \|U^\varepsilon(t_0; \delta^\varepsilon(\beta)) - U^0(t_0; \delta^\varepsilon(\beta))\|_{\mathbf{H}} + c\left\{\sum_{j=1}^{N_1} |\beta_j| \left\|\omega_{\varepsilon,j} - \frac{1}{\sqrt{|D|}}\omega_j\right\|_{\mathbf{H}}\right. \\ &\left. + \|p^\varepsilon(t_0; \delta^\varepsilon(\beta)) - p^0(t_0; \delta^\varepsilon(\beta))\|_{\mathbf{H}}\right\}\end{aligned}$$

for some positive constant c. On the other hand, it follows that

$$\begin{aligned}\|p^\varepsilon(t_0; \delta^\varepsilon(\beta)) - p^0(t_0; \delta^\varepsilon(\beta))\|_{\mathbf{H}} &\le c'\varepsilon\mu\|U^0(t_0; \delta^\varepsilon(\beta))\|_{0,\mathbf{H}_0} \\ &+ \|U^\varepsilon(t_0) - U^0(t_0)\|_{\mathbf{H}}\end{aligned}$$

for some positive constant c'. We thus obtain

$$\|\Phi_{t_0}^{*\varepsilon}(\beta) - \Phi_{t_0}^{*0}(\beta)\|_{\mathbf{H}} \le c_1\varepsilon\mu(|\beta| + \|U^0(t_0)\|_{0,\mathbf{H}_0}) + c_2\|U^\varepsilon(t_0) - U^0(t_0)\|$$

for some positive constants c_1 and c_2. Therefore, by using Lemma 6.2, Remark 6.5 and Lemma 6.6, the conclusion holds.

We next show that $\Phi^{*\varepsilon} \to \Phi^{*0}$ in $C^1(\mathbf{B}_{r_0}, (L^2(\Omega))^2)$ as $\varepsilon \downarrow 0$.

LEMMA 6.8. *For any $\eta > 0$, there are $t_0 = t_0(\eta) > 0$ and $\varepsilon_7(N_1) > 0$ such that for $0 < \varepsilon \le \varepsilon_7(N_1)$,*

$$\begin{aligned}\|D\Phi^{*\varepsilon}(\beta)\tilde{\beta} - \mathrm{D}\Phi^{*0}(\beta)\tilde{\beta}\|_{\mathbf{H}} &\le \frac{\eta}{4}|\tilde{\beta}| \\ &+ \|\mathrm{D}\Phi_{t_0}^{*\varepsilon}(\pi_\varepsilon p^\varepsilon(t_0; \delta^\varepsilon(\beta)))\tilde{\beta} - \mathrm{D}\Phi_{t_0}^{*0}(\pi_0 p^0(t_0; \delta^\varepsilon(\beta)))\tilde{\beta}\|_{\mathbf{H}}\end{aligned}$$

for $\beta, \tilde{\beta} \in \mathbf{R}^{N_1}$, $\beta \in \mathbf{B}_{r_0}$.

PROOF. Since $\Phi_{t_0}^{*0}$ belongs to C^1, we find that $\mathrm{D}\Phi_{t_0}^{*0}$ is uniformly continuous from $\mathbf{B}_{2r_0}$ into $\mathscr{L}(\mathbf{R}^{N_1}, \mathbf{W}^0)$. Thus for any $\eta > 0$, there is $\nu = \nu(\eta) > 0$ such that if $\beta^1, \beta^2 \in \mathbf{B}_{2r_0}$ and $|\beta^1 - \beta^2| \le \nu$, then

$$\|\mathrm{D}\Phi_{t_0}^{*\varepsilon}(\beta^1) - \mathrm{D}\Phi_{t_0}^{*0}(\beta^2)\|_{\mathscr{L}(\mathbf{R}^{N_1}, \mathbf{W}^0)} \le \frac{\eta}{8} \tag{6.10}$$

holds. On the other hand, noting by Theorem 4.3, Lemmas 6.6 and 6.7 that there are positive constants c_1, c_2 and c_3 such that

$$|\beta - \pi_0 p^0(t_0; \delta^\varepsilon(\beta))| = \|\pi_0^{-1}\beta - p^0(t_0; \delta^\varepsilon(\beta))\|_{0,\mathbf{H}_0}$$
$$\leq c_1 \|\pi_\varepsilon^{-1}\beta - \pi_0^{-1}\beta\|_{\mathbf{H}} + c_2 \|p^\varepsilon(t_0; \delta^\varepsilon(\beta)) - p^0(t_0; \delta^\varepsilon(\beta))\|_{\mathbf{H}}$$
$$\leq c_3\sqrt{\varepsilon} \qquad \text{for } \beta \in \mathbf{B}_{r_0},$$

we can choose $\varepsilon_7(N_1)$ sufficiently small such that

$$|\beta - \pi_0 p^0(t_0; \delta^\varepsilon(\beta))| \leq \nu.$$

The conclusion follows immediately from (6.10) and Lemma 6.2.

In order to estimate $\|D\Phi_{t_0}^{*\varepsilon}(\pi_\varepsilon p^\varepsilon(t_0; \delta^\varepsilon(\beta)))\tilde{\beta} - D\Phi_{t_0}^{*0}(\pi_0 p^0(t_0; \delta^\varepsilon(\beta)))\tilde{\beta}\|_{\mathbf{H}}$, we will give the precise expressions of $D\Phi_{t_0}^{*\varepsilon}$ and $D\Phi_{t_0}^{*0}$. We note that $\tilde{\rho}(t) = \frac{\partial}{\partial\beta} p^\varepsilon(t; \beta)\tilde{\beta}$ and $\tilde{\sigma}(t) = \frac{\partial}{\partial\beta} q^\varepsilon(t; \beta)\tilde{\beta}$ satisfy the equations

$$(6.11) \qquad \begin{cases} \rho_t = D\mathscr{F}_\varepsilon(p^\varepsilon(t; \beta),\ q^\varepsilon(t; \beta))(\rho, \sigma) \\ \sigma_t = -\mathbf{B}_\varepsilon\sigma + D\mathscr{G}_\varepsilon(p^\varepsilon(t; \beta),\ q^\varepsilon(t; \beta))(\rho, \sigma) \end{cases}$$

with the initial data $\tilde{\rho}(0) = \pi_\varepsilon^{-1}\tilde{\beta}$ and $\tilde{\sigma}(0) = 0$, and that the backward uniqueness of solutions of (6.11) holds. Let us show that $\frac{\partial}{\partial\beta} p^\varepsilon(t; \beta) \in \mathscr{L}(\mathbf{R}^{N_1}, \mathbf{V}^\varepsilon)$ is invertible for any $t > 0$. In fact, if there is $t_1 > 0$ such that $\frac{\partial}{\partial\beta} p^\varepsilon(t_1; \beta)\tilde{\beta} = 0$, since $\tilde{\sigma}(0) = 0$, $\tilde{\rho}(0) = \pi_\varepsilon^{-1}\beta$ which imply $\|\tilde{\sigma}(0)\|_{\mathbf{H}} \leq \|\tilde{\rho}(0)\|_{\mathbf{H}}$, we have by Lemma 5.7 that $\|\tilde{\sigma}(t_1)\|_{\mathbf{H}} \leq \|\tilde{\rho}(t_1)\|_{\mathbf{H}}$ which implies $\tilde{\sigma}(t_1) = 0$. The uniqueness of solutions of (6.11) shows that $\tilde{\rho}(t) = \tilde{\sigma}(t) = 0$ in $[0, t_1]$. Especially, $\tilde{\rho}(0) = \pi_\varepsilon^{-1}\tilde{\beta} = 0$ gives $\tilde{\beta} = 0$. It turns out that $\frac{\partial}{\partial\beta} p^\varepsilon(t; \beta)$ is invertible. Since $q^\varepsilon(t'; \beta) = \Phi_{t'}^{*\varepsilon}(\pi_\varepsilon p^\varepsilon(t'; \beta))$ for $t' > 0$, we have

$$(6.12) \quad D\Phi_{t'}^{*\varepsilon}(\pi_\varepsilon p^\varepsilon(t'; \beta)) = \left[\frac{\partial}{\partial\beta} q^\varepsilon(t'; \beta)\right]\left[\frac{\partial}{\partial\beta} p^\varepsilon(t'; \beta)\right]^{-1} \pi_\varepsilon^{-1} \qquad \text{for } \varepsilon \geq 0.$$

We simply write $p^\varepsilon(t; \delta^\varepsilon(\beta))$, $q^\varepsilon(t; \delta^\varepsilon(\beta))$ and $p^0(t; \delta^\varepsilon(\beta))$, $q^0(t; \delta^\varepsilon(\beta))$ as $p^\varepsilon(t)$, $q^\varepsilon(t)$ and $p^0(t)$, $q^0(t)$ as before. Let $(\rho^\varepsilon(t; \tilde{\beta}), \sigma^\varepsilon(t; \tilde{\beta}))$ be the solution of the equations

$$(6.13) \qquad \begin{cases} \rho_t = D\mathscr{F}_\varepsilon(p^\varepsilon(t), q^\varepsilon(t))(p, \sigma) \\ \sigma_t = -\mathbf{B}_\varepsilon\sigma + D\mathscr{G}_\varepsilon(p^\varepsilon(t), q^\varepsilon(t))(\rho, \sigma) \end{cases}$$

with the initial values

$$\sigma^\varepsilon(0; \tilde{\beta}) = 0,\ \rho^\varepsilon(0; \tilde{\beta}) = \pi_\varepsilon^{-1}\left[\frac{\partial}{\partial\gamma} p^\varepsilon(t_0; \gamma)\right]_{\gamma = \delta^\varepsilon(\beta)}^{-1} \pi_\varepsilon^{-1}\tilde{\beta} \qquad \text{for } \varepsilon > 0$$

and

$$\sigma^0(0;\tilde{\beta}) = 0,\ \rho^0(0;\tilde{\beta}) = \pi_0^{-1}\left[\frac{\partial}{\partial\gamma}p^0(t_0;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}^{-1}\pi_0^{-1}\tilde{\beta}.$$

We know that

$$\rho^\varepsilon(t;\tilde{\beta}) = \left[\frac{\partial}{\partial\gamma}p^\varepsilon(t;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}\left[\frac{\partial}{\partial\gamma}p^\varepsilon(t_0;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}^{-1}\pi_\varepsilon^{-1}\tilde{\beta},$$

$$\sigma^\varepsilon(t;\tilde{\beta}) = \left[\frac{\partial}{\partial\gamma}q^\varepsilon(t;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}\left[\frac{\partial}{\partial\gamma}p^\varepsilon(t_0;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}^{-1}\pi_\varepsilon^{-1}\tilde{\beta}$$

and

$$\rho^0(t;\tilde{\beta}) = \left[\frac{\partial}{\partial\gamma}p^0(t;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}\left[\frac{\partial}{\partial\gamma}p^0(t_0;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}^{-1}\pi_0^{-1}\tilde{\beta},$$

$$\sigma^0(t;\tilde{\beta}) = \left[\frac{\partial}{\partial\gamma}q^0(t;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}\left[\frac{\partial}{\partial\gamma}p^0(t_0;\gamma)\right]_{\gamma=\delta^\varepsilon(\beta)}^{-1}\pi_0^{-1}\tilde{\beta}.$$

Therefore, it follows from the above formula and (6.12) that

$$\rho^\varepsilon(t_0;\tilde{\beta}) = \pi_\varepsilon^{-1}\tilde{\beta} \qquad \text{for } \varepsilon \geq 0,$$

$$\sigma^\varepsilon(t_0;\tilde{\beta}) = \mathrm{D}\Phi_{t_0}^{*\varepsilon}(\pi_\varepsilon p^\varepsilon(t_0;\delta^\varepsilon(\beta)))\tilde{\beta} \qquad \text{for } \varepsilon > 0$$

and

$$\sigma^0(t_0;\tilde{\beta}) = \mathrm{D}\Phi_{t_0}^{*0}(\pi_0 p^0(t_0;\delta^\varepsilon(\beta)))\tilde{\beta}.$$

It is sufficient to estimate $\|\sigma^\varepsilon(t_0;\tilde{\beta}) - \sigma^0(t_0;\tilde{\beta})\|_{\mathbf{H}}$ in order to estimate $\|\mathrm{D}\Phi_{t_0}^{*\varepsilon}(\pi_\varepsilon p^\varepsilon(t_0;\delta^\varepsilon(\beta)))\tilde{\beta} - \mathrm{D}\Phi_{t_0}^{*0}(\pi_0 p^0(t_0;\delta^\varepsilon(\beta)))\tilde{\beta}\|_{\mathbf{H}}$.

LEMMA 6.9. *Let* $|\tilde{\beta}| \leq 1$. *Then for* $0 \leq t \leq t_0$ *and* $0 \leq \varepsilon \leq \varepsilon_7$,

(6.14) $$\|\sigma^\varepsilon(t;\tilde{\beta})\|_{\mathbf{H}} \leq \|\rho^\varepsilon(t;\tilde{\beta})\|_{\mathbf{H}} \leq e^{(\mu+c_1')t_0},$$

(6.15) $$\|\rho^\varepsilon(t;\tilde{\beta})\|_{\varepsilon,\mathbf{H}} \leq \sqrt{\mu}e^{(\mu+c_1')t_0}$$

and

(6.16) $$\|\sigma^\varepsilon(t;\tilde{\beta})\|_{\varepsilon,\mathbf{H}} \leq c_2' e^{(\mu+c_1')t_0}$$

hold for some positive constants c_1' *and* c_2'.

PROOF. Since $\sigma^\varepsilon(0,\tilde{\beta}) = 0$, $\rho^\varepsilon(0;\tilde{\beta}) \neq 0$ so that $\|\sigma^\varepsilon(0;\tilde{\beta})\|_{\mathbf{H}} \leq \|\rho^\varepsilon(0;\tilde{\beta})\|_{\mathbf{H}}$, we have by Lemma 5.7 that $\|\sigma^\varepsilon(t;\tilde{\beta})\|_{\mathbf{H}} \leq \|\rho^\varepsilon(t;\tilde{\beta})\|_{\mathbf{H}}$. A similar argument to the one in Mallet-Paret and Sell [30] shows that

$$\|\rho^\varepsilon(t;\tilde{\beta})\|_{\mathbf{H}} \leq \|\rho^\varepsilon(t_0;\tilde{\beta})\|_{\mathbf{H}} e^{(\mu+c_1')t_0} \leq e^{(\mu+c_1')t_0} \qquad \text{in } [0, t_0]$$

for some positive constant c_1', which implies that (6.14) holds. (6.15) is obvious since $\| \mathbf{B}_\varepsilon^{1/2} \rho^\varepsilon(t; \tilde{\beta}) \|_{\mathbf{H}} \le \sqrt{\mu} \| \rho^\varepsilon(t; \tilde{\beta}) \|_{\mathbf{H}}$ holds. (6.16) immediately follows from the Grownwall inequality by taking the inner product of the second equation of (6.13) with $\mathbf{B}_\varepsilon \sigma^\varepsilon(t; \tilde{\beta})$.

LEMMA 6.10. *Let* $|\tilde{\beta}| \le 1$. *Then for* $0 < \varepsilon \le \varepsilon_7$,

$$\| \sigma^\varepsilon(t_0; \tilde{\beta}) - \sigma^0(t_0; \tilde{\beta}) \|_{\mathbf{H}} \le c_3' \varepsilon^{\kappa_1} \tag{6.17}$$

holds for some positive constants c_3' *and* κ_1.

PROOF. Let

$$\begin{aligned} \psi^0(t) &= \rho^0(t; \tilde{\beta}) + \sigma^0(t; \tilde{\beta}), \\ \bar{\rho}(t) &= \rho^\varepsilon(t; \tilde{\beta}) - \mathbf{P}^\varepsilon \psi^0(t), \\ \bar{\sigma}(t) &= \sigma^\varepsilon(t; \tilde{\beta}) - \mathbf{Q}^\varepsilon \psi^0(t). \end{aligned}$$

We know that

$$\| \sigma^\varepsilon(t_0; \tilde{\beta}) - \sigma^0(t_0; \tilde{\beta}) \|_{\mathbf{H}} \le \| \bar{\sigma}(t) \|_{\mathbf{H}} + \| (\mathbf{P}^\varepsilon - \mathbf{P}^0) \psi^0(t) \|_{\mathbf{H}}$$

and by Lemma 6.3 and (6.14) that

$$\| \sigma^\varepsilon(t_0; \tilde{\beta}) - \sigma^0(t_0; \tilde{\beta}) \|_{\mathbf{H}} \le c_1 \varepsilon + \| \bar{\sigma}(t_0) \|_{\mathbf{H}} \tag{6.18}$$

for some positive constant c_1. Thus it is sufficient to estimate $\| \bar{\sigma}(t_0) \|_{\mathbf{H}}$. Note by Theorem 4.3, Lemma 6.3 and (6.14) that

$$\begin{aligned} \| \bar{\rho}(t_0) \|_{\mathbf{H}} = \| \pi_\varepsilon^{-1} \tilde{\beta} - \mathbf{P}^\varepsilon(\pi_0^{-1} \tilde{\beta} + \sigma^0(t_0; \tilde{\beta})) \|_{\mathbf{H}} &\le \| \pi_\varepsilon^{-1} \tilde{\beta} - \mathbf{P}^\varepsilon \pi_0^{-1} \tilde{\beta} \|_{\mathbf{H}} \\ &+ \| (\mathbf{P}^\varepsilon - \mathbf{P}^0) \sigma^0(t_0; \tilde{\beta}) \|_{\mathbf{H}} \le c_2 \varepsilon \end{aligned}$$

for some positive constant c_2. Then if $\| \bar{\sigma}(t_0) \|_{\mathbf{H}} \le \| \bar{\rho}(t_0) \|_{\mathbf{H}}$, (6.17) is clear. We next consider the case when $\| \bar{\rho}(t_0) \|_{\mathbf{H}} < \| \bar{\sigma}(t_0) \|_{\mathbf{H}}$. Let $\bar{\mathbf{I}}(\bar{\rho}(t), \bar{\sigma}(t)) = (\| \bar{\sigma}(t) \|_{\mathbf{H}}^2 - \| \bar{\rho}(t) \|_{\mathbf{H}}^2)/2$ and suppose that $\bar{\mathbf{I}}(\bar{\rho}(t), \bar{\sigma}(t)) > 0$ for $t \in (t_1, t_0]$ with $t_1 \ge 0$ and $\bar{\mathbf{I}}(\bar{\rho}(t_1), \bar{\sigma}(t_1)) = 0$. (If $\bar{\mathbf{I}}(\bar{\rho}(t), \bar{\sigma}(t)) > 0$ in $[0, t_0]$, let $t_1 = 0$). Using the argument used in the proof of Lemma 6.6 on (6.13), we have

$$\begin{aligned} &\langle \psi_t^0(t), w \rangle_{\mathbf{H}} + \mathbf{b}_\varepsilon(\psi^0(t), w) \\ &\quad = -2\mathrm{D}\theta(\| \sqrt{|D|} p^0(t) \|_{0,\mathbf{H}_0}^2) \langle \sqrt{|D|} p^0(t), \rho^0(t; \beta) \rangle_{0,\mathbf{H}_0} \langle F_0(U^0(t)), w \rangle_{\mathbf{H}} \\ &\quad - \theta(\| \sqrt{|D|} p^0(t) \|_{0,\mathbf{H}_0}^2) \langle \mathrm{D}F_0(U^0(t)) \psi^0(t), w \rangle_{\mathbf{H}} \\ &\quad - d_1 \sum_{j=1}^{N_1} \left\langle \frac{r'(s) \partial}{r(s) \partial s} (\psi^0)_1, z_j \frac{\partial}{\partial z_j} (w)_1 \right\rangle_{\mathbf{H}} - d_2 \sum_{j=1}^{N_1} \left\langle \frac{r'(s) \partial}{r(s) \partial s} (\psi^0)_2, z_j \frac{\partial}{\partial z_j} (w)_2 \right\rangle_{\mathbf{H}}, \end{aligned}$$

where $\psi^0(t) = ((\psi^0)_1, (\psi^0)_2) \in (H^1(0, L))^2$, $w = ((w)_1, (w)_2) \in (H^1(\Omega))^2$. Letting $w = \bar{\sigma}(t)$ and $w = \bar{\rho}(t)$ in the above formula, respectively, we have

$$\bar{\mathbf{I}}'(\bar{\rho}(t_1), \bar{\sigma}(t_1)) = \frac{d}{dt}\bar{\mathbf{I}}(\bar{\rho}(t), \bar{\sigma}(t))|_{t=t_1}$$
$$= \langle \bar{\sigma}_t(t_1), \bar{\sigma}(t_1)\rangle_{\mathbf{H}} - \langle \bar{\rho}_t(t_1), \bar{\rho}(t_1)\rangle_{\mathbf{H}} = -\langle \mathbf{B}_\varepsilon\bar{\sigma}, \bar{\sigma}\rangle_{\mathbf{H}} + \langle \mathbf{B}_\varepsilon\bar{\rho}, \bar{\rho}\rangle_{\mathbf{H}}$$
$$+ 2\mathrm{D}\theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle p^\varepsilon, \bar{\rho}\rangle_{\mathbf{H}}\langle F_\varepsilon(U^\varepsilon), \bar{\rho} - \bar{\sigma}\rangle_{\mathbf{H}}$$
$$+ \theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle \mathrm{D}F_\varepsilon(U^\varepsilon)(\bar{\rho} + \bar{\sigma}), \bar{\rho} - \bar{\sigma}\rangle_{\mathbf{H}} + G_1,$$

where

$$G_1 = 2\mathrm{D}\theta(\|\sqrt{|D|}p^0\|_{0,\mathbf{H}_0}^2)\langle\sqrt{|D|}p^0, \psi^0\rangle_{0,\mathbf{H}_0}\langle F_0(U^0), \bar{\sigma} - \bar{\rho}\rangle_{\mathbf{H}}$$
$$- 2\mathrm{D}\theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle p^\varepsilon, \psi^0\rangle_{\mathbf{H}}\langle F_\varepsilon(U^\varepsilon), \bar{\sigma} - \bar{\rho}\rangle_{\mathbf{H}}$$
$$+ \theta(\|\sqrt{|D|}p^0\|_{0,\mathbf{H}_0}^2)\langle \mathrm{D}F_0(U^0)\psi^0, \bar{\sigma} - \bar{\rho}\rangle_{\mathbf{H}} - \theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle \mathrm{D}F_\varepsilon(U^\varepsilon)\psi^0, \bar{\sigma} - \bar{\rho}\rangle_{\mathbf{H}}$$
$$+ d_1\sum_{j=1}^{N_1}\left\langle \frac{r'(s)\partial}{r(s)\partial s}(\psi^0)_1,\ z_j\frac{\partial}{\partial z_j}(\bar{\sigma} - \bar{\rho})_1\right\rangle_{\mathbf{H}}$$
$$+ d_2\sum_{j=1}^{N_1}\left\langle \frac{r'(s)\partial}{r(s)\partial s}(\psi^0)_2,\ z_j\frac{\partial}{\partial z_j}(\bar{\sigma} - \bar{\rho})_2\right\rangle_{\mathbf{H}}.$$

By using the gaps of eigenvalues of $\mathbf{B}_\varepsilon$, we have

$$\bar{\mathbf{I}}'(\bar{\rho}(t_1), \bar{\sigma}(t_1)) \leq -\|\bar{\sigma}(t_1)\|_{\mathbf{H}}^2 + |G_1|. \tag{6.19}$$

Recall that $U^\varepsilon(t)$ and $U^0(t)$ are uniformly bounded in $(L^\infty(\Omega))^2$ for $0 < \varepsilon \leq \varepsilon_7$ and $t \geq 0$. Then by using Lemmas 6.3 ~ 6.6 and 6.9, we have

$$|G_1| \leq c_3\varepsilon^{\kappa_2}\|\bar{\sigma}(t_1)\|_{\mathbf{H}} + c_4\sqrt{\varepsilon}$$

for some positive constants c_3, c_4 and κ_2. Substituting it into (6.19) and using $\bar{\mathbf{I}}'(\bar{\rho}(t_1), \bar{\sigma}(t_1)) \geq 0$, we obtain

$$\|\bar{\sigma}(t_1)\|_{\mathbf{H}} \leq c_5\varepsilon^{\kappa_3} \qquad \text{for some } c_5, \kappa_3 > 0. \tag{6.20}$$

Also, by (6.13), we have

$$\frac{1}{2}\frac{d}{dt}\|\bar{\sigma}(t)\|_{\mathbf{H}}^2 = -\langle \mathbf{B}_\varepsilon\bar{\sigma}, \bar{\sigma}\rangle_{\mathbf{H}} - 2\mathrm{D}\theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle p^\varepsilon, \bar{\rho}\rangle_{\mathbf{H}}\langle F_\varepsilon(U^\varepsilon), \bar{\sigma}\rangle_{\mathbf{H}}$$
$$- \theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle \mathrm{D}F_\varepsilon(U^\varepsilon)(\bar{\rho} + \bar{\sigma}), \bar{\sigma}\rangle_{\mathbf{H}} + G_2,$$

where

$$G_2 = 2\mathrm{D}\theta(\|\sqrt{|D|}p^0\|_{0,\mathbf{H}_0}^2)\langle\sqrt{|D|}p^0, \psi^0\rangle_{0,\mathbf{H}_0}\langle F_0(U^0), \bar{\sigma}\rangle_{\mathbf{H}}$$
$$- 2\mathrm{D}\theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle p^\varepsilon, \psi^0\rangle_{\mathbf{H}}\langle F_\varepsilon(U^\varepsilon), \bar{\sigma}\rangle_{\mathbf{H}}$$
$$+ \theta(\|\sqrt{|D|}p^0\|_{0,\mathbf{H}_0}^2)\langle \mathrm{D}F_0(U^0)\psi^0, \bar{\sigma}\rangle_{\mathbf{H}} - \theta(\|p^\varepsilon\|_{\mathbf{H}}^2)\langle \mathrm{D}F_\varepsilon(U^\varepsilon)\psi^0, \bar{\sigma}\rangle_{\mathbf{H}}$$

$$+ d_1 \sum_{j=1}^{N_1} \left\langle \frac{r'(s)\partial}{r(s)\partial s}(\psi^0)_1,\ z_j \frac{\partial}{\partial z_j}(\bar{\sigma})_1 \right\rangle_{\mathbf{H}}$$

$$+ d_2 \sum_{j=1}^{N_1} \left\langle \frac{r'(s)\partial}{r(s)\partial s}(\psi^0)_2,\ z_j \frac{\partial}{\partial z_j}(\bar{\sigma})_2 \right\rangle_{\mathbf{H}}.$$

It follows from the modification of F_ε, Remark 5.8 and $\|\bar{\rho}(t)\|_{\mathbf{H}} \leq \|\bar{\sigma}(t)\|_{\mathbf{H}}$ for $t_1 \leq t \leq t_0$ that there is some positive constant M_9 such that

$$\frac{1}{2}\frac{d}{dt}\|\bar{\sigma}(t)\|_{\mathbf{H}}^2 \leq -M_9\|\bar{\sigma}(t)\|_{\mathbf{H}}^2 + |G_2| \qquad \text{for } t \in [t_1, t_0] \tag{6.21}$$

holds. G_2 can be estimated in a similar way to G_1 so that there are some positive constants c_6 and c_7 such that

$$|G_2| \leq c_6 \varepsilon^{\kappa_2}\|\bar{\sigma}(t)\|_{\mathbf{H}} + c_7\sqrt{\varepsilon} \qquad \text{for } t \in [t_1, t_0]$$

holds. Substituting it into (6.21), we have

$$\frac{d}{dt}\|\bar{\sigma}(t)\|_{\mathbf{H}}^2 \leq -c_8\|\bar{\sigma}(t)\|_{\mathbf{H}}^2 + c_9\varepsilon^{\kappa_4} \qquad \text{for } t \in [t_1, t_0]$$

for some positive constants c_8, c_9 and κ_4. Therefore, we find by (6.20) that

$$\|\bar{\sigma}(t_0)\|_{\mathbf{H}} \leq c_{10}\varepsilon^{\kappa_5}$$

holds for some positive constants c_{10} and κ_5. So that (6.17) follows from the above inequality and (6.18). Thus, Lemma 6.10 is proved.

We know that $\Phi^{*\varepsilon} \to \Phi^{*0}$ in $C^1(\mathbf{B}_{r_0}, (L^2(\Omega))^2)$ as $\varepsilon \downarrow 0$ follows from Lemma 6.8 and Lemma 6.10. Thus, the proof of Theorem 6.1 is complete.

§7. Equilibrium solutions problem

We now investigate the relation between two solutions of the systems (3.2) ~ (3.5) and (3.6) ~ (3.9). Let U_ε^* and U_0^* be respectively equilibrium solutions of (3.2) with the boundary condition (3.3) or (3.4) or (3.5) and of (3.6) with the boundary condition (3.7) or (3.8) or (3.9). The linearized eigenvalue problems of (3.2) around U_ε^* and (3.6) around U_0^* are described by

$$-\mathbf{B}_\varepsilon W + \mathrm{D}F_\varepsilon(U_\varepsilon^*)W = -\mu W \tag{7.1}$$

and

$$-\mathbf{B}_0 W + \mathrm{D}F_0(U_0^*)W = -\mu W. \tag{7.2}$$

U_ε^* (resp. U_0^*) is said to be nondegenerate if (7.1) (resp. (7.2)) has no zero eigenvalue. We show the following result.

THEOREM 7.1. *Let* U_0^* *be a nondegenerate equilibrium solution of* (3.6) *with* (3.7) (*or* (3.8) *or* (3.9)) *satisfying* $0 < U_0^* < R^*$ *in* $(0, L)$. *Then there is* $\varepsilon_8 > 0$ *such that for* $0 < \varepsilon \le \varepsilon_8$ *there exists a nondegenerate equilibrium solution* U_ε^* *of* (3.2) *with* (3.3) (*or* (3.4) *or* (3.5)) *satisfying* $0 \le U_\varepsilon^*(s, z) \le R^*$ *in* Ω *such that* $U_\varepsilon^* \to U_0^*$ *in* $(L^2(\Omega))^2$ *and* $U_\varepsilon^*(s, z) \to U_0^*(s)$ *in* Ω *as* $\varepsilon \downarrow 0$. *Moreover, let the eigenvalues of* (7.2) *and* (7.1) *be* $\{\mu_i\}$ $(i = 1, 2, \cdots)$ $(\operatorname{Re}\mu_1 \le \operatorname{Re}\mu_2 \le \cdots)$ *and* $\{\mu_{i,\varepsilon}\}$ $(i = 1, 2, \cdots)$ $(\operatorname{Re}\mu_{1,\varepsilon} \le \operatorname{Re}\mu_{2,\varepsilon} \le \cdots)$. *Then*

(i) *If* $\operatorname{Re}\mu_1 > 0$, *then there is* $\mu^* > 0$ *such that* $\operatorname{Re}\mu_{1,\varepsilon} \ge \mu^*$ *for* $0 < \varepsilon \le \varepsilon_8$.

(ii) *If* $\operatorname{Re}\mu_m < 0$, $\operatorname{Re}\mu_{m+1} > 0$ *for some positive integer* m *and at least one of* $\{\mu_1, \cdots, \mu_m\}$ *is simple, then there exists an eigenvalue* $\mu_{m_1,\varepsilon}$ *such that* $\operatorname{Re}\mu_{m_1,\varepsilon} < 0$ *for* $0 < \varepsilon \le \varepsilon_8$.

PROOF. Let $U_0^* = p_0^* + q_0^* = \mathbf{P}_\mu^0 U_0^* + \mathbf{Q}_\mu^0 U_0^*$, we know that (p_0^*, q_0^*) is a nondegenerate equilibrium solution of the system

$$
\begin{cases} p_t = -\mathbf{B}_0 p + \mathbf{P}_\mu^0(F_0(p, q)), \\ q_t = -\mathbf{B}_0 q + \mathbf{Q}_\mu^0(F_0(p, q)) \end{cases} \tag{7.3}
$$

for appropriate constants K_4, R_1, R_2 and μ, and that $(p_0^*, q_0^*) \in \mathfrak{M}^0$, $q_0^* = \Phi^0(p_0^*)$ by Remark 5.10. We will show that p_0^* is a nondegenerate equilibrium solution of the equation

$$
\frac{dp}{dt} = \mathscr{F}_0(p, \Phi^0(p)) = \tilde{\mathscr{F}}_0(p). \tag{7.4}
$$

In fact, if p_0^* is degenerate, then we find that there exists $\bar{\rho} \in \mathbf{V}_\mu^0$ such that $\mathrm{D}_p \tilde{\mathscr{F}}_0(p_0^*)\bar{\rho} = 0$. That is,

$$
\mathrm{D}_p \mathscr{F}_0(p_0^*, \Phi^0(p_0^*))\bar{\rho} + \mathrm{D}_q \mathscr{F}_0(p_0^*, \Phi^0(p_0^*))\mathrm{D}\Phi^0(p_0^*)\bar{\rho} = 0. \tag{7.5}
$$

Since *graph* Φ^0 is invariant with respect to (7.3), we have

$$
\begin{cases} p_t = \mathscr{F}_0(p, \Phi^0(p)), \\ (\Phi^0(p))_t = -\mathbf{B}_0 \Phi^0(p) + \mathscr{G}_0(p, \Phi^0(p)), \end{cases}
$$

which implies

$$
\mathrm{D}\Phi^0(p)\mathscr{F}_0(p, \Phi^0(p)) = -\mathbf{B}_0 \Phi^0(p) + \mathscr{G}_0(p, \Phi^0(p)).
$$

The linear variational equation of the above formula is

$$\begin{aligned} &\mathrm{D}\Phi^0(p)(\mathrm{D}_p\mathscr{F}_0(p, \Phi^0(p))\rho + \mathrm{D}_q\mathscr{F}_0(p, \Phi^0(p))\mathrm{D}\Phi^0(p)\rho) \\ &\quad = -\mathbf{B}_0\mathrm{D}\Phi^0(p)\rho + \mathrm{D}_p\mathscr{G}_0(p, \Phi^0(p))\rho + \mathrm{D}_q\mathscr{G}_0(p, \Phi^0(p))\mathrm{D}\Phi^0(p)\rho \\ &\qquad\qquad \text{for } \rho\in\mathbf{V}_\mu^0. \end{aligned} \tag{7.6}$$

Letting $\tilde{\rho} = \bar{\rho}$, $\tilde{\sigma} = \mathrm{D}\Phi^0(p_0^*)\bar{\rho}$, we know by (7.5) that

$$\mathrm{D}\mathscr{F}_0(p_0^*, \Phi^0(p_0^*))(\tilde{\rho}, \tilde{\sigma}) = 0. \tag{7.7}$$

Also, letting $p = p_0^*$, $\rho = \bar{\rho} = \tilde{\rho}$ in (7.6), we know by (7.7) that

$$-\mathbf{B}_0\mathrm{D}\Phi^0(p_0^*)\tilde{\rho} + \mathrm{D}\mathscr{G}_0(p_0^*, \Phi^0(p_0^*))(\tilde{\rho}, \tilde{\sigma}) = 0. \tag{7.8}$$

Thus, (7.7) and (7.8) lead to that $(p_0^*, q_0^*) = (p_0^*, \Phi^0(p_0^*))$ is a degenerate equilibrium solution of (7.3). However, It is a contradiction to our hypothesis, so that p_0^* is a nondegenerate equilibrium solution of (7.4).

Since $\Phi^{*\varepsilon} \to \Phi^{*0}$ in $C^1(\mathbf{B}_{r_0}, (L^2(\Omega))^2)$ and $\mathscr{F}_\varepsilon{}^* \to \mathscr{F}^*_0$ in $C^1(\mathbf{B}_{r_0}, \mathbf{R}^{N_1})$ as $\varepsilon \downarrow 0$, we know by the Implicit Function Theorem that there is $\bar{\varepsilon}_1 > 0$ such that for $0 < \varepsilon \le \bar{\varepsilon}_1$, there exists an equilibrium solution p_ε^* of (6.1), satisfying $p_\varepsilon^* \to p_0^*$ in $(L^2(\Omega))^2$ as $\varepsilon \downarrow 0$. It is natural to consider that an element of $(L^2(0, L))^2$ is taken as the one in $(L^2(\Omega))^2$. Letting $U_\varepsilon^* = p_\varepsilon^* + \Phi^\varepsilon(p_\varepsilon^*)$, we know that U_ε^* is an equilibrium solution of (5.8) and $U_\varepsilon^* \to U_0^*$ in $(L^2(\Omega))^2$ as $\varepsilon \downarrow 0$. In order to show that $U_\varepsilon^*(s, z) \to U_0^*(s)$ in Ω as $\varepsilon \downarrow 0$, we will show that $U_0^* \in (C^1(0, L))^2$, $U_\varepsilon^* \in (C^1(\Omega))^2$ and there is $M^* > 0$ such that

$$\| U_0^* \|_{(C^1(0,L))^2} \le M^* \quad \text{and} \quad \| U_\varepsilon^* \|_{(C^1(\Omega))^2} \le M^* \qquad \text{for } 0 < \varepsilon \le \bar{\varepsilon}_1. \tag{7.9}$$

Because of $U_0^* \in (L^2(0, L))^2$, we know that U_0^* can be considered as an equilibrium solution of the integral form of (7.3)

$$U(t) = exp(-\mathbf{B}_0 t)U(0) + \int_0^t exp(-\mathbf{B}_0(t-s))F_0(U(s))\,ds$$

and $U_0^* \in \mathbf{X}^\alpha_{(0,L)} = \mathscr{D}(\mathbf{B}_0^\alpha)$ with $\alpha \in (3/4, 1)$. Also we have

$$\| U(t) \|_{\mathbf{X}^\alpha_{(0,L)}} \le \frac{ce^{-\mu_0 t}}{t^\alpha} \| U(0) \|_{0,\mathbf{H}_0} + c\int_0^t \frac{e^{-\mu_0(t-s)}}{(t-s)^\alpha} \| F_0(U(s)) \|_{0,\mathbf{H}_0}\,ds$$

for some positive constant c (Henry [19]). By using $\mathbf{X}^\alpha_{(0,L)} \subset (C^\nu(0, L))^2$ with $0 \le \nu < 2\alpha - 1/2$, we know that $U_0^* \in (C^1(0, L))^2$ and $\| U_0^* \|_{(C^1(0,L))^2} \le M_{10}$ for some positive constant M_{10}.

Next we discuss U_ε^* which can be considered as an equilibrium solution of the integral form of (5.8)

$$U(t) = exp(-\mathbf{B}_\varepsilon t)U(0) + \int_0^t exp(-\mathbf{B}_\varepsilon(t-s))F_\varepsilon(U(s))\,ds. \tag{7.10}$$

Since $U_\varepsilon^* \to U_0^*$ in $(L^2(\Omega))^2$ as $\varepsilon \downarrow 0$, we know that $U_\varepsilon^* \in (L^2(\Omega))^2$ and there are $\bar{\varepsilon}_2 > 0$ and $\bar{M}_1 > 0$ such that $\| U_\varepsilon^* \|_{\mathbf{H}} \le \bar{M}_1$ for $0 < \varepsilon \le \bar{\varepsilon}_2$. It follows from $U_\varepsilon^* = \mathbf{B}_\varepsilon^{-1} F_\varepsilon(U_\varepsilon^*)$ and the boundedness of $\mathbf{B}_\varepsilon^{-1}$ that $\| U_\varepsilon^* \|_{(H^2(\Omega))^2} \le c_1 \bar{M}_1$ holds for some positive constant c_1.

When $N \le 3$, since $H^2(\Omega) \subset L^q(\Omega)$ for $1 < q < \infty$, we have $U_\varepsilon^* \in (L^q(\Omega))^2$ for $q \ge 5$. We take $\mathbf{B}_\varepsilon$ as an operator in $(L^q(\Omega))^2$ with $q \ge 5$ and let $\mathbf{X}_q^\alpha = \mathscr{D}(\mathbf{B}_\varepsilon^\alpha)$ with its norm $\| U \|_{\alpha,q} = \| \mathbf{B}_\varepsilon^\alpha U \|_{(L^q(\Omega))^2}$, where $\alpha \in ((q + N + 1)/2q, 1)$. Then by using

$$\| U(t) \|_{\alpha,q} \le \frac{c_2 \, exp(-\mu_0 t)}{t^\alpha} \| U(0) \|_{(L^q(\Omega))^2} + c_2 \int_0^t \frac{exp(-\mu_0(t-s))}{(t-s)^\alpha} \| F_\varepsilon(U(s)) \|_{(L^q(\Omega))^2} ds$$

for some positive constant c_2 independently of ε (Henry [19]), we have $\| U_\varepsilon^* \|_{\alpha,q} \le \bar{M}_2$ for some positive constant $\bar{M}_2$. Since $\mathbf{X}_q^\alpha \subset (C^1(\Omega))^2$, we know that $\| U_\varepsilon^* \|_{(C^1(\Omega))^2} \le \bar{M}_3$ holds for some positive constant $\bar{M}_3$ independently of ε.

We now consider the case when $N > 3$. By the Sobolev Imbedding Theorem, we know $H^2(\Omega) \subset L^p(\Omega)$ with $1/p \ge 1/2 - 2/(N+1)$. When $N < 5$, using $U_\varepsilon^* \in (H^2(\Omega))^2 \subset (L^{p_1}(\Omega))^2$ with $1/(N+1) > 1/p_1 \ge 1/2 - 2/(N+1)$, and arguing as before with replacing (q, α) by (p_1, α_1) with $\alpha_1 \in ((p_1 + N + 1)/2p_1, 1)$, we know that $U_\varepsilon^* \in \mathbf{X}_{p_1}^{\alpha_1} \subset (C^1(\Omega))^2$ and $\| U_\varepsilon^* \|_{(C^1(\Omega))^2} \le \bar{M}_4$ hold for some positive constant $\bar{M}_4$. Otherwise, setting $1/p_1 = 1/2 - 2/(N+1)$, $\alpha_1 = (N+1)/(N+2)$, we have $U_\varepsilon^* \in \mathbf{X}_{p_1}^{\alpha_1} \subset (L^{p_2}(\Omega))^2$ with $1/p_2 \ge 1/p_1 - 2\alpha_1/(N+1) = 1/2 - 2/(N+1) - 2/(N+2)$. When $6 \le N < 9$, using $U_\varepsilon^* \in (L^{p_2}(\Omega))^2$ with $1/(N+1) > 1/p_2 \ge 1/2 - 2/(N+1) - 2/(N+2)$, and arguing as before with replacing (q, α) by (p_2, α_2) with $\alpha_2 \in ((p_2 + N + 1)/2p_2, 1)$, we know that $U_\varepsilon^* \in \mathbf{X}_{p_2}^{\alpha_2} \subset (C^1(\Omega))^2$ and $\| U_\varepsilon^* \|_{(C^1(\Omega))^2} \le \bar{M}_5$ hold for some positive constant $\bar{M}_5$. Otherwise, setting $1/p_2 = 1/2 - 2/(N+1) - 2/(N+2)$, $\alpha_2 = (N+1)/(N+2)$, we have $U_\varepsilon^* \in \mathbf{X}_{p_2}^{\alpha_2} \subset (L^{p_3}(\Omega))^2$ with $1/p_3 \ge 1/p_2 - 2\alpha_2/(N+1) = 1/2 - 2/(N+1) - 4/(N+2)$. When $9 \le N \le 12$, using $U_\varepsilon^* \in (L^{p_3}(\Omega))^2$ with $1/(N+1) > 1/p_3 \ge 1/2 - 2/(N+1) - 4/(N+2)$, and arguing as before with replacing (q, α) by (p_3, α_3) with $\alpha_3 \in ((p_3 + N + 1)/2p_3, 1)$, we know that $U_\varepsilon^* \in \mathbf{X}_{p_3}^{\alpha_3} \subset (C^1(\Omega))^2$ and $\| U_\varepsilon^* \|_{(C^1(\Omega))^2} \le \bar{M}_6$ hold for some positive constant $\bar{M}_6$. Otherwise, setting $1/p_3 = 1/2 - 2/(N+1) - 4/(N+2)$, $\alpha_3 = (N+1)/(N+2)$, we have $U_\varepsilon^* \in \mathbf{X}_{p_3}^{\alpha_3} \subset (L^{p_4}(\Omega))^2$ with $1/p_4 \ge 1/p_3 - 2\alpha_3/(N+1) = 1/2 - 2/(N+1) - 6/(N+2)$. Taking the above arguments step by step at most k-times where $1/(N+1) > 1/2 - 2/(N+1) - 2(k-1)/(N+2)$, we arrive at $\| U_\varepsilon^* \|_{(C^1(\Omega))^2} \le M^*$ for some positive constant M^*. By (7.9), it can be easily shown that $U_\varepsilon^*(s, z) \to U_0^*(s)$ in Ω as $\varepsilon \downarrow 0$ which implies that $0 \le U_\varepsilon^*(s, z) \le R^*$ in Ω.

We next consider the eigenvalue problems of (7.1) and (7.2). First, suppose that (i) does not hold. Then there exists $\{\varepsilon_n\}$ with $\varepsilon_n \downarrow 0$ as $n \to \infty$ such that $\operatorname{Re}\mu_{1,\varepsilon_n} \to 0$ as $n \to \infty$. Let ω_{1,ε_n} be the eigenfunction corresponding to μ_{1,ε_n} normalized in $(\mathbf{H}_\varepsilon(\Omega))^2$. Since $0 \le U_\varepsilon^*(s, z) \le R^*$ for sufficiently small ε and

$$\mathbf{B}_{\varepsilon_n}\omega_{1,\varepsilon_n} = \mathrm{D}F_{\varepsilon_n}(U_{\varepsilon_n}^*)\omega_{1,\varepsilon_n} + \mu_{1,\varepsilon_n}\omega_{1,\varepsilon_n}, \tag{7.11}$$

we know that $|\mu_{1,\varepsilon_n}|$ is uniformly bounded for large n. Moreover, since

$$\bar{c}_1 \|\omega_{1,\varepsilon_n}\|_{\varepsilon,\mathbf{H}}^2 \le \mathbf{b}_{\varepsilon_n}(\omega_{1,\varepsilon_n}, \omega_{1,\varepsilon_n}) = \langle \mathbf{B}_{\varepsilon_n}\omega_{1,\varepsilon_n}, \omega_{1,\varepsilon_n}\rangle_{\mathbf{H}},$$

we also know that $\|\omega_{1,\varepsilon_n}\|_{\varepsilon,\mathbf{H}}$ is uniformly bounded for large n. Therefore, it follows from the compactness of $\mathbf{B}_\varepsilon^{-1}$ that there exists a subsequence of $\{\varepsilon_n\}$ (denoted also by ε_n) such that for some $\mu^* \in \mathbf{C}$ and $\omega \in (H^1(\Omega))^2$, $\mu_{1,\varepsilon_n} \to \mu^*$ and $\omega_{1,\varepsilon_n} \to \omega^*$ strongly in $(H^1(\Omega))^2$ as $n \to \infty$. Here we know that $\operatorname{Re}\mu^* = 0$ and $\|\omega^*\|_{\mathbf{H}} \neq 0$. It follows from the definition of $\|\cdot\|_{\varepsilon,\mathbf{H}}$ that $\omega^*(s, z) = \omega^*(s)$ does not depend on z, that is, $\omega^* \in (H^1(0, L))^2$ and $\|\omega^*\|_{0,\mathbf{H}_0} \neq 0$. By using a similar argument to the one in Lemma 4.6, we know that

$$\|(\mathbf{B}_\varepsilon^{-1} - \mathbf{B}_0^{-1})w\|_{\varepsilon,\mathbf{X}} \le \bar{c}_6\varepsilon\|w\|_{0,\mathbf{H}_0} \quad \text{for } w \in (\mathbf{H}_0(0, L))^2,\ 0 < \varepsilon \le \varepsilon_6 \tag{7.12}$$

holds for some positive constants ε_6 and $\bar{c}_6$. Thus the limit $\varepsilon \downarrow 0$ in (7.11) leads to

$$\omega^* = \mathbf{B}_0^{-1}(\mathrm{D}F_0(U_0^*)\omega^* + \mu^*\omega^*).$$

Thus it turns out that μ^* is an eigenvalue of (7.2) with $\operatorname{Re}\mu^* = 0$ and ω^* is its eigenfunction. This is a contradiction to our hypothesis, which implies that (i) holds. A similar argument to the above can also show that U_ε^* is nondegenerate.

For the proof of (ii), let μ_{m_2} and ω_{m_2} be a simple eigenvalue with $\operatorname{Re}\mu_{m_2} < 0$ and its eigenfunction normalized in $(\mathbf{H}_0(0, L))^2$, respectively. In a similar manner to the one in the Step I of the proof of Lemma 4.2, we will show that $\mathbf{B}_\varepsilon$ has an eigenvalue $\mu_{m_1,\varepsilon}$ in the small neighborhood of μ_{m_2} for sufficiently small ε, which satisfies $\mu_{m_1,\varepsilon} \to \mu_{m_2}$ as $\varepsilon \downarrow 0$. We define the operators $\tilde{\mathscr{J}}_0$ and $\tilde{\mathscr{J}}_\varepsilon$ by

$$\begin{aligned}\tilde{\mathscr{J}}_0 : \mathbf{C} \times \mathbf{X}_0 &\longrightarrow \mathbf{C} \times \mathbf{X}_0 \quad \text{with} \quad \tilde{\mathscr{J}}_0(\tau, w)\\ &= (\tilde{\zeta}_0(w - \omega_{m_2}),\ w - \tau\mathbf{B}_0^{-1}w - \mathbf{B}_0^{-1}\mathrm{D}F_0(U_0^*)w)\end{aligned}$$

and

$$\begin{aligned}\tilde{\mathscr{J}}_\varepsilon : \mathbf{C} \times \mathbf{X}_\varepsilon &\longrightarrow \mathbf{C} \times \mathbf{X}_\varepsilon \quad \text{with} \quad \tilde{\mathscr{J}}_\varepsilon(\tau, w)\\ &= (\tilde{\zeta}_\varepsilon(w - \omega_{m_2}),\ w - \tau\mathbf{B}_\varepsilon^{-1}w - \mathbf{B}_\varepsilon^{-1}\mathrm{D}F_\varepsilon(U_\varepsilon^*)w),\end{aligned}$$

where $\tilde{\zeta}_0(w) = \mu_{m_2}^{-1}\mathbf{b}_0(w, \omega_{m_2})$ and $\tilde{\zeta}_\varepsilon(w) = \mu_{m_2}^{-1}\mathbf{b}_\varepsilon(w, \omega_{m_2})$ and the norms of $\mathbf{C} \times \mathbf{X}_0$ and $\mathbf{C} \times \mathbf{X}_\varepsilon$ are defined by $\|(\tau, w)\|_{00} = |\tau| + \|w\|_{0,\mathbf{X}}$ and $\|(\tau, w)\|_{\varepsilon\varepsilon} = |\tau| + \|w\|_{\varepsilon,\mathbf{X}}$, respectively. It is known that $\tilde{\mathscr{J}}_0(\mu_{m_2}, \omega_{m_2}) = 0$ and that if τ and w satisfy $\tilde{\mathscr{J}}_\varepsilon(\tau, w) = 0$, then τ is an eigenvalue of (7.1) and w is its eigenfunction. Noting that Lemmas 4.4, 4.5 and 4.6 hold for $w \in \mathbf{X}_\varepsilon$, $\mathbf{B}_\varepsilon$, we know that Lemmas 4.7 and 4.8 also hold for $\tilde{\mathscr{J}}_0$ and $\tilde{\mathscr{J}}_\varepsilon$. Therefore, by choosing $\bar{\varepsilon}_3$ and η_{m_2} sufficiently small, we find that $\tilde{\mathscr{J}}_\varepsilon(\tau, w) = 0$ has only one solution $(\mu_{m_1,\varepsilon}, \omega_{m_1,\varepsilon})$ in $\{(\tau, w) \in \mathbf{C} \times \mathbf{X}_\varepsilon | \|(\tau, w) - (\mu_{m_2}, \omega_{m_2})\|_{\varepsilon\varepsilon} \le \eta_{m_2}\}$ for $0 < \varepsilon \le \bar{\varepsilon}_3$, satisfying $\mu_{m_1,\varepsilon} \to \mu_{m_2}$ as $\varepsilon \downarrow 0$. Thus it turns out that $\mu_{m_1,\varepsilon}$ is the eigenvalue of (7.1) with $\operatorname{Re}\mu_{m_1,\varepsilon} < 0$ for sufficiently small ε. The proof of Theorem 7.1 is complete.

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