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Structure of the probability contents inner boundary of some family of three-parameter distributions

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1. Introduction

Let $F(x)$ be a strictly increasing and continuously differentiable distribution function (d.f.) on the real line $\boldsymbol{R}$, and let $h(x)$(resp. $\tilde{h}(x)$) be a continuous and strictly increasing function on $\boldsymbol{R}_+ = (0, \infty)$(resp. $\boldsymbol{R}$) with $h(\boldsymbol{R}_+) = \boldsymbol{R}$(resp. $\tilde{h}(\boldsymbol{R}) = \boldsymbol{R}$). Define a transformation $t(x, \theta)(\theta = (\alpha, \beta, \lambda) \in \boldsymbol{R}_+ \times \boldsymbol{R} \times [-\infty, \infty))$ by

$$t(x, \theta) = \begin{cases} \alpha\tilde{h}(x) - \beta, & \lambda = -\infty, \\ \alpha h(x - \lambda) - \beta, & \lambda \neq -\infty. \end{cases}$$

Let Θ be a nonempty subset of $\boldsymbol{R}_+ \times \boldsymbol{R} \times [-\infty, \infty)$ and put $\mathscr{F}(\Theta) = \{F(t(x, \theta)); \theta \in \Theta\}$, being called a family of three-parameter d.f.'s which are positive only to the right of a shifted origin. The family $\mathscr{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R})$ with $h(x) = \log x$ was considered in Finney [4].

Suppose that:

(i) We have N different kinds of experiments on some characteristic X.

(ii) The transformed variable $t(X, \theta)$ has a d.f. F.

(iii) In the ith experiment, n_i objects are tested and information available for each characteristic $X_{ij} (1 \leq j \leq n_i)$ is only that its value lies in a proper subinterval $\mathscr{C}_{ij}$ of $\boldsymbol{R}$ with nonempty interior.

The collection $\mathscr{C} \equiv \{\mathscr{C}_{ij}; 1 \leq i \leq N, 1 \leq j \leq n_i\}$ is called a *pooled interval-censored* (p.i.c.) *data*. When $N = 1$, the p.i.c. data $\mathscr{C}$ is simply called an *interval-censored* (i.c.) *data*. The i.c. data $\mathscr{C}$ is called a *grouped data* if each $\mathscr{C}_{1j}$ belongs to a set of mutually disjoint intervals whose union is equal to $\boldsymbol{R}$. The p.i.c. data $\mathscr{C}$ is called a *binary response data* if each $\mathscr{C}_{ij}$ belongs to a set of mutually disjoint two intervals, depending only on i, whose union is equal to $\boldsymbol{R}$.

There are various kinds of method for estimating the unknown true parameter θ_0 based on the p.i.c. data $\mathscr{C}$ (cf. [2], [9], [13]). In these methods, an estimate $\hat{\theta}$ of the unknown true parameter θ_0 is defined by an optimal solution of a minimizing problem. Hence there arises a problem whether such an estimate $\hat{\theta}$ exists or not. No well-founded argument has been given for this problem (cf. [1], [3], [4], [5], [12]). To solve this problem, Nakamura ([6], [9], [10]) proposed a unified method, called the *probability contents boundary* (PCB) *analysis.* Arrange all finite end points of $\mathscr{C}_{ij}$'s in order of magnitude and denote them by $x_1, \ldots, x_m$. The set $\{x_i\}$ of these m end points is called an *window.* Put $\boldsymbol{F}(\Theta) = \{(F(t(x_1, \theta)), \ldots, F(t(x_m, \theta))); \theta \in \Theta\}$ and

$$\partial \boldsymbol{F}(\Theta) = Cl(\boldsymbol{F}(\Theta)) - \boldsymbol{F}(\Theta).$$

The set $\partial \boldsymbol{F}(\Theta)$ is called the *probability contents inner boundary* (PCIB) of the family $\mathscr{F}(\Theta)$ through the window $\{x_i\}$. In the PCB analysis, it is important to analyze the structure of the PCIB $\partial \boldsymbol{F}(\Theta)$, since the explicit representation of the structure of the PCIB $\partial \boldsymbol{F}(\Theta)$ is very useful for deriving practical criteria for the existence of some kinds of estimate such as maximum likelihood estimate or least square estimate (cf. [10], [11]). The purpose of this paper is to give the explicit representation of the structure of the PCIB $\partial \boldsymbol{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R})$ through the window $\{x_i\}$.

In Section 2, a family $\mathscr{F}(\Theta)$ of three-parameter d.f.'s which are positive only to the right of a shifted origin, this being one of the unknown parameters, is introduced. The explicit representation of the structure of the PCIB $\partial \boldsymbol{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R})$ are stated without proof. In Section 3, fundamental properties of the PCIB $\partial \boldsymbol{F}(\Theta)$ are prepared to prove results given in Section 2. In Section 4, the proofs of results stated in Section 2 are given.

2. Structure of the PCIB $\partial \boldsymbol{F}(\Theta)$

We shall determine the structure of the PCIB $\partial \boldsymbol{F}(\Theta)$ through the window $\{x_i\}$ for $\Theta = \boldsymbol{R}_+ \times \boldsymbol{R} \times [-\infty, \infty)$ and for $\Theta = \boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R}$. Nakamura ([6], [7], [8]) determined the structure of the PCIB for many kinds of families of d.f.'s. However, they do not cover the three-parameter families $\mathscr{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R})$ and $\mathscr{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times [-\infty, \infty))$. Let us make the following two conditions:

(A.1) The function $h(s)$ (resp. $\tilde{h}(s)$) is twice continuously differentialbe on $\boldsymbol{R}_+$(resp. $\boldsymbol{R}$) such that

(i) $h'(s) > 0$ and $h''(s) < 0$ on $\boldsymbol{R}_+$,

(ii) $h(s) = o(h'(s))\,(s \to +0)$,

(iii) $\tilde{h}'(s) > 0$ and $\tilde{h}''(s) \leq 0$ on $\boldsymbol{R}$.

(A.2) There exist a function $a(s)$ on $\boldsymbol{R}_+$, a positive function $b(s)$ on $\boldsymbol{R}_+$, a differentiable function $c(s)$ on $\boldsymbol{R}_+$ and a function $w(s)$ on $\boldsymbol{R}$ such that

(i) $c'(s) > 0$ and $c(s) = o(1)\,(s \to +0)$,

(ii) for every fixed $x \in \boldsymbol{R}$,

$$R(x, s) = (h(x + 1/s) - a(s))/b(s) - \tilde{h}(x) - w(x)c(s)$$

is differentiable on the positive part of a neighbourhood of 0,

(iii) for every fixed $x \in \boldsymbol{R}$, $R(x, s) = o(1)(s \to +0)$ and $R'(x, s) = o(c'(s))(s \to +0)$.

In order to give an explicit representation of the structure of the PCIB, let us put $\boldsymbol{a}_i = (\overbrace{0,\dots,0}^{i}, \overbrace{1,\dots,1}^{m-i})$, $0 \le i \le m$; $\boldsymbol{b}_i(z) = (\overbrace{0,\dots,0}^{i-1}, z, \overbrace{1,\dots,1}^{m-i})$, $1 \le i \le m$; $0 < z < 1$ and $\boldsymbol{c}_i(z, z') = (\overbrace{0,\dots,0}^{i-1}, z, \overbrace{z',\dots,z'}^{m-i})$, $1 \le i \le m-1$; $0 \le z < z' < 1$.

Now we can state our main results.

THEOREM 2.1. *Let conditions* (A.1) *and* (A.2) *be satisfied. Then*

$$\begin{aligned}\partial \boldsymbol{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times [-\infty, \infty)) = {} & (\textstyle\bigcup_{i=0}^{m-1} \{\boldsymbol{a}_i\}) \cup \{z\boldsymbol{a}_0;\, 0 < z < 1\} \\ & \cup (\textstyle\bigcup_{i=1}^{m-1} \{\boldsymbol{b}_i(z);\, 0 < z < 1\}) \\ & \cup (\textstyle\bigcup_{i=1}^{m-2} \{\boldsymbol{c}_i(0, z');\, 0 < z' < 1\}) \\ & \cup (\textstyle\bigcup_{i=0}^{m-3} \{\boldsymbol{c}_{i+1}(z, z');\, 0 < z < z' < 1\}).\end{aligned}$$

THEOREM 2.2. *Let conditions* (A.1) *and* (A.2) *be satisfied. Then*

$$\begin{aligned}\partial \boldsymbol{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R}) = {} & (\textstyle\bigcup_{i=0}^{m-1} \{\boldsymbol{a}_i\}) \cup \{z\boldsymbol{a}_0;\, 0 < z < 1\} \\ & \cup (\textstyle\bigcup_{i=1}^{m-1} \{\boldsymbol{b}_i(z);\, 0 < z < 1\}) \\ & \cup (\textstyle\bigcup_{i=1}^{m-2} \{\boldsymbol{c}_i(0, z');\, 0 < z' < 1\}) \\ & \cup (\textstyle\bigcup_{i=0}^{m-3} \{\boldsymbol{c}_{i+1}(z, z');\, 0 < z < z' < 1\}) \\ & \cup \boldsymbol{F}(\boldsymbol{R}_+ \times \boldsymbol{R} \times \{-\infty\}).\end{aligned}$$

Consider the maximum likelihood estimation for example. Roughly speaking, the PCB analysis asserts that a necessary and sufficient condition for the existence of a maximum likelihood estimate (MLE) is that the supremum of the log-likelihood $L(z)$ over $\boldsymbol{F}(\Theta)$ is greater than the supremum of the log-likelihood $L(z)$ over $\partial \boldsymbol{F}(\Theta)$ (see [6], [9], [10], [11] for detailed discussions on the PCB analysis). To find a practical criterion for the existence of an MLE, we have to evaluate the value of the supremum of the log-likelihood $L(z)$ over $\partial \boldsymbol{F}(\Theta)$. Hence the explicit representation of the PCIB plays an important role in this stage.

3. Fundamental properties of the PCIB $\partial \boldsymbol{F}(\Theta)$

To state fundamental properties of the PCIB $\partial \boldsymbol{F}(\Theta)$, we prepare some notation and results. For notational simplicity, put $\Theta_0 = \boldsymbol{R}_+ \times \boldsymbol{R} \times [-\infty, \infty)$,

$\Theta_1 = \boldsymbol{R}_+ \times \boldsymbol{R} \times \boldsymbol{R}$, $\Theta_2 = \boldsymbol{R}_+ \times \boldsymbol{R} \times \{-\infty\}$ and $F(x, \theta) = F(t(x, \theta))$, and define

$$\boldsymbol{F}(\theta) = (F(x_1, \theta), \dots, F(x_m, \theta)),\ \theta \in \boldsymbol{R}_+ \times \boldsymbol{R} \times [-\infty, \infty).$$

Recall that $\mathscr{F}(\Theta_i) = \{F(t(x, \theta)); \theta \in \Theta_i\}$, $i = 0, 1, 2$. Hereafter the symbol "$\lim_n$" is used instead of "$\lim_{n\to\infty}$" and conditions (A.1) and (A.2) are assumed to be satisfied.

The following result, due to Nakamura [8], is useful.

PROPOSITION 3.1. *The set* $\bigcap_{j=1}^{2} \{\theta \in \boldsymbol{R}_+ \times \boldsymbol{R} \times [s, t]; u_j \le F(t(x_{i_j}, \theta)) \le u_j'\}$ *is compact for every set of pairs* (u_j, u_j') *with* $0 < u_j \le u_j' < u_{j+1} < 1$, $j = 1, 2$, (i_1, i_2) *with* $1 \le i_1 < i_2 \le m$ *and* (s, t) *with* $s = t = -\infty$ *or* $-\infty < s \le t < x_{i_1}$.

We give four properties of $\partial \boldsymbol{F}(\Theta)$.

LEMMA 3.1. *Let* $1 \le i < j \le m$ *and let* $\{\theta_n = (\alpha_n, \beta_n, \lambda_n)\}$ *be a sequence in* Θ_0 *such that* $\lim_n \lambda_n = -\infty$, $t_i = \lim_n t(x_i, \theta_n)$ *and* $t_j = \lim_n t(x_j, \theta_n)$. *If* $-\infty < t_i \le t_j < \infty$ *or* $-\infty < t_i < t_j = \infty$, *then*

$$\lim_n t(x, \theta_n) = t_i + \frac{\tilde{h}(x) - \tilde{h}(x_i)}{\tilde{h}(x_j) - \tilde{h}(x_i)}(t_j - t_i)$$

for all $x \in \boldsymbol{R}$.

PROOF. Choose $x \in \boldsymbol{R}$. We may assume that $\lambda_n < \min(x, x_1)$ for all $n = 1, 2, \dots$. Define $r_n(x) = (t(x, \theta_n) - t(x_i, \theta_n))/(t(x_j, \theta_n) - t(x_i, \theta_n))$, and put $s_n = -1/\lambda_n$ if $\lambda_n \ne -\infty$ and $s_n = 0$ if $\lambda_n = -\infty$. In case $s_n = 0$,

$$r_n(x) = \frac{\tilde{h}(x) - \tilde{h}(x_i)}{\tilde{h}(x_j) - \tilde{h}(x_i)}.$$

In case $s_n > 0$, by (ii) of (A.2),

$$\begin{aligned} r_n(x) &= \frac{h(x + 1/s_n) - h(x_i + 1/s_n)}{h(x_j + 1/s_n) - h(x_i + 1/s_n)} \\ &= \frac{\tilde{h}(x) - \tilde{h}(x_i) + (w(x) - w(x_i))c(s_n) + R(x, s_n) - R(x_i, s_n)}{\tilde{h}(x_j) - \tilde{h}(x_i) + (w(x_j) - w(x_i))c(s_n) + R(x_j, s_n) - R(x_i, s_n)}, \end{aligned}$$

By (i) and (iii) of (A.2), we obtain $\lim_n r_n(x) = (\tilde{h}(x) - \tilde{h}(x_i))/(\tilde{h}(x_j) - \tilde{h}(x_i))$. This and the relation

$$t(x, \theta_n) = t(x_i, \theta_n) + r_n(x)(t(x_j, \theta_n) - t(x_i, \theta_n)) \tag{3.1}$$

prove the lemma.

REMARK. Here we adopt the computational rule: $0 \cdot \infty = \infty \cdot 0 = 0$.

LEMMA 3.2. *Let $1 \leq i < j \leq m$ and let $\{\theta_n = (\alpha_n, \beta_n, \lambda_n)\}$ be a sequence in Θ_0 such that $\lambda_n < x_i$ for all n, $\lim_n \lambda_n = x_i$, $t_i = \lim_n t(x_i, \theta_n)$ and $t_j = \lim_n t(x_j, \theta_n)$. If $-\infty < t_i \leq t_j < \infty$ or $-\infty < t_i < t_j = \infty$, then*

$$\lim_n t(x, \theta_n) = t_j \qquad \text{for all } x > x_i.$$

PROOF. Let $r_n(x)$ be the same as in the proof of Lemma 3.1. It can be easily seen that $\lim_n r_n(x) = 1$, since $\lim_{s \searrow 0} h(s) = -\infty$. This, together with the relation (3.1), proves the lemma.

LEMMA 3.3. *Let $z = (z_1, \ldots, z_m) \in \partial \boldsymbol{F}(\Theta_0)$. Then there exists no triple (i_1, i_2, i_3) with $1 \leq i_1 < i_2 < i_3 \leq m$ such that $0 < z_{i_1} = z_{i_2} < z_{i_3} < 1$.*

PROOF. Since $z \in \partial \boldsymbol{F}(\Theta_0)$, we can choose a sequence $\{\theta_n = (\alpha_n, \beta_n, \lambda_n)\}$ in Θ_0 such that $\lim_n \boldsymbol{F}(\theta_n) = z$ and $\lim_n \lambda_n = \hat{\lambda}$. Assume that there exists a triple (i_1, i_2, i_3) such that $1 \leq i_1 < i_2 < i_3 \leq m$ and $0 < z_{i_1} = z_{i_2} < z_{i_3} < 1$. For simplicity, put $x'_j = x_{i_j}$ and $v_j = z_{i_j}$, $1 \leq j \leq 3$. Denote by $F^{-1}(z)$ the inverse function of $F(x)$. It is obvious that $\lim_n t(x'_j, \theta_n) = F^{-1}(v_j)$, $1 \leq j \leq 3$, since $t(x'_j, \theta_n) = F^{-1}(F(t(x'_j, \theta_n)))$. The inequalities $0 < v_1 < 1$ mean that $-\infty \leq \hat{\lambda} \leq x'_1$ and $\lambda_n < x'_1$ for sufficiently large n. If $\hat{\lambda} = x'_1$, then, by Lemma 3.2, $t_2 = t_3$. This is a contradiction. If $\hat{\lambda} = -\infty$, then, by Lemma 3.1, $t_2 = t_1 + (\tilde{h}(x_2) - \tilde{h}(x_1))(\tilde{h}(x_3) - \tilde{h}(x_1))^{-1}(t_3 - t_1)$. This contradicts the fact $t_1 = t_2$. Consider the case $-\infty < \hat{\lambda} < x'_1$. Choose a positive number δ so that $2\delta < \min(x'_1 - \hat{\lambda},\ v_1,\ v_3 - v_2,\ 1 - v_3)$. Put $\Theta' = \{\theta \in \boldsymbol{R}_+ \times \boldsymbol{R} \times [\hat{\lambda} - \delta, \hat{\lambda} + \delta];\ v_1 - \delta \leq F(x'_1, \theta) \leq v_1 + \delta\} \cap \{\theta \in \boldsymbol{R}_+ \times \boldsymbol{R} \times [\hat{\lambda} - \delta, \hat{\lambda} + \delta];\ v_3 - \delta \leq F(x'_3, \theta) \leq v_3 + \delta\}$. By Proposition 3.1, Θ' is compact. Since $\theta_n \in \Theta'$ for sufficiently large n, there exists $\theta' \in \boldsymbol{R}_+ \times \boldsymbol{R} \times [\hat{\lambda} - \delta, \hat{\lambda} + \delta]$ such that $\boldsymbol{F}(\theta') = z$. This contradicts $z \in \partial \boldsymbol{F}(\Theta_0)$.

LEMMA 3.4. *Let $z = (z_1, \ldots, z_m) \in \partial \boldsymbol{F}(\Theta_0)$ with $z_m = 1$. Then there exists no pair (i_1, i_2) such that $1 \leq i_1 < i_2 \leq m - 1$ and $0 < z_{i_1} \leq z_{i_2} < 1$.*

PROOF. Since $z \in \partial \boldsymbol{F}(\Theta_0)$, we can choose a sequence $\{\theta_n = (\alpha_n, \beta_n, \lambda_n)\}$ in Θ_0 such that $\lim_n \boldsymbol{F}(\theta_n) = z$ and $\lim_n \lambda_n = \hat{\lambda}$. Assume that there exists a pair (i_1, i_2) such that $1 \leq i_1 < i_2 \leq m - 1$ and $0 < z_{i_1} \leq z_{i_2} < 1$. For simplicity, put $x'_j = x_{i_j}$ and $v_j = z_{i_j}$, $j = 1, 2$. Denote by $F^{-1}(z)$ the inverse function of $F(x)$. It is obvious that $t_j \equiv \lim_n t(x'_j, \theta_n) = F^{-1}(v_j)$, $j = 1, 2$, since $t(x'_j, \theta_n) = F^{-1}(F(x'_j, \theta_n))$. The inequalities $0 < v_1 < 1$ imply that $-\infty \leq \hat{\lambda} \leq x'_1$ and $\lambda_n < x'_1$ for sufficiently large n. If $\hat{\lambda} = x'_1$, then by Lemma 3.2, $\lim_n t(x_m, \theta_n) = F^{-1}(v_2)$. This contradicts $z_m = 1$. If $\hat{\lambda} = -\infty$, then, by Lemma 3.1, $\lim_n t(x_m, \theta_n)$ is finite. This also contradicts $z_m = 1$. Consider the case $-\infty < \hat{\lambda} < x'_1$. From relations

$$\alpha_n = \frac{t(x_2', \theta_n) - t(x_1', \theta_n)}{h(x_2' - \lambda_n) - h(x_1' - \lambda_n)} \quad \text{and} \quad \beta_n = \alpha_n h(x_1' - \lambda_n) - t(x_1', \theta_n),$$

it follows that

$$\hat{\alpha} = \lim_n \alpha_n = \frac{F^{-1}(v_2) - F^{-1}(v_1)}{h(x_2' - \hat{\lambda}) - h(x_1' - \hat{\lambda})}$$

and

$$\hat{\beta} = \lim_n \beta_n = \hat{\alpha} h(x_1' - \hat{\lambda}) - F^{-1}(v_1).$$

Hence $-\infty < \lim_n t(x_m, \theta_n) = t(x_m, (\hat{\alpha}, \hat{\beta}, \hat{\lambda})) < \infty$, which contradicts $z_m = 1$. This completes the proof.

4. Proofs of Theorems 2.1 and 2.2

In this section we shall prove Theorems 2.1 and 2.2. To do this, we prepare some notation and results. Let us put

$$\boldsymbol{d}_{ijk}(z, z') = (\overbrace{0, \ldots, 0}^{i}, \overbrace{z, \ldots, z}^{j}, \overbrace{z', \ldots, z'}^{k}, \overbrace{1, \ldots, 1}^{m-i-j-k}),$$

$$0 \le z \le z' \le 1;\ i \ge 0;\ j \ge 0;\ k \ge 0;\ i + j + k \le m,$$

$$\begin{aligned} \mathscr{A}_3 = &(\textstyle\bigcup_{j+k=m; j,k \ge 1} \{\boldsymbol{d}_{0jk}(z, z');\ 0 \le z \le z' \le 1\}) \\ &\cup (\textstyle\bigcup_{i+j+k=m; i,j,k \ge 1} \{\boldsymbol{d}_{ijk}(z, z');\ 0 < z < z' \le 1\}) \\ &\cup (\textstyle\bigcup_{j+k<m; j,k \ge 1} \{\boldsymbol{d}_{0jk}(z, z');\ 0 \le z < z' < 1\}) \\ &\cup (\textstyle\bigcup_{i+j+k<m; i,j,k \ge 1} \{\boldsymbol{d}_{ijk}(z, z');\ 0 < z < z' < 1\}), \end{aligned}$$

where the union over the null index set is the empty set (ϕ).

The following result, due to Nakamura [8], is useful to represent the structure of the PCIB $\partial \boldsymbol{F}(\Theta_0)$ explicitly.

PROPOSITION 4.1. *Let Θ be a subset of Θ_0. The relation $\partial \boldsymbol{F}(\Theta) \subset \mathscr{A}_3$ holds if*

$$Cl(\boldsymbol{F}(\textstyle\bigcap_{j=1}^{3} \{\theta \in \Theta;\ u_j \le F(t(x_{i_j}, \theta)) \le u_j'\})) \subset \boldsymbol{F}(\Theta)$$

for every set of pairs (u_j, u_j'), $1 \le j \le 3$, with $0 < u_j \le u_j' < u_{j+1} < 1$ and of triples (i_1, i_2, i_3) with $1 \le i_1 < i_2 < i_3 \le m$.

The following lemma gives some information about the structure of $\partial \boldsymbol{F}(\Theta_0)$ and $\partial \boldsymbol{F}(\Theta_1)$.

LEMMA 4.1. *The following relations hold*:

(i) $\partial F(\Theta_0) \subset \mathscr{A}_3$.

(ii) $\partial F(\Theta_1) = \partial F(\Theta_0) \cup F(\Theta_2)$.

PROOF. Proof of (i): Note that $F(\Theta_0) = \{F(\theta); \theta \in \Theta_0\}$ and $\partial F(\Theta_0) = Cl(F(\Theta_0)) - F(\Theta_0)$. Let $0 < u_j \leq u_j' < u_{j+1} < 1$, $1 \leq j \leq 3$, and let $1 \leq i_1 < i_2 < i_3 \leq m$. Put $\Omega = \bigcap_{j=1}^3 \{\theta \in \Theta_0; u_j \leq F(x_j', \theta) \leq u_j'\}$, where $x_j' = x_{i_j}$. Choose a sequence $\{z_n\}$ in $F(\Omega)$ such that $\lim_n z_n = z$ and choose $\theta_n = (\alpha_n, \beta_n, \lambda_n)$ in Ω so that $z_n = F(\theta_n)$, $n = 1, 2, \ldots$. Without loss of generality, we may assume that $\lim_n \theta_n = \hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\lambda}) \in [0, \infty] \times \bar{R} \times \bar{R}$ and $\lim_n (t(x_1', \theta_n), t(x_2', \theta_n), t(x_3', \theta_n)) = (t_1, t_2, t_3) \in R^3$. Note that $-\infty < t_1 < t_2 < t_3 < \infty$. From $\theta_n \in \Omega$, it follows that $\lambda_n < x_1'$ for all n. Consider the case $-\infty < \hat{\lambda} \leq x_1'$. By Proposition 3.1, the set $\Omega_1 \equiv \bigcap_{j=2}^3 \{\theta \in R_+ \times R \times [\hat{\lambda} - 1, (x_1' + x_2')/2]; u_j \leq F(x_j', \theta) \leq u_j'\}$ is compact. Hence $\hat{\theta} \in \Theta_1$, since $\theta_n \in \Omega_1$ for sufficiently large n. Thus $z = F(\hat{\theta}) \in F(\Omega_1) \subset F(\Theta_0)$. Consider the case $\hat{\lambda} = -\infty$. By choosing a suitable subsequence of $\{\theta_n\}$, we may assume that $\lambda_n = -\infty$ for all n or $\lambda_n \neq -\infty$ for all n. If $\lambda_n = -\infty$ for all n, then $\theta_n \in \bigcap_{j=1}^2 \{\theta \in \Theta_2; u_j \leq F(x_j', \theta) \leq u_j'\}$ for all n. By Proposition 3.1, $\hat{\theta} \in \Theta_2 \subset \Theta_0$. Hence $z = F(\hat{\theta}) \in F(\Theta_0)$. Assume that $\lambda_n \neq -\infty$ for all n. By Lemma 3.1,

$$\lim_n t(x, \theta_n) = \alpha^* \tilde{h}(x) - \beta^* \qquad \text{for every } x \in R,$$

where $\alpha^* = (t_2 - t_1)/(\tilde{h}(x_2') - \tilde{h}(x_1'))$ and $\beta^* = \alpha^* \tilde{h}(x_1') - t_1$. Hence $z = F((\alpha^*, \beta^*, -\infty)) \in F(\Theta_2)$. Thus $Cl(F(\Omega)) \subset F(\Theta_0)$. From Proposition 4.1, the relation (i) follows.

Proof of (ii): Recall that $\partial F(\Theta_i) = Cl(F(\Theta_i)) - F(\Theta_i)$, $i = 1, 2$. To show the inclusion $\partial F(\Theta_0) \cup F(\Theta_2) \subset \partial F(\Theta_1)$, define a path $\theta(\lambda) = (\alpha(\lambda), \beta(\lambda), \lambda)$ on $(-\infty, x_1)$ by

$$\alpha(\lambda) = \frac{t_2 - t_1}{h(x_2 - \lambda) - h(x_1 - \lambda)},$$

$$\beta(\lambda) = \alpha(\lambda) h(x_1 - \lambda) - t_1,$$

for every pair $(t_1, t_2) \in R \times R$ with $t_1 \leq t_2$. It is easily seen that, with $\alpha = (t_2 - t_1)/(\tilde{h}(x_2) - \tilde{h}(x_1))$ and $\beta = \alpha \tilde{h}(x_1) - t_1$,

$$\lim_{\lambda \to -\infty} t(x, \theta(\lambda)) = \alpha \tilde{h}(x) - \beta \qquad \text{for all } x \in R.$$

This implies that $F(\Theta_2) \subset Cl(F(\Theta_1))$. Because of $m \geq 3$ and (A.1), $F(\Theta_1) \cap F(\Theta_2) = \phi$ and hence $F(\Theta_2) \subset \partial F(\Theta_1)$. Let $z \in \partial F(\Theta_0)$. There exists a sequence $\{z_n\}$ in $F(\Theta_0)$ such that $\lim_n z_n = z$. Choose $\theta_n = (\alpha_n, \beta_n, \lambda_n) \in \Theta_0$, so that $F(\theta_n) = z_n$, $n = 1, 2, \ldots$. If $\lambda_n \neq -\infty$ for infinitely many n, then $\theta_n \in \Theta_1$ for infinitely many n and hence $z \in \partial F(\Theta_1)$. Consider the case where $\lambda_n = -\infty$

for sufficiently many n. In this case, we may assume that $\lambda_n = -\infty$ for all n. Since $\theta_n \in \Theta_2$ and $\boldsymbol{F}(\Theta_2) \subset \partial \boldsymbol{F}(\Theta_1)$, we have $z \in Cl(\boldsymbol{F}(\Theta_1))$. Hence $z \in \partial \boldsymbol{F}(\Theta_1)$. This proves the desired inclusion. Note that $\boldsymbol{F}(\Theta_0) = \boldsymbol{F}(\Theta_1) \cup \boldsymbol{F}(\Theta_2)$. Let $z \in \partial \boldsymbol{F}(\Theta_1)$. There exists a sequence $\{z_n\}$ in $\boldsymbol{F}(\Theta_1)$ such that $\lim_n z_n = z$. If $z \notin \boldsymbol{F}(\Theta_0)$, then $z \in \partial \boldsymbol{F}(\Theta_0)$. If $z \in \boldsymbol{F}(\Theta_0)$, then $z \in \boldsymbol{F}(\Theta_2)$, since $z \notin \boldsymbol{F}(\Theta_1)$. This proves the converse inclusion.

By the relation $\{\boldsymbol{b}_m(z); 0 \le z < 1\} \cup \{\boldsymbol{d}_{m-211}(z, z'); 0 < z < z' < 1\} \subset \boldsymbol{F}(\Theta_0)$ and by Lemmas 3.3 and 3.4, we have

LEMMA 4.2. *The following relations hold*:

(i)
$$\begin{aligned} \partial \boldsymbol{F}(\Theta_0) \cap (\textstyle\bigcup_{j+k=m; j,k \ge 1} \{\boldsymbol{d}_{0jk}(z, z'); 0 \le z \le z' \le 1\}) \\ \subset (\textstyle\bigcup_{i=0}^{m-1} \{\boldsymbol{a}_i\}) \cup \{\boldsymbol{b}_1(z); 0 < z < 1\} \cup \{z\boldsymbol{a}_0; 0 < z < 1\} \\ \cup (\textstyle\bigcup_{i=1}^{m-2} \{\boldsymbol{c}_i(0, z'); 0 < z' < 1\}) \\ \cup \{\boldsymbol{c}_1(z, z'); 0 < z < z' < 1\}. \end{aligned}$$

(ii)
$$\begin{aligned} \partial \boldsymbol{F}(\Theta_0) \cap (\textstyle\bigcup_{i+j+k=m; i,j,k \ge 1} \{\boldsymbol{d}_{ijk}(z, z'); 0 < z < z' \le 1\}) \\ \subset (\textstyle\bigcup_{i=2}^{m-2} \{\boldsymbol{b}_i(z); 0 < z < 1\}) \\ \cup (\textstyle\bigcup_{i=1}^{m-3} \{\boldsymbol{c}_{i+1}(z, z'); 0 < z < z' < 1\}). \end{aligned}$$

(iii)
$$\begin{aligned} \partial \boldsymbol{F}(\Theta_0) \cap (\textstyle\bigcup_{j+k<m; j,k \ge 1} \{\boldsymbol{d}_{0jk}(z, z'); 0 \le z < z' < 1\}) \\ \subset \textstyle\bigcup_{j=2}^{m-1} \{\boldsymbol{b}_j(z'); 0 < z' < 1\}. \end{aligned}$$

(iv)
$$\partial \boldsymbol{F}(\Theta_0) \cap (\textstyle\bigcup_{i+j+k<m; i,j,k \ge 1} \{\boldsymbol{d}_{ijk}(z, z'); 0 < z < z' < 1\}) = \phi.$$

With the aid of Lemmas 4.1 and 4.2, we have

LEMMA 4.3. *The following relation holds*:

$$\begin{aligned} \partial \boldsymbol{F}(\Theta_0) \subset (\textstyle\bigcup_{i=0}^{m-1} \{\boldsymbol{a}_i\}) \cup \{z\boldsymbol{a}_0; 0 < z < 1\} \\ \cup (\textstyle\bigcup_{i=1}^{m-1} \{\boldsymbol{b}_i(z); 0 < z < 1\}) \\ \cup (\textstyle\bigcup_{i=1}^{m-2} \{\boldsymbol{c}_i(0, z'); 0 < z' < 1\}) \\ \cup (\textstyle\bigcup_{i=0}^{m-3} \{\boldsymbol{c}_{i+1}(z, z'); 0 < z < z' < 1\}). \end{aligned}$$

We shall represent the structure of $\partial \boldsymbol{F}(\Theta_0)$ explicitly. For each pair (x, x') with $-\infty < x < x' < \infty$ and for each pair (t, t') with $-\infty < t < t' < \infty$, define a path $\theta(\lambda) = (\alpha(\lambda), \beta(\lambda), \lambda)(\lambda \in (-\infty, x))$ by

$$\alpha(\lambda) = \alpha(\lambda; x, x', t, t') = \frac{t' - t}{h(x' - \lambda) - h(x - \lambda)}, \tag{4.1}$$

$$\beta(\lambda) = \beta(\lambda; x, x', t, t') = \alpha(\lambda) h(x - \lambda) - t. \tag{4.2}$$

Note that $t(x, \theta(\lambda)) = t$ and $t(x', \theta(\lambda)) = t'$ for all $\lambda \in (-\infty, x)$.

Now we are in position to prove Theorems 2.1 and 2.2.

PROOF OF THEOREM 2.1: Put $x_0 = -\infty$. To show $\{\boldsymbol{a}_0, \ldots, \boldsymbol{a}_{m-1}\} \subset \partial \boldsymbol{F}(\Theta_0)$, choose (α_i, λ_i), $0 \leq i \leq m-1$, so that $x_i < \lambda_i < x_{i+1}$ and $\alpha_i > 0$. It is easy to see that $\lim_{\beta \to -\infty} \boldsymbol{F}((\alpha_1, \beta, \lambda_i)) = \boldsymbol{a}_i$, and that $\boldsymbol{a}_i \notin \boldsymbol{F}(\Theta_0)$. Hence $\boldsymbol{a}_i \in \partial \boldsymbol{F}(\Theta_0)$ for all $i = 0, \ldots, m-1$.

To prove $\{z\boldsymbol{a}_0; 0 < z < 1\} \subset \partial \boldsymbol{F}(\Theta_0)$, put $x' = x_1$ and $t' = F^{-1}(z)$. Choose x and t so that $-\infty < x < x'$ and $-\infty < t < t'$. Let $\theta(\lambda)$ be the path defined by (4.1) and (4.2). Lemma 3.2 shows that $\lim_{\lambda \to x} t(x_i, \theta(\lambda)) = t'$ for all $i = 1, \ldots, m$. Hence $\lim_{\lambda \to x} \boldsymbol{F}(\theta(\lambda)) = z\boldsymbol{a}_0$. Noting that $z\boldsymbol{a}_0 \notin \boldsymbol{F}(\Theta_0)$, we have the desired inclusion.

To prove $\bigcup_{i=1}^{m-1} \{\boldsymbol{b}_i(z); 0 < z < 1\} \subset \partial \boldsymbol{F}(\Theta_0)$, let $1 \leq i \leq m-1$ and $0 < z < 1$. Choose λ so that $x_{i-1} < \lambda < x_i$ and put $\beta(\lambda) = \alpha h(x_i - \lambda) - F^{-1}(z)$. Then $t(x, (\alpha, \beta(\lambda), \lambda)) = F^{-1}(z) + \alpha(h(x - \lambda) - h(x_i - \lambda))$ for all $x > \lambda$. Hence $\lim_{\alpha \to \infty} \boldsymbol{F}((\alpha, \beta(\lambda), \lambda)) = \boldsymbol{b}_i(z)$. Since $\boldsymbol{b}_i(z) \notin \boldsymbol{F}(\Theta_0)$, the desired inclusion is established.

To prove $\bigcup_{i=0}^{m-3} \{\boldsymbol{c}_{i+1}(z, z'); 0 < z < z' < 1\} \subset \partial \boldsymbol{F}(\Theta_0)$, let $0 \leq i \leq m-3$, $x_i < \lambda < x_{i+1}$ and $0 < z < z' < 1$. Put $x = x_{i+1}$, $x' = x_{i+2}$, $t = F^{-1}(z)$ and $t' = F^{-1}(z')$. Let $\theta(\lambda)$ be the path defined by (4.1) and (4.2). Then $t(x, \theta(\lambda)) = t$ and $t(x', \theta(\lambda)) = t'$ for all $\lambda < x$. By Lemma 3.2, $\lim_{\lambda \to x} \boldsymbol{F}(\theta(\lambda)) = \boldsymbol{c}_{i+1}(z, z')$. The desired inclusion follows from $\boldsymbol{c}_{i+1}(z, z') \notin \boldsymbol{F}(\Theta_0)$. By the same argument as above, we can prove $\bigcup_{i=1}^{m-2} \{\boldsymbol{c}_i(0, z'); 0 < z' < 1\} \subset \partial \boldsymbol{F}(\Theta_0)$. This completes the proof.

PROOF OF THEOREM 2.2: Theorem 2.2 follows immediately from Lemma 4.1 and Theorem 2.1.

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