J. Sci. Hiroshima Univ. Ser. A-I
25 (1961) 95–112

A Stroboscopic Method in the Cylindrical Phase Space

Masataka Yorinaga

(Received March 6, 1961)

1. Introduction

In this paper, we are concerned with a real system of $n+1$ nonlinear differential equations of the form as follows:

$$(1.1)\qquad \begin{cases} \dfrac{dx_i}{dt}=\varepsilon X_i(x, \theta, t, \varepsilon) & (i=1, 2,\cdots, n), \\ \dfrac{d\theta}{dt}=\Theta(x, \theta, \varepsilon)+\varepsilon\Psi(x, \theta, t, \varepsilon), & \end{cases}$$

where

1° *ε is a parameter such that $|\varepsilon|\ll 1$,*

2° *$X_i(x, \theta, t, \varepsilon)$ $(i=1, 2,\cdots, n)$, $\Theta(x, \theta, \varepsilon)$ and $\Psi(x, \theta, t, \varepsilon)$ are twice continuously differentiable with respect to (x, θ, ε) in the domain*

$$D:\ |x|=\sum_{i=1}^{n}|x_i|<M,\ -\infty<\theta,\ t<+\infty,\ |\varepsilon|<\delta,$$

3° *$X_i(x, \theta, t, \varepsilon)$ $(i=1, 2,\cdots, n)$ and $\Psi(x, \theta, t, \varepsilon)$ are continuous with respect to t in the domain D and are periodic in t with period $T_0>0$,*

4° *$X_i(x, \theta, t, \varepsilon)$ $(i=1, 2,\cdots, n)$, $\Theta(x, \theta, \varepsilon)$ and $\Psi(x, \theta, t, \varepsilon)$ are periodic in θ with period 2π,*

5° *$\Theta(x, \theta, 0)\neq 0$ for any $(x, \theta)\in D$.*

The system of the form (1.1) cannot have any periodic solution of the proper sense, because $\theta(t)$ is monotonous due to the assumption 5°. But it may have a solution such that

$$(1.2)\qquad \begin{cases} x_i(t+lT_0)=x_i(t) & (i=1, 2,\cdots, n), \\ \theta(t+lT_0)=\theta(t)+2m\pi, & \end{cases}$$

where l and m are integers. Such a solution represents a closed curve in the cylindrical phase space, namely the space consisting of the points (x, θ), θ being considered modulus 2π. So the solution satisfying the condition (1.2) can be called a periodic solution in the cylindrical phase space. In the sequel, for brevity, we shall call a periodic solution in the cylindrical phase space simply a periodic solution.

The problem as to seeking a periodic solution of (1.1) and deciding its stability can be solved by extending the so-called stroboscopic method due to N. Minorsky [3, 4, 5, 7].

In this paper, the method to seek a periodic solution of (1.1) and to decide its stability will be described. And the two-dimensional case will be discussed more in detail and, from its consequence, there will be derived the results of A. M. Кац [1] and W. S. Loud [2] as the special cases of our results.

2. Existence of a periodic solution

Let

$$(2.1)\qquad \begin{cases} x_i = x_i(u, \varphi, t, \varepsilon) & (i=1, 2, \cdots, n), \\ \theta = \theta(u, \varphi, t, \varepsilon) \end{cases}$$

be the solution of (1.1) such that

$$(2.2)\qquad \begin{cases} x_i(u, \varphi, 0, \varepsilon) = u_i & (i=1, 2, \cdots, n), \\ \theta(u, \varphi, 0, \varepsilon) = \varphi, \end{cases}$$

where $|u| = \sum_{i=1}^{n} |u_i| < M$. Then, as is seen from the form of (1.1), such a solution exists in any finite interval containing $t=0$ provided $|\varepsilon|$ is sufficiently small. Further, by the assumption 2°, in such an interval, for any sufficiently small $|\varepsilon|$, the functions $x_i(u, \varphi, t, \varepsilon)$ $(i=1, 2, \cdots, n)$ and $\theta(u, \varphi, t, \varepsilon)$ can be written as follows:

$$(2.3)\qquad \begin{cases} x_i(u, \varphi, t, \varepsilon) = x_i^{(0)}(u, \varphi, t) + \varepsilon x_i^{(1)}(u, \varphi, t) + p_i(u, \varphi, t, \varepsilon) & (i=1, 2, \cdots, n), \\ \theta(u, \varphi, t, \varepsilon) = \theta_0(u, \varphi, t) + \varepsilon\theta_1(u, \varphi, t) + \eta(u, \varphi, t, \varepsilon), \end{cases}$$

where $p_i(u, \varphi, t, \varepsilon) = o(\varepsilon)$ $(i=1, 2, \cdots, n)$ and $\eta(u, \varphi, t, \varepsilon) = o(\varepsilon)$ as $\varepsilon \to 0$.

In this expression, by the initial condition (2.2), it must be that

$$(2.4)\qquad \begin{cases} x_i^{(0)}(u, \varphi, 0) = u_i & (i=1, 2, \cdots, n), \\ \theta_0(u, \varphi, 0) = \varphi, \end{cases}$$

$$(2.5)\qquad \begin{cases} x_i^{(1)}(u, \varphi, 0) = p_i(u, \varphi, 0, \varepsilon) = 0 & (i=1, 2, \cdots, n), \\ \theta_1(u, \varphi, 0) = \eta(u, \varphi, 0, \varepsilon) = 0. \end{cases}$$

On the other hand, the substitution of (2.3) into (1.1) yields the differential equations as follows:

$$(2.6)\quad \begin{cases} \dfrac{dx_i^{(0)}}{dt}=0 \qquad (i=1, 2,\cdots, n), \\[2mm] \dfrac{d\theta_0}{dt}=\Theta(x^{(0)}, \theta_0, 0), \\[2mm] \dfrac{dx_i^{(1)}}{dt}=X_i(x^{(0)}, \theta_0, t, 0) \qquad (i=1, 2,\cdots, n), \\[2mm] \dfrac{d\theta_1}{dt}=\displaystyle\sum_{j=1}^{n}\frac{\partial\Theta}{\partial x_j}(x^{(0)}, \theta_0, 0)x_j^{(1)}+\frac{\partial\Theta}{\partial\theta}(x^{(0)}, \theta_0, 0)\theta_1 \\[2mm] \qquad\qquad +\dfrac{\partial\Theta}{\partial\varepsilon}(x^{(0)}, \theta_0, 0)+\Psi(x^{(0)}, \theta_0, t, 0). \end{cases}$$

These equations can be solved successively by quadrature under the initial conditions (2.4) and (2.5).

In fact, from the first equations follows

$$x_i^{(0)}(u, \varphi, t)=u_i \qquad (i=1, 2,\cdots, n).$$

Consequently, substituting this into the second, we have

$$(2.7)\qquad \Xi(u, \theta_0)=t+\Xi(u, \varphi),$$

where

$$\Xi(u, \theta)=\int_0^{\theta}\frac{d\theta'}{\Theta(u, \theta', 0)}.$$

Since $\dfrac{\partial\Xi}{\partial\theta}(u, \theta)=\dfrac{1}{\Theta(u, \theta, 0)}\neq 0$, (2.7) can be solved as

$$\theta_0=\theta_0(u, \varphi, t),$$

which is a desired solution of the second of (2.6). Then the third equations of (2.6) are readily solved as

$$(2.8)\qquad x_i^{(1)}=\int_0^t X_i(u, \theta_0(u, \varphi, t'), t', 0)dt' \qquad (i=1, 2,\cdots, n).$$

Since $\Theta(u, \theta_0(u, \varphi, t), 0)$ is a solution of the linear homogeneous equation

$$\frac{d\bar{\theta}_1}{dt}=\frac{\partial\Theta}{\partial\theta}(u, \theta_0(u, \varphi, t), 0)\bar{\theta}_1,$$

the fourth equation of (2.6) is solved by the method of variation of constants as follows:

$$(2.9)\qquad \theta_1(u, \varphi, t)=\Theta(u, \theta_0(u, \varphi, t), 0)\int_0^t\frac{1}{\Theta(u, \theta_0(u, \varphi, t'), 0)}$$

$$\times\left\{\sum_{j=1}^{n}\frac{\partial\theta}{\partial x_j}(u,\ \theta_0(u,\ \varphi,\ t'),\ 0)\int_0^{t'}X_j(u,\ \theta_0(u,\ \varphi,\ t''),\ t'',\ 0)dt''\right.$$

$$\left.+\frac{\partial\theta}{\partial\varepsilon}(u,\ \theta_0(u,\ \varphi,\ t'),\ 0)+\Psi(u,\ \theta_0(u,\ \varphi,\ t'),\ t',\ 0)\right\}dt'.$$

Now let us seek a periodic solution of the form (2.1).

As is readily seen from the periodicity of the right-hand sides of (1.1), the necessary and sufficient condition that the solution (2.1) may be periodic is

$$\begin{cases} x_i(u,\ \varphi,\ L,\ \varepsilon)=u_i & (i=1,\ 2,\cdots,\ n), \\ \theta(u,\ \varphi,\ L,\ \varepsilon)=\varphi+2m\pi, \end{cases} \tag{2.10}$$

where $L=lT_0$. This condition can be written by (2.3), (2.4) and (2.5) as follows:

$$\begin{cases} x_i^{(1)}(u,\ \varphi,\ L)+o(1)=0 & (i=1,\ 2,\cdots,\ n), \\ \theta_0(u,\ \varphi,\ L)+\varepsilon\theta_1(u,\ \varphi,\ L)+o(\varepsilon)=\varphi+2m\pi. \end{cases} \tag{2.11}$$

When $\varepsilon=0$, the above condition is reduced to

$$\begin{cases} x_i^{(1)}(u,\ \varphi,\ L)=0 & (i=1,\ 2,\cdots,\ n), \\ \theta_0(u,\ \varphi,\ L)=\varphi+2m\pi. \end{cases} \tag{2.12}$$

The latter condition is rewritten by (2.7) as follows:

$$\Xi(u,\ \varphi+2m\pi)-\Xi(u,\ \varphi)=L,$$

which can be rewritten as

$$m\Omega(u)=L, \tag{2.13}$$

where

$$\Omega(u)=\int_0^{2\pi}\frac{d\theta}{\Theta(u,\ \theta,\ 0)}. \tag{2.14}$$

Thus the condition (2.12) can be replaced by

$$\begin{cases} x_i^{(1)}(u,\ \varphi,\ L)=0 & (i=1,\ 2,\cdots,\ n), \\ m\Omega(u)=L. \end{cases} \tag{2.15}$$

Then, from derivation of (2.12), it is evident that, if there exists no real value of $(u,\ \varphi)$ satisfying (2.15), there exists no real value of $(u,\ \varphi)$ satisfying (2.11), or, in other words, there exists no periodic solution of the initial equation (1.1).

When there exists a set of real values of $(u,\ \varphi)$ satisfying (2.15), let it be $(u^{(0)},\ \varphi_0)$. Then $(u^{(0)},\ \varphi_0)$ evidently satisfies (2.12). Now, by the assumption 2°, the left-hand sides of (2.11) are continuously differentiable with respect to u,

φ and ε. Therefore, if the Jacobian J of the functions

$$x_i^{(1)}(u, \varphi, L) \quad \text{and} \quad \xi(u, \varphi, L)=\theta_0(u, \varphi, L)-\varphi-2m\pi$$

does not vanish for $u=u^{(0)}$ and $\varphi=\varphi_0$, there exists a unique set of real values $(\tilde{u}, \tilde{\varphi})$ satisfying (2.10) such that

$$(\tilde{u}, \tilde{\varphi})=\{u(\varepsilon), \varphi(\varepsilon)\} \in C_\varepsilon^1 \quad \text{and} \quad \{u(0), \varphi(0)\}=(u^{(0)}, \varphi_0),$$

or in other words, there exists a periodic solution of the initial equation (1.1) lying near the periodic solution

$$x_i=u_i^{(0)}\ (i=1, 2, \cdots, n), \quad \theta=\theta_0(u^{(0)}, \varphi_0, t)$$

of the unperturbed system.

For the derivatives of $\theta_0(u, \varphi, L)$ with respect to u_i $(i=1, 2, \cdots, n)$ and φ, we can derive the simple formulas from (2.7) as follows.

In fact, from (2.7), we have:

$$-\int_0^{\theta_0} \frac{\frac{\partial \Theta}{\partial u_i}}{\Theta^2} d\theta+\frac{1}{\Theta} \frac{\partial \theta_0}{\partial u_i}=-\int_0^{\varphi} \frac{\frac{\partial \Theta}{\partial u_i}}{\Theta^2} d\theta \qquad (i=1, 2, \cdots, n), \tag{2.16}$$

$$\frac{1}{\Theta(u, \theta_0, 0)} \frac{\partial \theta_0}{\partial \varphi}=\frac{1}{\Theta(u, \varphi, 0)}. \tag{2.17}$$

From (2.16), it readily follows that

$$\frac{\partial \theta_0}{\partial u_i}=\Theta(u, \theta_0, 0)\int_{\varphi}^{\theta_0}\left[\frac{\partial \Theta(u, \theta, 0)}{\partial u_i}/\Theta^2(u, \theta, 0)\right]d\theta \qquad (i=1, 2, \cdots, n), \tag{2.18}$$

which implies

$$\begin{aligned} \frac{\partial \theta_0}{\partial u_i}(u^{(0)}, \varphi_0, L) &= \Theta(u^{(0)}, \varphi_0, 0) \int_{\varphi_0}^{\varphi_0+2m\pi}\left[\frac{\partial \Theta(u, \theta, 0)}{\partial u_i}/\Theta^2(u, \theta, 0)\right]_{u=u^{(0)}} d\theta \\ &= -m\Theta(u^{(0)}, \varphi_0, 0) \frac{\partial \Omega}{\partial u_i}(u^{(0)}) \qquad (i=1, 2, \cdots, n). \end{aligned} \tag{2.19}$$

The relation (2.17) evidently implies

$$\frac{\partial \theta_0}{\partial \varphi}(u^{(0)}, \varphi_0, L)=1. \tag{2.20}$$

Thus we see that

$$J=\begin{vmatrix} \frac{\partial x_i^{(1)}}{\partial u_j}(u^{(0)}, \varphi_0, L) & \frac{\partial x_i^{(1)}}{\partial \varphi}(u^{(0)}, \varphi_0, L) \\ -m\Theta(u^{(0)}, \varphi_0, 0)\frac{\partial \Omega}{\partial u_j}(u^{(0)}) & 0 \end{vmatrix} \tag{2.21}$$

$$(i\downarrow, j\rightarrow).$$

The above results are summarized as

Theorem 1. *Given a real system*

$$(1.1)\qquad \begin{cases} \dfrac{dx_i}{dt}=\varepsilon X_i(x,\ \theta,\ t,\ \varepsilon) & (i=1,\ 2,\cdots,\ n),\\[2ex] \dfrac{d\theta}{dt}=\Theta(x,\ \theta,\ \varepsilon)+\varepsilon\Psi(x,\ \theta,\ t,\ \varepsilon), \end{cases}$$

such that the right-hand sides satisfy the assumptions 1°, 2°, 3°, 4° *and* 5° *of* §1. *When there exists no real value of* $(u,\ \varphi)$ *satisfying* (2.15), *there exists no periodic solution of* (1.1). *When there exists a set of real values* $(u,\ \varphi)=(u^{(0)},\ \varphi_0)$ *satisfying* (2.15), *if* J *given by* (2.21) *does not vanish, there exists a periodic solution of* (1.1) *lying near the periodic solution*

$$x_i=u_i^{(0)}\ (i=1,\ 2,\cdots,\ n),\ \theta=\theta_0(u^{(0)},\ \varphi_0,\ t)$$

of the unperturbed system.

Remark In our problem, due to the presence of the equation

$$\frac{d\theta}{dt}=\Theta(x,\ \theta,\ \varepsilon)+\varepsilon\Psi(x,\ \theta,\ t,\ \varepsilon),$$

there appears the additional condition (2.13) compared with the case for which the usual stroboscopic method can be applied directly.

3. Stability of the periodic solution

As is well known, the periodic solution obtained in the preceding paragraph is stable if the iteration of the transformation

$$(3.1)\qquad \begin{cases} r'_i=x_i(\tilde{u}+r,\ \tilde{\varphi}+\sigma,\ L,\ \varepsilon)-\tilde{u} & (i=1,\ 2,\cdots,\ n),\\ \sigma'=\theta(\tilde{u}+r,\ \tilde{\varphi}+\sigma,\ L,\ \varepsilon)-\tilde{\varphi}-2m\pi \end{cases}$$

converges for sufficiently small $|r|$ and $|\sigma|$.

But, since $(\tilde{u},\ \tilde{\varphi})$ is a value satisfying (2.11), the above transformation can be written as follows:

$$(3.2)\qquad \begin{cases} r'_i=\displaystyle\sum_{j=1}^{n}\frac{\partial x_i}{\partial u_j}(\tilde{u},\ \tilde{\varphi},\ L,\ \varepsilon)r_j+\frac{\partial x_i}{\partial\varphi}(\tilde{u},\ \tilde{\varphi},\ L,\ \varepsilon)\sigma+o(\rho), & (i=1,\ 2,\cdots,\ n),\\[2ex] \sigma'=\displaystyle\sum_{j=1}^{n}\frac{\partial\theta}{\partial u_j}(\tilde{u},\ \tilde{\varphi},\ L,\ \varepsilon)r_j+\frac{\partial\theta}{\partial\varphi}(\tilde{u},\ \tilde{\varphi},\ L,\ \varepsilon)\sigma+o(\rho), \end{cases}$$

where $\rho=\sum_{i=1}^{n}|r_i|+|\sigma|$. Since

$$x_i(u,\ \varphi,\ L,\ \varepsilon),\ \theta(u,\ \varphi,\ L,\ \varepsilon)\in C^2_{u,\varphi,\varepsilon}\qquad (i=1,\ 2,\cdots,\ n)$$

by the assumption 2° of §1, it readily follows from (2.3) that

$$
(3.3)\qquad \begin{cases} \dfrac{\partial x_i}{\partial u_j}(\tilde{u}, \tilde{\varphi}, L, \varepsilon)=\delta_{ij}+\varepsilon\dfrac{\partial x_i{}^{(1)}}{\partial u_j}(\tilde{u}, \tilde{\varphi}, L)+O(\varepsilon), \\ \dfrac{\partial x_i}{\partial \varphi}(\tilde{u}, \tilde{\varphi}, L, \varepsilon)=\varepsilon\dfrac{\partial x_i{}^{(1)}}{\partial \varphi}(\tilde{u}, \tilde{\varphi}, L)+o(\varepsilon), \\ \dfrac{\partial \theta}{\partial u_j}(\tilde{u}, \tilde{\varphi}, L, \varepsilon)=-m\Theta(u^{(0)}, \varphi_0, 0)\dfrac{\partial \Omega}{\partial u_j}(u^{(0)})+O(\varepsilon) \end{cases}
$$
$$(i, j=1, 2, \cdots, n).$$

The expression for $\dfrac{\partial \theta}{\partial \varphi}(\tilde{u}, \tilde{\varphi}, L, \varepsilon)$ is obtained in the following way.

In fact, from (2.17),

$$\Theta(u, \varphi, 0)\frac{\partial \theta_0}{\partial \varphi}=\Theta(u, \theta_0, 0).$$

Differentiating both sides of this relation, we have

$$\frac{\partial \Theta}{\partial u_i}(u, \varphi, 0)\frac{\partial \theta_0}{\partial \varphi}+\Theta(u, \varphi, 0)\frac{\partial^2 \theta_0}{\partial u_i \partial \varphi}$$
$$=\frac{\partial \Theta}{\partial u_i}(u, \theta_0, 0)+\frac{\partial \Theta}{\partial \theta}(u, \theta_0, 0)\frac{\partial \theta_0}{\partial u_i} \qquad (i=1, 2, \cdots, n)$$

and

$$\frac{\partial \Theta}{\partial \theta}(u, \varphi, 0)\frac{\partial \theta_0}{\partial \varphi}+\Theta(u, \varphi, 0)\frac{\partial^2 \theta_0}{\partial \varphi^2}=\frac{\partial \Theta}{\partial \theta}(u, \theta_0, 0)\frac{\partial \theta_0}{\partial \varphi}.$$

Then, since $\theta_0=\varphi_0+2m\pi$ for $(u, \varphi)=(u^{(0)}, \varphi_0)$ and $t=L$, we see from (2.19) and (2.20) that

$$
\begin{cases} \dfrac{\partial^2 \theta_0}{\partial u_i \partial \varphi}(u^{(0)}, \varphi_0, L)=-m\dfrac{\partial \Theta}{\partial \theta}(u^{(0)}, \varphi_0, 0)\dfrac{\partial \Omega}{\partial u_i}(u^{(0)}) \qquad (i=1, 2, \cdots, n), \\ \dfrac{\partial^2 \theta_0}{\partial \varphi^2}(u^{(0)}, \varphi_0, L)=0. \end{cases}
$$

From this together with (2.19) and (2.20), we find that

$$\frac{\partial \theta}{\partial \varphi}(\tilde{u}, \tilde{\varphi}, L, \varepsilon)=1-m\frac{\partial \Theta}{\partial \theta}(u^{(0)}, \varphi_0, 0)\sum_{j=1}^{n}\frac{\partial \Omega}{\partial u_j}(u^{(0)})(\tilde{u}_j-u_j{}^{(0)})$$
$$+\varepsilon\frac{\partial \theta_1}{\partial \varphi}(\tilde{u}, \tilde{\varphi}, L)+o(|\tilde{u}-u^{(0)}|+|\tilde{\varphi}-\varphi_0|)+o(\varepsilon).$$

However, since $(\tilde{u}, \tilde{\varphi})=(u(\varepsilon), \varphi(\varepsilon))\in C_\varepsilon^1$ satisfy (2.11), it readily follows from the latter of (2.11) that

$$\sum_{j=1}^{n} \frac{\partial \Omega}{\partial u_j}(u^{(0)})(\tilde{u}_j - u_j^{(0)}) = \varepsilon \frac{\theta_1(u^{(0)}, \varphi_0, L)}{m\Theta(u^{(0)}, \varphi_0, 0)} + o(\varepsilon).$$

Thus we see that

$$\frac{\partial \theta}{\partial \varphi}(\tilde{u}, \tilde{\varphi}, L, \varepsilon) = 1 + \varepsilon \left\{ -\frac{\theta_1(u^{(0)}, \varphi_0, L)}{\Theta(u^{(0)}, \varphi_0, 0)} \frac{\partial \Theta}{\partial \theta}(u^{(0)}, \varphi_0, 0) + \frac{\partial \theta_1}{\partial \varphi}(u^{(0)}, \varphi_0, L) \right\} + o(\varepsilon). \tag{3.4}$$

Since $(\tilde{u}, \tilde{\varphi}) = (u(\varepsilon), \varphi(\varepsilon)) \in C_\varepsilon^1$ and $(u(0), \varphi(0)) = (u^{(0)}, \varphi_0)$, the expressions (3.3) are rewritten as follows:

$$\begin{cases} \dfrac{\partial x_i}{\partial u_j}(\tilde{u}, \tilde{\varphi}, L, \varepsilon) = \delta_{ij} + \varepsilon \dfrac{\partial x_i}{\partial u_j}(u^{(0)}, \varphi_0, L) + o(\varepsilon), \\ \dfrac{\partial x_i}{\partial \varphi}(\tilde{u}, \tilde{\varphi}, L, \varepsilon) = \varepsilon \dfrac{\partial x_i}{\partial \varphi}(u^{(0)}, \varphi_0, L) + o(\varepsilon), \\ \dfrac{\partial \theta}{\partial u_j}(\tilde{u}, \tilde{\varphi}, L, \varepsilon) = -m\Theta(u^{(0)}, \varphi_0, 0) \dfrac{\partial \Omega}{\partial u_j}(u^{(0)}) + O(\varepsilon), \end{cases} \tag{3.5}$$

$$(i, j = 1, 2, \cdots, n).$$

Thus, substituting (3.4) and (3.5) into (3.2), we see that the matrix A of the coefficients of the linear parts of the transformation (3.2) becomes

$$A = \begin{pmatrix} \delta_{ij} + \varepsilon \dfrac{\partial x_i}{\partial u_j} + o(\varepsilon) & \varepsilon \dfrac{\partial x_i}{\partial \varphi} + o(\varepsilon) \\ -m\Theta(u, \varphi, 0) \dfrac{\partial \Omega}{\partial u_j} + O(\varepsilon) & 1 + \varepsilon \left\{ -\dfrac{\theta_1}{\Theta(u, \varphi, 0)} \dfrac{\partial \Theta}{\partial \theta}(u, \varphi, 0) + \dfrac{\partial \theta_1}{\partial \varphi} \right\} + o(\varepsilon) \end{pmatrix} \tag{3.6}$$

$$(i \downarrow, j \rightarrow)$$

for $(u, \varphi) = (u^{(0)}, \varphi_0)$ and $t = L$.

Then, since the characteristic roots of A are the multipliers of the variation equations, we have

Theorem 2. *The pericdic solution whose existence is affiirmed in Theorem 1 is stable if the characteristic roots of the matrix A are all less than unity in absolute value.*

4. Two-dimensional case

In this paragraph the two-dimensional case is discussed more in detail.

Let the given system of two equations satisfying all the assumptions of §1 be

$$(4.1)\qquad \begin{cases} \dfrac{dx}{dt}=\varepsilon X(x,\ \theta,\ t,\ \varepsilon) \\[2ex] \dfrac{d\theta}{dt}=\Theta(x,\ \theta,\ \varepsilon)+\varepsilon\Psi(x,\ \theta,\ t,\ \varepsilon). \end{cases}$$

Then, by §2, the equations by which the first approximations $(u^{(0)},\ \varphi_0)$ of the initial values of a periodic solution are determined become

$$(4.2)\qquad \begin{cases} x^{(1)}(u,\ \varphi,\ L)=0, \\ m\Omega(u)=L, \end{cases}$$

where

$$x^{(1)}(u,\ \varphi,\ L)=\int_0^L X[u,\ \theta_0(u,\ \varphi,\ t),\ t,\ 0]dt.$$

Also the Jacobian J of (2.21) becomes

$$J=\det\begin{pmatrix} \dfrac{\partial x^{(1)}}{\partial u}(u^{(0)},\ \varphi_0,\ L) & \dfrac{\partial x^{(1)}}{\partial \varphi}(u^{(0)},\ \varphi_0,\ L) \\[2ex] -m\Theta(u^{(0)},\ \varphi_0,\ 0)\ \Omega'(u^{(0)}) & 0 \end{pmatrix}$$

But, by (4.2) and (2.17),

$$\frac{\partial x^{(1)}}{\partial\varphi}(u^{(0)},\ \varphi_0,\ L)=\int_0^L \frac{\partial X}{\partial\theta}[u^{(0)},\ \theta_0(u^{(0)},\ \varphi_0,\ t),\ t,\ 0]\frac{\partial\theta_0}{\partial\varphi}(u^{(0)},\ \varphi_0,\ t)dt,$$

$$=\int_0^L \frac{\partial X}{\partial\theta}[u^{(0)},\ \theta_0(u^{(0)},\ \varphi_0,\ t),\ t,\ 0]\ \frac{\Theta[u^{(0)},\ \theta_0(u^{(0)},\ \varphi_0,\ t),\ 0]}{\Theta(u^{(0)},\ \varphi_0,\ 0)}\ dt$$

Consequently it follows that

$$(4.3)\qquad J=m\Omega'(u^{(0)})\int_0^L \frac{\partial X}{\partial\theta}[u^{(0)},\ \theta_0(u^{(0)},\ \varphi_0,\ t),\ t,\ 0]\Theta[u^{(0)},\ \theta_0(u^{(0)},\ \varphi_0,\ t),\ 0]dt.$$

Thus, from Theorem 1 follows

Theorem 3. *The real system* (4.1) *has a periodic solution if* (4.2) *has a real solution* $(u^{(0)},\ \varphi_0)$ *and* J *given by* (4.3) *does not vanish.*

For detailed study of stability, the assumption 2° about the differentiability of the right-hand sides of (4.1) is not sufficient. So, for (4.1), the assumption 2° is replaced by the stronger assumption:

2'° $X(x,\ \theta,\ t,\ \varepsilon),\ \Theta(x,\ \theta,\ \varepsilon),\ \Psi(x,\ \theta,\ t,\ \varepsilon)\in C^3_{x,\theta,\varepsilon}[D]$.

Then

$$x(u,\varphi,t,\varepsilon),\ \theta(u,\varphi,t,\varepsilon)\in C^3_{u,\varphi,\varepsilon},$$

consequently (3.5) and (3.4) can be written as follows:

$$\begin{cases}\dfrac{\partial x}{\partial u}(\tilde{u},\tilde{\varphi},L,\varepsilon)=1+\varepsilon\dfrac{\partial x^{(1)}}{\partial u}(u^{(0)},\varphi_0,L)+O(\varepsilon^2),\\[2ex] \dfrac{\partial x}{\partial \varphi}(\tilde{u},\tilde{\varphi},L,\varepsilon)=\varepsilon\dfrac{\partial x^{(1)}}{\partial \varphi}(u^{(0)},\varphi_0,L)+O(\varepsilon^2),\\[2ex] \dfrac{\partial \theta}{\partial u}(\tilde{u},\tilde{\varphi},L,\varepsilon)=-m\Theta(u^{(0)},\varphi_0,0)\Omega'(u^{(0)})+O(\varepsilon),\\[2ex] \dfrac{\partial \theta}{\partial \varphi}(\tilde{u},\tilde{\varphi},L,\varepsilon)=1+\varepsilon\Big\{-\dfrac{\theta_1(u^{(0)},\varphi_0,L)}{\Theta(u^{(0)},\varphi_0,0)}\dfrac{\partial\Theta}{\partial\theta}(u^{(0)},\varphi_0,0)\\ \qquad\qquad +\dfrac{\partial\theta_1}{\partial\varphi}(u^{(0)},\varphi_0,L)\Big\}+O(\varepsilon^2).\end{cases}$$

Then the matrix A given by (3.6) becomes

$$A=E+\begin{pmatrix}\varepsilon a+O(\varepsilon^2) & \varepsilon b+O(\varepsilon^2)\\ -(\delta+O(\varepsilon)) & \varepsilon d+O(\varepsilon^2)\end{pmatrix}\tag{4.4}$$

$$=E+\tilde{A},$$

where E is a unit matrix and

$$\begin{cases}a=\dfrac{\partial x^{(1)}}{\partial u}(u^{(0)},\varphi_0,L),\\[2ex] b=\dfrac{\partial x^{(1)}}{\partial \varphi}(u^{(0)},\varphi_0,L),\\[2ex] \delta=m\Theta(u^{(0)},\varphi_0,0)\Omega'(u^{(0)}),\\[2ex] d=-\dfrac{\theta_1(u^{(0)},\varphi_0,L)}{\Theta(u^{(0)},\varphi_0,0)}\dfrac{\partial\Theta}{\partial\theta}(u^{(0)},\varphi_0,0)+\dfrac{\partial\theta_1}{\partial\varphi}(u^{(0)},\varphi_0,L).\end{cases}$$

As is readily seen from (4.4), the characteristic equation of $\tilde{A}$ is of the form

$$\mu^2-2\mu\frac{\varepsilon}{2}(a+d+O(\varepsilon))+\varepsilon(b\delta+O(\varepsilon))=0.\tag{4.5}$$

Solving this quadratic equation, we have

$$\mu=\pm\sqrt{-\varepsilon b\delta}+\frac{\varepsilon}{2}(a+d)+o(\varepsilon).$$

Consequently the characteristic roots λ of the matrix A becomes

$$\lambda=1\pm\sqrt{-\varepsilon b\delta}+\frac{\varepsilon}{2}(a+d)+o(\varepsilon). \tag{4.6}$$

Now, for the periodic solution affirmed in Theorem 3,

$$J=b\delta\neq 0,$$

consequently, from (4.6), it follows that

1° when $\varepsilon b\delta<0$, in their absolute values, one of λ's is less than 1 and the other is greater than 1 provided $|\varepsilon|$ is sufficiently small;

2° when $\varepsilon b\delta>0$,

$$|\lambda|=1+\frac{\varepsilon}{2}(a+d+b\delta)+o(\varepsilon),$$

consequently, provided $|\varepsilon|$ is sufficiently small,

$$\text{(i)}\ |\lambda|>1 \text{ when } \varepsilon(a+d+b\delta)>0,$$

$$\text{(ii)}\ |\lambda|<1 \text{ when } \varepsilon(a+d+b\delta)<0.$$

Thus we have

Theorem 4. *The periodic solution of* (4.1) *affirmed in Theorem* 3 *is*
(i) *conditionally stable when* $\varepsilon J<0$,
(ii) *stable when* $\varepsilon J>0$ *and* $\varepsilon(a+d+J)<0$,
(iii) *unstable when* $\varepsilon J>0$ *and* $\varepsilon(a+d+J)>0$.
The stability is undecided when $\varepsilon J>0$ *and* $\varepsilon(a+d+b\delta)=0$.

The quantities J and $a+d+J$ are expressed in terms of $\theta_0=\theta_0(u^{(0)}, \varphi_0, t)$ as follows:

$$J=b\delta=-\int_0^L \theta_t\frac{\partial X}{\partial\theta}(u^{(0)}, \theta_0, t, 0)dt\cdot\int_0^L\frac{1}{\theta_t}\frac{\partial\theta}{\partial x}(u^{(0)}, \theta_0, t, 0)dt, \tag{4.7}$$

$$a+d+J=\int_0^L\theta_t\left[\frac{\partial}{\partial x}\left(\frac{X}{\theta}\right)+\frac{\partial}{\partial\theta}\left(\frac{\hat{\Psi}}{\theta}\right)\right]_{\substack{x=u^{(0)}\\ \theta=\theta_0\\ \varepsilon=0}}dt, \tag{4.8}$$

where

$$\begin{cases} \theta_0=\theta_0(t)\overset{\text{def}}{=}\theta_0(u^{(0)}, \varphi_0, t), \\ \theta_t=\Theta(u^{(0)}, \theta_0, 0), \\ \hat{\Psi}=\hat{\Psi}(x, \theta, t, \varepsilon)\overset{\text{def}}{=}\dfrac{\partial\Theta(x, \theta, \varepsilon)}{\partial\varepsilon}+\hat{\Psi}(x, \theta, t\varepsilon). \end{cases} \tag{4.9}$$

In fact, the formula (4.7) is derived in the following way.
From (2.8) and (2.17), it readily follows that

$$b=\frac{1}{\theta_0}\int_0^L \theta_t \frac{\partial X}{\partial \theta}(u^{(0)}, \theta_0, t, 0)dt. \tag{4.10}$$

On the other hand, from (2.14),

$$m\Omega'(u^{(0)})=-\int_0^{2m\pi}\frac{1}{\Theta^2(u^{(0)}, \theta, 0)}\frac{\partial \Theta}{\partial x}(u^{(0)}, \theta, 0)d\theta.$$

This can be rewritten as

$$m\Omega'(u^{(0)})=-\int_0^L \frac{1}{\Theta_t}\frac{\partial \Theta}{\partial x}(u^{(0)}, \theta_0, 0)dt,$$

because $d\theta_0=\Theta_t\, dt$ and θ_0 is periodic in t modulus $2m\pi$ with period L. Then, by the definition of δ,

$$\delta=-\Theta_0\int_0^L \frac{1}{\Theta_t}\frac{\partial \Theta}{\partial x}(u^{(0)}, \theta_0, 0)dt. \tag{4.11}$$

From (4.11) follows readily (4.7).

The formula (4.8) is derived in the following way.

First, from (2.8) and (2.18) follows

$$a=a_1+a_2, \tag{4.12}$$

where

$$\begin{cases} a_1=\displaystyle\int_0^L \frac{\partial X}{\partial x}(u^{(0)}, \theta_0, t, 0)dt \\ a_2=\displaystyle\int_0^L \Theta_t \frac{\partial X}{\partial \theta}(u^{(0)}, \theta_0, t, 0)\left[\int_0^t \frac{1}{\Theta_\tau}\frac{\partial \Theta}{\partial x}(u^{(0)}, \theta_0(\tau), 0)d\tau\right]dt. \end{cases} \tag{4.13}$$

On the other hand, since

$$\theta_0(u^{(0)}, \varphi, L)=\varphi+2m\pi$$

for any φ as is seen from (2.13), from (2.9), we see that

$$\begin{aligned} &\theta_1(u^{(0)}, \varphi, L) \\ &=\Theta(u^{(0)}, \varphi, 0)\int_0^L \frac{1}{\Theta_t}\frac{\partial \Theta}{\partial x}(u^{(0)}, \theta_0, 0)\left[\int_0^t X(u^{(0)}, \theta_0(\tau), \tau, 0)\, d\tau\right]dt \\ &\quad+\Theta(u^{(0)}, \varphi, 0)\int_0^L \frac{1}{\Theta_t}\hat{\Psi}(u^{(0)}, \theta_0, t, 0)dt, \end{aligned} \tag{4.14}$$

but, here alone, θ_0 means

$$\begin{cases} \theta_0=\theta_0(u^{(0)}, \varphi, t), \\ \Theta_t=\Theta(u^{(0)}, \theta_0, 0)=\Theta[u^{(0)}, \theta_0\,(u^{(0)}, \varphi, t), 0]. \end{cases}$$

Then, making use of (2.17), we have:

(4.15)
$$\frac{\partial\theta_1}{\partial\varphi}(u^{(0)}, \varphi_0, L)=\sum_{i=1}^{7}d'_i,$$

where

(4.16)
$$\begin{cases} d'_1=\dfrac{\partial\theta}{\partial\theta}(u^{(0)}, \varphi_0, 0)\displaystyle\int_0^L \frac{1}{\Theta_t}(u^{(0)}, \theta_0, 0)\Big[\int_0^t X(u^{(0)}, \theta_0(\tau), \tau, 0)d\tau\Big]dt, \\ d'_2=-\displaystyle\int_0^L \frac{1}{\Theta_t}\frac{\partial\theta}{\partial\theta}(u^{(0)}, \theta_0, 0)\frac{\partial\theta}{\partial x}(u^{(0)}, \theta_0, 0)\Big[\int_0^t X(u^{(0)}, \theta_0(\tau), \tau, 0)d\tau\Big]dt \\ d'_3=\displaystyle\int_0^L \frac{\partial^2\theta}{\partial\theta\partial x}(u^{(0)}, \theta_0, 0)\Big[\int_0^t X(u^{(0)}, \theta_0(\tau), \tau, 0)d\tau\Big]dt, \\ d'_4=\displaystyle\int_0^L \frac{1}{\Theta_t}\frac{\partial\theta}{\partial x}(u^{(0)}, \theta_0, 0)\Big[\int_0^t \Theta_\tau\frac{\partial X}{\partial\theta}(u^{(0)}, \theta_0(\tau), \tau, 0)\,d\tau\Big]dt, \\ d'_5=\dfrac{\partial\theta}{\partial\theta}(u^{(0)}, \varphi_0, 0)\displaystyle\int_0^L \frac{1}{\Theta_t}\hat{\Psi}(u^{(0)}, \theta_0, t, 0)dt, \\ d'_6=-\displaystyle\int_0^L \frac{1}{\Theta_t}\frac{\partial\theta}{\partial\theta}(u^{(0)}, \Theta_0, 0)\,\hat{\Psi}(u^{(0)}, \theta_0, t, 0)dt, \\ d'_7=\displaystyle\int_0^L \frac{\partial\hat{\Psi}}{\partial\theta}(u^{(0)}, \theta_0, t, 0)dt. \end{cases}$$

But, since

$$\frac{d}{dt}\Big[\frac{1}{\Theta_t}\frac{\partial\theta}{\partial X}(u^{(0)}, \theta_0, 0)\Big]$$
$$=-\frac{1}{\Theta_t}\frac{\partial\theta}{\partial\theta}(u^{(0)}, \theta_0, 0)\frac{\partial\theta}{\partial x}(u^{(0)}, \theta_0, 0)+\frac{\partial^2\theta}{\partial\theta\partial x}(u^{(0)}, \theta_0, 0),$$

it readily follows by integration by parts that

(4.17)
$$d'_2+d'_3=-\int_0^L \frac{1}{\Theta_t}\frac{\partial\theta}{\partial x}(u^{(0)}, \theta_0, 0)\,X(u^{(0)}, \theta_0, t, 0)\,dt,$$

for

$$\int_0^L X(u^{(0)}, \theta_0, t, 0)dt=x^{(1)}(u^{(0)}, \varphi_0, L)=0$$

by (2.15).

Also, by integration by parts, it readily follows that

(4.18)
$$a_2+d'_4+J=0.$$

Thus, by the definition of d, we have:

$$a+d+J=a_1+(d'_2+d'_3)+d'_6+d'_7.$$

This implies (4.8) evidently.

5. Example

As an example, let us apply our method to the second order differential equation

$$x''+g(x)=\varepsilon f(x, x', t), \tag{5.1}$$

where

1° *ε is a parameter such that* $|\varepsilon| \ll 1$;

2° *$g(x)$ and $f(x, x', t)$ are four times continuously differentiable with respect to (x,x') in the domain:*

$$D: -\infty < x,\ x',\ t < +\infty;$$

3° *$f(x, x',t)$ is continuous with respect to t in the domain D and is periodic in t with period $T_0>0$;*

4° *$g(0)=0$, $g'(0)\neq 0$ and $xg(x)>0$ for any $x\neq 0$*[1];

The equation of the form (5.1) was discussed by A. M. Кац [1] for the analytic case and also by W. S. Loud [2] in detail for special case where $f=f(t)$.

At first, by means of the transformation employed by M. Urabe [6] for discussion of the unperturbed equation

$$x''+g(x)=0, \tag{6.2}$$

we shall reduce the equation (5.1) to the system of the form (4.1).

Writing (5.1) in a simultaneous form as

$$\begin{cases} \dfrac{dx}{dt}=y, \\[2ex] \dfrac{dy}{dt}=-g(x)+\varepsilon f(x, y, t), \end{cases} \tag{5.3}$$

we consider the transformation of (x, y) to (X, y) where

$$X=X(x)\overset{\text{def}}{=\!=}\eta\sqrt{2\int_0^x g(u)du} \qquad \left(\eta=\frac{x}{|x|}\right). \tag{5.4}$$

For the function $X(x)$, it readily follows from the assumptions 2° and 4° that

$$X(x)\in C_x^4 \text{ and } X'(0)=\sqrt{g'(0)}>0. \tag{5.5}$$

Consequently (5.3) can be solved reversely in x as

---

1) The assumption that $xg(x)>0$ for $x\neq 0$ implies $g'(0)>0$.

(5.6) $$x=x(X)\in C_x^4.$$

Then, since

$$\frac{dX(x)}{dt}=\frac{g(x)}{X(x)}\frac{dx}{dt}$$

by the definition (5.4), we see that the system (5.3) is transformed to the system

(5.7) $$\begin{cases} \dfrac{dX}{dt}=\dfrac{h(X)}{X}y, \\ \dfrac{dy}{dt}=-h(X)+\varepsilon F(X, y, t), \end{cases}$$

where

(5.8) $$\begin{cases} h(X)=g[x(X)]\in C_x^4, \\ F(X, y, t)=f[x(X), y, t]\in C_{x,y}^4. \end{cases}$$

Further, let us put

$$X=R\cos\theta,\ y=R\sin\theta,$$

then, after simple calculations, we see that the system (5.7) is reduced to the system

(5.9) $$\begin{cases} \dfrac{dR}{dt}=\varepsilon\Phi(R, \theta, t)\sin\theta, \\ \dfrac{d\theta}{dt}=-\dfrac{h(X)}{X}+\varepsilon\dfrac{1}{R}\Phi(R, \theta, t)\cos\theta, \end{cases}$$

where

$$\Phi(R, \theta, t)=F(R\cos\theta, R\sin\theta, t)\in C_{R,\theta}^4.$$

But, from (5.7),

$$\frac{h(X)}{X}\in C_X^3$$

and moreover, from the assumption 4° and (5.4),

$$\frac{h(X)}{X}=\frac{g(x)}{X(x)}>0 \text{ for any } X.$$

Thus we see that the system (5.8) is the system of the form (4.1).

For the system (5.9), $\theta_0=\theta_0(u, \varphi, t)$ is a solution of the equation

(5.10) $$\frac{d\theta}{dt}=-\frac{h(u\cos\theta)}{u\cos\theta}$$

such that

(5.11) $$\theta_0(u,\varphi,0)=\varphi.$$

And the equations by which the first approximations $(u^{(0)}, \varphi_0)$ of the initial values of the periodic solution are determined become

(5.12) $$\begin{cases} \displaystyle\int_0^L \Phi[u,\theta_0(u,\varphi,t),t]\sin\theta_0 dt=0, \\ \displaystyle m\int_0^{2\pi}\frac{u\cos\theta}{h(u\cos\theta)}d\theta+L=0. \end{cases}$$

The quantities J and $a+d+J$ are easily found by means of (4.7) and (4.8) after elementary calculations as follows:

(5.13) $$J=\int_0^L\left[\frac{h(R\cos\theta)}{R\cos\theta}\left(\frac{\partial\Phi}{\partial\theta}\sin\theta+\Phi\cos\theta\right)\right]_{\substack{R=u^{(0)}\\ \theta=\theta_0}}dt \times\int_0^L\left[\frac{h'(R\cos\theta)R\cos\theta-h(R\cos\theta)}{Rh(R\cos\theta)}\right]_{\substack{R=u^{(0)}\\ \theta=\theta_0}}dt,$$

(5.14) $$a+d+J=\int_0^L\left[\frac{\partial\Phi}{\partial R}\sin\theta+\frac{1}{R}\frac{\partial\Phi}{\partial\theta}\cos\theta\right]_{\substack{R=u^{(0)}\\ \theta=\theta_0}}dt,$$

where θ_0 means

$$\theta_0=\theta_0(u^{(0)},\varphi_0,t).$$

Now evidently the unperturbed system (5.2) has a periodic solution $x=x_0(t)$ with period $T_1=l\ T_0$ such that the values $(u^{(0)}, \varphi_0)$ determined by

(5.15) $$\begin{cases} X[x_0(0)]=u^{(0)}\cos\varphi_0, \\ x'_0(0)=u^{(0)}\sin\varphi_0 \end{cases}$$

satisfy (5.12), if and only if there exists a real solution $(u^{(0)}, \varphi_0)$ of (5.12) such that $u^{(0)}>0$.

Thus, from Theorms 3 and 4, we have

Theorem 5. *The equation* (5.1) *has a periodic solution if the unperturbed equation* (5.2) *has a periodic solution* $x=x_0(t)$ *with period* T_1 *commensurable with* T_0 *such that the values* $(u^{(0)}(>0), \varphi_0)$ *determined by* (5.15) *satisfy* (5.12) *and moreover if* J *given by* (5.13) *does not vanish for these values* $(u^{(0)}, \varphi_0)$. *Such a periodic solution is conditionally stable when* $\varepsilon J<0$, *and it is stable or unstable according as* $\varepsilon(a+d+J)$ *given by* (5.14) *is negative or positive* $\varepsilon J>0$.

In the special case where $f(x, x'\ t)=f(t)$ as in the case studied by W. S.

Loud [2],

$$\Phi(R,\ \theta,\ t)=f(t),$$

consequently the first equation of (5.12) turns out

$$\int_0^L f(t)x'_0(t)dt=0, \tag{5.16}$$

because

$$y=x'_0(t)=u^{(0)}\sin\theta_0. \tag{5.17}$$

Further, since

$$x''_0(t)=-h(u^{(0)}\cos\theta_0)$$

follows from (5.17) and (5.10), in the present case, (5.13) and (5.14) turn out

$$J=-\frac{1}{(u^{(0)})^2}\int_0^L f(t)x''_0(t)dt\int_0^L\left[\frac{u^{(0)}\cos\theta_0 h'(u^{(0)}\cos\theta_0)}{h(u^{(0)}\cos\theta_0)}-1\right]dt,$$

$$a+d+J=0.$$

In the present case, for simplicity, let us assume further

$$\Omega'(u^{(0)})\neq 0. \tag{5.18}$$

Then, as is seen from derivation of (4.11), (5.18) is equivalent to

$$\int_0^L\left[\frac{u^{(0)}\cos\theta_0 h'(u^{(0)}\cos\theta_0)}{h(u^{(0)}\cos\theta_0)}-1\right]dt\neq 0,$$

consequently the condition that $J\neq 0$ is equivalent to

$$\int_0^L f(t)x''_0(t)\neq 0. \tag{5.19}$$

Thus we have

Theorem 6. *The equation* (5.1) *where* $f(x,\ x',\ t)=f(t)$ *has a periodic solution if the unperturbed system* (5.2) *has a periodic solution* $x=x_0(t)$ *with period* T *commensurable with* T_0 *such that it satisfies* (5.16), (5.18) *and* (5.19) *and the values* $u^{(0)}$ *determined by* (5.15) *satisfy the latter* (5.12). *Such a periodic solution is conditionally stable when* $\varepsilon J<0$.

When $\varepsilon J>0$, the stability of the periodic solution affirmed in this theorem is not yet decided by our theorems. For decision of the stability, the calculation of the terms of the order higher than those calculated here will be needed.

In conclusion, the author whishes to express his hearty gratitude to Prof. M. Urabe for his kind guidance and constant advice.

References

[1] A. M. Кац: *Вынужденные колебания нелинейных систем с одной степенью свободы, близких к консервативиным.* Прикл. Мат. и Мех. XIX (1955), 13-32.

[2] W. S. LOUD: *Periodic solutions of* $x''+cx'+g(x)=\varepsilon f(t)$ Memoirs of the Amer. Math. Soc., no. 31 (1958).

[3] N. MINORSKY: *La méthod stroboscopique et ses applications.* Bull. Soc. France Mec., 4(1954), 15-26.

[4] N. MINORSKY: *The Theory of Oscillations.* Surveys in Applied Mathematics II, New York, (1958), 111-206.

[5] M. URABE: *Reduction of Periodic System to Autonomous One by Means of One-Parameter Group of Transformations.* J. Sci. Hiroshima Univ., Ser. A., 20(1956), 13-35.

[6] M. URABE: *Potential forces which yield periodic motions of a fixed period* (To appear in J. Math. and Meah. (1961)).

[7] M. YORINAGA: *A Generalization of the Stroboscopic Method.* J. Sci. Hiroshima Univ., Ser. A., 23 (1960), 367-375.

Department of Mathematics,
Faculty of Science,
Hiroshima University.