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On Cartan Subgroups of Linear Groups

Shigeaki Tôgô

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Introduction

Let k be an algebraically closed field of arbitrary characteristic and let $GL(n, k)$ be the group of all automorphisms of an n-dimensional vector space V over k. As usual, we introduce the Zariski topology on the space of all endomorphisms of V. For a subgroup G of $GL(n, k)$, we denote by G^* the closure of G in $GL(n, k)$. Then G^* is the smallest algebraic subgroup of $GL(n, k)$ containing G. In [8], by considering the fact that, for any connected complex linear Lie group H, the derived group of a group H^* is contained in H, we introduced the notions of D^∞-subgroups and C^∞-subgroups of $GL(n, k)$ in the following way. A subgrup G of $GL(n, k)$ is called a *D^∞-group* (resp. *C^∞-group*) provided

$$D^\infty G^* \subset G \quad (\text{resp. } C^\infty G^* \subset G)$$

where $D^\infty G^*$ (resp. $C^\infty G^*$) is the intersection of all members of the series of the derived groups $D^i G^*$ (resp. the descending central series $C^i G^*$) of a group G^*.

In [7 and 8], we introduced two kinds of "splittability" into subgroups of $GL(n, k)$. It is well known that an element x of $GL(n, k)$ can be decomposed into the Jordan product, that is, x is uniquely expressed as $x=x_s x_u$ in such a way that x_s is semisimple, x_u is unipotent and $x_s x_u = x_u x_s$. A subgroup G of $GL(n, k)$ is called splittable [7] provided every element of G can be decomposed into the Jordan product in G. Then a connected D^∞-subgroup of $GL(n, k)$ is splittable if and only if one of its maximal solvable connected subgroups is splittable [8, Theorem 4.9]. A D^∞-subgroup of $GL(n, k)$ is called to have the (S)-property provided one of its maximal solvable connected subgroups, say R, satisfies the condition that $R=TR_u$ for any maximal torus (that is, any maximal connected commutative subgroup consisting of semisimple elements) T and for the invariant subgroup R_u of all unipotent elements of R (see [8, Definitions 7.1 and 7.2]). These two kinds of "splittability" are possessed by an algebraic linear group [1, (9.2) and (12.9)] and are equivalent for a connected C^∞-group [8, Theorem 11.4]. But each of them does not imply the other for a connected D^∞-group generally [9, Examples 1 and 2].

A *Cartan subgroup* of a group G is a maximal nilpotent subgroup H such that any invariant subgroup of finite index of H is of finite index in its normalizer in G [3, p. 199]. C. Chevalley [3, Chapitre VI] and A. Borel [1, Chapitre V] investigated Cartan subgroups of a connected algebraic linear group and, in [8, Sections 9 and 12], we studied more generally Cartan subgroups of a connected D^∞-group having the splittability and the (S)-property.

The main purpose of this paper is to study Cartan subgroups of a connected D^∞-subgroup of $GL(n, k)$ satisfying two conditions which are respectively weaker than the splittability and the (S)-property.

In Section 1, we shall give some definitions and some fundamental properties of subgroups of $GL(n, k)$. In Sections 2 and 3, we shall generalize the fundamental results, known as the structure theorems, on a connected nilpotent and a connected solvable algebraic subgroups of $GL(n, k)$ given in [1, Theorems 11.1 and 12.9] to a connected nilpotent splittable subgroup and a connected solvable splittable C^∞-subgroup of $GL(n, k)$. Namely, a connected nilpotent splittable group is the direct product of a unique maximal torus and the invariant subgroup of all its unipotent elements (Theorem 2.4), and a connected solvable splittable C^∞-group has the (S)-property (Theorem 3.5). In [8] we proved these results by using the corresponding results of algebraic linear groups, but we shall give the proofs of these results which cover the proofs of the algebraic cases.

In Section 4, being based on the results of Sections 2 and 3, we shall show the conjugacy of maximal solvable connected subgroups of a connected D^∞-group G and a result on the connection of the maximal solvable connected subgroups of G and those of G^* (Theorem 4.3), and we shall also show the conjugacy of maximal tori of a connected C^∞-group H (Theorem 4.6) and the connectedness of the centralizer of a torus of H (Theorem 4.8). In Section 5, we shall recall some known facts on the relation of the splittability and the (S)-property (Theorem 5.3).

In Section 6, we shall introduce the following two conditions for a connected D^∞-subgroup G of $GL(n, k)$:

(a) For one of the maximal solvable connected subgroups R of G, the closure of any maximal torus of R is a maximal torus of R^*.

(b) All maximal nilpotent connected subgroups of G are splittable.

The (S)-property implies (a) but not conversely. The splittability implies (b) and, if G is a connected C^∞-group, (b) implies the splittability (Proposition 6.15). We shall prove that, for a connected D^∞-group G satisfying (a) and (b), a subgroup H of G is a Cartan subgroup of G, if and only if H is the centralizer of a maximal torus of G, and only if H is the intersection of G and a Cartan subgroup of G^* (Theorem 6.9), which is a generalization of [1, Theorem 20.4 and 8, Theorem 9.3]. We shall also prove that, for a connected C^∞-group G satisfying (a), a subgroup H of G is the centralizer of a maximal torus of G, if and only if H is the intersection of G and a Cartan subgroup of G^*, and only if H is a Cartan subgroup of G (Theorem 6.12), which is a generalization of [8, Theorem 12.2].

1. Preliminaries

We here recall some definitions and fundamental properties of linear groups given in [1 and 8], and we note some lemmas and notations which will be used through the paper.

1.1. Let k be an algebraically closed field of arbitrary characteristic and let $GL(n, k)$ be the group of all automorphisms of an n-dimensional vector space V over k. Let $M(n, k)$ be the space of all endomorphisms of V, which may be considered as the space of all square matrices of degree n with coefficients in k. The elements of $GL(n, k)$ are the non-singular matrices of $M(n, k)$. We introduce the Zariski topology on $M(n, k)$ as usual. For a subgroup G of $GL(n, k)$, the closure of G in $GL(n, k)$ is the smallest algebraic subgroup of $GL(n, k)$ containing G. We call the closure of G in $GL(n, k)$ the closure of G for simplicity and denote it by G^*. We mean by the dimension of G the dimension of G^*. We always denote by e the identity automorphism in $GL(n, k)$.

1.2. *For a subgroup G of $GL(n, k)$, we denote by G_0 the connected component of the identity element e of G. Then G_0 is an invariant closed subgroup of finite index of G. G_0 is the unique closed connected subgroup of finite index of G.*

$$G_0 = G \cap (G^*)_0 \quad \textit{and} \quad (G_0)^* = (G^*)_0.$$

G is connected if and only if G is irreducible, and if and only if G^* is connected.

In fact, as is well known, G is the union of a finite number of the irreducible closed subsets of G. If we denote by M_i $(i=1, 2, \cdots, m)$ the irreducible components of G such that $M_i \neq M_j$ for $i \neq j$, and if we denote by M_i^- the closure of M_i in $GL(n, k)$, then

$$G^* = \bigcup_{i=1}^{m} M_i^-$$

is the irredundant decomposition into the irreducible closed subsets of G^* and $M_i = G \cap M_i^-$ (e.g., see [5, pp. 35–36]). By using the fact that G (resp. G^*) is a group, it can be easily seen that these M_i (resp. M_i^-) are disjoint (see [10, (2.1)]). Therefore G is connected if and only if G is irreducible, and if and only if G^* is connected. Let M_1 contain e. Then it is immediate that M_1 is invariant by $x \to x^{-1}$, $x \to yxy^{-1}$ with y in G, $x \to xz$ and $x \to zx$ with z in M_1. Hence M_1 is an invariant subgroup of G and M_i's with $i \neq 1$ are the cosets of M_1. Thus $M_1 = G_0$ and $M_1^- = (G^*)_0$. Finally, if M is a connected closed subgroup of finite index of G, then $M \subset G_0$ and M is of finite index in G_0, whence $M = G_0$.

1.3. *If M and N are subgroups of $GL(n, k)$ and if M normalizes (resp. centralizes) N, then M^* normalizes (resp. centralizes) N^*.*

In fact, if $xNx^{-1} = N$ for any x in M, then $x\,N^*x^{-1} = N^*$ since $y \to xyx^{-1}$ $(y \in GL(n, k))$ is continuous. The normalizer of N^* is algebraic, whence $yN^*y^{-1} = N^*$ for any y in M^*. If M centralizes N, let f be the mapping $(x, y) \to xyx^{-1}y^{-1}$

of $GL(n, k)\times GL(n, k)$ into $GL(n, k)$. Then $f^{-1}(e)$ is algebraic and contains $M\times N$, whence it contains the closure $M^*\times N^*$. Therefore M^* centralizes N^*.

1.4. Let G be a subgroup of $GL(n, k)$. For any x and y in G, we denote by $[x, y]$ the commutator $xyx^{-1}y^{-1}$ of x and y, and for subsets M and N of G, we denote by $[M, N]$ the group generated by the commutators $[x, y]$ with x in M and y in N. We define inductively the series of derived groups

$$D^iG=[D^{i-1}G, D^{i-1}G] \qquad (D^0G=G, i=0, 1, 2,\dots),$$

and the descending central series

$$C^iG=[G, C^{i-1}G] \qquad (C^0G=G, i=0, 1, 2,\dots).$$

Put

$$D^\infty G=\bigcap_i D^iG \quad \text{and} \quad C^\infty G=\bigcap_i C^iG.$$

G is called solvable (resp. nilpotent) provided there exists j such that $D^jG=\{e\}$ (resp. $C^jG=\{e\}$).

If G is algebraic, then we have

$$D^\infty G=D^jG \quad \text{and} \quad C^\infty G=C^jG$$

for a sufficiently large integer j. This is immediate by considering the dimension of D^iG and C^iG.

(1) *G is solvable (resp. nilpotent, commutative) if and only if G^* is solvable (resp. nilpotent, commutative).*

(2) *If G is connected, then D^iG and C^iG for $i\geq 0$ are all connected.*

In fact, if H and L are invariant subgroups of G, let f be a mapping of $GL(n, k)\times GL(n, k)$ into $GL(n, k)$ defined by $f(x, y)=[x, y]$. Then $f^{-1}([H, L]^*)$ is algebraic and contains $H\times L$, whence it contains the closure $H^*\times L^*$. Hence $[H^*, L^*]\subset[H, L]^*$. Since H^* and L^* are invariant subgroups of G^* by (1.3), it is known that $[H^*, L^*]$ is algebraic (see [4, 3–04]). Therefore $[H^*, L^*]=[H, L]^*$. If H and L are furthermore connected, then H^* and L^* are irreducible by (1.2). Then it is known [2, p. 122] that $[H^*, L^*]$ is irreducible. By (1.2) we see that $[H, L]$ is connected. (1) and (2) are immediate from these facts.

1.5. *Let M and N be subgroups of $GL(n, k)$ such that M is contained in the normalizer of N. Then MN is a group and $(MN)^*=M^*N^*$. If M and N are connected, then MN is connected.*

In fact, it is clear that MN forms a group. First suppose that M and N are connected. Then they are irreducible, whence $M\times N$ is irreducible. Let f be a mapping of $GL(n, k)\times GL(n, k)$ into $GL(n, k)$ defined by $f(x, y)=xy$. As the image of $M\times N$ by f, we see that MN is irreducible and therefore connected. Next suppose that M and N are not necessarily connected. Since M^* is in the normalizer of N^* by (1.3), $(M^*)_0$ is in the normalizer of $(N^*)_0$. Since $(M^*)_0(N^*)_0$

is the image of $(M^*)_0 \times (N^*)_0$ by f and forms a group, it is known that $(M^*)_0(N^*)_0$ is algebraic. $(M^*)_0(N^*)_0$ is connected by the first case. But M^*N^* is a group which is the union of a finite number of the sets $x(M^*)_0(N^*)_0 y$ with x in M^* and y in N^*. Therefore M^*N^* is algebraic, whence $(MN)^* = M^*N^*$.

1.6. Let G be a subgroup of $GL(n, k)$. G is called [8, Definitions 4.1 and 6.2] a *D^∞-group* (resp. *C^∞-group*) provided

$$D^\infty G^* \subset G \quad (\text{resp. } C^\infty G^* \subset G).$$

If G is a D^∞-group, then we have $D^\infty G^* = D^j G^* \subset G$ for some integer j, whence it follows that

$$D^\infty G^* = D^\infty G = D^j G.$$

An element x of $GL(n, k)$ can be uniquely expressed as $x = x_s x_u$, where x_s is semisimple, x_u is unipotent, and $x_s x_u = x_u x_s$ [2, p. 71 and p. 184]. x_s and x_u are called the semisimple and unipotent components of x respectively, and $x_s x_u$ is called the Jordan product decomposition of x. G is called *splittable* [7, p. 299] provided the semisimple and unipotent components of any element of G belong to G.

We call a subgroup G of $GL(n, k)$ a *torus* provided G is commutative, is connected and consists of semisimple elements. By using Zorn's lemma, we see that any subgroup of $GL(n, k)$ has maximal tori.

1.7. Let G be a connected subgroup of $GL(n, k)$. Let A be a connected subgroup of G and let B be a connected invariant subgroup of G. G is called the *semi-direct product* of B by A provided $G = AB$, $A^* \cap B^* = \{e\}$ and the mapping τ: $(a, b) \rightarrow ab$ of $A^* \times B^*$ into G^* is birational. If A is furthermore an invariant subgroup of G, then G is called the *direct product* of A and B.

1.8. Let M be a commutative subset of $M(n, k)$. Then there exists an element x of $GL(n, k)$ such that

$$xMx^{-1} = \begin{pmatrix} N_1 & & 0 \\ & \ddots & \\ 0 & & N_t \end{pmatrix}$$

where each N_i has the unique characteristic root and the coefficients 0 under the principal diagonal. All semisimple elements of xMx^{-1} are diagonal [6, Lemma 1 or 1, (6.4)].

1.9. Let $D(n)$ denote the set of all diagonal elements of $GL(n, k)$. Then a connected algebraic subgroup of $D(n)$ of dimension m is isomorphic to the direct product of m copies of the group consisting of all elements of k except 0 [1, (7.5)]. By making use of this fact and (1.8), it can be proved that, for an element x of $GL(n, k)$, the semisimple and unipotent components of x belong to the smallest algebraic group containing x [1, (8.4)]. Therefore every algebraic subgroup of $GL(n, k)$ is splittable. If a subgroup G of $GL(n, k)$ is splittable, then every closed subgroup of G is splittable as the intersection of two splittable groups.

1.10. Let G be a connected solvable algebraic subgroup of $GL(n, k)$. Let a complete subvariety W of k^m be a space of transformations for G, that is, let there exist an everywhere defined rational mapping F: $(x, P)\rightarrow x(P)$ such that

$$x(x'(P))=(xx')(P) \quad \text{and} \quad e(P)=P$$

for any x, x' in G and any P in W. Then there exists a point of W fixed by G [1, (15.7) or 4, 5–14].

As an immediate consequence of this fact, we have Lie-Kolchin's theorem that, for a connected solvable algebraic subgroup G of $GL(n, k)$, there exists an element x of $GL(n, k)$ such that xGx^{-1} is in triangular form [1, (16.4)]. The theorem is true for any connected solvable subgroup of $GL(n, k)$, since its closure is a connected solvable algebraic group by (1.2) and (1.4).

1.11. Let G be a subgroup of $GL(n, k)$. We call a representation f of G *rational* provided there exists a representation f^* of G^* whose restriction to G is f and whose restriction to $(G^*)_0$ is an everywhere defined rational mapping. For simplicity, we shall sometimes write f in place of f^*.

Let f be a rational representation of G. Then it is known [2, p. 122] that, if G is especially an algebraic group, then $f(G)$ is an algebraic group. From this fact and the continuity of f, we have generally

$$f(G^*)=f(G)^*.$$

If G is a D^∞-group (resp. C^∞-group), then $f(G)$ is a D^∞-group (resp. C^∞-group). Indeed, if G is a D^∞-group, then by (1.4) there exists an integer j such that

$$D^\infty G^*=D^jG^* \quad \text{and} \quad D^\infty f(G^*)=D^jf(G^*),$$

whence

$$D^\infty f(G)^*=D^jf(G^*)=f(D^jG^*)=f(D^\infty G^*)\subset f(G),$$

that is, $f(G)$ is a D^∞-group. Similarly we have the statement for C^∞-groups.

If x is a semisimple (resp. unipotent) element of G, then $f(x)$ is a semisimple (resp. unipotent) element of $f(G)$ [1, (9.5)]. Therefore, if G is splittable, then $f(G)$ is splittable.

It is well known that, for an algebraic subgroup H of $GL(n, k)$ and an invariant closed subgroup N of H, there exists a rational representation of H with N as its kernel [1, (5.10)].

1.12. Let G be a subgroup of $GL(n, k)$ and let f be a rational representation of G. If G is connected, then $f(G)$ is connected. Indeed, G^* is irreducible by

(1.2), whence $f(G^*)$ is irreducible [2, p. 121]. Since $f(G^*)=f(G)^*$ by (1.11), it follows from (1.2) that $f(G)$ is connected.

Let f^* be a rational representation of G^* whose restriction to G is f, and let N' and N be respectively the kernels of f^* and f. If $f(G)$ is connected, and if

(1) *N is connected and $N=N'$, or*

(2) *N is connected and $N^*=N'$, or*

(3) *N' is connected,*

then G is connected.

In fact, (1) implies (2) and (2) implies (3). Suppose that $f(G)$ is connected and that (3) N' is connected. Since $f(G^*)=f(G)^*$, $f(G^*)$ is a connected algebraic group. But $f((G^*)_0)$ is a connected closed subgroup of finite index of $f(G^*)$. Hence, by (1.2), we have

$$f(G^*)=f((G^*)_0).$$

Since $N'\subset(G^*)_0$, it follows that $G^*=(G^*)_0$, that is, G^* is connected. Hence, by (1.2), G is connected.

1.13. For a subgroup G of $GL(n, k)$, the set of all semisimple (resp. unipotent) elements of G is called the semisimple (resp. unipotent) part of G and is denoted by G_s (resp. G_u). We denote by $Z(G)$ the center of G. Let M be a subset of $GL(n, k)$. We denote by $n(M)$ (resp. $n^*(M)$) the normalizer of M in G (resp. G^*) and by $z(M)$ (resp. $z^*(M)$) the centralizer of M in G (resp. G^*). Their connected components of the identity element e are called respectively the connected normalizer and the connected centralizer of M in G (resp. G^*).

1.14. *Let G be a triangular subgroup of $GL(n, k)$. Let H be a subgroup of G consisting of semisimple elements. Then H is commutative and $n(H)=z(H)$.*

In fact, if x is in $n(H)$, then, for any y in H, $xyx^{-1}y^{-1}$ is in H and therefore semisimple. Since G is triangular, it is unipotent. Therefore $xyx^{-1}y^{-1}=e$, whence x is in $z(H)$. It is now evident that H is commutative.

2. Nilpotent groups

We begin by generalizing [1, Theorem 9.1]:

LEMMA 2.1. *Let G be a connected commutative splittable subgroup of $GL(n, k)$. Then G_s and G_u are connected closed subgroups of G, and G is the direct product of G_s and G_u. $(G_s)^*=(G^*)_s$ and $(G_u)^*=(G^*)_u$.*

PROOF. It is immediate that G_s and G_u are subgroups of G, $G_s\cap G_u=\{e\}$, and $G=G_sG_u$. Since the fact that a matrix is unipotent can be expressed by algebraic condition, $(G_u)^*$ consists of unipotent elements. Therefore $(G_u)^*\cap G=G_u$ and G_u is a closed subset of G. By (1.8), there exists an element x of $GL(n, k)$ such that $G'=xGx^{-1}$ is in triangular form in such a way that $G'_s\subset D(n)$.

Therefore

$$(G'_s)^* \cap G' \subset D(n) \cap G' = G'_s,$$

whence G'_s is a closed subset of G'. Since $G_s = x^{-1}G'_s x$, it follows that G_s is a closed subset of G. By (1.5), we have $G^* = (G_s)^*(G_u)^*$. Since G^* is splittable and commutative by (1.4) and (1.9), $(G^*)_s$ and $(G^*)_u$ are closed subgroups of G^* and $G^* = (G^*)_s(G^*)_u$. Hence $(G_s)^* \subset (G^*)_s$ and $(G_u)^* \subset (G^*)_u$, from which it follows that

$$(G_s)^* = (G^*)_s \quad \text{and} \quad (G_u)^* = (G^*)_u.$$

Since $G'^* = xG^*x^{-1}$ and since

$$y_s = x^{-1}(xyx^{-1})_s x \quad \text{and} \quad y_u = x^{-1}(xyx^{-1})_u x \quad (y \in G^*),$$

$y \to y_s$ and $y \to y_u$ are rational representations of G^*. As their images of a connected group G, G_s and G_u are connected. It is now immediate that G is the direct product of G_s and G_u, completing the proof.

As an immediate consequence of the lemma, we have

COROLLARY 2.2. *A subgroup of $GL(n, k)$ is a torus if and only if its closure is a torus.*

LEMMA 2.3. *Let G be a triangular subgroup of $GL(n, k)$. Then G_u is a closed invariant subgroup of G. If G is connected and splittable, then G_u is connected.*

PROOF. Since $D^1G \subset G_u$, G_u is an invariant subgroup of G. Since $(G_u)^*$ consists of unipotent matrices, we have $G_u = G \cap (G_u)^*$, that is, G_u is a closed subset of G. Now suppose that G is connected and splittable. Let f^* be a rational representation of G^* with $(D^1G)^*$ as its kernel, and let f be the restriction of f^* to G. Then $f(G)$ is connected, commutative and splittable by (1.11). Therefore it follows from Lemma 2.1 that $f(G)_u$ is connected. By (1.11) we have $f(G_u) = f(G)_u$. The kernel of the restriction of f^* to $(G_u)^*$ is $(D^1G)^*$ and therefore connected by (1.4). Hence, by (1.12) we see that G_u is connected, completing the proof.

We can now prove the following theorem generalizing [1, Theorem 11.1]:

THEOREM 2.4. *Let G be a connected splittable nilpotent subgroup of $GL(n, k)$. Then G_s is a connected central closed subgroup of G, G_u is a connected invariant closed subgroup of G, and G is the direct product of G_s and G_u, and we have*

$$(G_s)^* = (G^*)_s \quad \textit{and} \quad (G_u)^* = (G^*)_u.$$

PROOF. If G is commutative, the theorem follows from Lemma 2.1. Hence we may assume that G is not commutative. Suppose that the theorem is proved for any connected splittable nilpotent group whose dimension is less than that of G, and we prove that G_s is a connected central closed subgroup of

G. Let C be the connected component of the identity element of the center $Z(G)$ of G. Then dim $C>0$. By Lemma 2.1, C_s and C_u are connected closed subgroups of C and C is the direct product of C_s and C_u. If $C_s \neq \{e\}$ (resp. $C_u \neq \{e\}$), then there exists a rational representation f^* of G^* with $(C_s)^*$ (resp. $(C_u)^*$) as its kernel since $(C_s)^*$ (resp. $(C_u)^*$) is a connected central subgroup of G^* by (1.3). Let f be the restriction of f^* to G and put $G'=f(G)$. Then the kernel of f is C_s (resp. C_u) and G' is a connected splittable nilpotent group whose dimension is less than dim G. Therefore, by our supposition, G' is the direct product of the connected invariant closed subgroups G'_s and G'_u.

In the case where $C_s \neq \{e\}$, $f^{-1}(G'_s)$ is an invariant subgroup of G, which is a closed subset of G by the continuity of f. Since the kernel of f is C_s, we have $f^{-1}(G'_s)=G_s$. Hence G_s is an invariant subgroup of G. Since G can be put in triangular form (1.10), G_s is central in G by (1.14). $f(G_s)$ is connected and the kernel of the restriction of f^* to $(G_s)^*$ is connected since it is equal to $(C_s)^*$. Hence by (1.12) we see that G_s is connected. Thus G_s is a connected central closed subgroup of G.

In the case where $C_u \neq \{e\}$, let s be any semisimple element of G and let x be any element of G. Then $f(s)$ is semisimple by (1.11). By our supposition, we see that $f(s)$ is central in G', whence

$$xsx^{-1}=su \qquad \text{with } u \text{ in } C_u.$$

Since su is clearly the Jordan product decomposition of xsx^{-1}, we see that $xsx^{-1}=s$, that is, s is in $Z(G)$. Thus G_s is the semisimple part of $Z(G)$. It follows that G_s forms a group. Since $Z(G)$ can be triangulated in such a way that $Z(G)_s \subset D(n)$ by (1.8), $(G_s)^*$ consists of semisimple elements. It follows that G_s is a closed subset of G. $f(G_s)$ is connected since it is equal to G'_s by (1.11), and the kernel of the restriction of f^* to $(G_s)^*$ is equal to $\{e\}$. Hence, by (1.12), we see that G_s is connected. Thus G_s is a connected central closed subgroup of G.

Since G may be triangulated, by Lemma 2.3 we see that G_u is a connected invariant closed subgroup of G. Since G is splittable, we have $G=G_sG_u$ and therefore $G^*=(G_s)^*(G_u)^*$ by (1.5). G^* is connected, splittable and nilpotent by (1.2), (1.4) and (1.9). Therefore, as proved above, $(G^*)_s$ is a central closed subgroup of G^* and $G^*=(G^*)_s(G^*)_u$. Hence $(G_s)^* \subset (G^*)_s$ and $(G_u)^* \subset (G^*)_u$, from which it follows that

$$(G_s)^*=(G^*)_s \text{ and } (G_u)^*=(G^*)_u.$$

Let τ be a mapping of $(G_s)^* \times (G_u)^*$ into G^* defined by $\tau(s, u)=su$. Since $(G_s)^*$ centralizes $(G_u)^*$, it is immediate that there exists an element x of $GL(n, k)$ such that $x(G_s)^*x^{-1}$ is diagonal and $x(G_u)^*x^{-1}$ is in triangular form with coefficients 1 on the principal diagonal. Then s is the diagonal part of $\tau(s, u)=g$. Therefore u is rationally expressed by g, whence τ^{-1} is a rational mapping. Hence G is the direct product of G_s and G_u. Thus the theorem is proved.

COROLLARY 2.5. *Let G be a connected nilpotent subgroup of $GL(n, k)$. Then G_s is a central subgroup of G. G is splittable if and only if G contains a torus T such that $G=TG_u$.*

PROOF. Let G_1 be the smallest splittable subgroup of $GL(n, k)$ containing G. Since $G\subset G_1\subset G^*$, by (1.2) and (1.4) we see that G_1 is connected and nilpotent. By Theorem 2.4, $(G_1)_s$ is central in G_1. Hence G_s is a central subgroup of G. If $G=TG_u$ with T a torus, then it follows that T is central in G. Hence it is immediate that G is splittable. The converse is evident by Theorem 2.4.

LEMMA 2.6. *Let G be an algebraic torus of $GL(n, k)$ and let H be a subgroup of G. If an automorphism α of finite order m of G induces the identity on H and G/H, then α is the identity automorphism* [1, (11.5)].

PROOF. As a consequence of (1.9), we see that, for an integer q which is prime to the characteristic of k, the set of all elements of order $q^i (i=1, 2, 3,\cdots)$ of G is dense in G. Hence it suffices to prove that, for any integer r which is prime to m, α induces the identity automorphism on the subgroup G_r of G consisting of the elements of order r. If x is in G_r, then $\alpha(x)=xz$ with z in $H\cap G_r$. Therefore $\alpha^i(x)=xz^i$ $(i=1, 2, 3,\cdots)$, whence $z^m=e$. Since m is prime to r, we have $z=e$, that is, α is the identity on G_r. This completes the proof.

By using Theorem 2.4 and Lemma 2.6, we prove the following

PROPOSITION 2.7. *Let G be a nilpotent subgroup of $GL(n, k)$. Then the semisimple part of G_0 is in the center of G.*

PROOF. Since G^* is nilpotent by (1.4), it suffices to prove the theorem when G is algebraic. Suppose that G is algebraic and let x be any element of G. Since G_0 is splittable, by Theorem 2.4 G_{0s} is a connected central closed subgroup of G_0 and therefore is invariant in G. Put $\alpha(s)=xsx^{-1} (s \in G_{0s})$. If we denote by m an integer such that x^m is in G_0, then α is an automorphism of order m of G_{0s}. $(C^iG)_{0s}$ is an algebraic torus by Theorem 2.4 and α induces the identity automorphism on $(C^iG)_{0s}/(C^{i+1}G)_{0s}$. Hence, by using Lemma 2.6, we see that α is the identity on G_{0s}. Thus x centralizes G_{0s}, which shows that $G_{0s}\subset Z(G)$. The proof is complete.

3. Solvable groups

PROPOSITION 3.1. *Let G be a connected solvable subgroup of $GL(n, k)$. Then G_u is a closed invariant subgroup of G. If G is splittable, or if G contains a torus T such that $G=TG_u$, then G_u is connected and $(G_u)^*=(G^*)_u$.*

PROOF. By Lie-Kolchin's theorem (1.10) and (1.11), we may suppose that G is triangular. The first part follows from Lemma 2.3. Since G^* is connected and triangular, $(G^*)_u$ is a closed subgroup of G^*, whence $(G_u)^*\subset(G^*)_u$.

If G is splittable, G_u is connected by Lemma 2.3. By (1.3), $(G_u)^*$ is a connected invariant subgroup of G^*, whence there exists a rational representation f of G^* with $(G_u)^*$ as its kernel. If we put $G'=f(G)$, then G' is a splittable connected solvable group. Therefore G' can be triangulated by (1.10) and consists of semisimple elements by (1.11), whence it follows from (1.14) that G' is commutative. Thus G' is a torus and therefore G'^* is a torus by Corollary 2.2. Now, for any element x of $(G^*)_u$, we have $f(x)=e$, whence x is in $(G_u)^*$. Thus $(G^*)_u \subset (G_u)^*$ and therefore $(G^*)_u=(G_u)^*$.

If $G=TG_u$ with a torus T, by (1.5) we have $G^*=T^*(G_u)^*$. Since $(G_u)^* \subset (G^*)_u$, it follows that

$$(G^*)_u=(T^* \cap (G^*)_u)(G_u)^*=(G_u)^*.$$

Since G^* is connected and splittable, Lemma 2.3 tells us that $(G^*)_u$ is connected. Hence, by (1.2), we see that G_u is connected, completing the proof.

LEMMA 3.2. *Let G be a subgroup of $GL(n, k)$ and let T be a torus of G. Then the set of all elements of T whose centralizers in G are equal to the centralizer of T in G contains a non-empty open subset of T.*

PROOF. By (1.8) we may suppose that T is diagonal. An element $x=(x_{ij})$ of $GL(n, k)$ centralizes an element $t=(t_i)$ of T if and only if $x_{ij}=0$ for $t_i \neq t_j$. Let U be the set of all elements $s=(s_i)$ of T such that $s_i \neq s_j$ if $t_i \neq t_j$ for some t in T. Then any element of U has the centralizer equal to that of T in $GL(n, k)$ and therefore in G. Thus we have the statement.

LEMMA 3.3. *Let G be an algebraic subgroup of $GL(n, k)$. Let N be a connected commutative closed invariant subgroup of G consisting of unipotent elements, and let x be a semisimple element of G. Let F_x be the set of all elements of N commuting with x, and let M_x be the image of N by the mapping $f_x: y \rightarrow [x, y]$. Then $N=F_xM_x$, $F_x \cap M_x=\{e\}$, F_x is connected, and f_x is a mapping of M_x onto M_x* [1, (9.8)].

PROOF. f_x is a rational representation of N whose kernel is F_x. Suppose that $[x, a]$ with a in N is an element of $F_x \cap M_x$. Then, if we put $f'(y)=[y, a]$ for y in the smallest algebraic group H containing x, it can be seen that f' is a rational representation of H into F_x. Hence $f'(x)$ is semisimple and therefore $f'(x)=e$. Thus $F_x \cap M_x=\{e\}$. Since M_x is connected and $f_x(M_x) \subset M_x$, we have $f_x(M_x)=M_x$ by considering the dimension. $(F_x)_0M_x$ is a closed connected subgroup of finite index of N and therefore, by (1.2), we have $N=(F_x)_0M_x$. It follows that $F_x=(F_x)_0$, whence F_x is connected and $N=F_xM_x$, completing the proof.

LEMMA 3.4. *Let G be a connected solvable subgroup of $GL(n, k)$ whose unipotent part G_u is commutative. Suppose that G contains a torus T such that $G=TG_u$, and let x be a semisimple element of G. Then there exists an element u of $(G^*)_u$ such that uxu^{-1} is in T.*

PROOF. Since G^* is connected and solvable, by Proposition 3.1 we see that

$(G^*)_u$ is a connected closed invariant subgroup of G^* and $(G_u)^*=(G^*)_u$. It follows from (1.4) that $(G^*)_u$ is commutative. We can write $x=tz$ with t in T and z in G_u. By applying Lemma 3.3 for t^{-1} and $(G^*)_u$, we have

$$x=tfm \qquad \text{with } t \text{ in } T,\ f \text{ in } F_{t^{-1}}, \text{ and } m \text{ in } M_{t^{-1}}$$

and $y\rightarrow[t^{-1}, y]$ is a mapping of $M_{t^{-1}}$ onto $M_{t^{-1}}$. Therefore there exists an element u of $(G^*)_u$ such that $[t^{-1}, u]=m^{-1}$. Now we have $utu^{-1}=tm^{-1}$, whence

$$uxu^{-1}=utu^{-1}fm=tf.$$

Since $tf=ft$, tf is the Jordan product decomposition of uxu^{-1}, whence $uxu^{-1}=t$. Thus uxu^{-1} is in T, completing the proof.

We now generalize the structure theorem of connected solvable algebraic subgroups of $GL(n, k)$ [1, Theorem 12.9] in the following

THEOREM 3.5. *Let G be a connected solvable splittable C^∞-subgroup of $GL(n, k)$. Then maximal tori of G are conjugate by the elements of $C^\infty G^*$. For any maximal torus T of G, G is the semi-direct product of G_u by T.*

PROOF. If G is nilpotent, the statement is proved in Theorem 2.4. Therefore we suppose that $C^\infty G^*\neq\{e\}$ and $G\neq G_u$. We prove the theorem by induction on dim G. Let f^* be a rational representation of G^* with $C^\infty G^*$ as its kernel and let f be the restriction of f^* to G. Then $f(G)$ is a connected nilpotent splittable group. By Theorem 2.4, we see that $f(G)$ is the direct product of $f(G)_s$ and $f(G)_u$. Let T_1 and T_2 be maximal tori of G. Then $f(T_i)\subset f(G)_s$ $(i=1, 2)$.

Suppose that $f(G)_u\neq\{e\}$. Put $H=f^{-1}(f(G)_s)$. Since $f(G)_s$ is a connected central closed subgroup of $f(G)$, H is connected by (1.12) and is obviously a connected closed C^∞-subgroup of G. H is splittable as a closed subgroup of a splittable group G. And dim $H<$ dim G. Hence, by induction hypothesis, T_1 and T_2 are conjugate by an element of $C^\infty H^*$ and therefore of $C^\infty G^*$. Since $f(G_u)=f(G)_u$ by (1.11), we have $G=HG_u$ and therefore

$$G=T_iH_uG_u=T_iG_u \qquad (i=1, 2).$$

Now suppose that $f(G)_u=\{e\}$. Then $f(G)$ is a torus by (1.14) and therefore

$$C^\infty G^*=G_u.$$

We shall first prove that, for any maximal torus T of G, $f(T)$ is dense in $f(G)$. Assume that $f(T)$ is not dense in $f(G)$. Put $M=f^{-1}(f(T))$. Then M is connected by (1.12) and is obviously a solvable C^∞-subgroup of G. Since G is splittable, it is immediate that M is splittable. For any element x of G, we have $x^{-1}Tx\subset M$ since $f(G)_s$ is central in $f(G)$ and contains $f(T)$. Thus T and $x^{-1}Tx$ are maximal tori of M. Since dim $M<$ dim G, by induction hypothesis there exists an element m of M such that

$$m^{-1}(x^{-}Tx)m=T,$$

whence xm belongs to $n(T)$. Since $n(T)=z(T)$ by (1.10) and (1.14), it follows that xm is in $z(T)$. Therefore $G=z(T)M$. Since $M=TM_u$ by induction hypothesis, we have

$$G=z(T)TM_u=z(T)G_u.$$

Since $z(T)_0$ is a closed subset of $z(T)$ by (1.2) and $z(T)$ is a closed subset of G by (1.3), $z(T)_0$ is a closed subset of G. Since G_u is algebraic, it is immediate that $z(T)_0G_u$ is a closed subgroup of finite index of G, whence by (1.2) we have

$$G=z(T)_0G_u.$$

$z(T)_0$ is splittable as a closed subgroup of a splittable group G. By (1.3) we have

$$C^\infty(z(T)_0)^*\subset C^\infty G^*\cap(z(T)_0)^*\subset G\cap(z(T)_0)^*=z(T)_0,$$

that is, $z(T)_0$ is a C^∞-group. If dim $z(T)_0<$dim G, we have $z(T)_0=T(z(T)_0)_u$ by induction hypothesis, whence $G=TG_u$ and therefore $f(G)=f(T)$, which is a contradiction. Therefore dim $z(T)_0=$dim G. We have $(z(T)_0)^*=G^*$. Since $z(T)_0$ is a closed subset of G, it follows that

$$G=z(T)_0=z(T).$$

Then T^* is an invariant subgroup of G^*. Hence there exists a rational representation g^* of G^* with T^* as its kernel. Let g be the restriction of g^* to G and put $G'=g(G)$. Then dim $G'<$dim G. By induction hypothesis, we have

$$G'=S'(G')_u \qquad \text{for a maximal torus } S' \text{ of } G'.$$

$(g^{-1}(S'))_0$ is a subgroup of G and consists of semisimple elements since the kernel of g is T. It follows from (1.14) that $(g^{-1}(S'))_0$ is a torus of G, which obviously contains T. By the maximality of T, we have $(g^{-1}(S'))_0=T$. S' is therefore a finite group, whence $S'=\{e'\}$ (e' the identity element of G'). Therefore we have

$$G'=(G')_u=g(G_u).$$

It follows that $G=TG_u$, whence $f(G)=f(T)$, which contradicts our assumption. Thus we conclude that $f(T)$ is dense in $f(G)$.

Now, since f is continuous, we see that

$$f^*(G^*)=f(G)^*=f(T_i)^*=f^*(T_i^*).$$

Hence $G^*=T_i^*G_u$ and therefore

$$G=(G\cap T_i^*)G_u=T_iG_u \qquad (i=1, 2).$$

To prove the conjugacy of T_1 and T_2, we consider the following two cases separately.

(1) In the case where $G_u = C^\infty G^*$ is not commutative, let g^* be a rational representation of G^* with $D^1 G_u$ as its kernel, and let g be the restriction of g^* to G. Then

$$g(G) = g(T_i) g(G_u) = g(T_i) g(G)_u .$$

It follows that $g(T_i)$ is a maximal torus of $g(G)$. Since $g(G)$ is a connected solvable splittable C^∞-group and since $C^\infty g(G)^* = g(C^\infty G^*)$, by induction hypothesis there exists an element x of $C^\infty G^*$ such that

$$g(x) g(T_1) g(x^{-1}) = g(T_2).$$

Hence $x T_1 x^{-1} \subset g^{-1}(g(T_2))$. By (1.12) we see that $g^{-1}(g(T_2))$ is connected. Thus it is a connected splittable C^∞-group which has smaller dimension than G. Therefore, by induction hypothesis, there exists an element y of $C^\infty G^*$ such that $y(x T_1 x^{-1}) y^{-1} = T_2$. Thus T_1 and T_2 are conjugate by an element of $C^\infty G^*$.

(2) In the case where $G_u = C^\infty G^*$ is commutative, by Lemma 3.2 we see that there exists an element t_1 of T_1 such that $z(t_1) = z(T_1)$. Since $G = T_2 G_u$, it follows from Lemma 3.4 that there exists an element u of G_u such that $u t_1 u^{-1}$ is in T_2. Therefore

$$z(u T_1 u^{-1}) = z(u t_1 u^{-1}) \supset T_2 ,$$

whence $u T_1 u^{-1}$ and T_2 generate a torus of G. By the maximality of T_1 and T_2, we conclude that $u T_1 u^{-1} = T_2$.

Thus it remains only to prove that G is the semi-direct product of G_u by any maximal torus T. By Lie-Kolchin's theorem, we may suppose that G is in triangular form. Since $D(n)$ is a maximal torus of the triangular subgroup of $GL(n, k)$, by the fact proved above there exists an element a of $GL(n, k)$ such that $a G^* a^{-1}$ is triangular and $a T^* a^{-1} \subset D(n)$. Since $G = T G_u$, we have

$$G^* = T^* (G_u)^* = T^* (G^*)_u$$

by Proposition 3.1. The mapping τ: $(t, u) \to tu$ of $T^* \times (G^*)_u$ into G^* is an injective rational mapping. It is clear that τ is surjective. t is the diagonal part of $\tau(t, u) = x$ and therefore u is rationally expressed by x. Hence τ is an everywhere defined birational mapping. Thus G is the semi-direct product of G_u by T. The theorem is completely proved.

Corollary 3.6. *Let G be a connected solvable C^∞-subgroup of $GL(n, k)$. Then maximal tori of G are conjugate by the elements of $C^\infty G^*$.*

Proof. Let G_1 be the smallest splittable group containing G. Since $G \subset G_1 \subset G^*$, G_1 is a connected solvable splittable C^∞-group. For maximal tori T_1 and T_2 of G, let Q_1 and Q_2 be maximal tori of G_1 containing T_1 and T_2 respectively. Then we have

$$T_1=(G\cap Q_1)_0 \text{ and } T_2=(G\cap Q_2)_0.$$

By Theorem 3.5, there exists an element x of $C^\infty G^*$ such that $xQ_1x^{-1}=Q_2$. Therefore

$$x(G\cap Q_1)x^{-1}=G\cap xQ_1x^{-1}=G\cap Q_2,$$

whence $xT_1x^{-1}=T_2$.

COROLLARY 3.7. *Let G be a connected solvable splittable C^∞-subgroup of $GL(n, k)$ and let f be a rational representation of G. If T is a maximal torus of G, then $f(T)$ is a maximal torus of $f(G)$; and conversely.*

PROOF. If T is a maximal torus of G, then we have $G=TG_u$ by Theorem 3.5, whence

$$f(G)=f(T)f(G_u)=f(T)f(G)_u.$$

Since $f(T)$ is a torus, it follows that $f(T)$ is a maximal torus of $f(G)$. Conversely, let T' be a maximal torus of $f(G)$. Since $f(G)$ is a connected solvable splittable C^∞-group and $f(C^\infty G^*)=C^\infty f(G)^*$, it follows from Theorem 3.5 that there exists an element x of $C^\infty G^*$ such that

$$T'=f(x)f(T)f(x)^{-1}.$$

Hence $T'=f(xTx^{-1})$, where xTx^{-1} is obviously a maximal torus of G, completing the proof.

LEMMA 3.8. *Let G be a connected solvable C^∞-subgroup of $GL(n, k)$. Let T be a maximal torus of G.*

(1) *Let f be a rational representation of G. Then $f(z(T))$ is the centralizer of $f(T)$ in $f(G)$.*

(2) $G=z(T)\,(C^\infty G^*)$.

PROOF. Put $C=z(T)$ and let C' be the centralizer of $f(T)$ in $f(G)$. Then it is clear that $f(C)\subset C'$. Conversely, let $f(x)$ with x in G be an element of C'. Put $H=(f^{-1}(f(T)))_0$. Then it is immediate that H is a C^∞-group. Since $T\subset H$, we have $f(H)=f(T)$, whence $H\subset TN$ where N denotes the kernel of f. We have $xTx^{-1}\subset H$. Therefore, by Corollary 3.6, there exists an element h of H such that $xTx^{-1}=h^{-1}Th$. If we write

$$h=tn \qquad \text{with } t \text{ in } T \text{ and } n \text{ in } N,$$

then $xTx^{-1}=n^{-1}Tn$, whence $(nx)T(nx)^{-1}=T$, that is, nx normalizes T. It follows from (1.14) that nx centralizes T. Since $f(x)=f(nx)$, $f(x)$ is in $f(C)$. Therefore we have $C'\subset f(C)$, whence $C'=f(C)$.

Let g be a rational representation of G with $C^\infty G^*$ as its kernel. Then

$g(G)$ is nilpotent and connected. Therefore, by Corollary 2.5, the centralizer of $g(T)$ in $g(G)$ is $g(G)$. By the first part, we have $g(z(T))=g(G)$. Hence $G=z(T)(C^{\infty}G^{*})$, completing the proof.

PROPOSITION 3.9. *Let G be a connected solvable splittable C^{∞}-subgroup of $GL(n, k)$ and let x be a semisimple element of G. Then x is contained in a torus of G.*

PROOF. We prove the proposition by induction on dim G. Let T be a maximal torus of G. Then $G=TG_{u}$ by Theorem 3.5. If G_{u} is commutative, by Lemma 3.4 there exists an element u of G^{*} such that x is in uTu^{-1}. By using the formula in Lemma 3.8, we may suppose that u is in $C^{\infty}G^{*}$ and therefore in G. Thus uTu^{-1} is a torus of G containing x. If G_{u} is not commutative, let f^{*} be a rational representation of G^{*} with $D^{1}(G^{*})_{u}$ as its kernel and let f be the restriction of f^{*} to G. Then $f(G)$ is a connected solvable splittable C^{∞}-group whose unipotent part is commutative. By induction hypothesis, $f(x)$ is in a maximal torus of $f(G)$, which we can write $f(T_{1})$ with T_{1} a maximal torus of G by Corollary 3.7. Put $H=f^{-1}(f(T_{1}))$. Then the kernel of the restriction of f^{*} to H^{*} is equal to $D^{1}(G^{*})_{u}$ and therefore connected by (1.4) and Proposition 3.1. Hence, by (1.12), H is connected. It is immediate that H is a splittable C^{∞}-group. Therefore, by induction hypothesis, we see that x is in a torus of H. The proof is complete.

COROLLARY 3.10. *Let G be a connected solvable splittable C^{∞}-subgroup of $GL(n, k)$. Let S be a torus of G and let x be a semisimple element of G centralizing S. Then there exists a torus of G containing x and S.*

PROOF. Denote by H the connected component of e of the centralizer of x in G. Then H contains x by Proposition 3.9 and S by hypothesis. By using (1.3) we see that the centralizer of x in G is a closed subset of G, from which it follows by (1.2) that H is a closed subset of G. Hence it follows easily that H is a splittable C^{∞}-group. By applying Theorem 3.5 to H, we see that x is contained in every maximal torus of H. Hence x is contained in any maximal torus of H containing S.

LEMMA 3.11. *Let G be a connected solvable C^{∞}-subgroup of $GL(n, k)$ and let S be a torus of G. Then $z^{*}(S)=z^{*}(S^{*})=z(S)^{*}$.*

PROOF. By using (1.3), it is immediate that $z^{*}(S)=z^{*}(S^{*})$. Let T be a maximal torus of G containing S. Then $z(T)\subset z(S)$. By Lemma 3.8, we have

$$G=z(T)\,(C^{\infty}G^{*}),$$

whence

$$G=z(S)\,(C^{\infty}G^{*}) \text{ and } G^{*}=z(S)^{*}\,(C^{\infty}G^{*}).$$

Since $z(S)^{*}\subset z^{*}(S)$ by (1.3), we have

$$z^*(S)=z(S)^*(C^\infty G^* \cap z^*(S))=z(S)^*,$$

completing the proof.

PROPOSITION 3.12. *Let G be a connected solvable C^∞-subgroup of $GL(n, k)$ and let S be any torus of G. Then $z(S)$ is connected.*

PROOF. First we consider the case where G is splittable and S is a maximal torus of G. We prove the assertion by induction on dim G. Let m be the smallest integer such that $D^m G=\{e\}$. Put $N=D^{m-1}G$. Then N is connected by (1.4). Let f^* be a rational representation of G^* with N^* as its kernel and let f be the restriction of f^* to G. Then $f(G)$ is a connected solvable splittable C^∞-group by (1.11). Put $C=z(S)$. Then we see by Corollary 3.7 that $f(S)$ is a maximal torus of $f(G)$ and by Lemma 3.8 that $f(C)$ is the centralizer of $f(S)$ in $f(G)$. Hence, by induction hypothesis, $f(C)$ is connected. We have $C^*=z^*(S^*)$ by Lemma 3.11. Therefore, by Lemma 3.2, there exists an element x of S^* whose centralizer in G^* is equal to C^*. Hence, by Lemma 3.3, we see that $C^* \cap N^*$ is connected, that is, the kernel of the restriction of f^* to C^* is connected. By (1.12) we see that C is connected.

Secondly, we consider the case where G is splittable and S is any torus of G. Let T be a maximal torus of G containing S. Then, by Theorem 3.5, we have $G=TG_u$. Therefore $z(S)=Tz(S)_u$. Since G_u is closed and connected by Proposition 3.1, SG_u is a connected C^∞-group by (1.5). From the splittability of G, it is immediate that SG_u is splittable. It is clear that S is a maximal torus of SG_u and that the centralizer of S in SG_u is $Sz(S)_u$. Hence, by the first case above, $Sz(S)_u$ is connected. It follows from Proposition 3.1 that its unipotent part $z(S)_u$ is connected. Therefore, by (1.5), it is immediate that $z(S)$ is connected.

Finally, suppose that G is not splittable and S is any torus of G. Then, by Lemma 3.11, we have $z^*(S)=z(S)^*$. By the second case above, we see that $z^*(S)$ is connected. Therefore it follows from (1.2) that $z(S)$ is connected. The proof is complete.

4. Some properties of maximal solvable connected subgroups and maximal tori

LEMMA 4.1. *Let G be a connected algebraic subgroup of $GL(n, k)$ and let R be a maximal solvable connected subgroup of G. Then G/R is a complete variety.*

We omit the proof (see [1, (16. 10)]).

LEMMA 4.2. *Let G be a connected algebraic subgroup of $GL(n, k)$ and let f be a rational representation of G. If R is a maximal solvable connected subgroup of G, then $f(R)$ is a maximal solvable connected subgroup of $f(G)$* [1, (22.3)].

PROOF. Put $G'=f(G)$ and $R'=f(R)$. Let ρ be the projection of G' onto

G'/R'. Then $\rho\circ f$ is an everywhere defined rational mapping of G onto G'/R' which is constant on the cosets of R in G. Hence $\rho\circ f$ induces an everywhere defined rational mapping of G/R onto G'/R'. Since G/R is complete by Lemma 4.1, G'/R' is complete. By (1.10), R' is a maximal solvable connected subgroup of G'.

We now show some fundamental properties of maximal solvable connected subgroups of D^∞-groups in the following

THEOREM 4.3. *Let G be a connected D^∞-subgroup of $GL(n, k)$. Then*:

(1) *Maximal solvable connected subgroups of G are conjugate by the elements of $D^\infty G$.*

(2) *Any maximal solvable connected subgroup of G is the intersection of G and a maximal solvable connected subgroup of G^*, and its closure is a maximal solvable connected subgroup of G^*; and conversely.*

(3) *For any maximal solvable connected subgroup R of G, we have $G=R(D^\infty G)$.*

PROOF. Since G is a connected D^∞-group, $D^\infty G$ is equal to $D^\infty G^*$ and therefore is a connected invariant closed subgroup of G^*. Take a rational representation f of G^* with $D^\infty G$ as its kernel. Then $f(G^*)$ is solvable and connected. Let S be a maximal solvable connected subgroup of G^*. Then, by Lemma 4.2, $f(S)$ is a maximal solvable connected subgroup of $f(G^*)$, whence $f(G^*)=f(S)$ and therefore $G^*=S(D^\infty G)$. Hence

$$G=(G\cap S)(D^\infty G) \quad\text{and}\quad G^*=(G\cap S)^*(D^\infty G).$$

Since $(G\cap S)^*\subset S$, it follows that

$$S=(G\cap S)^*(S\cap D^\infty G)=(G\cap S)^*.$$

By (1.2) $G\cap S$ is connected. It is now immediate that $G\cap S$ is a maximal solvable connected subgroup of G. Conversely, let R be a maximal solvable connected subgroup of G. Take a maximal solvable connected subgroup S' of G^* containing R. Then, as above, we see that $G\cap S'$ is connected, $S'=(G\cap S')^*$ and $G=(G\cap S')(D^\infty G)$. By the maximality of R, we have $R=G\cap S'$, whence $R^*=S'$ and $G=R(D^\infty G)$. (2) and (3) are proved.

Let R_1 and R_2 be maximal solvable connected subgroups of G. Since $R_1{}^*$ is a maximal solvable connected subgroup of G^* by (2), $G^*/R_1{}^*$ is complete by Lemma 4.1. Hence, by (1.10), there exists a point of $G^*/R_1{}^*$, say $xR_1{}^*$, which is fixed by $R_2{}^*$, from which it follows that $xR_1{}^*x^{-1}=R_2{}^*$. By using (3), we may suppose that x is in $D^\infty G$. By taking the intersection with G, we have $xR_1x^{-1}=R_2$. Thus the theorem is proved.

COROLLARY 4.4. *Let G be a connected D^∞-subgroup of $GL(n, k)$. Let H be a connected solvable subgroup of G and let x be an element of G centralizing H. Then there exists a maximal solvable cennected subgroup of G containing H and x.*

PROOF. By Theorem 4.3, it suffices to prove the statement when G is algebraic. We write the proof given in [1, (18.2)]. Suppose that G is algebraic. Then it is known [1, (17.6)] that the union of all maximal solvable connected subgroups of G coincides with G. Let R be a maximal solvable connected subgroup of G. Since G/R is a complete variety by Lemma 4.1, the set F of all fixed points of x on G/R is closed and non-empty by (1.10) and the fact that x is contained in a solvable closed subgroup of G. It follows that F is invariant by H. Hence, by (1.10), there exists a fixed point, say yR, of H on G/R. Thus yR is fixed by H and x, from which it follows that H and x are contained in yRy^{-1}, completing the proof.

COROLLARY 4.5. *Let G be a connected D^{∞}-subgroup of $GL(n, k)$ and let R be a maximal solvable connected subgroup of G. Then:*

(1) $Z(G)=Z(R)$.

(2) *If R is nilpotent, $G=R$.*

PROOF. By Theorem 4.3, (1.3) and (1.4), it suffices to prove the statement when G is algebraic. We write the proof given in [4, 6–10, 11]. Let z be any element of $Z(R)$. Put $f(x)=xzx^{-1}(x \in G)$. Then f is constant on the cosets of R, whence f induces an everywhere defined rational mapping of G/R into G. Since G/R is complete by Lemma 4.1, the image of G/R is a complete affine variety and therefore reduces to a point. Hence z is in $Z(G)$. Thus we have $Z(R)=Z(G)$.

We prove (2) by induction on dim R. If dim $R=0$, then $R=\{e\}$. Since G/R is complete by Lemma 4.1, $G=\{e\}$. If dim $R>0$, put $H=Z(R)_0$. Then dim $H>0$. By (1) we have $H=Z(G)_0$, whence H is an invariant closed subgroup of G. Take a rational representation f of G with H as its kernel. By Lemma 4.2, $f(R)$ is a maximal solvable connected subgroup of $f(G)$. Therefore $f(R)=f(G)$ by induction hypothesis, whence $R=G$, completing the proof.

By using these results, we show some properties of maximal tori of C^{∞}-groups:

THEOREM 4.6. *Let G be a connected C^{∞}-subgroup of $GL(n, k)$. Then:*

(1) *Maximal tori of G are conjugate by the elements of G.*

(2) *Any maximal torus of any maximal solvable connected subgroup of G is a maximal torus of G.*

PROOF. Let T_1 and T_2 be maximal tori of G. Take maximal solvable connected subgroups R_1 and R_2 of G containing T_1 and T_2 respectively. Then, by Theorem 4.3, we have

$$xR_1x^{-1}=R_2 \qquad \text{with } x \text{ in } D^{\infty}G.$$

Hence xT_1x^{-1} is a maximal torus of R_2. Since R_2 is obviously a C^{∞}-group, Corollary 3.6 tells us that

$$y(xT_1x^{-1})y^{-1}=T_2 \qquad \text{with } y \text{ in } C^\infty R_2^*.$$

Thus T_1 and T_2 are conjugate by an element of $C^\infty G^*$.

Let R be a maximal solvable connected subgroup of G and let T be a maximal torus of R. Take a maximal torus T' of G and a maximal solvable connected subgroup R' of G containing T$'$. Then, as above, by using Theorem 4.3 and Corollary 3.6 we see that T and T' are conjugate by an element of G. Hence T is a maximal torus of G. Thus the theorem is proved.

As an immediate consequence of the theorem we have

COROLLARY 4.7. *Let G be a connected C^∞-subgroup of $GL(n, k)$. Then any maximal torus of G^* contains a maximal torus of G.*

THEOREM 4.8. *Let G be a connected C^∞-subgroup of $GL(n, k)$ and let S be a torus of G. Then $z(S)$ is connected.*

PROOF. Let x be any element of $z(S)$. Then, by Corollary 4.4, there exists a maximal solvable connected subgroup R of G containing x and S. Since R is a closed subset of G, R is a C^∞-group. It follows from Proposition 3.12 that the centralizer of S in R is connected and therefore contained in $z(S)_0$. Hence x is in $z(S)_0$. Thus $z(S)=z(S)_0$, that is, $z(S)$ is connected.

COROLLARY 4.9. *Let G be a connected splittable C^∞-subgroup of $GL(n, k)$ and let T be a maximal torus of G. Then $z(T)$ is nilpotent and contained in any maximal solvable connected subgroup of G containing T.*

PROOF. By Theorem 4.8, $z(T)$ is connected. Let R be a maximal solvable connected subgroup of $z(T)$ containing T. By using (1.3), it is immediate that $z(T)$ is a closed C^∞-subgroup of G and therefore R is also a closed C^∞-subgroup of G. R is splittable as a closed subgroup of a splittable group G. Therefore, by Theorem 3.5, we have

$$R=TR_u=T\times R_u,$$

whence R is nilpotent. By Corollary 4.5, we see that $z(T)$ is nilpotent.

Now there exists a maximal solvable connected subgroup S of G containing $z(T)$. Let S' be any maximal solvable connected subgroup of G containing T. Then, by Theorem 4.3, we have $S=xS'x^{-1}$ with x in G. Since S is a C^∞-group, by Corollary 3.6 we have

$$yTy^{-1}=xTx^{-1} \qquad \text{with } y \text{ in } S.$$

Hence

$$z(T)=x^{-1}yz(T)y^{-1}x\subset x^{-1}ySy^{-1}x=x^{-1}Sx=S',$$

completing the proof.

COROLLARY 4.10. *Let G be a connected C^∞-subgroup of $GL(n, k)$ and let S be* *any torus of* G^*. *Then* $z(S)$ *is connected,*

$$z(S)^*=z^*(S)=z^*(S^*) \quad and \quad G=z(S)(C^\infty G^*).$$

PROOF. By using (1.3), it is immediate that $z^*(S)=z^*(S^*)$. Let Q be a maximal torus of G^* containing S. Then $z^*(Q)\subset z^*(S)$. By Theorem 4.8 and Corollary 4.9, $z^*(Q)$ is connected and nilpotent. Take a maximal solvable connected subgroup R of G^* containing $z^*(Q)$. Let f be a rational representation of G^* with $C^\infty G^*$ as its kernel. Then, by Lemma 3.8, $f(z^*(Q))$ is equal to the centralizer of $f(Q)$ in $f(R)$. But, since $f(R)$ is connected and nilpotent, by Corollary 2.5 we see that $f(Q)$ is central in $f(R)$. Therefore $f(z^*(Q))=f(R)$, whence $R\subset z^*(Q)(C^\infty G^*)$. Since $G^*=R(D^\infty G)$ by Theorem 4.3, it follows that

$$G^*=z^*(Q)(C^\infty G^*)=z^*(S)(C^\infty G^*).$$

Since $C^\infty G^*\subset G$, we have

$$G=(G\cap z^*(S))(C^\infty G^*)=z(S)(C^\infty G^*),$$

whence $G^*=z(S)^*(C^\infty G^*)$. Since $z(S)^*\subset z^*(S)$, we have

$$z^*(S)=z(S)^*(z^*(S)\cap C^\infty G^*)=z(S)^*.$$

Thus we have $z(S)^*=z^*(S)=z^*(S^*)$. But, by Theorem 4.8, we see that $z^*(S)$ is connected. Hence $z(S)$ is connected by (1.2). This completes the proof.

5. Splittability and (S)-property

LEMMA 5.1. *Let* $G=HN$ *be a subgroup of* $GL(n, k)$ *such that* H *is a subgroup of* G *and* N *is an algebraic invariant subgroup of* G. *If* H *is splittable, then* G *is splittable.*

PROOF. By (1.3), N is an invariant subgroup of G^*. Hence there exists a rational representation f of G^* with N as its kernel. Suppose that H is splittable. For any element x of G, there exists an element y of H such that $f(x)=f(y)$, whence

$$f(x_s)=f(x)_s=f(y)_s=f(y_s)$$

and therefore

$$x_s=y_sz \qquad \text{with } z \text{ in } N.$$

Since y_s is in H, x_s is in G. Thus G is splittable, completing the proof.

5.2. A connected D^∞-subgroup G of $GL(n, k)$ is splittable if and only if a maximal solvable connected subgroup of G is splittable. This follows from Lemma 5.1 by using the formula (3) in Theorem 4.3. Corresponding to the splittability, we introduced another kind of "splittability", the (S)-property, for a D^∞-group. Namely, a D^∞-subgroup (resp. C^∞-subgroup) G of $GL(n, k)$ is called an *SD^∞-group* (resp. *SC^∞-group*) [8, Definitions 7.2 and 11.1] or to *have the (S)-property* provided G satisfies the following condition:

(S) There exists a maximal solvable connected subgroup R of G such that $R=TR_u$ for any maximal torus T and the invariant subgroup R_u of all unipotent elements of R.

We note that, on the definition of the (S)-property above, if $R=TR_u$ with T a torus of R, that is, R is the semi-direct product of R_u by a torus T in the group-theoretic sense, then R is the semi-direct product of R_u by T in the sense of (1.7). This was verified in the last part of the proof of Theorem 3.5 by using Lie-Kolchin's theorem.

By virtue of Theorem 4.3 (1), it is clear that, if a D^∞-group G has the (S)-property, then all the maximal solvable connected subgroups of G have the (S)-property.

On the connection between the (S)-property and the splittability, it is known that, for a connected D^∞-group G, the (S)-property of G does not imply the splittability of G and conversely the splittability of G does not imply the (S)-property of G [9, Examples 1 and 2], but they are equivalent for a connected C^∞-group or more generally for a connected D^∞-group H such that $C^\infty H^*$ normalizes H_s [8, Theorem 11.4 and 9, Theorem 5.1]. For our convenience, we write the proof of the following

THEOREM 5.3. *Let G be a connected C^∞-subgroup of $GL(n, k)$. Then G has the (S)-property if and only if G is splittable.*

PROOF. If G is splittable, any maximal solvable connected subgroup of G is a splittable C^∞-group as a closed subgroup of G. Hence it follows from Theorem 3.5 that G has the (S)-property.

Conversely, if G has the (S)-property, let R be a maximal solvable connected subgroup of G. For any element x of R, x_s and x_u are in R^*. By Proposition 3.9, there exists a maximal torus Q of R^* containing x_s. Since R is a C^∞-group, Q contains a maximal torus T of R by Corollary 4.7. Since $R=TR_u$ by the (S)-property,

$$x=tu \qquad \text{with } t \text{ in } T \text{ and } u \text{ in } R_u.$$

Hence $t^{-1}x_s=ux_u{}^{-1}$, which is semisimple and unipotent. Therefore $t^{-1}x_s=ux_u{}^{-1}=e$, that is, $x_s=t$ and $x_u=u$. Thus R is splittable. By using Theorem 4.3 (3) and Lemma 5.1, we see that G is splittable. The proof is complete.

6. Cartan subgroups

DEFINITION 6.1. *Let G be a group. A subgroup H of G is called a Cartan subgroup of G provided H is maximal nilpotent and any invariant subgroup of* *finite index of H is of finite index in its normalizer* [3, p. 199].

The following are immediate from the definition: If H is a Cartan subgroup of a group G and if L is a subgroup of G containing H, then H is a Cartan subgroup of L. For groups G_1 and G_2, the Cartan subgroups of $G_1 \times G_2$ are the products of the Cartan subgroups of G_1 and the Cartan subgroups of G_2. A Cartan subgroup of G contains the center of G [3, pp. 200–202].

It is known that a connected algebraic linear group, a connected splittable SD^∞-group and a connected C^∞-group have Cartan subgroups [3, p. 208 and 1, (20.5); 8, Proposition 8.10; 10, Theorem 5.4]. It is also known that, if G is a connected algebraic linear group or a connected splittable SD^∞-group or, more generally, a connected splittable D^∞-group satisfying the condition (a) (see (6.4) below), then a subgroup H of G is a Cartan subgroup of G if and only if H is the centralizer of a maximal torus of G [1, (20.8); 8, Theorem 9.3; 9, Theorem 4.8]. We showed that, if G is a connected splittable C^∞-group, a subgroup H of G is a Cartan subgroup of G if and only if H is the intersection of G and a Cartan subgroup of G^*[8, Theorem 12.2].

In this section, we generalize these results to a more general subgroup G of $GL(n, k)$. Especially, we study the interrelation between the following three kinds of subgroups of G:

(1) A Cartan subgroup of G.

(2) The centralizer of a maximal torus of G.

(3) The intersection of G and a Cartan subgroup of G^*.

We begin with the following

LEMMA 6.2. *Let G be a subgroup of $GL(n, k)$ and let H be a closed subgroup of G. Then the following conditions are equivalent*:

(1) *Any subgroup M of finite index of H is of finite index in $n(M)$.*

(2) *Any invariant subgroup N of finite index of H is of finite index in $n(N)$.*

(3) *H_0 is of finite index in $n(H_0)$.*

(4) *H_0 is the connected component of the identity element of $n(H_0)$.*

PROOF. It is evident that (1) implies (2) and (2) implies (3). Suppose that H_0 is of finite index in $n(H_0)$. Since H is a closed subset of G and H_0 is a closed subset of H by (1.2), H_0 is a closed subset of G. Hence H_0 is a connected closed subgroup of finite index of $n(H_0)$. By (1.2) we have $H_0 = n(H_0)_0$. Thus (3) implies (4).

Suppose that $H_0 = n(H_0)_0$, and let M be any subgroup of finite index of H. Then it is immediate that $M^* \cap G$ is a closed subgroup of finite index of H. Therefore, by using (1.2), we see that

$$H_0 \subset M^* \cap G \subset H,$$

whence by (1.3) we have

$$n(M) \subset n(M^* \cap G) \subset n(H_0).$$

Since M is of finite index in H, $M\cap H_0$ is of finite index in H_0. But, by our supposition, H_0 is of finite index in $n(H_0)$. Hence $M\cap H_0$ is of finite index in $n(H_0)$ and therefore M is of finite index in $n(M)$. Thus (4) implies (1), completing the proof.

PROPOSITION 6.3. *Let G be a subgroup of $GL(n, k)$. A subgroup H of G is a Cartan subgroup of G if and only if H is maximal nilpotent and H satisfies one of the equivalent conditions in Lemma* 6.2.

PROOF. A maximal nilpotent subgroup of G is a closed subset of G since its closure is nilpotent by (1.4). Hence the proposition follows from Lemma 6.2.

6.4. We now introduce the following two conditions for a connected D^{∞}-subgroup G of $GL(n, k)$, which are respectively weaker than the (S)-property and the splittability of G:

(a) *For one of the maximal solvable connected subgroups of G, say R, the closure of any maximal torus of R is a maximal torus of R^*.*

(b) *All maximal nilpotent connected subgroups of G are splittable or equivalently have the (S)-property.*

The equivalence of two kinds of "splittability" in the condition (b) follows from Theorem 5.3.

It is to be noted that there exists a connected D^{∞}-group satisfying the condition (a) which does not have the (S)-property [9, Example 1].

LEMMA 6.5. *Let G be a connected D^{∞}-subgroup of $GL(n, k)$ satisfying the condition* (a). *Then, for any maximal solvable connected subgroup R of G, the closure of any maximal torus of R is a maximal torus of R^*.*

PROOF. Let R' be a maximal solvable connected subgroup of G satisfying the condition (a). Let T be any maximal torus of R. Then, by Theorem 4.3, there exists an element x of G such that $R=xR'x^{-1}$. If we write $T'=x^{-1}Tx$, then T' is a maximal torus of R'. It follows that

$$R^*=xR'^*x^{-1} \text{ and } T^*=xT'^*x^{-1}.$$

Since T'^* is a maximal torus of R'^*, T^* is a maximal torus of R^*.

PROPOSITION 6.6. *Let G be a connected D^{∞}-subgroup of $GL(n, k)$ satisfying the condition* (a). *Then*:

(1) *Any maximal torus of G is the intersection of G and a maximal torus of G^*, and its closure is a maximal torus of G^*.*

(2) *Any maximal torus of any maximal solvable connected subgroup of G is a maximal torus of G.*

PROOF. Let T be a maximal torus of G. Take a maximal solvable connected subgroup R of G containing T. By Lemma 6.5, T^* is a maximal torus of R^*. Since R^* is a maximal solvable connected subgroup of G^* by Theorem 4.3, it follows from Theorem 4.6 that T^* is a maximal torus of G^*. Since $G\cap T^*$ is a torus by Corollary 2.2, we have $T=G\cap T^*$ by the maximality of T.

Let T' be any maximal torus of a maximal solvable connected subgroup R' of G. Then T'^* is a maximal torus of R'^* by Lemma 6.5. Since R'^* is a maximal solvable connected subgroup of G^* by Theorem 4.3, it follows from Theorem 4.6 that T'^* is a maximal torus of G^*. Since R' is a closed subset of G and T' is a closed subset of R' by (1), T' is a closed subset of G. By using Corollary 2.2, it is now easy to see that T' is a maximal torus of G. The proof is complete.

Lemma 6.7. *Let G be a connected subgroup of $GL(n, k)$. Let N be a commutative invariant subgroup of G consisting of semisimple elements. Then N is contained in the center of G.*

Proof. N^* is an invariant subgroup of G^* by (1.3) and is commutative by (1.4). By (1.8) we may suppose that N is in diagonal form. Hence N^* consists of semisimple elements. Let q be any positive integer and let N^*_q be the set of all elements of order q of N^*. Then we assert that N^*_q is central in G^*. In fact, it is immediate that $z^*(N^*_q)$ is of finite index in G^*. Therefore $z^*(N^*_q)_0$ is of finite index in G^*. Since $z^*(N^*_q)$ is a closed subset of G^*, $z^*(N^*_q)_0$ is also a closed subset of G^*. By (1.2), we have $z^*(N^*_q)_0=G^*$, that is, N^*_q is central in G^*, as was asserted. Now we have $N^*_q\subset Z(G^*)\cap N^*$. Thus any element of finite order of N^* is contained in $Z(G^*)\cap N^*$. As an easy consequence of (1.9), we see that the set of all elements of finite order of N^* is dense in N^*. Hence we have $N^*\subset Z(G^*)\cap N^*$ and therefore $N^*\subset Z(G^*)$. It follows that $N\subset Z(G)$, completing the proof.

Lemma 6.8. *Let G be a connected nilpotent subgroup of $GL(n, k)$ and let H be a connected proper subgroup of G. Then H is properly contained in its connected normalizer* [8, Lemma 9.2].

Proof. We prove the lemma by induction on dim G. If G is commutative or if H does not contain $Z(G)_0$, the lemma is clearly true. If $Z(G)_0\subset H$, let f^* be a rational representation of G^* with the closure of $Z(G)_0$ as its kernel and let f be the restriction of f^* to G. Then $f(G)$ is a connected nilpotent group whose dimension is less than dim G, and $f(H)$ is a connected proper subgroup of $f(G)$. By induction hypothesis, $f(H)$ is different from its normalizer N' in $f(G)$. Put $N=f^{-1}(N')$. Then, since the kernel of the restriction of f^* to N^* is connected and $f(N)$ is connected, by (1.12) we see that N is connected. Since N is different from H and normalizes H, we conclude that H is properly contained in $n(H)_0$.

We are now in a position to prove the following

Theorem 6.9. *Let G be a connected D^∞-subgroup of $GL(n, k)$ satisfying the* *condition* (a). *Then*:

(1) *G has Cartan subgroups. The centralizer of a maximal torus of G is a Cartan subgroup of G, which is the intersection of G and a Cartan subgroup of* G^* *and contained in one and only one Cartan subgroup of* G^*.

(2) *Suppose that G furthermore satisfies the condition* (b). *Then a subgroup H of G is a Cartan subgroup of G, if and only if H is the centralizer of a maximal torus of G, and only if H is the intersection of G and a Cartan subgroup of* G^*.

PROOF. (1) Let T be a maximal torus of G. Then, by Proposition 6.6, T^* is a maximal torus of G^*. By Corollary 4.9, we see that $z^*(T^*)$ is nilpotent. By using (1.3), we have $z(T)=G\cap z^*(T^*)$. Hence $z(T)$ is nilpotent. Suppose that H is a nilpotent subgroup of G containing $z(T)$. Then, by Proposition 2.7, we see that $T\subset Z(H)$, whence $H\subset z(T)$ and therefore $H=z(T)$. Thus $z(T)$ is maximal nilpotent. Now put $C=z(T)_0$. Then T is a unique maximal torus of C. Hence T is invariant in $n(C)$. By Lemma 6.7, we see that T is contained in the center of $n(C)_0$, which shows that $n(C)_0\subset C$, that is, $n(C)_0=C$. Thus, by Proposition 6.3, we conclude that $z(T)$ is a Cartan subgroup of G.

Since $z^*(T^*)$ is a Cartan subgroup of G^* as proved above, $z(T)$ is the intersection of G and a Cartan subgroup of G^*. If M is a Cartan subgroup of G^* containing $z(T)$, then T^* is contained in M since M is a closed subset of G^*. It follows from Proposition 2.7 that T^* is in the center of M, whence M is contained in $z^*(T^*)$. By the maximal nilpotency of M, we have $M=z^*(T^*)$, which shows that $z(T)$ is contained in a unique Cartan subgroup $z^*(T^*)$ of G^*.

(2) Suppose that G furthermore satisfies the condition (b). Let H be a Cartan subgroup of G. Then H_0 is a maximal nilpotent connected subgroup of G. In fact, if H_0 is properly contained in a connected nilpotent subgroup N of G, then it follows from Lemma 6.8 that H_0 is properly contained in the connected normalizer of H in N, which contradicts the second condition of a Cartan subgroup H by Proposition 6.3. By the condition (b), we now see that H_0 is splittable. It follows from Theorem 2.4 that $(H_0)_s$ is a unique maximal torus of H_0. Put $S=(H_0)_s$. Let R be a maximal solvable connected subgroup of G containing H_0 and let T be a maximal torus of R containing S. Let M be the connected centralizer of S in a group TR_u. Then we have

$$M=T(M\cap R_u)=TM_u.$$

M_u is connected by Proposition 3.1. By Theorem 2.4 we have

$$H_0=S\times(H_0)_u\subset M=TM_u.$$

By using Lemma 6.8 and the second condition of Cartan subgroups (the condition (4) in Lemma 6.2), we see that $(H_0)_u=M_u$, and therefore that $S=T$. Hence, by Proposition 6.6, S is a maximal torus of G and S^* is a maximal torus of G^*. Since $z^*(S^*)$ is nilpotent by Corollary 4.9, $z(S)$ is nilpotent as its subgroup. By Proposition 2.7, S is contained in the center of H, whence we have $H \subset z(S)$. By the maximal nilpotency of H, we conclude that $H = z(S)$. The other parts of (2) are proved in (1). Thus the theorem is completely proved.

COROLLARY 6.10. *Let G be a connected D^∞-subgroup of $GL(n, k)$ satisfying the conditions* (a) *and* (b). *Then any Cartan subgroup of G is a Cartan subgroup of any maximal solvable connected subgroup R of G containing its maximal torus T and is the intersection of R and $n(T)$.*

PROOF. By Theorem 6.9 and (1.14), it suffices to show that, if T is a maximal torus of a maximal solvable connected subgroup R of G, then $z(T) \subset R$. If T is a maximal torus of R, then T^* is a maximal torus of G^* by Proposition 6.6. Since R^* is a maximal solvable connected subgroup of G^* by Theorem 4.3, we have $z^*(T^*) \subset R^*$ by Corollary 4.9. Hence, by using (1.3) and Theorem 4.3, we have

$$z(T) = G \cap z^*(T^*) \subset G \cap R^* = R.$$

COROLLARY 6.11. *Let G be a subgroup of $GL(n, k)$. In each of the following cases, a subgroup H of G is a Cartan subgroup of G if and only if H is the centralizer of a maximal torus of G*:

(1) *G is a connected algebraic group* [1, (20.8)].

(2) *G is a connected splittable SD^∞-group* [8, Theorem 9.3].

(3) *G is a connected splittable D^∞-group satisfying the condition* (a) [9, Theorem 4.8].

(4) *G is a connected D^∞-group all of whose maximal solvable connected subgroups and maximal nilpotent connected subgroups have the* (S)*-property.*

PROOF. If G has the (S)-property, then G clearly satisfies the condition (a). Therefore the statement is immediate from Theorem 6.9.

It is to be noted that there exists a connected splittable D^∞-group satisfying the condition (a) which is not an SD^∞-group [9, Example 1].

By making use of Theorem 6.9, we can now prove the following

THEOREM 6.12. *Let G be a connected C^∞-subgroup of $GL(n, k)$. Then*:

(1) *G has Cartan subgroups. The intersection of G and a Cartan subgroup of G^* is a connected Cartan subgroup of G, whose closure is a Cartan subgroup of G^*. These Cartan subgroups are conjugate by the elements of G. A Cartan subgroup of G^* is the closure of a Cartan subgroup of G* [10, Theorem 5.4].

(2) *Suppose that G satisfies the condition* (a). *Then a subgroup H of G is the centralizer of a maximal torus of G, if and only if H is the intersection of G and a Cartan subgroup of G^*, and only if H is a Cartan subgroup of G.*

PROOF. (1) By Theorem 6.9, G^* has Cartan subgroups. Let C be a Cartan subgroup of G^*. Then, by Theorem 6.9, we have

$$C = z^*(Q) \qquad \text{with } Q \text{ a maximal torus of } G^*.$$

By Corollary 4.10, $G\cap C$ is connected and $C=(G\cap C)^*$. Let H be a nilpotent subgroup of G containing $G\cap C$. Then

$$C=(G\cap C)^*\subset H^*.$$

Since H^* is nilpotent by (1.4), by the maximal nilpotency of C we have $C=H^*$. It follows that $H\subset G\cap C$, whence $H=G\cap C$. Therefore $G\cap C$ is maximal nilpotent. Furthermore, since $C=(G\cap C)^*$, by (1.3) we have

$$n(G\cap C)_0\subset G\cap n^*(C)_0\subset n(G\cap C).$$

Since $n^*(C)_0=C$ and since $G\cap C$ is connected, it follows that

$$n(G\cap C)_0=G\cap C.$$

By Proposition 6.3, we see that $G\cap C$ is a Cartan subgroup of G.

Let C_1 and C_2 be Cartan subgroups of G^*. Then

$$C_i=z^*(Q_i) \qquad \text{with } Q_i \text{ a maximal torus of } G^* \ (i=1,2).$$

By Theorem 4.6, there exists an element x of $C^\infty G^*$ such that $Q_1=xQ_2x^{-1}$. Hence $C_1=xC_2x^{-1}$ and therefore

$$G\cap C_1=G\cap xC_2x^{-1}=x(G\cap C_2)x^{-1}.$$

(2) Suppose that G satisfies the condition (a). Let H be the centralizer of a maximal torus T of G. Then, by (1.3), we have $H=G\cap z^*(T^*)$. By Proposition 6. 6, T^* is a maximal torus of G^* and therefore, by Theorem 6.9, $z^*(T^*)$ is a Cartan subgroup of G^*. Thus H is the intersection of G and a Cartan subgroup of G^*. Conversely, let H be the intersection of G and a Cartan subgroup C of G^*. Then, by Theorem 6.9, we have

$$C=z^*(Q) \qquad \text{with } Q \text{ a maximal torus of } G^*.$$

But Q contains a maximal torus T' of G by Corollary 4.7. Hence $Q=T'^*$ by Proposition 6.6. Therefore we have

$$H=G\cap C=G\cap z^*(T'^*)=z(T').$$

The other part of (2) is proved in (1). Thus the theorem is completely proved.

COROLLARY 6.13. *Let G be a connected C^∞-subgroup of $GL(n, k)$. Let S be a torus of G and let x be a semisimple element of G centralizing S. Then there exists a Cartan subgroup of G containing x and S* [10, Proposition 5.5].

PROOF. x and S are contained in a maximal solvable connected subgroup R of G by Corollary 4.4 and therefore in a maximal torus Q of R^* by Corollary 3.10. Since R^* is a maximal solvable connected subgroup of G^* by Theorem 4.3, Q is a maximal torus of G^* by Theorem 4.6. Hence $z^*(Q)$ is a Cartan subgroup of G^* by Theorem 6.9. Therefore, by Theorem 6.12, $G\cap z^*(Q)$ is a Cartan subgroup of G, which contains x and S.

COROLLARY 6.14. *Let G be a connected complex linear Lie group. Then G has Cartan subgroups and the Cartan subgroups of G contain all semisimple elements of G.*

PROOF. We may suppose that G is a subgroup of $GL(n, \boldsymbol{C})$ with $\boldsymbol{C}$ the field of complex numbers. As is well known, the Euclidean topology is finer than the Zariski topology in $GL(n, \boldsymbol{C})$. Hence G is connected in the Zariski topology. As was shown in [8, Example 2], G is a C^∞-subgroup of $GL(n, \boldsymbol{C})$. Therefore by Theorem 6.12 we see that G has Cartan subgroups and by Corollary 6.13 that every semisimple element of G is contained in a Cartan subgroup of G.

On the connection between the condition (b) and the splittability, we have the following

PROPOSITION 6.15. *Let G be a connected C^∞-subgroup of $GL(n, k)$. Then G satisfies the condition* (b) *if and only if G is splittable, and only if G satisfies the condition* (a).

PROOF. Suppose that G satisfies the condition (b). Take a maximal torus Q of G^* and denote by H the centralizer of Q in G. Then H is a connected Cartan subgroup of G by Theorem 6.12. Hence H is a maximal nilpotent connected subgroup of G and therefore H is splittable by the condition (b). By Corollary 4.10, we have

$$G=H(C^\infty G^*).$$

It follows from Lemma 5.1 that G is splittable. Conversely, if G is splittable, then any maximal nilpotent connected subgroup of G is splittable since it is a closed subset of G by (1.2) and (1.4), whence G satisfies the condition (b).

A connected splittable C^∞-group has the (S)-property by Theorem 5.3 and therefore satisfies the condition (a). Thus the proposition is proved.

COROLLARY 6.16. *Let G be a connected splittable C^∞-subgroup of $GL(n, k)$. Then G has Cartan subgroups.*

(1) *Cartan subgroups of G are connected, are splittable, are conjugate by the elements of G, and contain all semisimple elements of G.*

(2) *Any Cartan subgroup of G is the centralizer of a maximal torus of G, and conversely.*

(3) *Any Cartan subgroup of G is the intersection of G and a Cartan subgroup of G^*, and conversely. The closure of any Cartan subgroup of G is a Cartan subgroup of G^*, and conversely.*

(4) *Any Cartan subgroup of a maximal solvable connected subgroup of G is a Cartan subgroup of G, and any Cartan subgroup of G is a Cartan subgroup of any maximal solvable connected subgroup of G containing its maximal torus.*

(5) *Any Cartan subgroup of G is algebraic if and only if G is algebraic.*

PROOF. By Proposition 6.15 we see that G satisfies the conditions (a) and (b).

We have (2) by Theorem 6.9 and we have (3) by Theorems 6.9 and 6.12. Any Cartan subgroup of G is splittable as a closed subgroup of a splittable group G. Hence we have (1) by (3), Theorem 6.12 (1) and Corollary 6.13.

Let R be any maximal solvable connected subgroup of G. Then R is a splittable C^∞-group as a closed subgroup of G. Hence, by (2), any Cartan subgroup H of R is the centralizer of a maximal torus T of R in R. But T is a maximal torus of G by Theorem 4.6. By Corollary 6.10 we see that H is the centralizer of T in G. It follows from (2) that H is a Cartan subgroup of G. Thus we have the first part of (4). The second part of (4) follows from Corollary 6.10.

Any Cartan subgroup H' of G is the centralizer of a maximal torus of G by (2). Hence it follows from Corollary 4.10 that

$$G=H'(C^\infty G^*).$$

Therefore, if H' is algebraic, then G is algebraic by (1.5). The converse is immediate from (1.4) and we have (5). The proof is complete.

REMARK 6.17. In Theorem 6.9 (2), we cannot assert that (1) the intersection of G and any Cartan subgroup of G^* is a Cartan subgroup of G and also that (2) every Cartan subgroup of G^* is the closure of a Cartan subgroup of G. The statements (1) and (2) are not necessarily true even for a connected D^∞-group having both the splittability and the (S)-property, although they are true for a connected C^∞-group as was shown in Theorem 6.12.

The goup in [9, Example 3] gives an example for these facts. Namely, let $\boldsymbol{C}$ be the field of complex numbers and let a be an element of $\boldsymbol{C}$ which is transcendental over the prime field. Let k_1 be the subring of $\boldsymbol{C}$ consisting of all rational functions of a with integral coefficients. Let G be the group of all matrices of the following forms:

$$g=\begin{pmatrix} a^r & b \\ 0 & 1 \end{pmatrix} \quad \text{with } r \text{ an integer and } b \text{ in } k_1.$$

Then G is a connected splittable solvable subgroup of $GL(2, \boldsymbol{C})$. Furthermore G has the (S)-property, since every maximal torus of G is generated by an element g with $r=1$. Let H be the subgroup of G^* which can be represented as

$$\left\{\begin{pmatrix} x & u(x-1) \\ 0 & 1 \end{pmatrix} : x(\neq 0) \text{ variable in } \boldsymbol{C}\right\}$$

with u an element of $\boldsymbol{C}$ which is not in k_1. Then it is easy to see that H is a maximal torus of G^*, H is equal to its centralizer in G^* and $G\cap H=\{e\}$. Therefore H is a Cartan subgroup of G^* whose intersection with G is not a Cartan subgroup of G and which is not the closure of any Cartan subgroup of G.

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Department of Mathematics,
Faculty of Science,
Hiroshima University