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Dimension Theory on Relatively Semi-orthocomplemented Complete Lattices

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Introduction

In my previous paper [11], the dimension theory of a relatively semi-orthocomplemented complete lattice (cf. [13]) with an equivalence relation has been developed by the axiomatic treatement, which enables us to unify the dimension theories of the von Neumann algebras and the continuous geometries. Our method is very similar to that of Loomis [8], but he treated only the case where the lattice is orthocomplemented. Let L be a relatively semi-orthocomplemented complete lattice where the semi-orthogonality "$\perp$" satisfies the following condition: $a_\delta \uparrow a$, $a_\delta \perp b \Rightarrow a \perp b$. It has been shown in [11] that if there is an equivalence relation in L satisfying certain axioms (denoted by $(2, \beta)$—$(2, \zeta)$ in [11]) then there exist the dimension functions with respect to this equivalence relation. In [12], this system of axioms was modified for the purpose of giving simple conditions for a Baer *-ring under which the lattice of projections of this ring has the dimension functions with respect to the algebraic equivalence (or the *-equivalence) introduced by Kaplansky. Indeed, these conditions are satisfied by the Baer *-rings considered by Kaplansky [6] and [7], and consequently by the AW*-algebras and the von Neumann algebras.

Now, we consider the projectivity of an upper-continuous complemented modular lattice for the purpose of generalizing the dimension theory of the continuous geometries. The systems of axioms given in [11] and [12] include the axiom of (complete or finite) additivity, but the above projectivity does not generally satisfy this axiom. For this reason, in this paper we shall give another system of axioms which is weaker than the systems in [12] and [8], and we shall develop the dimension theory on L, which not only covers the existing dimension theories of the Baer *-rings and the continuous geometries but also throws light on the dimension theory of upper-continuous complemented modular lattices.

In this paper, the system of axioms for equivalence relation is given as follows:

(A_1) $a \sim 0$ implies $a=0$;

(A_2) if $a \sim b_1 \dot{\cup} b_2$ then there exists a decomposition $a = a_1 \dot{\cup} a_2$ with $a_i \sim b_i$ $(i=1, 2)$;

(B) if we put $a=(a\cap b)\dot{\cup} a_1$, $b=(a\cap b)\dot{\cup} b_1$, $a\cup b = a_2 \dot{\cup} b = a \dot{\cup} b_2$ for any a, b, then there exist decompositions $a_1=a_1'\dot{\cup}a_1''$, $b_1=b_1'\dot{\cup}b_1''$ such that $a_1'\sim a_2$, $b_1'\sim b_2$, $a_1''\sim b_1''$ (see Fig. 1);

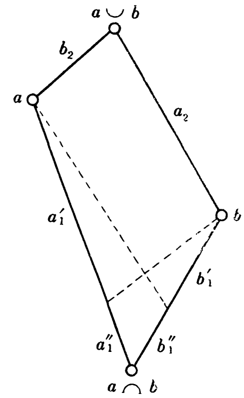

Fig. 1.

($C_\perp$) if $a=\dot{\bigvee}_\alpha a_\alpha$, $b=\dot{\bigvee}_\alpha b_\alpha$, $a_\alpha\sim b_\alpha$ for every α and $a\perp b$, then $a\sim b$;

(C_f) if $a=a_1\dot{\cup}a_2$, $b=b_1\dot{\cup}b_2$, $a_i\sim b_i$ $(i=1, 2)$, then $a\sim b$.

The projectivity in an upper-continuous complemented modular lattice does not generally satisfy (C_f), but we shall show (in §8) that (C_f) may be omitted if (B) is replaced by the following stronger axiom ($a_1''=b_1''=0$):

($\overline{B}$) If a and b are perspective then $a\sim b$.

(In the previous papers [11] and [12], the axioms (A_1), (A_2), ($\overline{B}$), ($C_\perp$), (C_f) are denoted by $(2, \beta)$, $(2, \gamma)$, $(2, \eta)$, $(2, \delta_2)$, $(2, \delta_1)$ respectively.)

The dimension function on L having an equivalence relation "$\sim$" is a mapping d on L into the set of non-negative continuous functions on the Boolean space Ω, representing the relative center Z_0 with respect to "$\sim$". It is characterized by the following axioms:

($1°$) If $a\sim b$ then $d(a)=d(b)$;

($2°$) if $a\perp b$ then $d(a\cup b)=d(a)+d(b)$;

($3°$) if $z\in Z_0$ then $d(z\cap a)=\chi(z)d(a)$, where $\chi(z)$ is the characteristic function of the compact, open subset of Ω corresponding to z;

($4°$) if $a>0$ then $d(a)>0$;

($5°$) if a is a finite element then $d(a)$ is finite valued except on a set of the first category.

Our main result is that if "$\sim$" satisfies the axioms (A_1), (A_2), (B), ($C_\perp$), (C_f) or the axioms (A_1), (A_2), ($\overline{B}$), ($C_\perp$) then we can construct on L the dimension functions with respect to "$\sim$".

In §1, some properties of relatively semi-orthocomplemented complete lattices are given. In §2, the relative center Z_0 with respect to an equivalence relation "$\sim$" in L is defined. And, supposing that "$\sim$" satisfies (A_1) and (A_2), we define the minimal element and the finite element, and it is proved that L can be decomposed into five summands, which are finite of types I and II, properly infinite of types I and II and of type III respectively.

In §§3–5, the axiom (B) is replaced by the following weaker one:

(B') If b is a complement of a and c is a semi-orthocomplement of a then $b\gtrsim c$.

Supposing that "$\sim$" satisfies (A_1), (A_2) and (B'), the following property (denoted by $(2, \varepsilon)$ in [11] and [12]) is proved in §3:

(B'') If $e(a)\cap e(b)\neq 0$, then there exist a_1, b_1 such that $0\neq a_1\leqq a$, $0\neq b_1\leqq b$, $a_1\sim b_1$.

If "$\sim$" satisfies (A_1), (A_2), (B'), ($C_\perp$) and (C_f), then we can prove the comparability theorems which play an important rôle in the dimension theory. But, in the case of AW*-algebras, it is not easy to show that the *-equivalence satisfies (B′), though (B″) is easily proved. For this reason, some results implied from the five axioms (A_1), (A_2), (B″), ($C_\perp$), (C_f) without (B′) are gathered in §4. These results will be useful not only in the following argument of our dimension theory but also in proving that the *-equivalence in an AW*-algebra satisfies (B′) (actually satisfies ($\bar{B}$)).

In §5, supposing that "$\sim$" satisfies moreover (B′), we prove the comparability theorems and also prove the complete additivity of "$\sim$" in the finite case.

Besides the comparability theorems, the following theorem is important in the dimension theory: If a and b are finite then so is $a\cup b$. In §6, we show that this theorem holds if and only if "$\sim$" satisfies moreover (B).

Supposing that "$\sim$" satisfies (A_1), (A_2), (B), ($C_\perp$), (C_f), our concluding theorems concerning the existence and other properties of the dimension functions can be proved in the same way as in [11], by using the results of these sections. These theorems are stated in §7 without proofs.

In §8, we consider the axiom ($\bar{B}$). We show that if "$\sim$" satisfies (A_1), (A_2), ($\bar{B}$), ($C_\perp$) then we can define a new equivalence relation "$\overset{+}{\sim}$" satisfying the above four axioms and moreover (C_f). And, it is proved that, for both the original and the new equivalence relations, the relative centers (resp. the minimal elements, the finite elements, the summands of each type, the dimension functions) are the same, though their definitions depend on the equivalence relations. This shows that (C_f) may be omitted from our system if (B) is replaced by ($\bar{B}$). In this case, we can prove the following theorem by the aid of ($\bar{B}$): If L is finite then it is an upper-continuous complemented modular lattice.

The examples of our axiomatic argument are given in §9 and §10.

It is shown in §9 that the projectivity in a complemented modular complete lattice satisfies the axioms (A_1), (A_2) and ($\bar{B}$), and that it satisfies moreover ($C_\perp$) if the lattice is upper-continuous or orthocomplemented (the semi-orthogonality is defined by the independence or the orthogonality). This implies that any upper-continuous complemented modular lattice and any orthocomplemented modular complete lattice have the dimension functions with respect to the projectivity. We note that Kaplansky's theorem: "Any orthocomplemented modular complete lattice is a continuous geometry" is a consequence of our dimension theory.

It is shown in §10 that the algebraic equivalence in the lattice of principal right ideals of any upper-continuous regular ring or the lattice of projections of any complete *-regular ring (Kaplansky [6]) satisfies (A_1), (A_2) ($\bar{B}$), ($C_\perp$) and (C_f), and that the *-equivalence in the lattice of projections of any AW*-algebra does also. Hence, these lattices have dimension functions with respect to the algebraic equivalence or the *-equivalence. These equivalences are defined by the algebraic structure, but our final result shows that each of them coincides with an equivalence relation (called semi-projectivity) defined by the lattice-structure.

§ 1 Semi-orthogonal relation

Let L be a lattice with 0. A *semi-orthogonal relation* "$\perp$" in L is a binary relation which satisfies the following axioms (see [13, §1]):

($\perp$1) $a\perp a$ implies $a=0$,

($\perp$2) $a\perp b$ implies $b\perp a$,

($\perp$3) $a\perp b$, $a_1\leqq a$ imply $a_1\perp b$,

($\perp$4) $a\perp b$, $a\cup b\perp c$ imply $a\perp b\cup c$.

A subset S of L is called a *semi-orthogonal family*, in notation $(a;\, a\in S)\perp$, if for any pair of disjoint finite subsets F_1, F_2 of S it holds that $\bigvee(a;\, a\in F_1)\perp\bigvee(a;\, a\in F_2)$. It is obvious that if S is a semi-orthogonal family then so is every subset of S and that if every finite subset of S is a semi-orthogonal family then so is S. The symbol $\dot{\cup}$ means the join of a semi-orthogonal family.

LEMMA 1.1. (i) *Let $a_i\in L$, $1\leqq i\leqq n$. If $a_1\cup\cdots\cup a_i\perp a_{i+1}$ for every $i=1,\cdots, n-1$, then $(a_i;\, 1\leqq i\leqq n)\perp$.*

(ii) *Let S be a subset of L. If $a_0\perp\bigvee(a;\, a\in F)$ whenever F is a finite subset of S and $a_0\in S-F$, then $(a;\, a\in S)\perp$.*

PROOF. The statement (i) can be proved by induction, because, if $(a_1,\cdots, a_i)\perp$ for $i<n$ then it is easy to prove $(a_1,\cdots, a_{i+1})\perp$ by the axiom ($\perp$4).

(ii) It follows from (i) that any finite subset of S is a semi-orthogonal family and hence so is S.

LEMMA 1.2. *Let L be complete. If S_α is a semi-orthogonal family for every $\alpha\in I$ and $\{\dot{\bigvee}(a;\, a\in S_\alpha);\, \alpha\in I\}$ is also, then $\bigvee(S_\alpha;\, \alpha\in I)$ is also a semi-orthogonal family.*

PROOF. In the case $I=\{1,2\}$, this lemma is easily proved by Lemma 1.1 (ii) and ($\perp$4). Hence, in the general case, any finite subset of $\bigvee(S_\alpha;\, \alpha\in I)$ is a semi-orthogonal family and then so is $\bigvee(S_\alpha;\, \alpha\in I)$.

Let L be a lattice with 0,1 and have a semi-orthogonal relation. $a'\in L$ is called a *semi-orthocomplement* of $a\in L$ if $a\perp a'$, $a\cup a'=1$ (or simply $a\dot{\cup}a'=1$). L is called to be *relatively semi-orthocomplemented* if for every $a, b\in L$ with $a\leqq b$ there exists $c\in L$ with $b\dot{\cup}c=a$ (c is called a relative semi-orthocomplement of b in a). The following statements are proved in [13, §2].

LEMMA 1.3. *Let L be a relatively semi-orthocomplemented lattice and Z be its center. An element of L is in Z if and only if it has a unique complement.*

THEOREM 1.1. *Let L be a relatively semi-orthocomplemented complete lat-*

tice.

(i) *The center Z of L is a complete Boolean sublattice of L.*

(ii) *Let $a_\delta \uparrow a$* (i.e. $\{a_\delta\}$ is an ascending set with the join a). *If $a_\delta \in Z$ for every δ or $b \in Z$, then $a_\delta \cap b \uparrow a \cap b$.*

§ 2. Axiom A for equivalence relation

Hereafter, let L be a relatively semi-orthocomplemented complete lattice and Z be its center.

Definition 2.1. We assume that there is an equivalence relation "$\sim$" in L. If $a \sim b_1 < b$ in L, we shall write $a \prec b$. The set $Z_0 = \{z \in Z;\ a \precsim z$ implies $a \leqq z\}$ is called a *relative center* with respect to the equivalence relation. Since $1 \in Z_0$ and since $z_\alpha \in Z_0$ for every α implies $\bigcap_\alpha z_\alpha \in Z_0$, for any $a \in L$ there is the smallest element $z \in Z_0$ such that $a \leqq z$. We shall denote it by $e(a)$. It is obvious that $a \sim b$ implies $e(a) = e(b)$.

Axiom A. We give the following axioms for equivalence relations in L.

(A_1) *$a \sim 0$ implies $a = 0$.*

(A_2) *If $a \sim b_1 \dot{\cup} b_2$ then there exists a decomposition $a = a_1 \dot{\cup} a_2$ with $a_i \sim b_i$* ($i = 1, 2$).

Sometimes, we shall replace (A_2) by the following stronger axiom:

($\bar{A}_2$) *If $a \sim \dot{\bigcup}_\alpha b_\alpha$ then there exists a decomposition $a = \dot{\bigcup}_\alpha a_\alpha$ with $a_\alpha \sim b_\alpha$ for every α.*

These axioms are satisfied if the following one is satisfied. ($L(0, a)$ denotes the lattice $\{x \in L;\ x \leqq a\}$.)

(A) *If $a \sim b$ then there is a lattice-isomorphism Φ of $L(0, a)$ onto $L(0, b)$ such that $\Phi(x) \sim x$ for every $x \in L(0, a)$ and that $x \perp y \Leftrightarrow \Phi(x) \perp \Phi(y)$ ($x, y \in L(0, a)$).*

In this section we assume that there is an equivalence relation "$\sim$" in L satisfying the axioms (A_1) and (A_2). The relative center has the following properties.

Lemma 2.1. (i) *Z_0 is a complete Boolean sublattice of L.*

(ii) $e(\bigcup_\alpha a_\alpha) = \bigcup_\alpha e(a_\alpha)$.

(iii) *If $z \in Z_0$ then $e(z \cap a) = z \cap e(a)$.*

(iv) *If $a \sim b$ and $z \in Z_0$ then $z \cap a \sim z \cap b$.*

Proof. (i) Since $z_\alpha \in Z_0$ implies $\bigcap_\alpha z_\alpha \in Z_0$, it suffices to show that $z \in Z_0$ implies $1 - z \in Z_0$. If $a \sim b \leqq 1 - z$, then by (A_2) there is $b_1 \leqq b$ with $b_1 \sim z \cap a$, and then it follows from $z \in Z_0$ that $b_1 \leqq z$. Hence $b_1 \leqq z \cap b = 0$, which implies $z \cap a = 0$ by (A_1). Therefore $a = (1 - z) \cap a \leqq 1 - z$.

(ii) It is obvious that $\bigcup_\alpha a_\alpha \leqq \bigcup_\alpha e(a_\alpha) \leqq e(\bigcup_\alpha a_\alpha)$. Since $\bigcup_\alpha e(a_\alpha) \in Z_0$ by (i), we have $\bigcup_\alpha e(a_\alpha) = e(\bigcup_\alpha a_\alpha)$.

(iii) It is obvious that $e(z \cap a) \leqq z \cap e(a)$. Since it follows from (ii) that $e(a) = e(z \cap a) \cup e((1 - z) \cap a) \leqq e(z \cap a) \cup (1 - z)$, we have $z \cap e(a) \leqq e(z \cap a)$.

(iv) Since $a=(z\cap a)\dot{\cup}((1-z)\cap a)$, it follows from (A_2) that there exists a decomposition $b=b_1\dot{\cup}b_2$ with $b_1\sim z\cap a$, $b_2\sim(1-z)\cap a$. Since z and $1-z$ are in Z_0, we have $b_1\leqq z$, $b_2\leqq 1-z$, which imply $z\cap b=(z\cap b_1)\cup(z\cap b_2)=b_1\sim z\cap a$.

DEFINITION 2.2. We shall write $a\ll b$ if for every $z\in Z_0$ either $z\cap a\prec z\cap b$ or $z\cap a=z\cap b=0$. Obviously, $a\ll b$ implies either $a\prec b$ or $a=b=0$. It follows from (A_1) and (A_2) that $a\precsim b\ll c\precsim d$ implies $a\ll d$ and it follows from Lemma 2.1 (iv) that $a\ll b$ implies $z\cap a\ll z\cap b$ for $z\in Z_0$. We shall write $a\lll b$ if for every $z\in Z_0$ either $z\cap a<z\cap b$ or $z\cap a=z\cap b=0$. $a\lll b$ implies $a\ll b$, implies either $a<b$ or $a=b=0$ and implies $z\cap a\lll z\cap b$ for $z\in Z_0$. $a\leqq b\lll c\leqq d$ implies $a\lll d$.

An element $a\in L$ is called to be *minimal* if $a\gg b$ implies $b=0$. It is obvious that if a is minimal and $a\succsim b$ then b is also minimal.

LEMMA 2.2. *The following statements are equivalent* (cf. [10], Theorem 3.1).

(α) *a is minimal.*

(β) *$a\ggg b$ implies $b=0$.*

(γ) *$b<a$ implies $e(b)<e(a)$.*

(δ) *$b\leqq a$ implies $b=e(b)\cap a$.*

(ε) *If b, $c\leqq a$ then $e(b\cap c)=e(b)\cap e(c)$.*

(ζ) *If $b\dot{\cup}c\leqq a$ then $e(b)\cap e(c)=0$.*

PROOF. The implication $(\alpha)\Rightarrow(\beta)$ is obvious. $(\beta)\Rightarrow(\gamma)$. It suffices to prove that if $b\leqq a$ and $e(b)=e(a)$ then $b=a$. Let $a=b\dot{\cup}c$. If $z\cap b\neq 0$ for $z\in Z_0$ then we have $z\cap a>z\cap c$ and if $z\cap b=0$ then since $z\cap e(a)=z\cap e(b)=e(z\cap b)=0$ by Lemma 2.1 we have $z\cap a=z\cap c=0$. Hence $a\ggg c$, which implies $c=0$ by (β). $(\gamma)\Rightarrow(\delta)$. Let $b\leqq a$ and $c=\{(1-e(b))\cap a\}\cup b$. Since $c\leqq a$ and since it follows from Lemma 2.1 that $e(c)=\{(1-e(b))\cap e(a)\}\cup e(b)=e(a)$, we have $c=a$ by (γ). Hence $e(b)\cap a=e(b)\cap c=b$. $(\delta)\Rightarrow(\alpha)$. If $a\gg b$, there is $c\leqq a$ with $c\sim b$ and then $a\gg c$. Since $e(c)\cap a=c$ by (δ) and $e(c)\cap c=c$, we have either $c\succ c$ or $c=0$. But $c\succ c$ does not hold, since $c\geqq c_1\sim c$ implies $c=e(c)\cap a=e(c_1)\cap a=c_1$ by (δ). Hence $c=0$ and then $b=0$. Therefore the four statements (α)—(δ) are equivalent.

$(\delta)\Rightarrow(\varepsilon)$. If b, $c\leqq a$ then it follows from (δ) that $b\cap c=e(b)\cap e(c)\cap a$, and hence $e(b\cap c)=e(b)\cap e(c)\cap e(a)=e(b)\cap e(c)$. $(\varepsilon)\Rightarrow(\zeta)$ is obvious. $(\zeta)\Rightarrow(\gamma)$. If $b<a$ then putting $a=b\dot{\cup}c$ we have $c\neq 0$. Since $e(b)\cap e(c)=0$ by (ζ), we have $e(b)<e(b)\cup e(c)=e(a)$. This completes the proof.

LEMMA 2.3. *If $a\in L$ is minimal, then $L(0, a)$ is distributive, that is, a is a D-element defined by Kaplansky* ([6], p. 538).

PROOF. It is obvious by (δ) of Lemma 2.2 that the mapping $b\rightarrow e(b)$ gives a lattice-isomorphism of $L(0, a)$ into Z_0. Hence $L(0, a)$ is distributive.

Later, we shall give a condition under which the converse of Lemma 2.3 holds (see Lemma 3.3).

DEFINITION 2.3. A subset $\{a_\alpha; \alpha \in I\}$ of L is called a *homogeneous family* if it is a semi-orthogonal family and $a_\alpha \sim a_\beta$ for every $\alpha, \beta \in I$. It is called an *n-homogeneous family* when the cardinal of I is n. A homogeneous family is called to be zero if all its elements are zero.

An element $a \in L$ is called to be *infinite* if $L(0, a)$ has a non-zero $\aleph_0$-homogeneous family, and otherwise called to be *finite*. $a \in L$ is called to be *simple* (Loomis [8], p. 6) if $L(0, a)$ has no non-zero 2-homogeneous family. We have the following implications: minimal$\Rightarrow$simple$\Rightarrow$finite. Because, the first implication follows from (ζ) of Lemma 2.2 and the second one is obvious. (We shall show in the next section that the minimal element and the simple element coincide under the condition that "$\sim$" satisfies the axiom (B$'$).)

If $a \prec a$ then using (A_2) repeatedly it is shown that a is infinite. Hence if a is finite then $a \prec a$ does not hold. If a is simple and $a \sim b$ then b is also simple by (A_2). (If ($\bar{A}_2$) is satisfied, the similar statement for finiteness holds.)

LEMMA 2.4. *Let $a \in L$ and $z_\alpha \in Z_0$. If $z_\alpha \frown a$ is minimal (resp. finite, simple) for every α, then so is $\bigcup_\alpha z_\alpha \frown a$.*

PROOF. If $z_\alpha \frown a$ is minimal for every α and $\bigcup_\alpha z_\alpha \frown a \gg b$ then we have $z_\alpha \frown b = 0$ for every α since $z_\alpha \frown a \gg z_\alpha \frown b$. Hence $b = \bigcup_\alpha z_\alpha \frown b = \bigcup_\alpha (z_\alpha \frown b) = 0$, which shows that $\bigcup_\alpha z_\alpha \frown a$ is minimal. If $z_\alpha \frown a$ is finite for every α and $L(0, \bigcup_\alpha z_\alpha \frown a)$ has an $\aleph_0$-homogeneous family $\{b_i; 1 \leqq i < \infty\}$, then since $\{z_\alpha \frown b_i; 1 \leqq i < \infty\}$ is an $\aleph_0$-homogeneous family in $L(0, z_\alpha \frown a)$ we have $z_\alpha \frown b_i = 0$. Hence $b_i = \bigcup_\alpha (z_\alpha \frown b_i) = 0$, which shows that $\bigcup_\alpha z_\alpha \frown a$ is finite. The statement for simple elements is proved similarly.

DEFINITION 2.4. L is called to be of *type I* if it has a minimal element a such that $e(a) = 1$; of *type II* if it has no non-zero minimal element and has a finite element b such that $e(b) = 1$; of *type III* if it has no non-zero finite element.

LEMMA 2.5. *There exists a unique decomposition $1 = z_{\mathrm{I}} \dot{\cup} z_{\mathrm{II}} \dot{\cup} z_{\mathrm{III}}$ in Z_0 such that the summands $L(0, z_{\mathrm{I}})$, $L(0, z_{\mathrm{II}})$ and $L(0, z_{\mathrm{III}})$ are of type I, type II and type III respectively.*

PROOF. Let $z_{\mathrm{I}} = \bigcup(e(a); a \text{ is minimal})$ and $z^* = \bigcup(e(b); b \text{ is finite})$. Since Z_0 is a complete Boolean lattice by Lemma 2.1, we may write $z_{\mathrm{I}} = \dot{\bigcup}_\alpha z_\alpha$ where $z_\alpha = e(a_\alpha)$ and a_α is minimal. Putting $a_0 = \bigcup_\alpha a_\alpha$, we have $e(a_0) = z_{\mathrm{I}}$ and since $z_\alpha \frown a_0 = a_\alpha$ is minimal it follows from Lemma 2.4 that a_0 is minimal. Similarly we have a finite element b_0 such that $e(b_0) = z^*$. Putting $z_{\mathrm{II}} = z^* - z_{\mathrm{I}}$ (since $z_{\mathrm{I}} \leqq z^*$) and $z_{\mathrm{III}} = 1 - z^*$, we have the desired decomposition. The uniqueness is obvious.

DEFINITION 2.5. A non-zero element $a \in L$ is called to be *properly infinite* if, for every $z \in Z_0$, $z \frown a$ is either infinite or zero. It is obvious that if $a \ll a$ and $a \neq 0$ then a is properly infinite. L is called to be *finite* (resp. *infinite*, *properly infinite*) if so is $1 \in L$.

LEMMA 2.6. *For any $a \in L$ there exist $e^f(a)$, $e^\infty(a) \in Z_0$ having the following properties*:

(1) $e^f(a) \dot{\cup} e^\infty(a) = e(a)$;

(2) $e^f(a) \cap a$ *is finite*;

(3) *if $e^\infty(a) \neq 0$ then $e^\infty(a) \cap a$ is properly infinite.*

Then $e^f(a)$, $e^\infty(a)$ are uniquely determined.

PROOF. Let $e^f(a) = \bigvee (z \in Z_0;\ z \leqq e(a),\ z \cap a$ is finite) and $e^\infty(a) = e(a) - e^f(a)$. It follows from Lemma 2.4 that $e^f(a) \cap a$ is finite. If $e^\infty(a) \neq 0$, then it is easy to show that $e^\infty(a) \cap a$ is properly infinite. The uniqueness is obvious.

COROLLARY. *There exist a unique decomposition $1 = e^f(1) \dot{\cup} e^\infty(1)$ in Z_0 such that $e^f(1)$ is finite and $e^\infty(1)$ is properly infinite or zero. If $e^\infty(1) \neq 0$, it is the largest properly infinite element.*

PROOF. The first statement directly follows from the lemma. If $a \in L$ is properly infinite, then $e^f(1) \cap a$ is zero since it is finite. Hence $a = e^\infty(1) \cap a \leqq e^\infty(1)$.

The following theorem is a direct consequence of Lemmas 2.5 and 2.6 by putting $z_{\mathrm{I}} \cap e^f(1) = z_{\mathrm{I}}^f$, $z_{\mathrm{I}} \cap e^\infty(1) = z_{\mathrm{I}}^\infty$, $z_{\mathrm{II}} \cap e^f(1) = z_{\mathrm{II}}^f$, $z_{\mathrm{II}} \cap e^\infty(1) = z_{\mathrm{II}}^\infty$ ($z_{\mathrm{III}} \cap e^f(1) = 0$).

THEOREM 2.1. *There exists a unique decomposition $1 = z_{\mathrm{I}}^f \dot{\cup} z_{\mathrm{I}}^\infty \dot{\cup} z_{\mathrm{II}}^f \dot{\cup} z_{\mathrm{II}}^\infty \dot{\cup} z_{\mathrm{III}}$ in Z_0 such that the summands $L(0, z_{\mathrm{I}}^f)$, $L(0, z_{\mathrm{I}}^\infty)$, $L(0, z_{\mathrm{II}}^f)$, $L(0, z_{\mathrm{II}}^\infty)$ and $L(0, z_{\mathrm{III}})$ are respectively finite of type I, properly infinite of type I, finite of type II, properly infinite of type II and of type III.*

§ 3. Axiom B

AXIOM B. We give the following axioms for equivalence relations in L.

($\bar{\mathrm{B}}$) *If a and b are perspective* (i.e. they have a common complement), *then $a \sim b$.*

(B) *If $a = (a \cap b) \dot{\cup} a_1$, $b = (a \cap b) \dot{\cup} b_1$, $a \cup b = a_2 \dot{\cup} b = a \dot{\cup} b_2$, then there exist decompositions $a_1 = a_1' \dot{\cup} a_1''$, $b_1 = b_1' \dot{\cup} b_1''$ such that $a_1' \sim a_2$, $b_1' \sim b_2$, $a_1'' \sim b_1''$.*

(B′) *If b is a complement of a and c is a semi-orthocomplement of a, then $b \gtrsim c$.*

(B″) *If $e(a) \cap e(b) \neq 0$, then there exist a_1, b_1 such that $0 \neq a_1 \leqq a$, $0 \neq b_1 \leqq b$, $a_1 \sim b_1$.*

It is obvious that ($\bar{\mathrm{B}}$) implies (B) (where $a_1'' = b_1'' = 0$) and that (B) implies (B′). Remark that (B′) is equivalent to the following statement: $a \cup b = a \dot{\cup} c$ implies $b \gtrsim c$. Because, if $a \cup b = a \dot{\cup} c$ and (B′) holds, then, putting $b = (a \cap b) \dot{\cup} b_1$, $(a \cup b) \dot{\cup} d = 1$, it is easy to show that b_1 is a complement of $a \dot{\cup} d$ and c is a semi-orthocomplement of $a \dot{\cup} d$, whence $c \lesssim b_1 \leqq b$ by (B′). The converse is obvious.

DEFINITION 3.1. Let L have an equivalence relation "$\sim$". Two elements

$a, b \in L$ are called to be *unrelated* if $a_1 \leqq a$, $b_1 \leqq b$, $a_1 \sim b_1$ imply $a_1 = b_1 = 0$. It is obvious that $e(a) \frown e(b) = 0 \Rightarrow a$ and b are unrelated $\Rightarrow a \frown b = 0$. The axiom (B$''$) is equivalent to the following statement: if a and b are unrelated then $e(a) \frown e(b) = 0$. Supposing that "$\sim$" satisfies (A_1) and (A_2), it is obvious that if a and b are unrelated and $b \gtrsim c$ then a and c are unrelated.

LEMMA 3.1. *Let "$\sim$" satisfy* (A_1) *and* (A_2). *The following statements are equivalent.*

(α) *"$\sim$" satisfies* (B$''$).

(β) *If two elements a and b are unrelated, then $b \leqq$ every complement of a.*

(γ) *If two elements a and b are unrelated, then $b \leqq$ every semi-orthocomplement of a.*

(δ) *For any $a \in L$, there is the largest element a' unrelated to a, and, a and a' are semi-orthogonal.*

PROOF. Let a' be a complement of a. If $e(a) \frown e(b) = 0$ then since $e(b) \frown a = 0$ we have $b \leqq e(b) = e(b) \frown a' \leqq a'$. Hence ($\alpha$) implies ($\beta$). ($\beta$)$\Rightarrow$($\gamma$) is trivial.

(γ)$\Rightarrow$(δ). Let S be the set of all elements unrelated to a, and put $a' = \bigvee (x;\, x \in S)$. Then we have $a' \in S$, because, if $a \gtrsim c \leqq a'$ and c' is a semi-orthocomplement of c, then since c and $x \in S$ are unrelated it follows from (γ) that $x \leqq c'$ for every $x \in S$, and hence $a' \leqq c'$, which implies $c = c \frown a' \leqq c \frown c' = 0$. Therefore a' is the largest element in S. We have $a \perp a'$ since $a' \leqq$ a semi-orthocomplement of a by (γ).

(δ)$\Rightarrow$(α). Let a and b be unrelated. It follows from (δ) that there is the largest element a' unrelated to a and that there is the largest element a'' unrelated to a'. Then $b \leqq a'$, $a \leqq a''$ and it follows from (δ) that $a' \perp a''$. We shall show that $a' \in Z_0$. Let c be a complement of a'. Then c and a' are unrelated, because, if $c \geqq x \lesssim a'$, then we have $x \leqq a'$ since x and a are unrelated, and hence $x \leqq a' \frown c = 0$. Hence $c \leqq a''$, and we put $c \dot\smile d = a''$. Then it follows from $a'' \perp a'$ that $d \perp c \dot\smile a' = 1$, which implies $d = 0$ and $c = a''$. Hence a' has a unique complement a'', and then $a' \in Z$ by Lemma 1.3. Since $x \lesssim a'$ implies $x \leqq a'$ as shown above, we have $a' \in Z_0$. Then we have $e(b) \leqq a'$ and then $e(a) \frown e(b) = e(a \frown e(b)) = 0$. This completes the proof.

REMARK 3.1. Let "$\sim$" satisfy (A_1) and (A_2).

(i) If (B$''$) is satisfied, it is easy to show that $e(a) = \bigvee (x \in L;\, x \lesssim a)$ and $1 - e(a) =$ the largest element unrelated to a. From the former equation it follows that $a \in Z_0$ if and only if $x \lesssim a$ implies $x \leqq a$ (a is invariant in the sense of Loomis [8]).

(ii) If (B$''$) is satisfied, it is easy to show that for any non-zero element $a \leqq z_{\mathrm{I}}$ there is a non-zero minimal element b with $b \leqq a$. This fact implies by Zorn's lemma that for any element $a \leqq z_{\mathrm{I}}$ there is a decomposition $a = \dot\bigvee_\alpha a_\alpha$ such that a_α is minimal for every α.

(iii) If every element of L has a unique semi-orthocomplement, that is, L is a relatively orthocomplemented complete lattice (see [10], Theorem 1), then (B″) is equivalent to the following axiom:

(B_0'') *If a and b are not orthogonal then there exist a_1, b_1 such that $0 \neq a_1 \leqq a$, $0 \neq b_1 \leqq b$, $a_1 \sim b_1$.*

Because, by the assumption, (γ) of Lemma 3.1 may be stated as follows: If two elements a and b are unrelated then $b \leqq$ the orthocomplement of a (which means that a and b are orthogonal). (B_0'') coincides with the axiom (D) of Loomis [8].

THEOREM 3.1. *If "$\sim$" satisfies (A_1) and (A_2), then (B′) is stronger than (B″).*

PROOF. Supposing that (B′) is satisfied, we shall prove (β) of Lemma 3.1. Let a and b be unrelated and a' be a complement of a. Putting $a' \cup b = a' \dot{\cup} c$, we have $c \lesssim b$ by (B′), and putting $(a' \cup b) \dot{\cup} d = 1$, we have $c \dot{\cup} d \lesssim a$ by (B′). Hence $a \gtrsim c \lesssim b$, which implies $c = 0$ and $b \leqq a'$.

LEMMA 3.2. *Let "$\sim$" satisfy* (A_1), (A_2) *and (B″). An element of L is minimal if and only if it is simple.*

PROOF. If a is simple and $b \dot{\cup} c \leqq a$, then since b and c are unrelated we have $e(b) \frown e(c) = 0$ by (B″). Hence a is minimal by Lemma 2.2. The converse is obvious.

LEMMA 3.3. (i) *Let "$\sim$" satisfy* (A_1), (A_2), (B″) *and moreover the following condition:*

(P) *If a, b are non-zero elements with $a \perp b$, $a \sim b$ then there exist non-zero elements $a_1 \leqq a$, $b_1 \leqq b$ such that a_1 and b_1 are perspective.*

Then it follows that

(P′) *For any $a \in L$, if b is in the center of $L(0, a)$ then $b = e(b) \frown a$.* (It is easily seen that $Z_0 = Z$, by putting $a = 1$. L is a Z_α-lattice defined by F. Maeda [10].)

(ii) *Let "$\sim$" satisfy* (A_1), (A_2) *and* (P′). *An element of L is minimal if and only if it is a D-element.*

PROOF. (i) Let b be in the center of $L(0, a)$, and put $a = b \dot{\cup} c$. If $e(b) \frown e(c) \neq 0$, it follows from (B″) and (P) that there exist non-zero elements $b_1 \leqq b$, $c_1 \leqq c$ such that b_1 and c_1 are perspective. Then, in $L(0, a)$, since b_1 and c_1 are perspective and since b is in the center, $b \frown b_1 = b_1$ is perspective to $b \frown c_1 = 0$, which implies $b_1 = 0$, a contradiction. Hence $e(b) \frown e(c) = 0$ and then $e(b) \frown a = (e(b) \frown b) \cup (e(b) \frown c) = b$.

(ii) If a is a D-element and $b \leqq a$, then since in $L(0, a)$ any element is in the center, it follows from (P′) that $b = e(b) \frown a$. Hence a is minimal by Lemma 2.2. The converse is given by Lemma 2.4.

§ 4. Axiom C

AXIOM C. We give the following axioms for equivalence relations in L.

($C_\perp$) *If* $a=\dot{\bigvee}_\alpha a_\alpha$, $b=\dot{\bigvee}_\alpha b_\alpha$, $a_\alpha\sim b_\alpha$ *for every* α *and* $a\perp b$, *then* $a\sim b$ *(complete additivity in the semi-orthogonal case).*

(C_f) *If* $a=a_1\dot{\cup}a_2$, $b=b_1\dot{\cup}b_2$, $a_i\sim b_i(i=1, 2)$, *then* $a\sim b$ *(finite additivity).*

LEMMA 4.1. *Let an equivalence relation "$\sim$" in* L *satisfy* (A_1), (A_2), (B'') *and* ($C_\perp$).

(i) *If* $a\perp b$, *then there exist decompositions* $a=a'\dot{\cup}a''$, $b=b'\dot{\cup}b''$ *such that* $a'\sim b'$, $e(a'')\cap e(b'')=0$.

(ii) *If* $a\perp b$, *then there exists a decomposition* $1=z_1\dot{\cup}z_2\dot{\cup}z_3$ *in* Z_0 *such that* $z_1\cap a\gg z_1\cap b$, $z_2\cap a\ll z_2\cap b$, $z_3\cap a\sim z_3\cap b$. *More simply, there exists* $z\in Z_0$ *such that* $z\cap a\gtrsim z\cap b$, $(1-z)\cap a\lesssim(1-z)\cap b$.

(iii) *If* $a=a_1\dot{\cup}a_2$, $b=b_1\dot{\cup}b_2$, $a\perp b$, $a\sim b$, $a_1\sim b_1$ *and if* a *is finite, then* $a_2\sim b_2$.

PROOF. (i) Consider pairs of semi-orthogonal families $\{a_\alpha\}$, $\{b_\alpha\}$ in $L(0, a)$, $L(0, b)$ respectively such that $a_\alpha\sim b_\alpha$ for every α. Among these there is a maximal pair $\{a_\alpha;\ \alpha\in I\}$, $\{b_\alpha;\ \alpha\in I\}$ by Zorn's lemma, and we put $a'=\dot{\bigvee}(a_\alpha;\ \alpha\in I)$, $b'=\dot{\bigvee}(b_\alpha;\ \alpha\in I)$ and $a=a'\dot{\cup}a''$, $b=b'\dot{\cup}b''$. Then since a'' and b'' are unrelated, it follows from (B'') that $e(a'')\cap e(b'')=0$. It follows from ($C_\perp$) that $a'\sim b'$, since $a'\perp b'$.

(ii) Put $z_1=e(a'')$, $z_2=e(b'')$ and $z_3=1-(z_1\dot{\cup}z_2)$. It follows from $z_3\cap a''=z_3\cap b''=0$ that $z_3\cap a=z_3\cap a'\sim z_3\cap b'=z_3\cap b$. We can see that $z_1\cap a\gg z_1\cap b$, because, if $z\cap z_1\cap a''\neq 0(z\in Z_0)$ then we have $z\cap z_1\cap a\succ z\cap z_1\cap b$ since $z_1\cap b''=0$, and if $z\cap z_1\cap a''=0$ then $z\cap z_1\cap a=z\cap z_1\cap b=0$ since $z\cap z_1=e(z\cap z_1\cap a'')=0$. Similarly we have $z_2\cap a\ll z_2\cap b$.

(iii) There is $z\in Z_0$ such that $z\cap a_2\gtrsim z\cap b_2$, $(1-z)\cap a_2\lesssim(1-z)\cap b_2$ by (ii). If $z\cap a_2\succ z\cap b_2$, then, since $z\cap a_1\sim z\cap b_1$, it follows from ($C_\perp$) that $z\cap a\succ z\cap b\sim z\cap a$, contradicting the finiteness of a. Hence $z\cap a_2\sim z\cap b_2$, and similarly we have $(1-z)\cap a_2\sim(1-z)\cap b_2$, since $(1-z)\cap a_2\prec(1-z)\cap b_2$ implies $(1-z)\cap a\prec(1-z)\cap b\sim(1-z)\cap a$. Therefore $a_2\sim b_2$.

LEMMA 4.2. *Let "$\sim$" satisfy* (A_1), (A_2), (B'') *and* ($C_\perp$). *If* $a\leqq z_{\mathrm{I}}$, *then there exists a decomposition* $1=\dot{\bigvee}_\alpha z_\alpha$ *in* Z_0 *such that each* $z_\alpha\cap a$ *is the join of a homogeneous family of minimal elements.*

PROOF. It suffices to show that, in Z_0, for any non-zero element $z\leqq z_{\mathrm{I}}$ there exists $0\neq z_0\leqq z$ such that $z_0\cap a$ is the join of a homogeneous family of minimal elements. In $L(0, z\cap a)$ there exists a maximal one $\{a_\beta\}$ among homogeneous families of minimal elements. Put $z\cap a=\dot{\bigvee}_\beta a_\beta\dot{\cup}b$. It follows from Lemma 4.1 that there is $z_1\in Z_0$ such that $z_1\cap a_\beta\gg z_1\cap b$, $(1-z_1)\cap a_\beta\lesssim(1-z_1)\cap b$. We have $z_1\cap b=0$ since a_β is minimal, and hence $z_1\cap z\cap a$ is the join of the homogeneous family $\{z_1\cap a_\beta\}$. To complete the proof, it suffices to show that $z_1\cap z\neq 0$. If we suppose that $z_1\cap z=0$, then we have a_β, $b\leqq z\leqq 1-z_1$ and hence $a_\beta \lesssim b$, which contradicts that $\{a_\beta\}$ is maximal.

FIFTH AXIOM FOR SEMI-ORTHOGONALITY. We give the following axiom for semi-orthogonal relation "$\perp$".

($\perp$5) *If $a_\delta \uparrow a$ and $a_\delta \perp b$ for every δ then $a \perp b$.*

If a complete lattice has a semi-orthogonal relation satisfying ($\perp$5), then it is obvious that semi-orthogonal families have the following property: If S is a semi-orthogonal family and S_1, S_2 are disjoint subsets (not necessarily finite) of S then $\bigvee(a;\ a \in S_1) \perp \bigvee(a;\ a \in S_2)$.

In the remainder of this section we assume that, in a relatively semi-orthocomplemented complete lattice L, the semi-orthogonal relation satisfies ($\perp$5).

LEMMA 4.3. *Let "$\sim$" satisfy* (A_1), (A_2), (B'') *and* ($C_\perp$). *If $a \leqq z_{\mathrm{II}} \dot\cup z_{\mathrm{III}}$, then for any finite n there exists an n-homogeneous family with the join a. If $a \leqq z_{\mathrm{III}}$, then there exists an $\aleph_0$-homogeneous family with the join a.*

PROOF. If $a=0$, the lemma is trivial. Let $0 \neq a \leqq z_{\mathrm{II}} \dot\cup z_{\mathrm{III}}$. Since $L(0, z_{\mathrm{II}} \dot\cup z_{\mathrm{III}})$ has no non-zero minimal element, it has no non-zero simple element by Lemma 3.2. Hence for any $0 \neq b \leqq z_{\mathrm{II}} \dot\cup z_{\mathrm{III}}$ and for any finite n, there exists a non-zero n-homogeneous family in $L(0, b)$. Using Zorn's lemma, we can get non-zero n-homogeneous families $\{a_i^\alpha;\ 1 \leqq i \leqq n\}$ $(\alpha \in I)$ such that the joins $a_\alpha = \dot\bigvee(a_i^\alpha;\ 1 \leqq i \leqq n)$ $(\alpha \in I)$ form a semi-orthogonal family and that $a = \dot\bigvee(a_\alpha;\ \alpha \in I)$. Putting $a_i = \dot\bigvee(a_i^\alpha;\ \alpha \in I)$ $(1 \leqq i \leqq n)$, since $(a_i^\alpha;\ 1 \leqq i \leqq n,\ \alpha \in I)\perp$ by Lemma 1.2, it follows from ($\perp$5) that $(a_i;\ 1 \leqq i \leqq n)\perp$. And then $\{a_i\}$ is a homogeneous family by ($C_\perp$) and its join is a. The second statement of the lemma can be proved similarly, because $L(0, z_{\mathrm{III}})$ has no non-zero finite element.

LEMMA 4.4. *Let "$\sim$" satisfy* ($C_\perp$) *and* (C_f). *If $\{a_\alpha;\ \alpha \in I\}$ is a homogeneous family and a subset J of I has the same cardinal as I, then $\dot\bigvee(a_\alpha;\ \alpha \in J) \sim \dot\bigvee(a_\alpha;\ \alpha \in I)$.*

PROOF. If I is finite, then the lemma is trivial since $J=I$. If I is infinite, then J can be divided into two parts J_1, J_2 such that both parts have the same cardinal as I. It follows from ($\perp$5) and ($C_\perp$) that $\dot\bigvee(a_\alpha;\ \alpha \in J_1) \sim \dot\bigvee(a_\alpha;\ \alpha \in J_2) \sim \dot\bigvee(a_\alpha;\ \alpha \in J_1 \cup (I-J))$ since J_1, J_2 and $J_1 \cup (I-J)$ have the same cardinal as I. We have $\dot\bigvee(a_\alpha;\ \alpha \in J) \sim \dot\bigvee(a_\alpha;\ \alpha \in I)$ by (C_f).

LEMMA 4.5. *Let "$\sim$" satisfy* (A_2), ($C_\perp$) *and* (C_f). *If $a \gtrsim b$ and $a \lesssim b$, then $a \sim b$.*

PROOF. $a \gtrsim b$ means that there is a_1 with $a \geqq a_1 \sim b$. Put $a = a_1 \dot\cup c_1$. Since $a \lesssim b \sim a_1$, there is a_2 with $a \sim a_2 \leqq a_1$ and by (A_2) there is a decomposition $a_2 = a_3 \dot\cup c_3$ with $a_3 \sim a_1$, $c_3 \sim c_1$. Putting $a_1 = a_2 \dot\cup c_2$, we have a decomposition $a_3 = a_4 \dot\cup c_4$ with $a_4 \sim a_2$, $c_4 \sim c_2$. Repeating this, we have sequences $\{a_n\}$, $\{c_n\}$ such that $a_n = a_{n+1} \dot\cup c_{n+1}$, $a_{n+1} \sim a_{n-1}$, $c_{n+1} \sim c_{n-1}$. It is obvious that $(c_n;\ 1 \leqq n < \infty)\perp$. Putting $a_1=\dot{\bigcup}(c_n;\ 2\leqq n<\infty)\dot{\cup}d$, we have $a=\dot{\bigcup}(c_n;\ 1\leqq n<\infty)\dot{\cup}d$, and since $\dot{\bigcup}(c_n;\ n=1,3,5,\cdots)\sim\dot{\bigcup}(c_n;\ n=3,5,7,\cdots)$ by Lemma 4.4, we have $a\sim a_1\sim b$.

LEMMA 4.6. *Let "$\sim$" satisfy* (A_1), (A_2), $(C_\perp)$ *and* (C_f).

(i) *An element $a\in L$ is finite if and only if $a\sim a_1\leqq a$ implies $a_1=a$, in other words, $a\lessdot a$ does not hold.*

(ii) *If a is finite and $a\sim b$ then b is also finite.*

PROOF. (i) Let $\{b_i;\ 1\leqq i<\infty\}$ be an $\aleph_0$-homogeneous family in $L(0, a)$. Putting $a=\dot{\bigcup}(b_i;\ 1\leqq i<\infty)\dot{\cup}c$ and $a_1=\dot{\bigcup}(b_i;\ 2\leqq i<\infty)\dot{\cup}c$, we have $a_1\sim a=a_1\dot{\cup}b_1$ by Lemma 4.4. Hence if $a\lessdot a$ does not hold then $b_1=0$, which implies that a is finite. The converse is obvious. The statement (ii) is implied from (i), since $a\sim b$ and $b\lessdot b$ imply $a\lessdot a$.

REMARK 4.1. Let "$\sim$" satisfy (A_1), (A_2), (B''), $(C_\perp)$ and (C_f). Using (B'') and Lemma 4.6 (ii), it is easy to show that for any non-zero element $a\leqq z_{\text{II}}$ there is a non-zero finite element b with $b\leqq a$. Hence, for any element $a\leqq z_{\text{II}}$ there is a decomposition $a=\dot{\bigcup}_\alpha a_\alpha$ such that a_α is finite for every α.

LEMMA 4.7. *Let "$\sim$" satisfy* (A_1), (A_2), (B''), $(C_\perp)$ and (C_f). *If $a\in L$ is properly infinite, then for any $n\leqq\aleph_0$ there exists an n-homogeneous family $\{a_i\}$ with the join a such that $a_i\sim a$.*

PROOF. We shall prove that for any non-zero element $z\in Z_0$ there exists $z_0\in Z_0$ with $0\neq z_0\leqq z$ such that $z_0\frown a$ is the join of an $\aleph_0$-homogeneous family. If $z\frown a=0$, this is trivial. If $z\frown a\neq 0$, then $z\frown a$ is infinite by the assumption, i.e., there is an infinite homogeneous family $\{b_\alpha;\ \alpha\in I\}$ in $L(0, z\frown a)$. We can suppose that this family is maximal. Put $z\frown a=\dot{\bigcup}(b_\alpha;\ \alpha\in I)\dot{\cup}b'$. By Lemma 4.1, there is $z_1\in Z_0$ such that $z_1\frown b'\lesssim z_1\frown b_\alpha$, $(1-z_1)\frown b'\gtrsim(1-z_1)\frown b_\alpha$. Putting $z_0=z_1\frown z$, we have $z_0\frown a\neq 0$; because, if $z_1\frown z\frown a=0$ then $z_1\frown b'=z_1\frown b_\alpha=0$ and hence $b'\gtrsim b_\alpha$, which contradicts the maximality of $\{b_\alpha\}$. Since I is infinite, it can be divided into a countably infinite number of parts $I_i(1\leqq i<\infty)$ such that each I_i has the same cardinal as I. Put $c_1=(z_0\frown b')\dot{\cup}\dot{\bigcup}(z_0\frown b_\alpha;\ \alpha\in I_1)$ and $c_i=\dot{\bigcup}(z_0\frown b_\alpha;\ \alpha\in I_i)$ for $2\leqq i<\infty$. Then $z_0\frown a=\dot{\bigcup}(c_i;\ 1\leqq i<\infty)$ and since $z_0\frown b'\lesssim z_0\frown b_\alpha$ it follows from Lemmas 4.4 and 4.5 that $\{c_i;\ 1\leqq i<\infty\}$ is a homogeneous family.

It follows from the above result that there is a decomposition $1=\dot{\bigcup}_\beta z_\beta$ in Z_0 such that $z_\beta\frown a$ is the join of an $\aleph_0$-homogeneous family $\{d_i^\beta;\ 1\leqq i<\infty\}$. Putting $d_i=\dot{\bigcup}_\beta d_i^\beta$ we have an $\aleph_0$-homogeneous family $\{d_i;\ 1\leqq i<\infty\}$ with the join a. If $n\leqq\aleph_0$, $\{i\}$ can be divided into n parts N_j such that each N_j has the cardinal $\aleph_0$. Putting $a_j=\dot{\bigcup}(d_i;\ i\in N_j)$, we have $a=\dot{\bigcup}_j a_j$ and $a_j\sim a$ for every j by Lemma 4.4.

LEMMA 4.8. *Let "$\sim$" satisfy* (A_1), (A_2), (B''), $(C_\perp)$ *and* (C_f).

(i) *If a finite element a is the join of an n-homogeneous family of minimal elements* (n is necessarily finite), *then $L(0, a)$ has no non-zero $n+1$-homo-*

geneous family.

(ii) *If $\{a_\alpha\}$ is a semi-orthogonal family in $L(0, z_{\mathrm{I}})$ and its join is a finite element, then there exists a decomposition $1=\dot{\bigcup}_\beta z_\beta$ in Z_0 such that, in every $L(0, z_\beta)$, $z_\beta \cap a_\alpha$ is zero except a finite number of α.*

(iii) *If L is finite of type I, then it is continuous.*

PROOF. (i) Let $\{a_i;\ 1\leqq i\leqq n\}$ be the given homogeneous family of minimal elements with the join a and $\{b_j;\ 1\leqq j\leqq n+1\}$ be a homogeneous family in $L(0, a)$. If we suppose that $e(a_1)\cap e(b_1)\neq 0$, then by (B$''$) there exists a non-zero element $c\leqq a_1$ with $c\lesssim b_1$. Since a_1 is minimal it follows from (δ) of Lemma 2.2 that $e(c)\cap a_1=c$. Hence $e(c)\cap a_i\sim c\lesssim e(c)\cap b_j$ for every i, j, and hence $e(c)\cap a\lesssim \dot{\bigcup}(e(c)\cap b_j;\ 1\leqq j\leqq n)$. Since it follows from $c\neq 0$ that $e(c)\cap b_{n+1}\neq 0$, we have $e(c)\cap a< e(c)\cap a$, which contradicts the finiteness of a. Therefore $e(a_1)\cap e(b_1)=0$, and we have $b_1=0$ since $e(a_1)=e(a)\geqq e(b_1)$.

(ii) Put $a=\dot{\bigcup}_\alpha a_\alpha$. By Lemma 4.2, it suffices to show that if $z\cap a (z\in Z_0)$ is the join of a non-zero n-homogeneous family of minimal elements (n is necessarily finite) then there exists $z_0\in Z_0$ with $0\neq z_0\leqq z$ such that $z_0\cap a_\alpha=0$ except a finite number of a_α. Consider a non-zero homogeneous family $\{b_i\}$ satisfying the following condition:

$(*)$ For any b_i there is a_{α_i} with $b_i\leqq z\cap a_{\alpha_i}$.

It follows from (i) that the cardinal k of $\{b_i\}$ is smaller than n. Hence we can choose the family with the largest k. Putting $z_0=e(b_1)$, we can show that $z_0\cap a_\alpha=0$ when $\alpha\neq\alpha_1,\cdots\alpha_k$; because if we suppose $z_0\cap a_\alpha\neq 0$, then $e(b_i)\cap e(z\cap a_\alpha)=z_0\cap e(a_\alpha)\neq 0$ and then there is $c_0\leqq z\cap a_\alpha$ with $0\neq c_0\lesssim b_i$, which implies that there are $c_i\leqq b_i$ with $c_0\sim c_i$, and hence $\{c_0, c_1,\cdots c_k\}$ is a $k+1$-homogeneous family satisfying $(*)$, a contradiction.

(iii) Let $\{a_\rho;\ \rho<\Omega\}$ be a well-ordered ascending set with the join a (Ω is a limit ordinal), and we shall prove that $a_\rho\cap b\uparrow a\cap b$. We may assume that if ρ is a limit ordinal then $a_\rho=\bigcup(a_\gamma;\ \gamma<\rho)$. Putting $a_{\rho+1}=a_\rho\dot{\cup} c_\rho$ for every $\rho<\Omega$, we have $(c_\rho;\ \rho<\Omega)\perp$. It follows from (ii) that there is a decomposition $1=\dot{\bigcup}_\beta z_\beta$ in Z_0 such that, in every $L(0, z_\beta)$, $z_\beta\cap c_\rho=0$ except a finite number of c_ρ. Then, for every β, it is easy to show that there is $\rho(\beta)$ such that $z_\beta\cap a= z_\beta\cap a_{\rho(\beta)}$, and hence we have $z_\beta\cap a\cap b=z_\beta\cap a_{\rho(\beta)}\cap b\leqq z_\beta\cap\bigcup_\rho(a_\rho\cap b)\leqq z_\beta\cap a\cap b$. Therefore we have $a\cap b=\dot{\bigcup}_\beta(z_\beta\cap a\cap b)=\dot{\bigcup}_\beta(z_\beta\cap\bigcup_\rho(a_\rho\cap b))=\bigcup_\rho(a_\rho\cap b)$. Similarly $a_\rho\downarrow a$ implies $a_\rho\cap b\downarrow a\cap b$. This completes the proof.

REMARK 4.2. Let "$\sim$" satisfy (A_1), (A_2), (B$''$), ($C_\perp$) and (C_f). We can prove the following statements by the similar methods as in Lemmas 4.13, 4.14 and 5.1 of [5], but the details are omitted.

(i) Let $a=a_1\dot{\cup} a_2=b_1\dot{\cup} b_2$ and $a_1\sim b_1$. If a is finite and $a\lesssim$ a semi-orthocomplement of a (in other words, a belongs to a 2-homogeneous family) then $a_2\sim b_2$.

(ii) Let $0\neq a=\dot{\bigcup}(a_i;\ 1\leqq i<\infty)$ and put $b_n=\dot{\bigcup}(a_i;\ 1\leqq i\leqq n)$, $b'_n=\dot{\bigcup}(a_i;$

$n<i<\infty$) for $1\leqq n<\infty$. If a is finite and $a\lesssim$ a semi-orthocomplement of a, then it does not hold that $b_n\lesssim b'_n$ for all n.

(iii) Let "$\sim$" satisfy moreover $(\bar{\mathrm{A}}_2)$. If $a=\dot{\bigvee}(a_i;\ 1\leqq i<\infty)$, $b=\dot{\bigvee}(b_i;\ 1\leqq i<\infty)$ and $a_i\sim b_i$ for every i, then there exists a decomposition $1=\dot{\bigvee}_\beta z_\beta$ in Z_0 such that $z_\beta\cap a\sim z_\beta\cap b$ for every β.

LEMMA 4.9. *Let "$\sim$" satisfy* (A_1), (A_2), (B''), $(\mathrm{C}_\perp)$ *and* (C_f). *If* $a=\dot{\bigvee}(a_\alpha;\ \alpha\in I)$, $a\perp b$, $a\sim b$ *and if a is finite, then there exists a decomposition* $b=\dot{\bigvee}(b_\alpha;\ \alpha\in I)$ *such that* $a_\alpha\sim b_\alpha$ *for every* $\alpha\in I$. *In other words,* $(\bar{\mathrm{A}}_2)$ *holds if* $a\perp b$ *and a is finite.*

PROOF. If I is finite, the lemma is trivial. Otherwise, let I be well-ordered: $I=\{\rho;\ \rho<\Omega\}$, where Ω is a limit ordinal. We shall construct a semi-orthogonal family $\{b_\rho;\ \rho<\Omega\}$ in $L(0, b)$ with $b_\rho\sim a_\rho$ by transfinite induction. Suppose that $\{b_\gamma;\ \gamma<\rho\}$ has been constructed. Since $\dot{\bigvee}(b_\gamma;\ \gamma<\rho)\sim\dot{\bigvee}(a_\gamma;\ \gamma<\rho)$ by $(\mathrm{C}_\perp)$, putting $\dot{\bigvee}(b_\gamma;\ \gamma<\rho)\,\dot{\cup}\,b'_\rho=b$, it follows from Lemma 4.1 (iii) that $b'_\rho\sim\dot{\bigvee}(a_\gamma;\ \gamma\geqq\rho)$. Hence there is $b_\rho\leqq b'_\rho$ with $b_\rho\sim a_\rho$, and the construction is completed. Since $\dot{\bigvee}(b_\rho;\ \rho<\Omega)\sim a\sim b$ by $(\mathrm{C}_\perp)$ and since b is finite, we have $b=\dot{\bigvee}(b_\rho;\ \rho<\Omega)$.

Remark that the axiom (B') is not assumed in this section. It will be assumed in the following section.

§ 5. Comparability theorems

In this section, we assume that the semi-orthogonal relation in L satisfies $(\perp 5)$ and that the equivalence relation "$\sim$" in L satisfies the axioms (A_1), (A_2), (B'), $(\mathrm{C}_\perp)$ and (C_f).

LEMMA 5.1. *If each of $a, b\in L$ belongs to a 5-homogeneous family whose join is a finite element, then $a\cup b$ belongs to a 2-homogeneous family.*

PROOF. Putting $(a\cup b)\,\dot{\cup}\,d=1$, there exists $z\in Z_0$ such that $z\cap(a\cup b)\gtrsim z\cap d$, $(1-z)\cap(a\cup b)\lesssim(1-z)\cap d$ by Lemma 4.1 (ii). To prove the lemma, it suffices to show that $z\cap(a\cup b)=0$. Putting $a\cup b=a\,\dot{\cup}\,c$, it follows from Lemma 4.1 (ii) that the problem is reduced to the two cases: (i) $a\lesssim c$, (ii) $a\gtrsim c$. Case (i). By the assumption there is a homogeneous family $\{b, b_1, b_2, b_3, b_4\}$ whose join $\bar{b}$ is finite. Since $b\gtrsim c\gtrsim a$ by (B'), we have $z\cap d\lesssim z\cap(a\cup b)=(z\cap a)\,\dot{\cup}\,(z\cap c)\lesssim(z\cap b_1)\,\dot{\cup}\,(z\cap b_2)\sim(z\cap b_3)\,\dot{\cup}\,(z\cap b_4)$, and then $(z\cap b_1)\,\dot{\cup}\,(z\cap b_2)\,\dot{\cup}\,(z\cap b_3)\,\dot{\cup}\,(z\cap b_4)\gtrsim(z\cap(a\cup b))\,\dot{\cup}\,(z\cap d)=z\geqq z\cap\bar{b}$. Hence we have $z\cap b=0$ by the finiteness of $\bar{b}$, and then $z\cap a=0$ and $z\cap(a\cup b)=0$. Case (ii). By the assumption there is a homogeneous family $\{a, a_1, a_2, a_3, a_4\}$ whose join is finite. We have $(z\cap a)\,\dot{\cup}\,(z\cap c)\lesssim(z\cap a_1)\,\dot{\cup}\,(z\cap a_2)$ since $a\gtrsim c$. Hence it follows that $z\cap a=0$ in the same way as above, and then $z\cap c=0$ and $z\cap(a\cup b)=0$.

LEMMA 5.2. *If* $a=\dot{\bigvee}_\alpha a_\alpha$, $b=\dot{\bigvee}_\alpha b_\alpha$, $a_\alpha\sim b_\alpha$ *for every* α *and if each of* a, b *belongs to a* 5-*homogeneous family whose join is a finite element, then* $a\sim b$.

PROOF. It follows from Lemma 5.1 that there exists $c\in L$ such that $c\perp a\cup b$, $c\sim a$. Then, it follows from Lemma 4.9 that there is a decomposition $c=\dot{\bigvee}_\alpha c_\alpha$ such that $c_\alpha\sim a_\alpha$ for every α. We have $b\sim c$ since $b\perp c$ and $b_\alpha\sim c_\alpha$ for every α, and hence $a\sim b$.

LEMMA 5.3. *If* $a\in L$ *is minimal and* $e(a)\leqq e(b)$, *then* $a\lesssim b$. *If both of* a, $b\in L$ *are minimal and* $e(a)=e(b)$, *then* $a\sim b$.

PROOF. Let a be minimal and $e(a)\leqq e(b)$, and put $a\cup b=a\dot{\cup} c$. By Lemma 4.1 (ii), there exists $z\in Z_0$ such that $z\cap a\gg z\cap c$, $(1-z)\cap a\lesssim(1-z)\cap c$. We have $z\cap c=0$ since $z\cap a$ is minimal, and hence $z\cap a\geqq z\cap b$. It follows from (δ) of Lemma 2.2 that $z\cap b=e(z\cap b)\cap z\cap a=z\cap e(b)\cap a=z\cap a$. On the other hand, we have $(1-z)\cap a\lesssim(1-z)\cap b$ since $b\gtrsim c$ by (B$'$). Hence $a\lesssim b$. Let, moreover, b be minimal and $e(a)=e(b)$. Putting $(1-z)\cap a\sim c\leqq(1-z)\cap b$, it follows from ($\delta$) of Lemma 2.2 that $c=e(c)\cap(1-z)\cap b=(1-z)\cap e(a)\cap b=(1-z)\cap b$, which implies $a\sim b$.

LEMMA 5.4. *If* $a=\dot{\bigvee}_\alpha a_\alpha$, $b=\dot{\bigvee}_\alpha b_\alpha$, $a_\alpha\sim b_\alpha$ *for every* α, *and if* a_α, b_α *are minimal for every* α *and* a, b *are finite, then* $a\sim b$.

PROOF. It follows from Lemma 4.8 (ii) that there is a decomposition $1=\dot{\bigvee}_\gamma z_\gamma$ in Z_0 such that the set $I_\gamma=\{\alpha;\ z_\gamma\cap a_\alpha\neq 0\}$ is finite for every γ. We may suppose that $e(z_\gamma\cap a_\alpha)=z_\gamma$ for $\alpha\in I_\gamma$, by choosing a finer decomposition in Z_0. Then, it follows from Lemma 5.3 that $z_\gamma\cap a_\alpha\sim z_\gamma\cap a_\beta\sim z_\gamma\cap b_\alpha\sim z_\gamma\cap b_\beta$ for α, $\beta\in I_\gamma$. We divide each I_γ into six disjoint parts I_γ^i $(1\leqq i\leqq 5)$ and J_γ such that I_γ^i have the same cardinal and J_γ has the cardinal $\leqq 4$ (I_γ^i and J_γ may be empty). Putting $a_\gamma^i=\dot{\bigvee}(z_\gamma\cap a_\alpha;\ \alpha\in I_\gamma^i)$ and $b_\gamma^i=\dot{\bigvee}(z_\gamma\cap b_\alpha;\ \alpha\in I_\gamma^i)$, we have $a_\gamma^i\sim a_\gamma^j\sim b_\gamma^i\sim b_\gamma^j (1\leqq i, j\leqq 5)$ by (C_f), and putting $a^i=\dot{\bigvee}_\gamma a_\gamma^i$ and $b^i=\dot{\bigvee}_\gamma b_\gamma^i$, we have $a^i\sim a^j$ and $b^i\sim b^j$ by $(\mathrm{C}_\perp)$. Then, it follows from Lemma 5.2 that $a^i\sim b^i$ $(1\leqq i\leqq 5)$. Next, we denotes the elements $\{a_\alpha, b_\alpha;\ \alpha\in J_\gamma\}$ by $\{a_\gamma^{(\nu)}, b_\gamma^{(\nu)};\ 1\leqq\nu\leqq 4\}$ with $a_\gamma^{(\nu)}\sim b_\gamma^{(\nu)}$ ($a_\gamma^{(\nu)}$ and $b_\gamma^{(\nu)}$ may be zero simultaneously). Putting $a^{(\nu)}=\dot{\bigvee}_\gamma(z_\gamma\cap a_\gamma^{(\nu)})$ and $b^{(\nu)}=\dot{\bigvee}_\gamma(z_\gamma\cap b_\gamma^{(\nu)})$, it follows from Lemma 2.4 that $a^{(\nu)}$ and $b^{(\nu)}$ are minimal, and since $e(a^{(\nu)})=e(b^{(\nu)})$ we have $a^{(\nu)}\sim b^{(\nu)}$ by Lemma 5.2. Therefore, we conclude $a\sim b$, since $a=\dot{\bigvee}(a_i, a^{(\nu)};\ 1\leqq i\leqq 5,\ 1\leqq\nu\leqq 4)$ and $b=\dot{\bigvee}(b_i, b^{(\nu)}, 1\leqq i\leqq 5, 1\leqq\nu\leqq 4)$.

Now, we shall prove two comparability theorems.

THEOREM 5.1. *For* a, $b\in L$, *there exist decompositions* $a=a'\dot{\cup} a''$, $b=b'\dot{\cup} b''$ *such that* $a'\sim b'$, $e(a'')\cap e(b'')=0$.

PROOF. It follows from Theorem 2.1 that the problem is reduced to the following three cases: L is respectively finite of type I, finite of type II and properly infinite.

(i) If there are $a_1, b_1 \in L$ such that $a_1 \sim a$, $b_1 \sim b$, $a_1 \perp b_1$, then we have the desired decompositions. Because, it follows from Lemma 4.1 (i) that there exist decompositions $a_1 = a_1' \dot{\cup} a_1''$, $b_1 = b_1' \dot{\cup} b_1''$ such that $a_1' \sim b_1'$, $e(a_1'') \cap e(b_1'') = 0$, and then there exist decompositions $a = a' \dot{\cup} a''$, $b = b' \dot{\cup} b''$ such that $a^{(\nu)} \sim a_1^{(\nu)}$, $b^{(\nu)} \sim b_1^{(\nu)}$ ($\nu = 1, 2$).

(ii) If L is properly infinite, then by Lemma 4.7 there are $c_1, c_2 \in L$ such that $1 = c_1 \dot{\cup} c_2 \sim c_1 \sim c_2$, and then $a \precsim c_1$, $b \precsim c_2$. Hence, we have the desired decompositions by (i).

(iii) Let L be finite of type II. It follows from Lemma 4.3 that there exist decompositions $a = \dot{\bigvee}(a_i; 1 \leqq i \leqq 5)$, $b = \dot{\bigvee}(b_i; 1 \leqq i \leqq 5)$ such that $a_i \sim a_j$, $b_i \sim b_j$ $(1 \leqq i, j \leqq 5)$. It follows from Lemma 5.1 that there is $c \in L$ such that $a_1 \precsim c \perp b_1$, and hence, by (i), there exist decompositions $a_1 = a_1' \dot{\cup} a_1''$, $b_1 = b_1' \dot{\cup} b_1''$ such that $a_1' \sim b_1'$, $e(a_1'') \cap e(b_1'') = 0$. Then, there exist decompositions $a_i = a_i' \dot{\cup} a_i''$, $b_i = b_i' \dot{\cup} b_i''$ $(2 \leqq i \leqq 5)$ with $a_i^{(\nu)} \sim a_1^{(\nu)}$, $b_i^{(\nu)} \sim b_1^{(\nu)}$ $(\nu = 1, 2)$. Putting $a^{(\nu)} = \dot{\bigvee}_i a_i^{(\nu)}$, $b^{(\nu)} = \dot{\bigvee}_i b_i^{(\nu)}$, we have $a' \sim b'$ and $e(a'') \cap e(b'') = e(a_1'') \cap e(b_1'') = 0$.

(iv) Let L be finite of type I. In the same way as in the proof (i) of Lemma 4.1, we have decompositions $a = \dot{\bigvee}_\alpha a_\alpha \dot{\cup} a''$, $b = \dot{\bigvee}_\alpha b_\alpha \dot{\cup} b''$ such that a_α, b_α are minimal, $a_\alpha \sim b_\alpha$ for every α and that $e(a'') \cap e(b'') = 0$. Putting $a' = \dot{\bigvee}_\alpha a_\alpha$, $b' = \dot{\bigvee}_\alpha b_\alpha$, we have $a' \sim b'$ by Lemma 5.4.

THEOREM 5.2. *For $a, b \in L$, there exists a decomposition $1 = z_1 \dot{\cup} z_2 \dot{\cup} z_3$ in Z_0 such that $z_1 \cap a \gg z_1 \cap b$, $z_2 \cap a \ll z_2 \cap b$, $z_3 \cap a \sim z_3 \cap b$. More simply, there exists $z \in Z_0$ such that $z \cap a \succsim z \cap b$, $(1-z) \cap a \precsim (1-z) \cap b$.*

This theorem is implied from Theorem 5.1 in the same way as Lemma 4.1 (ii), and the following lemma is implied from this theorem in the same way as Lemma 4.1 (iii).

LEMMA 5.5. *Let $a_1 \dot{\cup} a_2 \sim b_1 \dot{\cup} b_2$ and let $a_1 \dot{\cup} a_2$ be finite. If $a_1 \sim b_1$ then $a_2 \sim b_2$.*

LEMMA 5.6. *Let $a, b \in L$. If there exists a decomposition $1 = \dot{\bigvee}_\alpha z_\alpha$ in Z_0 such that $z_\alpha \cap a \sim z_\alpha \cap b$ for every α and if a is finite, then $a \sim b$.*

PROOF. It follows from Theorem 5.2 that there exists a decomposition $1 = z_1 \dot{\cup} z_2 \dot{\cup} z_3$ in Z_0 such that $z_1 \cap a \gg z_1 \cap b$, $z_2 \cap a \ll z_2 \cap b$, $z_3 \cap a \sim z_3 \cap b$. If $z_\alpha \cap z_1 \cap a \succ z_\alpha \cap z_1 \cap b$, then $z_\alpha \cap z_1 \cap a \succ z_\alpha \cap z_1 \cap a$, contradicting the finiteness of a. Hence we have $z_\alpha \cap z_1 \cap a = z_\alpha \cap z_1 \cap b = 0$ for every α, and hence $z_1 \cap a = z_1 \cap b = 0$. Similarly, we have $z_2 \cap a = z_2 \cap b = 0$. Therefore $a = z_3 \cap a \sim z_3 \cap b = b$.

LEMMA 5.7. *Let L be of type I. If $a = \dot{\bigvee}_\alpha a_\alpha$, $b = \dot{\bigvee}_\alpha b_\alpha$, $a_\alpha \sim b_\alpha$ for every α and if a is finite, then $a \sim b$* (hence b is finite).

PROOF. It follows from Lemma 4.8 (ii) that there is a decomposition $1 = \dot{\bigvee}_\beta z_\beta$ in Z_0 such that, in every $L(0, z_\beta)$, $z_\beta \cap a_\alpha$ is zero except a finite number of a_α. Then $z_\beta \cap a \sim z_\beta \cap b$ by (C_f). Hence we have $a \sim b$ by Lemma 5.6.

THEOREM 5.3. (*Complete additivity in the finite case*) $a=\dot{\bigvee}_\alpha a_\alpha$, $b=\dot{\bigvee}_\alpha b_\alpha$, $a_\alpha \sim b_\alpha$ *for every* α *and if* a *and* b *are finite, then* $a\sim b$.

PROOF. If L is of type I, the statement holds by Lemma 5.7. Hence, it suffices to prove the theorem when L is of type II. Then, for every α there exists a 5-homogeneous family $\{a_\alpha^{(\nu)}; 1\leqq\nu\leqq 5\}$ with the join a_α by Lemma 4.3, and then by (A_2) there exists a 5-homogeneous family $\{b_\alpha^{(\nu)}; 1\leqq\nu\leqq 5\}$ with the join b_α such that $a_\alpha^{(\nu)}\sim b_\alpha^{(\nu)}(1\leqq\nu\leqq 5)$. Putting $a^{(\nu)}=\dot{\bigvee}_\alpha a_\alpha^{(\nu)}$ and $b^{(\nu)}=\dot{\bigvee}_\alpha b_\alpha^{(\nu)}$, it follows that $\{a^{(\nu)}; 1\leqq\nu\leqq 5\}$ and $\{b^{(\nu)}; 1\leqq\nu\leqq 5\}$ are homogeneous families with the joins a and b respectively. We have $a^{(\nu)}\sim b^{(\nu)}$ by Lemma 5.2, and hence $a\sim b$.

COROLLARY. *If "$\sim$" satisfies moreover* $(\bar{A}_2)$, *then it is completely additive.*

PROOF. If L is finite, the statement holds by the theorem. If L is properly infinite, then there are $c_1, c_2\in L$ with $1=c_1\dot{\cup}c_2\sim c_1\sim c_2$ and then there are $a', b'\in L$ with $a\sim a'\leqq c_1$, $b\sim b'\leqq c_2$. By $(\bar{A}_2)$ there exist decompositions $a'=\dot{\bigvee}_\alpha a'_\alpha$, $b'=\dot{\bigvee}_\alpha b'_\alpha$ such that $a'_\alpha\sim a_\alpha$, $b'_\alpha\sim b_\alpha$ for every α, and hence we have $a\sim a'\sim b'\sim b$ by $(C_\perp)$.

LEMMA 5.8. *Let L be finite. If* $a_\delta\uparrow a$ *and* $a_\delta\lesssim b$ *for every* δ, *then* $a\lesssim b$.

PROOF. We may assume that $\{\delta\}$ is a well-ordered set $\{\rho; \rho<\Omega\}$ where Ω is a limit ordinal (see [11], Lemma 3.2), and that if ρ is a limit ordinal then $a_\rho=\bigvee(a_\gamma; \gamma<\rho)$. Putting $a_{\rho+1}=a_\rho\dot{\cup}c_\rho$ for every $\rho<\Omega$, we have a semi-orthogonal family $\{c_\rho; \rho<\Omega\}$ with $\dot{\bigvee}(c_\gamma; \gamma<\rho)=a_\rho\lesssim b$, $\dot{\bigvee}(c_\gamma; \gamma<\Omega)=a$. Using Theorem 5.3, it is easy to show by transfinite induction that there exists a semi-orthogonal family $\{b_\rho; \rho<\Omega\}$ in $L(0, b)$ such that $b_\rho\sim c_\rho$ for every $\rho<\Omega$ (see [5], Lemma 6.4). Hence, we have $a\sim\dot{\bigvee}(b_\rho; \rho<\Omega)\leqq b$ by Theorem 5.3.

§ 6. The axiom (B)

LEMMA 6.1. *Let the semi-orthogonal relation in L satisfy* $(\perp 5)$ *and "$\sim$" be an equivalence relation in L satisfying* (A_1), (A_2), (B'), $(C_\perp)$ *and* (C_f). *The following five statements are equivalent.*

(α) "$\sim$" *satisfies* (B).

(β) *If* $a_1\dot{\cup}a_2=b_1\dot{\cup}b_2$, *then there exists* $z\in Z_0$ *such that* $z\cap a_1\lesssim z\cap b_1$, $(1-z)\cap a_2\gtrsim(1-z)\cap b_2$.

(γ) *If* $a_1\dot{\cup}a_2=b_1\dot{\cup}b_2$, $a_1\sim a_2$, $b_1\sim b_2$, *then* $a_1\sim b_1$. (In [11] this statement is denoted by $(2, \zeta)$.)

(δ) *If* $a_1\perp a_2$, $a_1\sim a_2$ *and if* a_1 *is finite, then* $a_1\dot{\cup}a_2$ *is finite.*

(ε) *If* $a=b\dot{\cup}c$, $b\gtrsim c$ *and if* a *is properly infinite, then* $a\sim b$.

PROOF. (α)$\Rightarrow$(β). Let $a_1\dot{\cup}a_2=b_1\dot{\cup}b_2=u$ and put $a_1=(a_1\cap b_1)\dot{\cup}c_1$, $b_1=(a_1\cap b_1)\dot{\cup}d_1$, $a_1\cup b_1=c_2\dot{\cup}b_1=a_1\dot{\cup}d_2$. It follows from (B) that there exist decompositions $c_1=c_1'\dot\cup c_1''$, $d_1=d_1'\dot\cup d_1''$ such that $c_1'\sim c_2$, $d_1'\sim d_2$, $c_1''\sim d_1''$. Put $u=(a_1\cup b_1)\dot\cup v$. We have $a_1\dot\cup d_2\dot\cup v=a_1\dot\cup a_2$, which implies $a_2\sim d_2\dot\cup v$ by (B') and Lemma 4.5. Similarly, $c_2\dot\cup b_1\dot\cup v=b_1\dot\cup b_2$ implies $b_2\sim c_2\dot\cup v$. It follows from Theorem 5.2 that there is $z\in Z_0$ such that $z\cap c_1'\lesssim z\cap d_1'$, $(1-z)\cap c_1'\gtrsim(1-z)\cap d_1'$. Then, we have $z\cap a_1=(z\cap a_1\cap b_1)\dot\cup(z\cap c_1')\dot\cup(z\cap c_1'')\lesssim(z\cap a_1\cap b_1)\dot\cup(z\cap d_1')\dot\cup(z\cap d_1'')=z\cap b_1$, and since $(1-z)\cap c_2\gtrsim(1-z)\cap d_2$ we have $(1-z)\cap a_2\sim((1-z)\cap d_2)\dot\cup((1-z)\cap v)\lesssim((1-z)\cap c_2)\dot\cup((1-z)\cap v)\sim(1-z)\cap b_2$.

$(\beta)\Rightarrow(\gamma)$. Let $a_1\dot\cup a_2=b_1\dot\cup b_2$, $a_1\sim a_2$, $b_1\sim b_2$. (β) implies that there is $z\in Z_0$ such that $z\cap a_1\lesssim z\cap b_1$, $(1-z)\cap a_1\lesssim(1-z)\cap b_1$, and hence $a_1\lesssim b_1$. Similarly we have $a_1\gtrsim b_1$, and hence $a_1\sim b_1$.

$(\gamma)\Rightarrow(\delta)$. Let $a=a_1\dot\cup a_2$, $a_1\sim a_2$ and let a_1 be finite. There are b_1, b_2 such that $e^\infty(a)\cap a=b_1\dot\cup b_2\sim b_1\sim b_2$ by Lemma 4.7, and hence $b_1\sim e^\infty(a)\cap a_1$ by (γ). We have $e^\infty(a)\cap a_1=0$, since b_1 is properly infinite and a_1 is finite. Therefore $e^\infty(a)\cap a=0$, which means that a is finite.

$(\delta)\Rightarrow(\varepsilon)$. Let $a=b\dot\cup c$, $b\gtrsim c$ and let a be properly infinite. For $z\in Z_0$, there are a_1, a_2 with $z\cap a=a_1\dot\cup a_2\sim a_1\sim a_2$ since $z\cap a$ is properly infinite or zero. Then, there are decompositions $a_i=b_i\dot\cup c_i (i=1,2)$ with $b_i\sim z\cap b$, $c_i\sim z\cap c$. If $z\cap b$ is finite, then b_1, b_2 are finite and $b_1\dot\cup b_2$ is also by (δ), and then $z\cap a$ is finite since $z\cap a\sim b_1\dot\cup c_1\lesssim b_1\dot\cup b_2$. Hence, $z\cap a=0$ and $z\cap b=0$, which shows that b is properly infinite. There are b', b'' with $b=b'\dot\cup b''\sim b'\sim b''$ and then $a=b\dot\cup c\lesssim b_1\dot\cup b_2=b$, which implies $a\sim b$.

$(\varepsilon)\Rightarrow(\alpha)$. Put $a=(a\cap b)\dot\cup a_1$, $b=(a\cap b)\dot\cup b_1$, $a\cup b=a\dot\cup b_2=a_2\dot\cup b$. Then, $a_1\gtrsim a_2$, $b_1\gtrsim b_2$ by (B'), and $(a\cap b)\dot\cup a_1\dot\cup b_2=a\cup b=(a\cap b)\dot\cup b_1\dot\cup a_2$ implies $a_1\dot\cup b_2\sim b_1\dot\cup a_2$ by (B') and Lemma 4.5. If $a_1\dot\cup b_2$ is finite, then, putting $a_1=a_1'\dot\cup a_1''$, $b_1=b_1'\dot\cup b_1''$ where $a_1'\sim a_2$, $b_1'\sim b_2$, we have $a_1''\sim b_1''$ by Lemma 5.5, and hence (α) holds. When $a_1\dot\cup b_2$ is properly infinite, the problem is reduced to the following three cases by Theorem 5.2: (i) $a_1\lesssim b_2$, (ii) $b_1\lesssim a_2$, (iii) $a_1\gtrsim b_2$, $b_1\gtrsim a_2$. Case (i). Since $a_2\lesssim a_1\lesssim b_2\lesssim b_1$, it follows from (ε) that $b_1\sim b_1\dot\cup a_2\sim a_1\dot\cup b_2\sim b_2$. Putting $a_1'=a_1'\dot\cup a_1''$ where $a_1'\sim a_2$, we have $a_1''\dot\cup b_2\sim a_1\dot\cup b_2$ since $a_1''\dot\cup b_2\geqq b_2\sim a_1\dot\cup b_2$, and hence $b_1\sim a_1''\dot\cup b_2$, which implies that there is a decomposition $b_1=b_1'\dot\cup b_1''$ such that $b_1'\sim b_2$, $b_1''\sim a_1''$. Therefore (α) holds. Case (ii). Since $b_2\lesssim b_1\lesssim a_2\lesssim a_1$, we can prove (α) in the similar way as (i). Case (iii). It follows from (ε) that $a_1\sim a_1\dot\cup b_2\sim b_1\dot\cup a_2\sim b_1$. Since a_1 is properly infinite, there are c_1, c_2 with $a_1=c_1\dot\cup c_2\sim c_1\sim c_2$. Then $c_1\sim a_1\gtrsim b_2$, and then we have a decomposition $c_1=a_1'\dot\cup c_1'$ with $a_1'\sim b_2$. Putting $c_1'\dot\cup c_2=a_1''$, we have $a_1=a_1'\dot\cup a_1''$, and $a_1''\sim a_1$ since $a_1''\geqq c_2\sim a_1$. Similarly, we have a decomposition $b_1=b_1'\dot\cup b_1''$ with $b_1'\sim a_2$, $b_1''\sim b_1$. Then $a_1''\sim b_1''$, and hence (α) holds.

In the remainder of this section, we assume that the semi-orthogonal relation in L satisfies $(\perp 5)$ and there is an equivalence relation "$\sim$" in L satisfying the axioms (A_1), (A_2), (B), $(C_\perp)$ and (C_f).

THEOREM 6.1. *If a and b are finite, then so is $a\cup b$.*

PROOF. Let c be the properly infinite part of $a\cup b$, i.e., $c=e^\infty(a\cup b)\cap(a\cup b)$.

There are c_1, c_2 with $c=c_1\dot{\cup}c_2\sim c_1\sim c_2$. Putting $a\cup b=a\dot{\cup}b_1$, we have $b\gtrsim b_1$ by (B′). Since $c_1\dot{\cup}c_2=(e^\infty(a\cup b)\cap a)\dot{\cup}(e^\infty(a\cup b)\cap b_1)$, it follows from ($\beta$) of Lemma 6.1 that there is $z\in Z_0$ such that $z\cap c_1\lesssim z\cap e^\infty(a\cup b)\cap a$, $(1-z)\cap c_2\lesssim(1-z)\cap e^\infty(a\cup b)\cap b_1$. Since a, b_1 are finite and since c_1, c_2 are properly infinite (or zero), we have $z\cap c_1=(1-z)\cap c_2=0$, which implies $z\cap c=(1-z)\cap c=0$ and $c=0$. This concludes that $a\cup b$ is finite.

LEMMA 6.2. *If $a_1\dot{\cup}a_2\sim b_1\dot{\cup}b_2$, $a_1\lesssim b_1$ and if a_1 is finite, then $a_2\gtrsim b_2$.*

PROOF. If $a_1\dot{\cup}a_2$ is finite, the statement follows from Lemma 5.5. Let $a_1\dot{\cup}a_2$ be properly infinite. There is $z\in Z_0$ with $z\cap a_1\gtrsim z\cap a_2$, $(1-z)\cap a_1\lesssim(1-z)\cap a_2$ by Theorem 5.2, and then $z\cap(a_1\dot{\cup}a_2)$ is finite by Theorem 6.1, which implies $z\cap(a_1\dot{\cup}a_2)=0$. Hence $a_1\lesssim a_2$, and it follows from (ε) of Lemma 6.1 that $a_2\sim a_1\dot{\cup}a_2\gtrsim b_2$.

LEMMA 6.3. *If each of a, $b\in L$ belongs to a 4-homogeneous family and if a is finite, then $a\cup b$ belongs to a 2-homogeneous family.*

PROOF. The problem is reduced to the following two cases by Theorem 5.1: (i) $a\lesssim b$, (ii) $a\gtrsim b$. Case (i). There is a 4-homogeneous family $\{b, b_1, b_2, b_3\}$ by the assumption. Putting $a\cup b=a_1\dot{\cup}b$, we have $a_1\lesssim a$, and putting $a_1\dot{\cup}b\dot{\cup}c=1$, $b\dot{\cup}b_1\dot{\cup}b_2\dot{\cup}b_3\dot{\cup}d=1$, we have $a_1\dot{\cup}c\sim b_1\dot{\cup}b_2\dot{\cup}b_3\dot{\cup}d$. Since $a_1\lesssim b_1$ and since a_1 is finite, we have $c\gtrsim b_2\dot{\cup}b_3\dot{\cup}d\gtrsim a_1\dot{\cup}b=a\cup b$ by Lemma 6.2. Hence $a\cup b$ belongs to a 2-homogeneous family. Case (ii). b is finite since $a\gtrsim b$. Hence we can prove the lemma in the similar way as Case (i).

LEMMA 6.4. *If $a=\dot{\bigcup}_\alpha a_\alpha$, $b=\dot{\bigcup}_\alpha b_\alpha$, $a_\alpha\sim b_\alpha$ for every α and if a is finite, then $a\sim b$* (hence b is finite).

PROOF. If L is of type I, then the statement holds by Lemma 5.7. If L is of type III, then $a=0$, and then $b=0$ since $b_\alpha=0$ for every α. Let L be of type II. The following statement is implied from Lemma 6.3 in the similar way as Lemma 5.2: If $a=\dot{\bigcup}_\alpha a_\alpha$, $b=\dot{\bigcup}_\alpha b_\alpha$, $a_\alpha\sim b_\alpha$ for every α, each of a, b belongs to a 4-homogeneous family and if a is finite then $a\sim b$. Hence the lemma can be proved in the similar way as Theorem 5.3.

LEMMA 6.5. *Let $a_\delta\uparrow a$. If $a_\delta\lesssim b$ for every δ and if b is finite, then $a\lesssim b$.*

PROOF. This can be proved in the similar way as Lemma 5.8, by the aid of Lemma 6.4.

The last lemma will be used in the proof of the complete additivity of dimension functions (Theorem 7.5).

§ 7. Dimension functions

Assume that the semi-orthogonal relation in L satisfies ($\perp 5$) and there is an equivalence relation "$\sim$" in L satisfying the axioms (A_1), (A_2), (B), ($C_\perp$) and (C_f).

The arguments of preceding sections §§ 2—6 implies that all the lemmas and theorems in § 2, § 3 of [11] hold, and hence the statements in this section can be proved by the same methods as in § 4, § 5 of [11]. Here the proofs are omitted.

DEFINITION 7.1. For any $a \in L$, the class $\{x \in L; x \sim a\}$ is denoted by $[a]$. The set $\{[a]; a \in L\}$ is denoted by $[L]$. It follows from Lemma 4.5 that $[L]$ is partially ordered if $[a] \leqq [b]$ is defined by $a \precsim b$, and it is easy to show that $[L]$ is a lattice, by Theorem 5.2. $[L]$ is called a *dimension lattice* of L. $[L]$ is totally ordered if and only if $Z_0 = \{0, 1\}$. If there exist $a_1 \in [a]$, $b_1 \in [b]$ with $a_1 \perp b_1$, then $[a]+[b]$ is defined by $[a_1 \dot{\cup} b_1]$. If $a \in L$ belongs to an n-homogeneous family with the join b, then $n[a]$ is defined by $[b]$ $(0 \cdot [a] = [0])$. We write $[a] \ll [b]$ if $a \ll b$.

LEMMA 7.1. *Let $c \in L$ be finite. For any $a \in L$ and for n, $0 \leqq n < \infty$, there exists a unique element $q_n(c, a) \in Z_0$ satisfying the following condition:*

$$z \leqq q_n(c, a) \textit{ if and only if } n[z \cap c] \leqq [z \cap a] \ (n[z \cap c] \text{ exists}), \textit{ where } z \in Z_0.$$

Then, putting $r_n(c, a) = q_n(c, a) - q_{n+1}(c, a)$, the following equations hold:

$$[r_n(c, a) \cap a] = n[r_n(c, a) \cap c] + [p] \textit{ with } [p] \ll [r_n(c, a) \cap c],$$
$$1 = q_0(c, a) = \dot{\bigvee}(r_n(c, a);\ 0 \leqq n < \infty) \dot{\cup} (e^\infty(a) \cap (1 - e(c))).$$

We use Theorems 5.2 and 6.1 in the proof of this lemma.

LEMMA 7.2. *Let $h \in L$ be a minimal element with $e(h) = z_{\mathrm{I}}$* ($[h]$ is uniquely determined by Lemma 5.3). *For $z \in Z_0$ with $z \leqq z_{\mathrm{I}}$, $z \leqq r_n(b, a)$ if and only if $n[z \cap h] = [z \cap a]$, and $r_0(h, a) = z_{\mathrm{I}} \cap (1 - e(a))$.*

COROLLARY. *There exists a unique decomposition $z_{\mathrm{I}}^f = \dot{\bigvee}(z_{\mathrm{I}}^{(n)};\ 1 \leqq n < \infty)$ in Z_0 such that $z_{\mathrm{I}}^{(n)}$ is the join of an n-homogeneous family of minimal elements.* (Put $z_{\mathrm{I}}^{(n)} = r_n(h, 1)$.)

DEFINITION 7.2. Let Ω be the representation space of the complete Boolean lattice Z_0 ([9], Kap. I, § 5). The compact open subset of Ω corresponding to $z \in Z_0$ is denoted by $E(z)$, and its characteristic function is denoted by $\chi(z)$. The complete lattice of all $[0, \infty]$-valued continuous functions on Ω is denoted by $\mathbf{Z}$. A mapping d of L into $\mathbf{Z}$ is called a *dimension function* on L if it satisfies the following conditions:

(1°) *If $a \sim b$ then $d(a) = d(b)$,*

(2°) *if $a \perp b$ then $d(a \cup b) = d(a) + d(b)$,*

(3°) *if $z \in Z_0$ then $d(z \cap a) = \chi(z) d(a)$,*

(4°) *if $a > 0$ then $d(a) > 0$,*

(5°) *if a is finite then $d(a)$ is finite valued a.e.* (a. e. means "except on a set of the first category").

It follows from (1°), (2°) that any dimension function defines an order- and addition-preserving mapping of $[L]$ into $\mathbf{Z}$. This mapping is one-to-one in the finite case, as stated below. If $Z_0=\{0, 1\}$, then since Ω is a one-point set, d is numerical valued ($\mathbf{Z}=[0, \infty]$).

Using Lemmas 7.1 and 7.2 we can prove the following theorem.

THEOREM 7.1. *(Existence) There exists a dimension function d on L such that $d(h)=\chi(z_{\mathrm{I}})$, $d(z_{\mathrm{II}}^f)=\chi(z_{\mathrm{II}}^f)$* ($h$ is a minimal element with $e(h)=z_{\mathrm{I}}$).

THEOREM 7.2. *Let d be a dimension function on L.*

(i) *$d(a)(\omega)\equiv 0$ on $\Omega-E(e(a))$ ($\omega \in \Omega$), $0<d(a)(\omega)<\infty$ a.e. on $E(e^f(a))$, $d(a)(\omega)\equiv\infty$ on $E(e^\infty(a))$. Especially, the converses of* (4°) *and* (5°) *hold.*

(ii) *If a (or b) is finite, then $a\succ b$ (resp. $\sim$, $\prec$) is equivalent to $d(a)>d(b)$ (resp. $=$, $<$).*

(iii) *$d(a\cup b)+d(a\cap b)\leqq d(a)+d(b)$ for any $a, b \in L$.*

The statement (iii) is easily implied from (B′). If "$\sim$" satisfies ($\bar{\mathrm{B}}$), then the equality $d(a\cup b)+d(a\cap b)=d(a)+d(b)$ holds.

THEOREM 7.3. *(Uniqueness in a certain sense) If d_1, d_2 are dimension functions on L, then there exists a function $f \in \mathbf{Z}$, $0<f(\omega)<\infty$ a.e. such that $d_1(a)=f\cdot d_2(a)$ for every $a \in L$.*

COROLLARY. *Let $z_{\mathrm{III}}=0$ and let d_0 be a dimension function on L. There exists a one-to-one correspondence between the dimension functions d on L and the functions $f \in \mathbf{Z}$ with $0<f(\omega)<\infty$ a.e., where the correspondence is given by the equation $d(a)=f\cdot d_0(a)$ for every $a \in L$.*

DEFINITION 7.3. A dimension function d on L is called to be *normalized* if $d(h)=\chi(z_{\mathrm{I}})$ and $d(z_{\mathrm{II}}^f)=\chi(z_{\mathrm{II}}^f)$, as in Theorem 7.1. A normalized dimension function is uniquely determined if $z_{\mathrm{II}}^\infty=0$. $\mathbf{Z}_L$ denotes the set of $f \in \mathbf{Z}$ such that

$$\begin{aligned} f(\omega)&=0, 1, \cdots, n \text{ on } E(z_{\mathrm{I}}^{(n)}),\\ &=0, 1, \cdots, \infty \text{ on } E(z_{\mathrm{I}}^{(\infty)}),\\ &\leqq 1 \text{ on } E(z_{\mathrm{II}}^f),\\ &=0, \infty \text{ on } E(z_{\mathrm{III}}). \end{aligned}$$

It is a complete sublattice of $\mathbf{Z}$.

THEOREM 7.4. *If d is a normalized dimension function, then the image of d is equal to $\mathbf{Z}_L$.*

COROLLARY. *$\{[a];\ a$ is finite$\}$ is lattice-isomorphic to $\{f \in \mathbf{Z}_L;\ f(\omega)<\infty$ a.e.$\}$. Especially if L is finite, then $[L]$ is lattice-isomorphic to the complete lattice $\mathbf{Z}_L$* (addition-preserving).

THEOREM 7.5. *Let d be a dimension function on L. If $a_\delta \uparrow a$ then $d(a_\delta)\uparrow d(a)$. Especially, d is completely additive, i.e., if $a=\dot{\bigcup}(a_\alpha;\ \alpha \in I)$ then $d(a)=\sum(d(a_\alpha);\ \alpha \in I)$ (=the join of all finite sums $\sum_i d(a_{\alpha_i})$).*

§ 8. The axiom $(\bar{\mathrm{B}})$

DEFINITION 8.1. Let "$\sim$" be an equivalence relation in L. We define a new relation "$\overset{+}{\sim}$" as follows: $a \overset{+}{\sim} b$ if there exist decompositions $a=\dot{\bigcup}(a_i;\ 1\leqq i\leqq n)$, $b=\dot{\bigcup}(b_i;\ 1\leqq i\leqq n)$ such that $a_i\sim b_i$ for every i.

We shall show that if "$\sim$" satisfies (A_2) and $(\bar{\mathrm{B}})$ then "$\overset{+}{\sim}$" is an equivalence relation.

LEMMA 8.1. *Let "$\sim$" satisfy* (A_2) *and* $(\bar{\mathrm{B}})$. *If* $\dot{\bigcup}(a_i;\ 1\leqq i\leqq m)=\dot{\bigcup}(b_j;\ 1\leqq j\leqq n)$, *then there exist decompositions* $a_i=\dot{\bigcup}(a_{ij};\ 1\leqq j\leqq n)$, $b_j=\dot{\bigcup}(b_{ij};\ 1\leqq i\leqq m)$ *such that* $a_{ij}\sim b_{ij}$ *for every* i, j.

PROOF. We shall prove the lemma by induction. Let $a=a_1\dot{\cup}a_2=b_1\dot{\cup}b_2$, and put $a_{11}=b_{11}=a_1\cap b_1$, $a_1=a_{11}\dot{\cup}a_{12}$, $b_1=b_{11}\dot{\cup}b_{21}$. Putting $a_1\cup b_1=a'\dot{\cup}b_1=a_1\dot{\cup}b'$, we have $a_{12}\sim a'$, $b_{21}\sim b'$ by $(\bar{\mathrm{B}})$. Putting $(a_1\cup b_1)\dot{\cup}c=a$, we have $a_1\dot{\cup}b'\dot{\cup}c=a_1\dot{\cup}a_2$, $a'\dot{\cup}b_1\dot{\cup}c=b_1\dot{\cup}b_2$ and hence $a_2\sim b'\dot{\cup}c$, $b_2\sim a'\dot{\cup}c$ by $(\bar{\mathrm{B}})$. By (A_2), there exist decompositions $a_2=a_{21}\dot{\cup}a_{22}$, $b_2=b_{12}\dot{\cup}b_{22}$ such that $a_{21}\sim b'$, $a_{22}\sim c$, $b_{12}\sim a'$, $b_{22}\sim c$. We have $a_{ij}\sim b_{ij}$ for every i, j, and hence the lemma holds when $m=n=2$. Suppose that the lemma holds when the number of a_i is $\leqq m$ and the number of b_j is $\leqq n$. Let $\dot{\bigcup}(a_i;\ 1\leqq i\leqq m+1)=\dot{\bigcup}(b_j;\ 1\leqq j\leqq n)$. Putting $c=a_m\dot{\cup}a_{m+1}$, we have the following decompositions by the assumption: $a_i=\dot{\bigcup}(a_{ij};\ 1\leqq j\leqq n)\ (1\leqq i\leqq m-1)$, $c=\dot{\bigcup}(c_j;\ 1\leqq j\leqq n)$, $b_j=\dot{\bigcup}(b_{ij};\ 1\leqq i\leqq m-1)\ \dot{\cup}\ d_j(1\leqq j\leqq n)$, where $a_{ij}\sim b_{ij}$ and $c_j\sim d_j$. And, since $a_m\dot{\cup}a_{m+1}=c=\dot{\bigcup}(c_j;\ 1\leqq j\leqq n)$, we have the following decompositions: $a_m=\dot{\bigcup}(a_{mj};\ 1\leqq j\leqq n)$, $a_{m+1}=\dot{\bigcup}(a_{m+1,j};\ 1\leqq j\leqq n)$, $c_j=c_{mj}\dot{\cup}c_{m+1,j}$, where $a_{mj}\sim c_{mj}$, $a_{m+1,j}\sim c_{m+1,j}$. Since $c_j\sim d_j$, there are decompositions $d_j=b_{mj}\dot{\cup}b_{m+1,j}$ such that $b_{mj}\sim c_{mj}$, $b_{m+1,j}\sim c_{m+1,j}$. Hence we have the desired decompositions of a_i, b_j. When $\dot{\bigcup}(a_i;\ 1\leqq i\leqq m)=\dot{\bigcup}(b_j;\ 1\leqq j\leqq n+1)$, we have the same argument as above. Therefore the lemma holds for any m, n.

THEOREM 8.1. (i) *Let "$\sim$" satisfy* (A_2) *and* $(\bar{\mathrm{B}})$. *Then "$\overset{+}{\sim}$" is an equivalence relation satisfying* (A_2), $(\bar{\mathrm{B}})$ *and* (C_f), *and the relative center with respect to "$\overset{+}{\sim}$" coincides with* Z_0 (=the relative center with respect to "$\sim$").

(ii) *Let "$\sim$" satisfy moreover* (A_1). *Then "$\overset{+}{\sim}$" also satisfies* (A_1). *If* $a\overset{+}{\sim}b$ *and* $z\in Z_0$ *then* $z\cap a\overset{+}{\sim}z\cap b$. *The notions of minimal element, simple element and type I do not change when "$\sim$" is replaced by "$\overset{+}{\sim}$".*

(iii) *Let "$\sim$" satisfy moreover* $(\mathrm{C}_\perp)$. *Then "$\overset{+}{\sim}$" also satisfies* $(\mathrm{C}_\perp)$. *If* $a\overset{+}{\sim}b$ *and* $a\perp b$ *then* $a\sim b$. *The notions of homogeneous family, finite element, properly infinite element, type II and type III do not change when "$\sim$" is replaced by "$\overset{+}{\sim}$".*

PROOF. (i) It is obvious that "$\overset{+}{\sim}$" is symmetric and reflexive. It is easy to prove by Lemma 8.1 that "$\overset{+}{\sim}$" is transitive, which concludes that it is an equivalence relation. It is obvious that "$\overset{+}{\sim}$" satisfies (C_f) and the relative center coincides with Z_0. Using Lemma 8.1, it is easy to show that "$\overset{+}{\sim}$" satisfies (A_2). "$\overset{+}{\sim}$" satisfies $(\bar{B})$ since $a\sim b$ implies $a\overset{+}{\sim}b$. The statements (ii), (iii) are easily proved.

Remark that the finiteness of $a\in L$ is equivalent to "$a\overset{+}{\sim}b\leqq a$ implies $a=b$" by Lemma 4.6 (i), but not equivalent to "$a\sim b\leqq a$ implies $a=b$".

We assume that the semi-orthogonality in L satisfies $(\perp 5)$ and that there is an equivalence relation "$\sim$" in L satisfying the axioms (A_1), (A_2), $(\bar{B})$ and $(C_\perp)$. Then, "$\overset{+}{\sim}$" is an equivalence relation satisfying (A_1), (A_2), $(\bar{B})$, $(C_\perp)$ and (C_f) by Theorem 8.1, and hence all the lemmas and theorems in the preceding sections (§§ 4–7) are available, replacing "$\sim$" by "$\overset{+}{\sim}$". Remark that the notion of dimension function (Definition 7.2) does not change when "$\sim$" is replaced by "$\overset{+}{\sim}$". Because, the condition "if $a\sim b$ then $d(a)=d(b)$" is equivalent to "if $a\overset{+}{\sim}b$ then $d(a)=d(b)$" by the aid of the condition "if $a\perp b$ then $d(a\cup b)=d(a)+d(b)$".

Furthermore, we shall prove some theorems by the aid of $(\bar{B})$. We have the following type of comparability theorem.

THEOREM 8.2. *For any a, $b\in L$, there exist decompositions $a=(a\cap b)\dot{\cup}a'\dot{\cup}a''$, $b=(a\cap b)\dot{\cup}b'\dot{\cup}b''$* such that $a'\sim b'$, $e(a'')\cap e(b'')=0$.

PROOF. Putting $a=(a\cap b)\dot{\cup}a_1$, $b=(a\cap b)\dot{\cup}b_1$, we have $a_1\cap b_1=a_1\cap a\cap b\cap b_1=0$. Then, putting $a_1\cup b_1=a_1\dot{\cup}c$, it follows from $(\bar{B})$ that $b_1\sim c$. It follows from Lemma 4.1 (i) that there exist decompositions $a_1=a'\dot{\cup}a''$, $c=c'\dot{\cup}c''$ such that $a'\sim c'$, $e(a'')\cap e(c'')=0$, and then there is a decomposition $b_1=b'\dot{\cup}b''$ such that $b'\sim c'$, $b''\sim c''$. Hence we have the desired decompositions.

This theorem implies directly that Theorem 5.1 holds if "$\sim$" is replaced by "$\overset{+}{\sim}$".

THEOREM 8.3. *If L is finite then it is an upper-continuous complemented modular lattice. If L is finite of type I then it is a continuous geometry of type I.*

PROOF. Replacing "$\sim$" by "$\overset{+}{\sim}$", we may assume that "$\sim$" satisfies (C_f) besides (A_1), (A_2), $(\bar{B})$, $(C_\perp)$. If $a\leqq c$, then $x=(a\cup b)\cap c$ and $y=a\cup(b\cap c)$ are perspective and $x\geqq y$. Hence, it follows from $(\bar{B})$ and the finiteness that $x=y$, which implies that L is modular. We shall show that $a_\delta\uparrow a$ implies $a_\delta\cap b\uparrow a\cap b$. Putting $a=(a\cap b)\dot{\cup}a'$, we have $a_\delta\lesssim(a_\delta\cap b)\dot{\cup}a'$; because, putting $a_\delta=(a_\delta\cap b)\dot{\cup}c_1$, $(a\cap b)\cup a_\delta=(a\cap b)\dot{\cup}c_2$, we have $c_1\sim c_2\lesssim a'$ by $(\bar{B})$. Hence $a_\delta\lesssim\bigcup_\delta(a_\delta\cap b)\dot{\cup}a'$ for every δ, which implies $a\lesssim\bigcup_\delta(a_\delta\cap b)\dot{\cup}a'$ by Lemma 5.8. We have $a=\bigcup_\delta(a_\delta\cap b)\dot{\cup}a'$ by the finiteness, and hence $a\cap b=\bigcup_\delta(a_\delta\cap b)$. Hence L is upper-continuous.

If L is finite of type I, then it is continuous by Lemma 4.8 (iii), and hence it is a continuous geometry. Furthermore, it follows from Remark 3.1 (ii) that 1 is the join of a set of minimal elements and it follows from Lemma 2.3

that our minimal element is minimal in the sense of continuous geometry (see [10], Remark 3.2). Hence, L is a continuous geometry of type I. This completes the proof.

In many examples, the equivalence relations satisfy ($\bar{\mathrm{B}}$) (see §9 and §10), but we remark that the equi-dimensionality in an affine geometry does not satisfy ($\bar{\mathrm{B}}$) though it satisfies (A), (B), ($\mathrm{C}_\perp$) and (C_f).

§ 9. Example 1. Projectivity in modular lattices

Let L be a relatively semi-orthocomplemented complete lattice and assume that L is modular.

Definition 9.1. Two elements a, $b \in L$ are called to be *independent* if $a \frown b = 0$. The independence satisfies the four axioms for semi-orthogonality since L is modular ([13], §1), but we shall distinguish it from the semi-orthogonality in L. A subset S of L is called an *independent family* if $\bigvee(a; a \in F_1) \frown \bigvee(a; a \in F_2) = 0$ for any pair of disjoint finite subsets F_1, F_2 of S. Concerning independent families, we have the similar arguments as Lemmas 1.1 and 1.2. S is called a *residually independent family* if $a \frown \bigvee(b \in S; b \neq a) = 0$ for every $a \in S$ (see Amemiya-Halperin [1], §3). It is obvious that any residually independent family is independent and any finite independent family is residually indendent.

In this section, the equivalence relation in L is defined by the projectivity, i.e., $a \sim b$ if there exists a finite sequence $\{a_0, a_1, \cdots a_n\}$ such that $a_0 = a$, $a_n = b$ and that a_{i-1} and a_i are perspective ($1 \leqq i \leqq n$). a, $b \in L$ are called to be $+$*-projective* if $a \overset{+}{\sim} b$ (Definition 8.1). It is obvious that the relative center coincides with the center Z and that the axioms (A_1) and ($\bar{\mathrm{B}}$) are satisfied. We shall show that (A_2) is also satisfied. If $a \sim b = b_1 \dot{\smile} b_2$, then there is a projective mapping T of $L(0, b)$ onto $L(0, a)$, which is a lattice-isomorphism since L is modular. Hence $Tb_1 \smile Tb_2 = Tb = a$, $Tb_1 \frown Tb_2 = 0$. Putting $a_1 = Tb_1$ and $a = a_1 \dot{\smile} a_2$, we have $a_i \sim b_i (i = 1, 2)$ since a_2 and Tb_2 are perspective. Hence (A_2) is satisfied. Therefore, all the lemmas and theorems in §2 are valid, especially, L can be decomposed into five summands, which are finite of type I, properly infinite of type I, finite of type II, properly infinite of type II and type III respectively. By the argument of §3, the minimal element coincides with the simple element, and it also coincides with the D-element since the projectivity satisfies the condition (P) of Lemma 3.3 (see [9], Kap. II, Satz 3.5).

Definition 9.2. A relatively semi-orthocomplemented complete lattice is called an *MD-lattice* if it is modular and the following conditions are satisfied:

(1) The semi-orthogonality satisfies the axiom ($\perp 5$) (see §4),

(2) if $a=\dot{\bigcup}_\alpha a_\alpha$, $b=\dot{\bigcup}_\alpha b_\alpha$, a_α and b_α are perspective for every α and $a\cap b=0$, then a and b are perspective.

We shall have two examples of the MD-lattice, i.e., the upper-continuous complented modular lattice and the orthocomplemented modular complete lattice.

EXAMPLE 1.1. Let L be an upper-continuous complemented modular lattice. The semi-orthogonality in L is defined by $a\cap b=0$ (the semi-orthogonality and the independence coincide). Since L is upper-continuous, the condition (1) is satisfied. We shall show that (2) is satisfied. There exists x_α such that $a_\alpha \dot{\cup} x_\alpha = b_\alpha \dot{\cup} x_\alpha = a_\alpha \cup b_\alpha$, since a_α and b_α are perspective. Since $a\cap b=0$, $\{a_\alpha, b_\alpha;\ \alpha\in I\}$ is a semi-orthogonal family by Lemma 1.2 and then so is $\{a_\alpha\cup b_\alpha;\ \alpha\in I\}$. Hence, $\{a_\alpha, x_\alpha;\ \alpha\in I\}$ and $\{b_\alpha, x_\alpha;\ \alpha\in I\}$ are also semi-orthogonal families by Lemma 1.2, and putting $x=\dot{\bigcup}_\alpha x_\alpha$, we have $a\cap x=b\cap x=0$ by ($\perp$5). Then a and b are perspective, since $a\cup x=b\cup x$. Therefore, L is an MD-lattice. In this case, the projectivity satisfies (A).

EXAMPLE 1.2. Let L be an orthocomplemented modular complete lattice. The semi-orthogonality in L is defined by the orthogonality. The condition (1) is satisfied since any element has a unique semi-orthocomplement (=orthocomplement). We shall show that (2) is satisfied. There exists x_α such that $a_\alpha\cup x_\alpha=b_\alpha\cup x_\alpha=a_\alpha\cup b_\alpha$, $a_\alpha\cap x_\alpha=b_\alpha\cap x_\alpha=0$, since a_α and b_α are perspective. Putting $x=\bigcup_\alpha x_\alpha$, we have $a\cup x=b\cup x$. Since L is modular, $a\cap x=(\bigcup(a_\alpha;\ \alpha\neq\beta)\cup a_\beta)\cap x=(\bigcup(a_\alpha;\ \alpha\neq\beta)\cup((\bigcup(a_\alpha;\ \alpha\neq\beta)\cup x)\cap a_\beta))\cap x$. But, since $\{\bigcup(a_\alpha;\ \alpha\neq\beta), \bigcup(b_\alpha;\ \alpha\neq\beta), a_\beta, b_\beta\}$ is an independent family, we have $\bigcup(a_\alpha\cup b_\alpha;\ \alpha\neq\beta)\cap(a_\beta\cup b_\beta)=0$, and hence $\{\bigcup(a_\alpha\cup x_\alpha;\ \alpha\neq\beta), a_\beta, x_\beta\}$ is an independent family, which implies that $(\bigcup(a_\alpha;\ \alpha\neq\beta)\cup x)\cap a_\beta=(\bigcup(a_\alpha\cup x_\alpha;\ \alpha\neq\beta)\cup x_\beta)\cap a_\beta=0$. Hence $a\cap x=\bigcup(a_\alpha;\ \alpha\neq\beta)\cap x$ for every β, which implies that $a\cap x\leqq\bigcap_\beta\bigcup(a_\alpha;\ \alpha\neq\beta)$. But, since $\bigcup(a_\alpha;\ \alpha\neq\beta)=a_\beta^\perp\cap a$ ($a_\beta^\perp$ is the orthocomplement of a_β), we have $\bigcap_\beta\bigcup(a_\alpha;\ a\neq\beta)=\bigcap_\beta a_\beta^\perp\cap a=(\bigcup_\beta a_\beta)^\perp\cap a=a^\perp\cap a=0$. Hence $a\cap x=0$ and similarly $b\cap x=0$, which concludes that a and b are perspective. Therefore L is an MD-lattice.

In an MD-lattice L, the projectivity "$\sim$" satisfies (A_1), (A_2) and ($\bar{B}$), and we shall show that it also satisfies ($C_\perp$) and that L has the dimension functions.

LEMMA 9.1. *In a modular lattice with zero,*

(i) *if both a and b are perspective to c and if $(a\cup c)\cap b=0$ then a and b are perspective* (Amemiya-Halperin [1], p. 484);

(ii) *if $\{a, b, c\}$ is an independent family and if a and b are perspective then $a\cup c$ and $b\cup c$ are perspective* (von Neumann [15], Part I, Theorem 3.5).

LEMMA 9.2. *If a relatively semi-orthocomplemented complete lattice L is modular, then for any residually independent family $\{a_\alpha;\ \alpha\in I\}$ in L there exists a semi-orthogonal family $\{b_\alpha;\ \alpha\in I\}$ such that a_α and b_α are perspective* *and that* $\dot{\bigcup}_\alpha b_\alpha = \bigcup_\alpha a_\alpha$.

PROOF. Let I be well-ordered: $I=\{\rho;\ \rho<\Omega\}$. Put $\bigcup(a_\gamma;\ \gamma<\rho)=c_\rho$ for $\rho \leqq \Omega$ $(c_1=0)$ and $c_{\rho+1}=c_\rho \dot{\cup} b_\rho$ for $\rho<\Omega$. Then, it is obvious that $\{b_\rho;\ \rho<\Omega\}$ is a semi-orthogonal family. Since $\{a_\gamma;\ \gamma\leqq\rho\}$ is a residually independent family with the join $c_{\rho+1}$, we have $c_\rho \cap a_\rho=0$, $c_\rho \cup a_\rho = c_{\rho+1}$. Hence a_ρ and b_ρ are perspective. It is easy to show that $\dot{\bigcup}_\rho b_\rho = c_\Omega = \bigcup_\rho a_\rho$.

COROLLARY. *If a relatively semi-orthocomplemented complete lattice L is modular and if the semi-orthogonality satisfies* $(\perp 5)$, *then the projectivity* "$\sim$" *satisfies* $(\bar{\mathrm{A}}_2)$.

PROOF. Let $a \sim \dot{\bigcup}_\alpha b_\alpha$ and let T be the projective mapping of $L(0, \dot{\bigcup}_\alpha b_\alpha)$ onto $L(0, a)$. Since $\{b_\alpha\}$ is residually independent by $(\perp 5)$, so is $\{Tb_\alpha\}$, and hence by the lemma there exists a semi-orthogonal family $\{a_\alpha\}$ with the join a such that a_α and Tb_α are perspective, whence $a_\alpha \sim b_\alpha$.

LEMMA 9.3. *Let L be an MD-lattice and $a, b \in L$. If $a \sim b$* (projective) *and $a \cap b=0$, then a and b are perspective.*

PROOF. Let $a \neq 0$ and let T be the projective mapping of $L(0, a)$ onto $L(0, b)$. By [9], Kap. II, Satz 3.5, there exists non-zero element a_1 in $L(0, a)$ such that T is a perspective mapping of $L(0, a_1)$ onto $L(0, Ta_1)$. Hence if we choose a maximal semi-orthogonal family $\{a_\alpha\}$ in $L(0, a)$ such that T is a perspective mapping of $L(0, a_\alpha)$ onto $L(0, Ta_\alpha)$ for every α, then we have $a=\dot{\bigcup}_\alpha a_\alpha$. Since $\{a_\alpha\}$ is residually independent by the condition (1) of the MD-lattice, $\{Ta_\alpha\}$ is also, and hence by Lemma 9.2 there exists a semi-orthogonal family $\{b_\alpha\}$ with the join b such that b_α and Ta_α are perspective. Since $(b_\alpha \cup Ta_\alpha) \cap a_\alpha \leqq b \cap a = 0$, a_α and b_α are perspective by Lemma 9.1 (i). Hence a and b are perspective by the condition (2) of the MD-lattice.

LEMMA 9.4. *In an MD-lattice, the projectivity satisfies the axioms* (A_1), $(\bar{\mathrm{A}}_2)$, $(\bar{\mathrm{B}})$, $(\mathrm{C}_\perp)$, *and the* $+$*-projectivity satisfies* (A_1), (A_2), $(\bar{\mathrm{B}})$, $(\mathrm{C}_\perp)$, (C_f).

PROOF. It was already shown that the projectivity satisfies (A_1), $(\bar{\mathrm{A}}_2)$ and $(\bar{\mathrm{B}})$. We shall show that $(\mathrm{C}_\perp)$ is satisfied. If $a=\dot{\bigcup}_\alpha a_\alpha$, $b=\dot{\bigcup}_\alpha b_\alpha$, $a_\alpha \sim b_\alpha$ and $a \perp b$, then a_α and b_α are perspective by Lemma 9.3, and hence a and b are perspective by the condition (2). Hence $a \sim b$. The second statement is implied from Theorem 8.1.

By this lemma, the arguments of §§ 4–6 are available for the $+$-projectivity. The arguments of § 7 and § 8 implies that

THEOREM 9.1. *If L is an MD-lattice and* "$\sim$" *is defined by the projectivity, then there exist dimension functions on L* (Definition 7.2) *with the properties stated in Theorems* 7.2—7.5.

Remark that Theorem 7.2 (ii) holds by the property "if $a \overset{+}{\sim} b$ and a is finite then $a\sim b$", which will be proved by Lemma 9.5 below. It follows from Theorem 8.3 that

THEOREM 9.2. *If an MD-lattice L is finite, i.e., L includes no infinite semi-orthogonal family of pairwise perspective non-zero elements, then L is upper-continuous. If L is finite of type I, then it is a continuous geometry of type I.*

COROLLARY. *(Kaplansky's theorem) Any orthocomplemented modular complete lattice is a continuous geometry.*

PROOF. It follows from Amemiya-Halperin [1], Appendix (p. 516) that any orthocomplemented modular complete lattice is finite. Hence, it is upper-continuous by the theorem, and is also lower-continuous by the duality.

REMARK 9.1. Let L be a finite MD-lattice. Since L is upper-continuous, any independent family in L is residually independent. Hence, it follows from Lemma 9.2, that L includes no infinte *independent* family of pairwise perspective non-zero elements.

In the case of the MD-lattice, we have the following type of comparability theorem.

THEOREM 9.3. *For any elements a, b of an MD-lattice, there exist decompositions $a=a'\dot{\cup}a''$, $b=b'\dot{\cup}b''$ such that a' and b' are perspective and that $e(a'')\cap e(b'')=0$.* (Remark that $Z_0=Z$.)

PROOF. By Theorem 8.2, there exist decompositions $a=(a\cap b)\dot{\cup}a'_1\dot{\cup}a''$, $b=(a\cap b)\dot{\cup}b'_1\dot{\cup}b''$ such that $a'_1\sim b'_1$, $e(a'')\cap e(b'')=0$. Since $a'_1\cap b'_1=0$, a'_1 and b'_1 are perspective by Lemma 9.3, and $\{a'_1, b'_1, a\cap b\}$ is an independent family since $(a'_1\cup(a\cap b))\cap b'_1\leqq a\cap b'_1=0$. Hence, putting $a'=(a\cap b)\dot{\cup}a'_1$, $b'=(a\cap b)\dot{\cup}b'_1$, a' and b' are perspective by Lemma 9.1 (ii).

This comparability theorem implies the following lemma.

LEMMA 9.5. *Let L be a relatively semi-orthocomplemented complete lattice where the semi-orthogonality satisfies ($\perp$5), and let "$\sim$" be an equivalence relation in L satisfying* (A_1), (A_2), ($\bar{B}$), ($C_\perp$) *and* (C_f).

(i) *If L is finite and $Z_0=Z$, then "$\sim$" coincides with the perspectivity.*

(ii) *Let "$\sim$" satisfy moreover the condition* (P′) *of Lemma* 3.3. *If $a\sim b$ and if a is finite, then a and b are perspective.*

PROOF. (i) Let L be finite and $Z_0=Z$. L is an upper-continuous complemented modular lattice by Theorem 8.3. Let $a\sim b$. It follows from Theorem 9.3 that there exist decompositions $a=a'\dot{\cup}a''$, $b=b'\dot{\cup}b''$ such that a' and b' are perspective and that $e(a'')\cap e(b'')=0$, since $Z_0=Z$. Since $e(a'')\cap b''=0$, we have $e(a'')\cap a\sim e(a'')\cap b=e(a'')\cap b'\sim e(a'')\cap a'$. By the finiteness, we have $e(a'')\cap a=e(a'')\cap a'$, which implies $a''=0$. Similarly we have $b''=0$, and hence a and b are perspective. The converse follows from ($\bar{B}$).

(ii) Let $a\sim b$ and a be finite. It follows from Theorem 6.1 that $a\cup b$ is finite. In $L'=L(0, a\cup b)$, "$\sim$" satisfies (A_1), (A_2), ($\overline{B}$), ($C_\perp$) and (C_f). It follows from (P′) that, in L', the relative center with respect to "$\sim$" coincides with the center. Hence a and b are perspective by (i).

Corollary. *Let "$\sim$" satisfy* (A_1), (A_2), ($\overline{B}$) *and* ($C_\perp$).

(i) *If L is finite and $Z_0=Z$, then "$\overset{+}{\sim}$", "$\sim$" and the perspectivity coincide.*

(ii) *Let "$\sim$" satisfy moreover* (P′). *If $a\overset{+}{\sim}b$ and if a is finite, then a and b are perspective* (hence $a\sim b$).

Returning to the MD-lattice, we get some conditions equivalent to the finiteness as follows.

Theorem 9.4. *Let L be an MD-lattice. The following four statements are equivalent.*

(α) *L is finite.*

(β) *The projectivity and the perspectivity coincide, in other words, the perspectivity is transitive.*

(γ) *The projectivity and the $+$-projectivity coincide, in other words, the projectivity is additive.*

(δ) *The projectivity is subtractive, i.e., if $a_1\dot{\cup}a_2\sim b_1\dot{\cup}b_2$ and $a_1\sim b_1$ then $a_2\sim b_2$.*

Proof. It follows from Corollary (i) of Lemma 9.5 that (α) impies (β) (and (γ)).

(β)$\Rightarrow$(δ). Since the projectivity "$\sim$" satisfies (A_2), it suffices to prove that $a_1\dot{\cup}a_2=b_1\dot{\cup}b_2=a$, $a_1\sim b_1$ imply $a_2\sim b_2$. It follows from (β) that a_1 and b_1 have a common complement c in $L(0, a)$. Hence $a_2\sim c\sim b_2$.

(δ)$\Rightarrow$(γ). Let $a_1\perp a_2$, $b_1\perp b_2$ and $a_i\sim b_i$ $(i=1,2)$. Putting $a_1\dot{\cup}a_2\dot{\cup}a'=b_1\dot{\cup}b_2\dot{\cup}b'=1$, it follows from ($\delta$) that $a_2\dot{\cup}a'\sim b_2\dot{\cup}b'$ and that $a'\sim b'$. Hence $a_1\dot{\cup}a_2\sim b_1\dot{\cup}b_2$ by (δ) again.

(γ)$\Rightarrow$(α). If $a\overset{+}{\sim}1$, then $a\sim 1$ by (γ), and then $a=1$ since 1 is the only element perspective to 1. Hence L is finite, by Lemma 4.6 (i).

Corollary. *If an upper-continuous complemented modular lattice is $\aleph_0$-lower-continuous, i.e., $a_i\cup b\downarrow a\cup b$ holds for any descending sequence $a_i\downarrow a$, then it is finite and the perspectivity, the projectivity and the $+$-projectivity coincide.*

Proof. The finiteness is proved in the same way as [9], Kap. IV, Satz 2.1. The last statement follows from the theorem.

Finally, we shall show an example given by Halperin [4], where the relative center does not generally coincide with the center.

Example 1.3. Let L be a continuous geometry. L is a finite MD-lattice where the semi-orthogonality is defined by the independence and where the theorem of the superposition of decompositions holds ([4], Theorem 3.1). Let G be a group of lattice-automorphisms of L. We denote $a \overset{G}{\sim} b$ if there exist decompositions $a=\dot{\bigvee}_\alpha a_\alpha$, $b=\dot{\bigvee}_\alpha b_\alpha$ such that, for every α, b_α is perspective to $T_\alpha a_\alpha$ for some $T_\alpha \in G$. Then, it follows from the theorem of the superposition of decompositions that "$\overset{G}{\sim}$" is an equivalence relation satisfying ($\bar{\mathrm{A}}_2$) ([4], Theorem 4.1). It is obvious that "$\overset{G}{\sim}$" satisfies (A_1), ($\bar{\mathrm{B}}$), ($\mathrm{C}_\perp$) and (C_f) (moreover it is completely additive obviously). It is easy to prove that the relative center with respect to "$\overset{G}{\sim}$" is equal to $\{z \in Z;\ Tz=z$ for every $T \in G\}$. Hence if L is irreducible under G then it has numerical dimension functions.

§ 10. Example 2. Baer rings and Baer *-rings

Let $\mathfrak{A}$ be a ring with unity. The set of all idempotents of $\mathfrak{A}$ is denoted by $I(\mathfrak{A})$, and the set of all principal right ideals $e\mathfrak{A}$ (denoted by $(e)_r$) generated by $e \in I(\mathfrak{A})$ is denoted by $\mathcal{R}_I(\mathfrak{A})$. If $\mathfrak{A}$ is a Baer ring, then $\mathcal{R}_I(\mathfrak{A})$ coincides with the set of all right annihilators and it forms a relatively semi-orthocomplemented complete lattice by [13], Theorem 4.

Two elements $e, f \in I(\mathfrak{A})$ are called to be *algebraically equivalent*, in notation $e \overset{a}{\sim} f$, if there exist $x, y \in \mathfrak{A}$ with $xy=e$, $yx=f$ (we may assume $x \in e\mathfrak{A}f$, $y \in f\mathfrak{A}e$). Since $(e_1)_r=(e_2)_r$ implies $e_1 \overset{a}{\sim} e_2$, we can define the algebraic equivalence in $\mathcal{R}_I(\mathfrak{A})$ as follows: $(e)_r \overset{a}{\sim} (f)_r$ if $e \overset{a}{\sim} f$.

It is easy to show that the relative center with respect to "$\overset{a}{\sim}$" coincides with the center Z of $\mathcal{R}_I(\mathfrak{A})$, since $(e)_r \in Z$ if and only if e is in the center of $\mathfrak{A}$ ([14], Theorem 2.1). "$\overset{a}{\sim}$" satisfies the axiom (A), since $(e)_r \overset{a}{\sim} (f)_r$ if and only if $(e)_r$ and $(f)_r$ are isomorphic right $\mathfrak{A}$-modules (Kaplansky [7], Chap. I, Lemma 1); and hence all the lemmas and theorem of § 2 are available. Especially, $\mathcal{R}_I(\mathfrak{A})$ can be decomposed into direct summands of five types by Theorem 2.1, and hence $\mathfrak{A}$ can be also.

$e \in I(\mathfrak{A})$ is called to be *abelian* (Kaplansky [7], Chap. I, Definition 4) if the idempotents of $e\mathfrak{A}e$ mutually commute. It is easy to show that $e \in I(\mathfrak{A})$ is abelian if and only if $(e)_r$ is a D-element, and hence it follows from Lemma 2.4 that if $(e)_r$ is minimal then e is abelian (see [10], Theorem 5.5). We know that the converse is valid if the condition (P′) in Lemma 3.3 is satisfied, in other words, if $\mathcal{R}_I(\mathfrak{A})$ is a Z_α-lattice. It is shown by [7], Chap. III, Exercise that if $\mathfrak{A}$ has no nilpotent ideals then $\mathcal{R}_I(\mathfrak{A})$ is a Z_α-lattice. Concerning this, we have the following remarks.

REMARK 10.1. (i) A ring $\mathfrak{A}$ with unity has no nilpotent ideals if and only if $x\mathfrak{A}x=0$ implies $x=0$. Proof. The "only if" part is obvious, since $x\mathfrak{A}x=0$ implies $(\mathfrak{A}x\mathfrak{A})^2=0$. Now, suppose that $x\mathfrak{A}x=0$ implies $x=0$. If $y \in \mathfrak{A}$ belongs to a nilpotent ideal, then $(\mathfrak{A}y\mathfrak{A})^{2^n}=0$ for some n. Since $(\mathfrak{A}y\mathfrak{A})^{2^{n-1}}\mathfrak{A}(\mathfrak{A}y\mathfrak{A})^{2^{n-1}}=(\mathfrak{A}y\mathfrak{A})^{2^n}$, we have $(\mathfrak{A}y\mathfrak{A})^{2^{n-1}}=0$ by the assumption, and repeating this, we have

$\mathfrak{A}y\mathfrak{A}=0$. Hence $\mathfrak{A}$ has no nilpotent ideals.

(ii) The center $\mathfrak{Z}$ of a Baer ring $\mathfrak{A}$ is also a Baer ring and the lattice $I(\mathfrak{Z})$ (isomorphic to $\mathcal{R}_I(\mathfrak{Z})$) is isomorphic to the center of $\mathcal{R}_I(\mathfrak{A})$ ([14], Theorem 2.1, Corollary 1). Since $I(\mathfrak{Z})$ is complete, for every $x\in\mathfrak{A}$ there is the smallest element $h\in I(\mathfrak{Z})$ such that $hx=x$. This element h is called the *central cover* of x (by Kaplansky [7]), and is denoted by $C(x)$. Remark that if $e\in I(\mathfrak{A})$ then $(C(e))_r=e((e)_r)$ in the sense of Definition 2.1.

The following eight statements are equivalent.

(α) $\mathfrak{A}$ has no nilpotent ideals.

(β) For any right ideal $\mathfrak{E}$, the right annihilator $(\mathfrak{E})^r$ of $\mathfrak{E}$ is a direct summund, i.e., there is $h\in I(\mathfrak{Z})$ such that $(h)_r=(\mathfrak{E})^r$.

(β') For any $e\in I(\mathfrak{A})$, the right annihilator $((e)_r)^r$ is a direct summund.

(γ) For $x, y\in\mathfrak{A}$, $x\mathfrak{A}y=0$ implies $y\mathfrak{A}x=0$.

(γ') For $e, f\in I(\mathfrak{A})$, $e\mathfrak{A}f=0$ implies $f\mathfrak{A}e=0$.

(δ) For $x, y\in\mathfrak{A}$, $x\mathfrak{A}y=0$ implies $C(x)C(y)=0$.

(δ') For $e, f\in I(\mathfrak{A})$, $e\mathfrak{A}f=0$ implies $C(e)C(f)=0$.

(ε) For $x, y\in\mathfrak{A}$, $x\mathfrak{A}y=y\mathfrak{A}x=0$ implies $C(x)C(y)=0$.

And, the following two statements are equivalent.

(ε') For $e, f\in I(\mathfrak{A})$, $e\mathfrak{A}f=f\mathfrak{A}e=0$ implies $C(e)C(f)=0$.

(ζ) If $e\in I(\mathfrak{A})$ and if f is a central idempotent of $e\mathfrak{A}e$ then there exists $h\in I(\mathfrak{Z})$ such that $f=he$. In other words, $\mathcal{R}_I(\mathfrak{A})$ is a Z_α-lattice.

Proof. First, we shall show that the six statements (β), (β'), (γ), (γ'), (δ), (δ') are equivalent. The implications (β)$\Rightarrow$(β'), (γ)$\Rightarrow$(γ'), (δ)$\Rightarrow$(δ') are trivial. We shall prove (β)$\Rightarrow$(δ). It follows from (β) that for $x\in\mathfrak{A}$ there exists $h\in I(\mathfrak{Z})$ such that $(h)_r=(x\mathfrak{A})^r$, and then it follows from $xh=0$ that $C(x)\leqq 1-h$. If $x\mathfrak{A}y=0$, then $y\in(h)_r$ and hence $C(y)\leqq h$. Therefore $C(x)C(y)=0$. The implication (β')$\Rightarrow$(δ') can be proved similarly. The implications (δ)$\Rightarrow$(γ) and (δ')$\Rightarrow$(γ') is obvious since $y\mathfrak{A}x=yC(y)\mathfrak{A}C(x)x=y\mathfrak{A}C(x)C(y)x$. We shall show that ($\gamma'$) $\Rightarrow$(β), which concludes the equivalence of the six statements. The right annihilator of a right ideal $\mathfrak{E}$ is of the form $(h)_r$, $h\in I(\mathfrak{A})$, since $\mathfrak{A}$ is a Baer ring. Since $\mathfrak{E}\mathfrak{A}h\subset\mathfrak{E}h=0$, we have $\mathfrak{A}h\subset(h)_r$, which implies $(1-h)\mathfrak{A}h=0$. It follows from (γ') that $h\mathfrak{A}(1-h)=0$. Hence $(1-h)xh=hx(1-h)=0$ for every $x\in\mathfrak{A}$, which implies $xh=hxh=hx$. Therefore $h\in\mathfrak{Z}$. Next, we shall prove the implications (δ)$\Rightarrow$(ε)$\Rightarrow$(α)$\Rightarrow$(γ). (δ)$\Rightarrow$(ε) is trivial. (ε) implies (α) by (i), since it follows from (ε) that $x\mathfrak{A}x=0$ implies $C(x)=0$ (hence $x=0$). (α) implies (γ), because if $x\mathfrak{A}y=0$ then $(\mathfrak{A}y\mathfrak{A}x\mathfrak{A})^2=0$, which implies $y\mathfrak{A}x=0$ by (α).

The equivalence of (ε') and (ζ) is proved as follows. (ε')$\Rightarrow$(ζ). If f is a central idempotent of $e\mathfrak{A}e$, then we have $(e-f)\mathfrak{A}f=(e-f)e\mathfrak{A}ef=(e-f)fe\mathfrak{A}e=0$ and similarly $f\mathfrak{A}(e-f)=0$. Since $e-f\in I(\mathfrak{A})$, it follows from (ε') that $C(f)(e-f)=0$, and hence $f=C(f)e$. (ζ)$\Rightarrow$(ε'). If $e\mathfrak{A}f=f\mathfrak{A}e=0$, then, putting $g=e+f$, we have $g\in I(\mathfrak{A})$ and $gxg=exe+fxf$ for every $x\in\mathfrak{A}$. Hence, $egxg=exe=gxge$, which means that e is a central idempotent of $g\mathfrak{A}g$. By (ζ), there exists $h\in I(\mathfrak{Z})$ such that $e=hg$. It follows from $he=e$ that $C(e)\leqq h$, and it follows from $hf=hg-he=0$ that $C(f)\leqq 1-h$. Therefore $C(e)C(f)=0$. This completes the proof.

From the trivial implication $(\varepsilon)\Rightarrow(\varepsilon')$ it follows that if a Baer ring $\mathfrak{A}$ has no nilpotent ideals then $\mathcal{R}_I(\mathfrak{A})$ is a Z_α-lattice and then "$(e)_r$ is minimal $\Leftrightarrow e$ is abelian".

It is obvious that the algebraic equivalence "$\overset{a}{\sim}$" in $\mathcal{R}_I(\mathfrak{A})$ satisfies the axiom (C_f), but "$\overset{a}{\sim}$" satisfies neither (B) nor $(C_\perp)$ in general. We shall give an equivalent condition to that "$\overset{a}{\sim}$" satisfies $(\overline{\mathrm{B}})$. The set of right (resp. left) idempotents of $x \in \mathfrak{A}$ is denoted by $RI(x)$ (resp. $LI(x)$), i.e., $RI(x)=\{e \in I(\mathfrak{A});\, (e)^r=(x)^r\}$, $LI(x)=\{e \in I(\mathfrak{A});\, (e)^l=(x)^l\}$ (see [14], § 4). Since $e_1, e_2 \in RI(x)$ (or $e_1, e_2 \in LI(x)$) implies $e_1 \overset{a}{\sim} e_2$, we shall write $RI(x) \overset{a}{\sim} LI(x)$ if there are $e \in RI(x)$, $f \in LI(x)$ with $e \overset{a}{\sim} f$. It follows from the proof of [14], Lemma 4.4 (i) that $RI(x) \overset{a}{\sim} LI(x)$ if and only if there exists a relatively regular element $u \in \mathfrak{A}$ such that $(x)^r=(u)^r$, $(x)^l=(u)^l$. (Especially, if x is relatively regular then $RI(x) \overset{a}{\sim} LI(x)$.)

LEMMA 10.1. *Let $\mathfrak{A}$ be a Baer ring. The algebraic equivalence "$\overset{a}{\sim}$" in $\mathcal{R}_I(\mathfrak{A})$ satisfies the axiom $(\overline{\mathrm{B}})$ if and only if $\mathfrak{A}$ satisfies the following condition*:
(B^a) $RI(x) \overset{a}{\sim} LI(x)$ *for every* $x \in I^2(\mathfrak{A})$, *where* $I^2(\mathfrak{A})=\{ef;\, e, f \in I(\mathfrak{A})\}$.

PROOF. Let "$\overset{a}{\sim}$" satisfy $(\overline{\mathrm{B}})$. If $x \in I^2(\mathfrak{A})$, $e \in RI(x)$, $f \in LI(x)$, then it follows from [14], Theorem 4.1 (i) that $(e)_r$ and $(f)_r$ are perspective in $\mathcal{R}_I(\mathfrak{A})$, whence $e \overset{a}{\sim} f$ by $(\overline{\mathrm{B}})$. Hence (B^a) holds. Conversely, let $\mathfrak{A}$ satisfy (B^a). If $(e)_r$ and $(f)_r$ are perspective, then it follows from [14], Theorem 4.1 (ii) that there are $x, y \in I^2(\mathfrak{A})$ such that $e \in RI(x)$, $f \in RI(y)$, $(x)^l=(y)^l$. We have $RI(x) \overset{a}{\sim} LI(x) = LI(y) \overset{a}{\sim} RI(y)$ by (B^a), which implies $(e)_r \overset{a}{\sim} (f)_r$.

REMARK 10.2. (i) It follows from [14], Lemma 4.1 that "$\overset{a}{\sim}$" in $\mathcal{R}_I(\mathfrak{A})$ satisfies (B') if and only if $\mathfrak{A}$ satisfies the following condition: $RI(x) \overset{a}{\gtrsim} LI(x)$ for every $x \in I^2(\mathfrak{A})$.

(ii) In [12] and [14], Baer rings (or Rickart rings) satisfying the following condition (stronger than (B^a)) are treated.

$$(\hat{\mathrm{B}}^a)\ RI(x) \overset{a}{\sim} LI(x) \textit{ for every } x \in \mathfrak{A}.$$

If a Baer ring $\mathfrak{A}$ satifies $(\hat{\mathrm{B}}^a)$, then it has no nilpotent ideals. Because, if $x\mathfrak{A}x=0$, then, since there is a relatively regular element u with $(u)^r=(x)^r$, $(u)^l=(x)^l$ by $(\hat{\mathrm{B}}^a)$, we have $u\mathfrak{A}u=0$, which implies $u=0$ by the relative regularity of u, and hence $x=0$.

(iii) By [14], Lemma 4.3, "$\overset{a}{\sim}$" in $\mathcal{R}_I(\mathfrak{A})$ has the following property: If $(e)_r \perp (f)_r$ and $(e)_r \overset{a}{\sim} (f)_r$, then $(e)_r$ and $(f)_r$ are perspective. It follows from Lemma 3.3 (i) that if $\mathfrak{A}$ satisfies (B^a), then $\mathcal{R}_I(\mathfrak{A})$ is a Z_α-lattice, because the above property implies (P).

(iv) In order that $\mathcal{R}_I(\mathfrak{A})$ have dimension functions with respect to "$\overset{a}{\sim}$", it suffices that the following three statements hold: (1) $\mathfrak{A}$ satisfies (B^a), (2) the semi-orthogonality in $\mathcal{R}_I(\mathfrak{A})$ satisfies $(\perp 5)$, (3) "$\overset{a}{\sim}$" satisfies the axiom $(C_\perp)$.

Example 2.1. (Complete regular ring, upper-continuous regular ring) If $\mathfrak{A}$ is a regular ring with unity, then $\mathcal{R}_I(\mathfrak{A})$ coincides with the set of all principal right ideals and it forms a complemented modular lattice. A regular ring $\mathfrak{A}$ with unity is called a *complete* (resp. *upper-continuous*, *continuous*) *regular ring* if the lattice $\mathcal{R}_I(\mathfrak{A})$ is complete (resp. upper-continuous, continuous). Let $\mathfrak{A}$ be a complete regular ring. It follows from [14], Lemma 1.3 that $\mathfrak{A}$ is a Baer ring and it follows from the last remark of [13] that the semi-orthogonality in $\mathcal{R}_I(\mathfrak{A})$ coincides with the independence (see Definition 9.1). Since any regular ring satisfies $(\hat{B}^a)$, the algebraic equivalence "$\overset{a}{\sim}$" in $\mathcal{R}_I(\mathfrak{A})$ satisfies the axioms (A) and $(\bar{B})$. On the other hand, the projectivity in $\mathcal{R}_I(\mathfrak{A})$ (denoted by "$\sim$") also satisfies (A) and $(\bar{B})$ since $\mathcal{R}_I(\mathfrak{A})$ is modular (see § 9). Obviously, $(e)_r \sim (f)_r$ implies $(e)_r \overset{a}{\sim} (f)_r$, and the converse is true if $(e)_r \perp (f)_r$ (Remark 10.2 (ii)). Hence, the minimal (resp. finite, properly infinite) elements are the same for both equivalences.

Let $\mathfrak{A}$ be an upper-continuous regular ring. Since the semi-orthogonality in $\mathcal{R}_I(\mathfrak{A})$ coincides with the independence, it satisfies $(\perp 5)$ by the upper-continuity of $\mathcal{R}_I(\mathfrak{A})$. "$\overset{a}{\sim}$" satisfies $(C_\perp)$ since the projectivity satisfies $(C_\perp)$ by Lemma 9.4. Hence, $\mathcal{R}_I(\mathfrak{A})$ has dimension functions with respect to "$\overset{a}{\sim}$". We shall show later that "$\overset{a}{\sim}$" coincides with the $+$-projectivity.

Let $\mathfrak{A}$ be a Baer *-ring. Then, the set $P(\mathfrak{A})$ of all projections of $\mathfrak{A}$ forms a lattice isomorphic to $\mathcal{R}_I(\mathfrak{A})$. Furthermore, $P(\mathfrak{A})$ has an orthogonal relation and is a relatively orthocomplemented complete lattice (see [14], § 6). As the canonical semi-orthogonality in $P(\mathfrak{A})$, let us take this orthogonality ($P(\mathfrak{A})$ has another semi-orthogonality, induced from $\mathcal{R}_I(\mathfrak{A})$). Then, it is not valid in general that the algebraic equivalence "$\overset{a}{\sim}$" satisfies (A). But, it satisfies (A_1) obviously and satisfies $(\bar{A}_2)$ by [12], Lemma 2.3, and hence all the lemmas and theorem of § 2 are available. In $\mathfrak{A}$, $xx^*=0$ implies $x=0$ by [7], Chap. III, Proposition 2. From this property and Remark 9.1 (i) it follows that $\mathfrak{A}$ has no nilpotent ideals ($x\mathfrak{A}x=0$ implies $xx^*xx^*=0$, whence $xx^*=0$ and $x=0$). Hence, $P(\mathfrak{A})$ is a Z_α-lattice and $e \in P(\mathfrak{A})$ is minimal if and only if e is abelian. ($e \in P(\mathfrak{A})$ is abelian if and only if the projections of $e\mathfrak{A}e$ mutually commute. See [7], Chap. III.) The semi-orthogonality in $P(\mathfrak{A})$ satisfies $(\perp 5)$ since it is defined by the orthogonality. It follows from [12], Lemma 2.8 that if "$\overset{a}{\sim}$" satisfies $(\bar{B})$ then it satisfies $(C_\perp)$ also. Hence, in order that $P(\mathfrak{A})$ have dimension functions with respect to "$\overset{a}{\sim}$" it suffices that $\mathfrak{A}$ satisfies (B^a) ([12], Theorem 2.1 (i)).

Example 2.2. (Complete *-regular ring) A complete *-regular ring $\mathfrak{A}$ (Kaplansky [6]) is a regular Baer *-ring, and vice versa. Since $\mathfrak{A}$ is regular, $(\hat{B}^a)$ is satisfied. Hence, in $P(\mathfrak{A})$, the algebraic equivalence "$\overset{a}{\sim}$" satisfies (A_1),

$(\bar{A}_2)$, $(\bar{B})$, $(C_\perp)$, (C_f), and $P(\mathfrak{A})$ has dimension functions with respect to "$\overset{a}{\sim}$". Since $P(\mathfrak{A})$ is finite by [6], Theorem 1, $P(\mathfrak{A})$ is upper-continuous and modular by Theorem 8.3, and it is lower-continuous by the duality. Therefore $P(\mathfrak{A})$ is a continuous geometry ([6], Theorem 3), and it follows from Lemma 9.5 (i) that "$\overset{a}{\sim}$" coincides with the perspectivity.

Let $\mathfrak{A}$ be a Baer *-ring. In $P(\mathfrak{A})$, we can define another equivalence relation as follows: $e, f \in P(\mathfrak{A})$ are called to be **-equivalent*, in notation $e \overset{*}{\sim} f$, if there exists $w \in \mathfrak{A}$ with $ww^* = e$, $w^*w = f$. An element $w \in \mathfrak{A}$ is called a *partial isometry* (Berberian [2], p. 500) if $ww^* \in P(\mathfrak{A})$ (or equivalently $w^*w \in P(\mathfrak{A})$). Any partial isometry w is relatively regular since $ww^*w = w$. The right (resp. left) projection of $x \in \mathfrak{A}$ is denoted by $RP(x)$ (resp. $LP(x)$). It is easy to show that $RP(x) \overset{*}{\sim} LP(x)$ if and only if there exists a partial isometry w such that $(x)^r = (w)^r$, $(x)^l = (w)^l$. It is obvious that the relative center with respect to "$\overset{*}{\sim}$" coincides with the center of $P(\mathfrak{A})$ and that "$\overset{*}{\sim}$" satisfies (A) and (C_f). It follows from [12], Lemma 2.7 that "$\overset{*}{\sim}$" satisfies $(C_\perp)$. Hence, in order that $P(\mathfrak{A})$ have dimension functions with respect to "$\overset{*}{\sim}$" it suffices that "$\overset{*}{\sim}$" satisfies (B).

LEMMA 10.2. *Let $\mathfrak{A}$ be a Baer *-ring. The following statements are equivalent.*

(α) *The *-equivalence "$\overset{*}{\sim}$" in $P(\mathfrak{A})$ satisfies* $(\bar{B})$.

(β) "$\overset{*}{\sim}$" *satisfies* (B).

(γ) "$\overset{*}{\sim}$" *satisfies* (B′).

(B*) $RP(x) \overset{*}{\sim} LP(x)$ *for every* $x \in P^2(\mathfrak{A})$, *where* $P^2(\mathfrak{A}) = \{ef;\ e, f \in P(\mathfrak{A})\}$.

PROOF. The equivalence of (α) and (B*) can be proved in the similar way as Lemma 10.1, by the aid of [14], Theorem 6.3. The implications (α)$\Rightarrow$(β) $\Rightarrow$(γ) are trivial. We can show that (B*) is equivalent to the condition: $RP(x) \overset{*}{\gtrsim} LP(x)$ for every $x \in P^2(\mathfrak{A})$; because, it follows from $RP(x) = LP(x^*)$ that the last condition implies $RP(x) \overset{*}{\lesssim} LP(x)$ for every $x \in P^2(\mathfrak{A})$, and hence implies (B*) by Lemma 4.5. Hence (B*) and (γ) are equivalent by [14], Lemma 6.4.

REMARK 10.3. In [12], Baer *-rings satisfying the following condition (stronger than (B*)) are treated.

$$(\hat{B}^*)\ RP(x) \overset{*}{\sim} LP(x) \text{ for every } x \in \mathfrak{A}.$$

This condition implies that "$\overset{*}{\sim}$" and "$\overset{a}{\sim}$" coincide ([12], Remark 2.2 (ii)). We shall show later that (B*) implies the same property.

EXAMPLE 2.3. (Von Neumann algebra, AW*-algebra) Let $\mathfrak{A}$ be a von Neumann algebra (=W*-algebra). It is obvious that $\mathfrak{A}$ is a Baer *-ring. By the polar decomposition theorem ([3], Appendice III), any element $x \in \mathfrak{A}$ can be written by the form $x = wr$, $w, r \in \mathfrak{A}$, where w is a partial isometry and $r^* = r$, $r^2 = x^*x$. From this fact it follows that $\mathfrak{A}$ satisfies $(\hat{B}^*)$ (see [2], Lemma 3.3).

But, it is difficult to prove that any AW*-algebra (=C*-algebra which is a Baer *-ring) satisfies ($\hat{\mathrm{B}}^*$). The outline of the proof given by Kaplansky [5] is as follows. It is easy to show that "$\overset{*}{\sim}$" satisfies (B″) ([5], Lemma 3.3), and hence it can be proved by Remark 4.2 (iii) that "$\overset{*}{\sim}$" is countably additive. It can be proved by this result that ($\hat{\mathrm{B}}^*$) is satisfied ([5], Theorem 5.2). Hence, if $\mathfrak{A}$ is an AW*-algebra, then the *-equivalence in $P(\mathfrak{A})$ satisfies the axioms (A), ($\bar{\mathrm{B}}$), ($\mathrm{C}_\perp$), (C_f), and $P(\mathfrak{A})$ has dimension functions with respect to "$\overset{*}{\sim}$". If $\mathfrak{A}$ is a finite AW*-algebra (which means that $P(\mathfrak{A})$ is finite), then, as in Example 2.2, it can be proved that $P(\mathfrak{A})$ is a continuous geometry and that "$\overset{*}{\sim}$" coincides with the perspectivity.

Finally, we shall give a lattice-theoretic characterization of the algebraic equivalences or the *-equivalences in these examples.

Definition 10.1. In a relatively complemented lattice, two elements a and b are called to be *semi-perspective* if there exist four elements a_1, a_2, b_1, b_2 such that $a=a_1\cup a_2$, $b=b_1\cup b_2$, $a_1\cap a_2=b_1\cap b_2=0$ and that a_i and b_i are perspective ($i=1,2$). a and b are called to be *semi-projective* if there exists a finite sequence $(a_0, a_1,\cdots, a_n)$ such that $a_0=a$, $a_n=b$ and that a_{i-1} and a_i are semi-perspective ($1\leqq i\leqq n$). The semi-projectivity is an equivalence relation weaker than the projectivity.

Theorem 10.1. *Let L be a relatively semi-orthocomplemented complete lattice where the semi-orthogonality satisfies ($\perp 5$). Let "$\sim$" be an equivalence relation in L satisfying the axioms* (A_1), (A_2), ($\bar{\mathrm{B}}$), ($\mathrm{C}_\perp$) *and* (C_f). *If "$\sim$" moreover satisfies the following condition: $a\sim b$, $a\perp b$ imply that a and b are perspective, then "$\sim$" coincides with the semi-projectivity.*

Proof. It is easily proved by ($\bar{\mathrm{B}}$) and (C_f) that if a and b are semi-projective then $a\sim b$. To prove the converse, supposing $a\sim b$, it suffices to show that a and b are semi-projective when a is either finite or properly infinite (Lemma 2.6). If a is finite, then, since the last condition in the theorem implies (P′) by Lemma 3.3, a and b are perspective by Lemma 9.5 (ii). Let a be properly infinite. By Lemma 4.7, there are a_1, a_2 such that $a=a_1\dot{\cup} a_2\sim a_1\sim a_2$. Putting $a\cup b=a\dot{\cup} b_1$, we have $b_1\lesssim b\sim a\sim a_2$, whence $a_1\dot{\cup} b_1\lesssim a_1\dot{\cup} a_2\sim a_1$. Hence, by Lemma 4.5, we have $a_1\dot{\cup} b_1\sim a_1\sim a_2$. Since $a_1\dot{\cup} b_1\perp a_2$, $a_1\dot{\cup} b_1$ and a_2 are perspective and so are a_2 and a_1. Hence $a\cup b$ and a are semi-perspective. Similarly $a\cup b$ and b are semi-perspective. Therefore, a and b are semi-projective.

Corollary 1. *In an MD-lattice, the $+$-projectivity and the semi-projectivity coincide.* (Use Lemmas 9.3 and 9.4.)

Corollary 2. *If $\mathfrak{A}$ is an upper-continuous regular ring, then, in $\mathcal{R}_I(\mathfrak{A})$, the algebraic equivalence, the $+$-projectivity and the semi-projectivity coincide.*

(See Remark 10.2 (iii) and Example 2.1)

COROLLARY 3. *In an upper-continuous complemented modular lattice, the $+$-projectivity ($=$semi-projectivity) satisfies the axiom* (A) *and is completely additive.*

PROOF. It suffices to prove the statement when the lattice L is either finite or properly infinite. If L is finite, the statement follows from Theorems 9.4 and 5.3. Let L be properly infinite. Since 1 is the join of a 4-homogeneous family by Lemma 4.7, there is a regular ring $\mathfrak{A}$ such that the lattice $\mathcal{R}_I(\mathfrak{A})$ is isomorphic to L (von Neumann [15], Theorem 14.1; [9], Kap. XI, Theorem 3.2). Then, $+$-projectivity coincides with the algebraic equivalence in $\mathcal{R}_I(\mathfrak{A})$ by Corollary 2, and hence it satisfies (A) and is completely additive by the corollary of Theorem 5.3.

COROLLARY 4. *If $\mathfrak{A}$ is a Baer *-ring satisfying* (B^a), *then, in $P(\mathfrak{A})$, the algebraic equivalence and the semi-projectivity coincide.*

It is easy to show that if a Baer *-ring satisfies (B^*) then it satisfies also (B^a). Hence we have the following result.

COROLLARY 5. *If $\mathfrak{A}$ is a Baer *-ring satisfying* (B^*) (especially $\mathfrak{A}$ is an AW*-algebra), *then in $P(\mathfrak{A})$, the *-equivalence, the algebraic equivalence and the semi-projectivity coincide.*

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