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On FC-solvable Groups

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In this note, we shall show that some properties which hold true in finite groups and solvable groups satisfying the maximal condition for subgroups (say S-groups, after K. A. Hirsch) may be available for FC-solvable groups satisfying the maximal condition for subgroups, and that for FC-solvable groups the problem concerning the finiteness of finitely generated and torsion groups holds affirmative. The proofs are based on a method of K. A. Hirsch which is used in the case of S-groups [2; 3], that is, a method which reduces the problem concerned to the case of some finite factor groups.

A finite system of subgroups of a group G,

$$E=G_0\subset G_1\subset G_2\subset\cdots\quad G_k=G$$

beginning with the unit subgroup and ending in G itself, is called an *FC-solvable series* of G if every subgroup G_i is a proper normal subgroup of G_{i+1} and every factor group G_{i+1}/G_i is an FC-group; $i=0,1,\cdots,k-1$. An FC-solvable series in which every subgroup G_i is a normal subgroup of G is called an *FC-solvable normal series* of G. And a group G is called *FC-solvable* if it has an FC-solvable series. After P. Hall, we denote simply the maximal condition for subgroups by Max.

Now, we proceed to prove the properties.

Proposition 1. *The FC-solvable groups satisfying Max are the finite extensions of torsion-free solvable groups.*

Proof. Let G be an FC-solvable group satisfying Max. It is sufficient to prove only in the case where the group G is infinite. From the result of A. M. Duguid and D. H. McLain ([1] Theorem 3), we can see that G has an FC-solvable normal series. Let $G_0\subset G_1\subset\cdots\subset G_k$ be an FC-solvable normal series of G. Among the factor groups $G_i/G_{i-1}(i=1,2,\cdots,k)$, at least one factor group is infinite. Let l be the smallest of the suffix i of infinite factor groups G_i/G_{i-1}. Then the factor group G_l/G_{l-1} is an infinite FC-group and the group G_{l-1} is finite. Therefore the group G_l is an infinite FC-group. Moreover, G_l is finitely generated, so its center has a finite index in it. Therefore, in the group G_l there exists a torsion-free abelian subgroup A which is normal in G. Let N be a maximal normal subgroup of G which is torsion-free and solvable and contains A. Then the factor group G/N is finite. For, suppose that the group G/N is infinite. The group G/N is also FC-solvable group satisfying Max, so G/N has a torsion-free, solvable and normal subgroup for the same reason as the above. Therefore G has a torsion-free normal subgroup which is solvable and contains N. This is a contradiction. Thus the proof is complete.

NOTE. This proposition is a more precise form than the theorem of A. M. Duguid and D. H. McLain ([1] Theorem 3).

LEMMA. *Let G be an FC-solvable group satisfying Max. If G is not nilpotent, then it has a normal subgroup of finite index whose factor group is not nilpotent.*

PROOF. By the above proposition, G is a finite extension of a solvable normal subgroup N. If G/N is not nilpotent, then the lemma holds true. If G/N is nilpotent, then G is a solvable group satisfying Max. Therefore, by the theorem K. A. Hirsch ([2] Theorem 3·24), G has a normal subgroup of finite index, whose factor group is not nilpotent. This completes the proof.

THEOREM 1. *Let G be an FC-solvable group satisfying Max. If all maximal subgroups are normal, then G is nilpotent.*

PROOF. The property that every maximal subgroup of G is noraml in G holds true in any factor group of G. Therefore, all finite factor groups of G are nilpotent. On the other hand, if G is not nilpotent, then by the above lemma, G has a finite factor group which is not nilpotent. This is a contradiction. Therefore G is nilpotent.

By the same process as the above, we can prove the following:

THEOREM 2. *Let G be an FC-solvable group satisfying Max. Let $\Phi(G)$ and $D(G)$ be the Frattini subgroup and the commutator subgroup of G respectively. If $\Phi(G)$ contains $D(G)$, then G is nilpotent.*

Further, by using the results of A. M. Duguid and D. H. McLain ([1] p. 395) and K. A. Hirsch ([3] p. 251), in the above lemma we can assume the normal subgroup of finite index to be characteristic in the group G. So, we can also prove the following:

PROPOSITION 2. *Let G be an FC-solvable group satisfying Max. The Frattini subgroup of G is nilpotent.*

REMARK. Finally we note another property of FC-solvable group. By using the result of B. H. Neumann ([5] p. 184) repeatedly, we can easily prove that finitely generated, torsion and FC-solvable groups are finite.

References

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