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Note on Formal Lie Groups (II)

Shigeaki Tôgô

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1. Let K be an algebraically closed field. For any algebraic subgroup of the general linear group $GL(n, K)$ we can associate a formal Lie group, and for any subgroup G of the formal Lie group $GL^*(n, K)$ associated with $GL(n, K)$ the algebraic hull $\mathfrak{A}(G)$ can be defined in $GL(n, K)$. On the base of such connection with algebraic linear groups, non-commutative formal Lie groups were investigated in [3] by making use of the properties of algebraic linear groups in [1]. In [4], we settled some questions raised in [3] on maximal solvable subgroups, maximal tori etc. of a subgroup of $GL^*(n, K)$.

The purpose of this note is to show some properties of formal Lie groups which follow from the results in [4].

The following theorem was proved by J. Dieudonné in [3]: In order that a formal Lie group G over an algebraically closed field K of characteristic $p>0$ be nilpotent, it is necessary and sufficient that it contain a unique maximal torus. We shall give another condition for G to be nilpotent and give another proof of the sufficiency part of the theorem by using [4, Th. 2], which allows us to make use of the corresponding theorem of algebraic linear groups. We shall also show some properties of maximal unipotent subgroups of a subgroup G of $GL^*(n, K)$. E.g., if a maximal torus and a maximal unipotent subgroup are associated with algebraic subgroups of $GL(n, K)$, then so is G.

2. We shall recall some definitions, results and notations on formal Lie groups in [3, Chap. III]. We denote by H^* the formal Lie group associated with an algebraic subgroup H of $GL(n, K)$. Let f be a rational homomorphism of H into an algebraic linear group H_1. Then there exists a corresponding homomorphism $\boldsymbol{f}$ of H^* into H_1^* and $\boldsymbol{f}(H^*)=f(H)^*$. If N is the kernel of f, then N^* is the kernel of $\boldsymbol{f}$. Given an element s of $GL(n, K)$, we denote by $\boldsymbol{a}_s$ the automorphism of $GL^*(n, K)$ corresponding to the inner automorphism of $GL(n, K)$ induced by s. If H is connected, then H^* is solvable (resp. nilpotent, commutative) if and only if H is solvable (resp. nilpotent, commutative). A formal Lie group over K is called representable provided it is isogenous to a subgroup of the formal Lie group $GL^*(n, K)$. The quotient group of a formal Lie group by its center is always representable. For a subgroup G of $GL^*(n, K)$, the algebraic hull $\mathfrak{A}(G)$ is solvable (resp. nilpotent, commutative) if and only if G is solvable (resp. nilpotent, commutative). $\mathfrak{A}(G)^*$ is denoted by $\mathfrak{A}^*(G)$. It is known that $DG=\mathfrak{A}^*(DG)=D(\mathfrak{A}(G))^*$. For a connected algebraic linear group H, we have $\mathfrak{A}(H^*)=H$. The subgroups of any formal Lie group form

a complete lattice. For its subgroups G_1 and G_2, we denote by $G_1 \wedge G_2$, $G_1 \vee G_2$ the g. l. b. and the l. u. b. of G_1 and G_2. If, for connected algebraic subgroups H_1 and H_2 of $GL(n, K)$, we denote by $H_1 \vee H_2$ the smallest algebraic subgroup of $GL(n, K)$ containing H_1 and H_2, then we have $(H_1 \vee H_2)^* = H_1^* \vee H_2^*$.

3. We first write the following results in [4], on which we essentially depend in developing our theorems.

Let K be an algebraically closed field and let G be a subgroup of $GL^*(n, K)$. Then:

(A) If S_1 and S_2 are maximal solvable subgroups (resp. maximal tori, Cartan subgroups) of G, then there exists an element s of $\mathfrak{A}(DG)$ such that $\boldsymbol{a}_s(S_1) = S_2$.

(B) The algebraic hull of any maximal solvable subgroup (resp. any maximal torus, any Cartan subgroup, the radical) of G is a maximal solvable connected subgroup (resp. a maximal torus, a Cartan subgroup, the radical) of $\mathfrak{A}(G)$ and conversely.

(C) G is associated with an algebraic subgroup of $GL(n, K)$ if and only if so is a maximal solvable subgroup (resp. a Cartan subgroup, the radical).

These results were proved by the author in [4, Th. 1, Th. 2 and Cor. 2, Th. 4 and Cor. 1], where G should obviously be a subgroup of $GL^*(n, K)$ as above although it was assumed to be a representable formal Lie group.

THEOREM 1. *Let G be a formal Lie group over an algebraically closed field K of characteristic $p>0$. Then G is nilpotent*

(1) *if and only if it has a unique maximal torus;*

(2) *if and only if a maximal solvable subgroup is nilpotent.*

The first statement is a theorem of J. Dieudonné [3, Th. 6]. We here give another proof of "if" part by using the result (B), which allows us to make use of the corresponding result of algebraic linear groups [2, Exposé 6, Cor. 2 to Th. 4]. Suppose that G has a unique maximal torus T. Put $G' = G/Z(G)$, where $Z(G)$ is the center of G. If $\boldsymbol{f}$ is the natural epimorphism of G onto G', then any maximal torus of G' is the image of a maximal torus of G by $\boldsymbol{f}$ [3, p. 379]. Therefore $\boldsymbol{f}(T)$ is the unique maximal torus T' of G'. Since G' is representable and since maximal tori are preserved by an isogeny, we may suppose that G' is a subgroup of $GL^*(n, K)$. Then, by (B) for maximal tori, we see that $\mathfrak{A}(T')$ is the unique maximal torus of $\mathfrak{A}(G')$. Therefore it follows that $\mathfrak{A}(G')$ is niloptent. Hence G' is nilpotent and therefore G is nilpotent.

To prove the second statement, suppose that a maximal solvable subgroup R of G is nilpotent. Put $R' = \boldsymbol{f}(R)$. Then it is easy to see that R' is a maximal solvable subgroup of G'. We may suppose that G' is a subgroup of $GL^*(n, K)$. Then, by virtue of (B), $\mathfrak{A}(R')$ is a maximal solvable connected subgroup of $\mathfrak{A}(G')$. Since $\mathfrak{A}(R')$ is nilpotent, it follows from the result of algebraic linear groups corresponding to (2) [2, Exposé 6, Cor. 2 to Th. 4] that $\mathfrak{A}(G')$ is nilpotent. Hence G' and therefore G is nilpotent.

THEOREM 2. *Let K be an algebraically closed field of characteristic $p>0$ and let G be a subgroup of $GL^*(n, K)$. Then:*

(1) *If H_1 and H_2 are maximal unipotent subgroups of G, then there exists an element s of $\mathfrak{A}(DG)$ such that $\boldsymbol{a}_s(H_1)=H_2$.*

(2) *The algebraic hull of any maximal unipotent subgroup H of G is a maximal unipotent subgroup of $\mathfrak{A}(G)$ and conversely. And we have $H=G\wedge\mathfrak{A}^*(H)$.*

(3) *G is associated with an algebraic subgroup of $GL(n, K)$ if and only if a maximal torus and a maximal unipotent subgroup are associated with algebraic subgroups of $GL(n, K)$.*

Let R be a maximal solvable subgroup of G. Then it is known that R has a largest unipotent subgroup R_u, which is normal in R and $R=T\vee R_u$ for any maximal torus T of R [3, Prop. 38]. If H is a unipotent subgroup of G containing R_u, take a maximal solvable subgroup R' of G containing H. Then $R_u \subset H \subset R'_u$. By (A) there exists an element s of $\mathfrak{A}(DG)$ such that $\boldsymbol{a}_s(R)=R'$, whence $\boldsymbol{a}_s(R_u)=R'_u$. Hence R_u and R'_u have the same dimension and therefore $R_u=H=R'_u$. Thus R_u is a maximal unipotent subgroup of G. The converse is easy and we have the following statement:

(α) Any maximal unipotent subgroup of G is the largest unipotent subgroup of a maximal solvable subgroup of G and conversely.

We can similarly prove the corresponding result for maximal unipotent subgroups of a connected algebraic subgroup of $GL(n, K)$, which we denote by (α').

Further we know the following fact [3, Cor. to Prop. 38]:

(β) If G is a solvable subgroup of $GL^*(n, K)$, then $\mathfrak{A}(G_u)$ is the largest unipotent subgroup of $\mathfrak{A}(G)$.

Now we have all the statements of the theorem as follows. (1) is immediate from (α) and the conjugation theorem (A) for maximal solvable subgroups. The first part of (2) follows from (α), (α'), (β) and (B) for maximal solvable subgroups. The second part of (2) is immediate from the first part. As for (3), let R be a maximal solvable subgroup of G. Then $R=T\vee R_u$ with T a maximal torus of R. If T and R_u are associated with algebraic linear groups, then we have

$$R=T\vee R_u=\mathfrak{A}^*(T)\vee\mathfrak{A}^*(R_u)=(\mathfrak{A}(T)\,\mathfrak{A}(R_u))^*,$$

whence R is associated with an algebraic linear group. Since T is a maximal torus of G [3, Prop. 34], (3) now follows immediately from (1), (α), the conjugation theorem (A) for maximal tori and (C) for maximal solvable subgroups.

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Department of Mathematics,
Faculty of Science,
Hiroshima University