J. Sci. Hiroshima Univ. Ser. A-I
25 (1961), 135-352

*On Potentials in Locally Compact Spaces.**)

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(Received September 18, 1961)

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*) The research in this paper was partially supported by National Science Foundation Grant NSF G-8154.

Introduction

In early 1940's Kametani ([1; 2; 3]) became interested in the fundamental work of Frostman [1] and initiated an attempt to study potentials with more general kernels. The several mathematicians in Japan like Ugaheri, Kunugui and Ninomiya joined him under the isolated circumstances from other countries, which were caused by the war. They tried to find general kernels which retain almost all properties of Newtonian potentials.

On the other hand, potentials were independently and vigorously investigated in France during the war, particularly by Brelot and H. Cartan, and the study of Newtonian potentials culminated in the works [5; 6] by H. Cartan. Some attempts to discuss general kernels, seen, for instance, in H. Cartan [4], flourished after the war in Deny [1] who treated distributions. A detailed story of the development is found in Brelot [1].

These people began to contact each other around 1950 and some of them published papers with the intention of seeking relations among the energy principle, the maximum principles, the existence of equilibrium measure and the possibility of sweeping-out process. We mention the works of H. Cartan and Deny [1] and Ninomiya [4; 5; 6] in this connection. However, it was not very far before 1955 that people started to seek a possible full generality in the theory of potentials.

In 1952 the present author began to be interested in capacity of product sets ([3]) and needed some results on potentials in a locally compact metric space. This led him to the study of potentials in a locally compact space. He tried to examine each of the known main properties of potentials under the possibly least conditions. The present paper is a result of his efforts although it covers only a part of the field.

It was a coincidence that Choquet started a similar study and, in particular, that both Choquet and the present author observed independently the fact that a very weak form of maximum principle means the continuity principle. The present author found that the boundedness principle satisfied on every compact set is actually equivalent to the continuity principle for a large class of kernels. This continuity principle is known as Evans-Vasilesco's theorem in the theory of Newtonian potentials and its importance had been recognized by many mathematicians. This principle and the continuous potentials still play important roles in many papers, e.g. those by Anger and Kishi. Soon Kishi, Fuglede and others joined us in investigating potentials in locally compact spaces and the investigation is still being actively conducted.

This paper consists of three chapters. In Chapter I we call many properties, well known in the case of Newtonian potentials, principles, seek relations among them, introduce local notions like Choquet [2] and study relations among them and also their relations with principles. Next, several topologies are defined on classes of measures, and the completeness is investigated, particularly, with respect to the strong topology. The remaining part of the chapter is devoted to refining some results on set functions related to capacity and on convergence theorems; these will be used in Chapter III. Chapter II is concerned with Gauss variation. Discussions are rather elementary and carried out in a general form. In the last three sections we generalize some theorems of Ninomiya. We close the paper with Chapter III in which we deal with the inner and outer Gauss variational problems. These are extensions of the problem of finding inner and outer capacitary distributions and studying their properties. Some notes and open questions are stated at the end of each chapter.

The author wishes to express his deepest gratitude to Professor N. Aronszajn who kindly enabled the author to join his seminar at the University of Kansas in 1959. Without this opportunity the publication of the present paper would have been further delayed. Appreciation is expressed to Professor T. Ogasawara of Hiroshima University who listened to talks by the author in a seminar which lasted for a year after his return from Kansas and gave him many valuable suggestions. Thanks are also due to Professor B. Fuglede who sent a manuscript of his paper [1] and his comments to the author. Subjects in his paper overlap partially with those of the present paper and some of his discussions influenced this paper considerably.

Chapter I. Fundamental notions

1.1. Potentials and exceptional sets.

We shall be concerned with a locally compact Hausdorff space Ω. We take for granted the definition of positive (=nonnegative) Radon measure μ on Ω, that of its (closed) support S_μ and the notion of integrals with respect to μ of μ-measurable functions; we refer to Bourbaki [1; 2] and Fuglede [1] for these notions. Any set containing S_μ is said to support μ. The class of all Radon measures will be denoted by $\mathscr{M}$ and a positive Radon measure will be called simply a *measure* hereafter. Whenever we consider an integral, we presume that the value is determined, finite or infinite ($\pm\infty$).

We take a lower semicontinuous function $\Phi(P, Q)$ defined on $\Omega\times\Omega$ and satisfying $-\infty<\Phi(P, Q)\leqq\infty$, and call it a *kernel*. Given a measure μ, we consider the integral

$$U^\mu(P)=\int\Phi(P, Q)d\mu(Q)=\int\Phi^+(P, Q)d\mu(Q)-\int\Phi^-(P, Q)d\mu(Q).$$

We shall call the set of points P for which $\int\Phi^-(P, Q)d\mu(Q)$ is defined and finite (i.e. μ-integrable) the *domain of definition* of the *potential* $U^\mu(P)$ of μ with kernel Φ. The class of measures, whose potentials are bounded from below on every compact set in Ω and not constantly equal to ∞, will be denoted by $\mathscr{M}_0$.

The kernel $\check{\Phi}(P, Q)=\Phi(Q, P)$ is called the *adjoint kernel* and

$$\check{U}^{\mu}(P)=\int\Phi(Q, P)d\mu(Q)$$

is called the *adjoint potential.* If $\Phi(P, Q)$ is symmetric, namely, if $\Phi(P, Q) =\Phi(Q, P)$, then $U^{\mu}(P)$ and $\check{U}^{\mu}(P)$ have the same domain of definition and $U^{\mu}(P) =\check{U}^{\mu}(P)$ there. For the kernel

$$\hat{\Phi}(P, Q)=\frac{\Phi(P, Q)+\Phi(Q, P)}{2}=\frac{\Phi(P, Q)+\check{\Phi}(P, Q)}{2}$$

the potential

$$\hat{U}^{\mu}(P)=\int\hat{\Phi}(P, Q)d\mu(Q)$$

has a domain of definition which contains the intersection of the domains of definition of $U^{\mu}(P)$ and $\check{U}^{\mu}(P)$.

In general, potentials are not lower semicontinuous in the domains of definition. However, in the special case that S_{μ} is compact or in the case that $\Phi(P, Q)\geqq 0$ in $\Omega\times\Omega$, the domain of definition is equal to Ω and $U^{\mu}(P)$ is lower semicontinuous in Ω. If, in addition, $U^{\mu}(P)\not\equiv\infty$, then $\mu\in\mathscr{M}_0$.

The *mutual energy* of two measures μ and ν is defined by

$$\begin{aligned}(\mu, \nu)&=\iint\Phi(P, Q)d\mu(Q)d\nu(P)\\ &=\iint\Phi^{+}(P, Q)d\mu(Q)d\nu(P)-\iint\Phi^{-}(P, Q)d\mu(Q)d\nu(P),\end{aligned}$$

provided that $\Phi^{-}(P, Q)$ is integrable with respect to the product measure $\mu\otimes\nu$. Then the points of Ω which do not belong to the domain of definition of $U^{\mu}(P)$ ($\check{U}^{\nu}(P)$ resp.) form a set of ν-measure (μ-measure resp.) zero and we have

$$(\mu, \nu)=\int U^{\mu}(P)d\nu(P)=\int\check{U}^{\nu}(P)d\mu(P).$$

We call (μ, μ) simply the *energy* of μ provided that it is defined. For a set $X\subset\Omega$, we put

$$\mathscr{E}_X=\{\mu\in\mathscr{M}_0;\ S_{\mu}\subset X,\ (\mu, \mu)\text{ is defined and finite}\}.$$

We write simply $\mathscr{E}$ for $\mathscr{E}_{\Omega}$.

We shall consider some set functions which are related to the classical notion of capacity. A measure will be called a unit measure if its total mass is equal to one. For a measure $\mu\neq 0$ we set

$$V(\mu)=\sup_{P\in S_\mu} U^\mu(P),$$

and, for a set $X\neq\emptyset$, we put

$$V_i(X)=\inf_\mu V(\mu),$$

where the infimum is taken with respect to the class of all unit measures μ with compact support $S_\mu\subset X$. For the empty set $\emptyset$, we put $V_i(\emptyset)=\infty$. We shall say that a property holds on a set $A\subset\Omega$ *p.p.p.* (or nearly everywhere) if the V_i-value of the exceptional set in A is infinite. We define also

$$V_e(X)=\sup V_i(G) \qquad \text{for open sets } G\supset X.$$

In the case of a Newtonian potential, the reciprocals of $V_i(X)$ and $V_e(X)$ are defined to be the inner and outer capacities of X respectively. We shall say that a property holds on $A\subset\Omega$ *q.p.* (or quasi everywhere) if the V_e-value of the exceptional set in A is infinite.

The corresponding set functions defined with respect to an adjoint kernel will be denoted by $\check{V}_i(X)$ and $\check{V}_e(X)$. Also $\hat{V}_i(X)$ and $\hat{V}_e(X)$ will correspond to $\hat{\Phi}$.

For the sake of later applications we prove

PROPOSITION 1. *Let $\{A_n\}$ be sets which are measurable for every measure on Ω and X be an arbitrary set. If $V_i(A_n\cap X)=\infty$ for each n, then $V_i(\bigcup_n A_n\cap X)=\infty$. In case $\Phi(P,Q)\geqq m>-\infty$ on $(\bigcup_n A_n\cap X)\times(\bigcup_n A_n\cap X)$, we have*

$$\frac{1}{V_i(\bigcup_n A_n\cap X)-m}\leqq\sum_n\frac{1}{V_i(A_n\cap X)-m}. \tag{1.1}$$

PROOF. We assume that $V_i(\bigcup_n A_n\cap X)<\infty$, and choose a unit measure μ with compact support $S_\mu\subset\bigcup_n A_n\cap X$ such that $V(\mu)<\infty$. Then $1=\mu(\Omega)\leqq\sum_n\mu(A_n)$ and, for some n, say for n_0, $\mu(A_{n_0})>0$. We take a compact set $K\subset A_{n_0}$ such that $\mu(K)>0$, and denote by μ_K the restriction of μ to K. We extend this restriction to the whole space by the value 0 and call this extension the restriction too; by a *restriction of a measure* we shall mean such an extension in this paper. It holds that

$$\begin{aligned}V(\mu)&=\sup_{S_\mu} U^\mu(P)=\sup_{S_\mu}(U^{\mu_K}(P)+U^{\mu-\mu_K}(P))\\&\geqq\sup_{S_\mu} U^{\mu_K}(P)+\inf_{S_\mu} U^{\mu-\mu_K}(P)\\&\geqq\sup_{S_{\mu_K}} U^{\mu_K}(P)+\inf_{P,Q\in S_\mu}\Phi(P,Q)\{\mu(\Omega)-\mu_K(\Omega)\},\end{aligned}$$

and hence $V(\mu_K)<\infty$. Taking $S_{\mu_K}\subset A_{n_0}\cap X$ into consideration, we conclude that $V_i(A_{n_0}\cap X)<\infty$. Consequently the assumption $V_i(A_n\cap X)=\infty$, $n=1, 2, \cdots$, implies $V_i(\bigcup_n A_n\cap X)=\infty$.

Next we assume that $\Phi(P, Q)\geqq m>-\infty$ on $(\bigcup_n A_n\cap X)\times(\bigcup_n A_n\cap X)$ and that $V_i(\bigcup_n A_n\cap X)<\infty$. Given $\varepsilon>0$, there exists a unit measure μ with compact $S_\mu\subset\bigcup_n A_n\cap X$ such that

$$V(\mu)=\sup_{S_\mu} U^\mu(P)<V_i(\bigcup_n A_n\cap X)+\varepsilon.$$

We can find a compact set $K_n\subset A_n$ such that $\mu(A_n-K_n)<\varepsilon/2^n$. We shall denote the restriction of μ to K_n by μ_n; $S_{\mu_n}\subset A_n\cap X$. It follows that

$$\begin{aligned} V_i(\bigcup_n A_n\cap X)+\varepsilon-m>V(\mu)-m&=\sup_{S_\mu}\int(\Phi-m)\,d\mu \\ &\geqq \sup_{S_{\mu_n}}\int(\Phi-m)\,d\mu_n=V(\mu_n)-m\mu_n(\Omega). \end{aligned}$$

If $\mu_n\not\equiv 0$,

$$\frac{V(\mu_n)}{\mu_n(\Omega)}\geqq V_i(A_n\cap X).$$

In any case we have

$$\mu_n(\Omega)\leqq\frac{V_i(\bigcup_n A_n\cap X)+\varepsilon-m}{V_i(A_n\cap X)-m}$$

and

$$1\leqq\sum_n\mu(A_n)\leqq\sum_n\mu_n(\Omega)+\varepsilon\leqq\sum_n\frac{V_i(\bigcup_n A_n\cap X)+\varepsilon-m}{V_i(A_n\cap X)-m}+\varepsilon.$$

Inequality (1.1) now follows.

PROPOSITION 2. *Let $\{X_n\}$ be a sequence of sets such that, for an open set $G_0\supset\bigcup_n X_n$, $\Phi(P, Q)\geqq m>-\infty$ on $G_0\times G_0$. Then we have*

$$\frac{1}{V_e(\bigcup_n X_n)-m}\leqq\sum_n\frac{1}{V_e(X_n)-m}, \tag{1.2}$$

and hence, if $V_e(X_n)=\infty$ for each $X_n\subset G_0$, then $V_e(\bigcup_n X_n)=\infty$.

PROOF. Given $\varepsilon>0$, choose an open set G_n such that $X_n\subset G_n\subset G_0$ and

$$\frac{1}{V_i(G_n)-m}\leqq\frac{1}{V_e(X_n)-m}+\frac{\varepsilon}{2^n}.$$

We have

$$\frac{1}{V_e(\bigcup_n X_n)-m} \leqq \frac{1}{V_i(\bigcup_n G_n)-m} \leqq \sum_n \frac{1}{V_i(G_n)-m}$$
$$\leqq \sum_n \frac{1}{V_e(X_n)-m} + \sum_n \frac{\varepsilon}{2^n}$$

by (1.1). From this follows (1.2).

1.2. Principles.

Let $\mathscr{F}$ be a class of functions defined in Ω. We shall define principles in this section and see relations among them in the next section.

($\mathrm{D}_{\mathscr{F}}$) *$\mathscr{F}$-relative domination principle*[1]. If $U^\mu(P) \leqq f(P)$ on S_μ for a measure $\mu \in \mathscr{E}$, $\mu \neq 0$, with compact support and for a function $f \in \mathscr{F}$, then

$$U^\mu(P) \leqq f(P) \qquad \text{in } \Omega.$$

($\mathrm{DV}_{\mathscr{F}}$) *$\mathscr{F}$-relative vicinal domination principle.* If the support S_μ of $\mu \in \mathscr{E}$, $\mu \neq 0$, is compact and $U^\mu(P) \leqq f(P)$ on S_μ for $f \in \mathscr{F}$, then there exists, for any $\varepsilon > 0$, a neighborhood $V(=V(\mu, f, \varepsilon))$ of S_μ such that

$$U^\mu(P) \leqq f(P) + \varepsilon \qquad \text{in } V.$$

($\mathrm{U}_{\mathscr{F}}$) *$\mathscr{F}$-relative Ugaheri's domination principle.* There is a constant $c > 0$ such that, whenever the support S_μ of $\mu \in \mathscr{E}$, $\mu \neq 0$, is compact and $U^\mu(P) \leqq f(P)$ on S_μ for $f \in \mathscr{F}$, we have

$$U^\mu(P) \leqq cf(P) \qquad \text{in } \Omega.$$

This may be called also the $\mathscr{F}$-relative dilated domination principle.

($\mathrm{U}_{\mathscr{F}})_c$ *$\mathscr{F}$-relative c-dilated domination principle.* This is the same as above but we specify the constant c.

($\mathrm{UV}_{\mathscr{F}}$) *$\mathscr{F}$-relative vicinal Ugaheri's domination principle.* There is a constant $c > 0$ such that, whenever the support S_μ of $\mu \in \mathscr{E}$, $\mu \neq 0$, is compact and $U^\mu(P) \leqq f(P)$ on S_μ, there exists, for any $\varepsilon > 0$, a neighborhood V of S_μ such that

$$U^\mu(P) \leqq cf(P) + \varepsilon \qquad \text{in } V.$$

($\mathrm{UV}_{\mathscr{F}})_c$ *$\mathscr{F}$-relative vicinal c-dilated domination principle.* This is the same as above but we specify the constant c.

($\mathrm{U}'_{\mathscr{F}}$) *$\mathscr{F}$-relative weak Ugaheri's domination principle.* For any compact set $K \subset \Omega$, there is a constant $c = c(K) > 0$ such that, whenever $U^\mu(P) \leqq f(P)$ on S_μ for $\mu \in \mathscr{E}_K$, $\mu \neq 0$, and $f \in \mathscr{F}$, we have

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1) Relative principles were first introduced in Choquet and Deny [3].

$$U^\mu(P) \leqq cf(P) \qquad \text{in } \Omega.$$

This may be called the $\mathscr{F}$-relative weak dilated domination principle.

($\mathrm{U'V}_{\mathscr{F}}$) *$\mathscr{F}$-relative vicinal weak Ugaheri's domination principle.* For any compact set $K \subset \Omega$, there is a constant $c=c(K)>0$ with the following property: Whenever $U^\mu(P) \leqq f(P)$ on S_μ for $\mu \in \mathscr{E}_K$, $\mu \not\equiv 0$, and $f \in \mathscr{F}$, we can find, for any $\varepsilon>0$, a neighborhood V of S_μ such that

$$U^\mu(P) \leqq cf(P)+\varepsilon \qquad \text{in } V.$$

($\mathrm{U'V}_{\mathscr{F}})_c$ *$\mathscr{F}$-relative vicinal weak c-dilated domination principle.* This is the same as above but we specify c.

($\mathrm{K}_{\mathscr{F}}$) *$\mathscr{F}$-relative Kishi's domination principle.* For any compact set $K \subset \Omega$, there is a constant $c=c(K) >0$ such that, whenever $U^\mu(P) \leqq f(P)$ on K for $\mu \in \mathscr{E}_K$ and $f \in \mathscr{F}$, we have

$$U^\mu(P) \leqq cf(P) \qquad \text{in } \Omega.$$

($\mathrm{KV}_{\mathscr{F}}$) *$\mathscr{F}$-relative vicinal Kishi's domination principle.* For any compact set $K \subset \Omega$, there is a constant $c=c(K)>0$ with the following property: Whenever $U^\mu(P) \leqq f(P)$ on K for $\mu \in \mathscr{E}_K$ and $f \in \mathscr{F}$, we can find, for any $\varepsilon>0$, a neighborhood V of K such that

$$U^\mu(P) \leqq cf(P)+\varepsilon \qquad \text{in } V.$$

($\mathrm{KV}_{\mathscr{F}})_c$ *$\mathscr{F}$-relative vicinal c-dilated Kishi's domination principle.* This is the same as above but we specify c.

($\mathrm{K}'_{\mathscr{F}}$) *$\mathscr{F}$-relative weak Kishi's domination principle.* For any compact set $K \subset \Omega$, there is a constant $c=c(K)>0$ such that, whenever $U^\mu(P)$ is continuous[2)] as a function on K for $\mu \in \mathscr{E}_K$ and $U^\mu(P) \leqq f(P)$ on K for $f \in \mathscr{F}$, we have

$$U^\mu(P) \leqq cf(P) \qquad \text{in } \Omega.$$

($\mathrm{K'V}_{\mathscr{F}}$) *$\mathscr{F}$-relative vicinal weak Kishi's domination principle.* For any compact set $K \subset \Omega$, there is a constant $c=c(K)>0$ with the following property: Whenever $U^\mu(P)$ is continuous as a function on K for $\mu \in \mathscr{E}_K$, and $U^\mu(P) \leqq f(P)$ on K for $f \in \mathscr{F}$, we can find, for any $\varepsilon>0$, a neighborhood V of K such that

$$U^\mu(P) \leqq cf(P)+\varepsilon \qquad \text{in } V.$$

($\mathrm{K'V}_{\mathscr{F}})_c$ *$\mathscr{F}$-relative vicinal weak c-dilated Kishi's domination principle.* This is the same as above but we specify the constant c.

($\mathrm{B}_{\mathscr{F}}$) *$\mathscr{F}$-relative upper boundedness princplie.* If the support S_μ of $\mu \in \mathscr{E}$, $\mu \not\equiv 0$, is compact and $U^\mu(P) \leqq f(P)$ on S_μ for $f \in \mathscr{F}$, then there is a constant $c>0$ which may depend on μ and f such that

2) By continuity we mean that the value is finite and continuous. If we allow ∞ or $-\infty$ or both, we shall say that the function is continuous in the extended sense.

$$U^{\mu}(P) \leqq cf(P) \qquad \text{in } \Omega.$$

($B'_{\mathscr{F}}$) *$\mathscr{F}$-relative weak upper boundedness principle.* If the support S_μ of $\mu \in \mathscr{E}$, $\mu \neq 0$, is compact, if $U^{\mu}(P)$ is continuous as a function on S_μ and if $U^{\mu}(P) \leqq f(P)$ for $f \in \mathscr{F}$, then there is a constant $c > 0$ which may depend on μ and f such that

$$U^{\mu}(P) \leqq cf(P) \qquad \text{in } \Omega.$$

If a principle is satisfied on every compact set, being considered as a space, we denote the principle with the subscript K. Constants will depend on each compact set in general. Principles ($D_{\mathscr{F}}$) and ($DV_{\mathscr{F}}$), however, will remain unchanged. We shall write down one example explicitly.

($K'V_{K,\mathscr{F}}$) *$\mathscr{F}$-relative vicinal weak Kishi's domination principle satisfied on every compact set.* Let K be any compact set in Ω. For any compact subset $K' \subset K$, there is a constant $c = c(K, K') > 0$ with the following property: Whenever $U^{\mu}(P)$ is continuous as a function on K' for $\mu \in \mathscr{E}_K$, and $U^{\mu}(P) \leqq f(P)$ on K' for $f \in \mathscr{F}$, we can find, for any $\varepsilon > 0$, a neighborhood V of K' in Ω such that

$$U^{\mu}(P) \leqq cf(P) + \varepsilon \qquad \text{in } V \frown K.$$

In special cases we shall use specific terminologies and notations. In case $\mathscr{F}$ consists of all finite constants, ($D_{\mathscr{F}}$) is called the *first maximum principle* and will be denoted by (F); it is called also Frostman's maximum principle. Similarly ($DV_{\mathscr{F}}$) will be called the vicinal first maximum principle[3] and denoted by (FV). In the other principles up to $(K'V_{\mathscr{F}})_c$ (also in the corresponding principles satisfied on every compact set) the word $\mathscr{F}$-relative will be omitted and the word domination will be replaced by the word maximum. In the notations the subscript $\mathscr{F}$ will be dropped. For instance, (U) will mean Ugaheri's maximum principle. We shall write simply (B) for ($B_{\mathscr{F}}$) and call this the upper boundedness principle. The corresponding changes will be made on other similar principles.

In case $\mathscr{F}$ consists of all potentials which are defined everywhere in Ω, ($D_{\mathscr{F}}$) will be called the domination principle and denoted by (D). This is called also the second maximum principle or Cartan's maximum principle. In the other principles up to $(K'V_{\mathscr{F}})_c$ the word $\mathscr{F}$-relative will be omitted. The principle ($DV_{\mathscr{F}}$) will be denoted by (DV) and, in the other notations, $\mathscr{F}$ will be replaced by d. For instance ($U_{\mathscr{F}}$) will be denoted by (U_d). We shall call ($B_{\mathscr{F}}$) the relative upper boundedness principle and denote it by (B_d). The corresponding changes will be made on the similar principles.

In case $\mathscr{F}$ consists of all potentials of measures of $\mathscr{E}$ with compact support we add the adjective 'restricted' and replace $\mathscr{F}$ by *. For instance, ($D_{\mathscr{F}}$) will be called the restricted domination principle and denoted by (D^*).

3) Choquet [2] called this *le principle du maximum local faible.*

We assume in the following definition that each function of $\mathscr{F}$ is positive in Ω.

($C_{\mathscr{F}}$) *$\mathscr{F}$-relative continuity principle.* If, for a μ with compact support and an $f \in \mathscr{F}$, the restriction of $U^{\mu}(P)/f(P)$ to S_{μ} can be defined and is continuous, then $U^{\mu}(P)/f(P)$ can be defined and is continuous in Ω.

In the special case when $\mathscr{F}$ consists only of the constant 1, the principle is called the *continuity principle.* Such a kernel is called *regular* by some mathematicians. This principle is closely related to the maximum and boundedness principles defined above.

We shall define quasicontinuity principles and discuss them on some other occasion; see Kishi [2; 3] and Ohtsuka [7] for these principles.

We give also

$(S_{\mathscr{F}})_c$ *$\mathscr{F}$-relative c-dilated sweeping-out principle* ($c>0$). For any compact set $K \subset \Omega$ and any $f \in \mathscr{F}$, there is a measure μ supported by K such that $U^{\mu}(P) \geqq f(P)$ p.p.p. on K and $U^{\mu}(P) \leqq cf(P)$ everywhere in Ω. We omit the word c-dilated and write $(S_{\mathscr{F}})$ if $c=1$.

In case $\mathscr{F}$ consists of potentials of measures with compact support, this principle will be called the c-dilated sweeping-out principle (the sweeping-out principle if $c=1$) and denoted by $(S)_c$ ((S) resp.). In case $\mathscr{F}$ consists of potentials of measures with compact support, we shall add the adjective 'restricted' and use the notations $(S^*)_c$ and (S^*).

For positive kernels the following principle coincides with $(S_{\mathscr{F}})_c$ for $\mathscr{F} \equiv \{1\}$.

$(E_q)_c$ *c-dilated equilibrium principle* ($c>0$). For any compact set $K \subset \Omega$, there are a constant $a \leqq \infty$ and a unit measure μ supported by K such that

$$U^{\mu}(P) \geqq \text{ const. } a \qquad \text{p.p.p. on } K$$

and

$$U^{\mu}(P) \leqq ca \qquad \text{in } \Omega.$$

For $c=1$ this principle is called the equilibrium principle and denoted by (E_q).

$(E)_c$ *c-energy principle* ($c>0$). The kernel is symmetric and, for any different $\mu, \nu \in \mathscr{E}$,

$$(\mu, \mu)+c(\nu, \nu)-2(\mu, \nu)>0$$

whenever (μ, ν) is defined. In case $c=1$, the principle is called the energy principle or the kernel is called strictly positive definite, and the principle is denoted by (E).

$(E^*)_c$ *Restricted c-energy principle* ($c>0$). The kernel is symmetric and, for any different $\mu, \nu \in \mathscr{E}$ with compact support,

$$(\mu, \mu)+c(\nu, \nu)-2(\mu, \nu)>0.$$

$(E')_c$ *Weak c-energy principle* ($c>0$). The kernel is symmetric and, for any

$\mu, \nu \in \mathscr{E}$,

$$(\mu, \mu)+c(\nu, \nu)-(\mu, \nu) \geqq 0$$

whenever (μ, ν) is defined. In case $c=1$, the kernel is called *of positive type* or positive definite, and the principle is denoted by (P). We obtain the same principle if we restrict $\mu, \nu \in \mathscr{E}$ to those having compact supports, because every integral is approximated by the integral taken on a compact set.

There are other principles defined and discussed by some writers; see Choquet and Deny [3], Ninomiya [8; 10] and Kishi [8]. However, we shall limit ourselves to the above principles in the present paper except in § 2.11.

1.3. Relations among principles.

Now our problems are

(1) to see relations among these principles,

(2) to characterize the class of kernels which satisfy principles.

We shall discuss (1) rather in details and refer occasionally to some known results about (2).

In case $\Phi(P, Q)$ is symmetric, finite outside the diagonal set of $\Omega \times \Omega$ and continuous in the extended sense that ∞ is allowed, we have the following diagram as was shown in Ohtsuka [4]; we put ? at the end because it is not a priori true for more general kernels:

$$(1.3)\quad \begin{tikzcd}
 & & & (\mathrm{B}) \arrow[rr, shift left] \arrow[dll, shift left, "/" marking] \arrow[dr, shift left, "/" marking] & & (\mathrm{B}') \arrow[ll, shift left, "(\quad)"] \arrow[dr, shift left] \arrow[dl, shift left, "/" marking] & & & \\
(\mathrm{F}) \arrow[r, shift left] & (\mathrm{U}) \arrow[l, shift left, "/" marking] \arrow[r, shift left] & (\mathrm{U}') \arrow[l, shift left, "/" marking] \arrow[ur, shift left] \arrow[drr, shift left] & & & & (\mathrm{B}_{\mathrm{K}}) \arrow[ul, shift left, "/" marking] \arrow[r, shift left] \arrow[dll, shift left, "/" marking] & (\mathrm{B}'_{\mathrm{K}}) \arrow[l, shift left] \arrow[r, shift left] & (\mathrm{C})? \arrow[l, shift left] \\
 & & & & (\mathrm{U}_{\mathrm{K}}) \arrow[ull, shift left, "/" marking] \arrow[ul, shift left, "/" marking] \arrow[ur, shift left, "/" marking] \arrow[urr, shift left] & & & &
\end{tikzcd}$$

Here $(\leftarrow)$ means that this relation is true if $-\infty < \inf \Phi(P, Q)$ on $\Omega \times \Omega$. Since the proof for this result is scattered in several short notes (Kishi [1], Choquet [2], Ohtsuka [2; 4]) and since we need to examine relations for kernels more general than those in Ohtsuka [2; 4], we shall start from the beginning. We divide the discussions into several steps.

(I) *Principles* $(\mathrm{D}_{\mathscr{F}})$, $(\mathrm{U}_{\mathscr{F}})$, $(\mathrm{U}'_{\mathscr{F}})$, $(\mathrm{K}_{\mathscr{F}})$, $(\mathrm{K}'_{\mathscr{F}})$, $(\mathrm{U}_{\mathrm{K},\mathscr{F}})$, $(\mathrm{K}_{\mathrm{K},\mathscr{F}})$, $(\mathrm{K}'_{\mathrm{K},\mathscr{F}})$, $(\mathrm{B}_{\mathscr{F}})$, $(\mathrm{B}'_{\mathscr{F}})$, $(\mathrm{B}_{\mathrm{K},\mathscr{F}})$, $(\mathrm{B}'_{\mathrm{K},\mathscr{F}})$. We first write obvious relations:

$$(1.4)\quad \begin{tikzcd}
 & & & (\mathrm{B}_{\mathscr{F}}) \arrow[r] \arrow[dr] & (\mathrm{B}'_{\mathscr{F}}) \arrow[dr] & \\
(\mathrm{D}_{\mathscr{F}}) \arrow[r] & (\mathrm{U}_{\mathscr{F}}) \arrow[r] & (\mathrm{U}'_{\mathscr{F}}) \arrow[ur] \arrow[dr] & & & (\mathrm{B}'_{\mathrm{K},\mathscr{F}}) \\
 & & & (\mathrm{U}_{\mathrm{K},\mathscr{F}}) \arrow[r] & (\mathrm{B}_{\mathrm{K},\mathscr{F}}) \arrow[ur] &
\end{tikzcd}$$

and

$$
(1.5)\qquad
\begin{array}{ccccc}
 & & (\mathrm{B}_{\mathscr{F}}) & \to & (\mathrm{B}'_{\mathscr{F}}) \\
 & & \uparrow & & \uparrow \\
(\mathrm{U}'_{\mathscr{F}}) & \to & (\mathrm{K}_{\mathscr{F}}) & \to & (\mathrm{K}'_{\mathscr{F}}) \\
 & & \downarrow & & \downarrow \\
 & & (\mathrm{K}_{\mathrm{K},\mathscr{F}}) & \to & (\mathrm{K}'_{\mathrm{K},\mathscr{F}}) \\
 & & \downarrow & & \downarrow \\
 & & (\mathrm{B}_{\mathrm{K},\mathscr{F}}) & \to & (\mathrm{B}'_{\mathrm{K},\mathscr{F}}).
\end{array}
$$

We shall establish exact diagrams in a special case and here we prove only two general facts.

LEMMA 1.1. *Let X be any set in Ω, K be a compact set in Ω, and $f(P)$ be a function defined on $K \cup X$ and bounded on K. Consider a kernel which is positive*[4] *on the diagonal set and assume that if the potential of any measure of $\mathscr{E}_K$ is continuous as a function on K, then it is nonnegative on $K \cup X$. Assume also that, whenever $U^{\nu}(P)$ is continuous as a function on K and $U^{\nu}(P) \leqq f(P)$ on K for $\nu \in \mathscr{E}_K$, $U^{\nu}(P) \leqq cf(P)$ on X with a finite constant $c = c(K, X, \nu) > 0$. Then there is a finite constant $c' = c'(K, X) > 0$ not depending on μ such that, whenever $U^{\mu}(P)$ is continuous as a function on K and $U^{\mu}(P) \leqq f(P)$ on K for $\mu \in \mathscr{E}_K$, $U^{\mu}(P) \leqq c' f(P)$ on X.*

PROOF. Assume, to the contrary, that, for every integer n, there exist μ_n with $S_{\mu_n} \subset K$ and $P_n \in X$ such that $U^{\mu_n}(P) \leqq f(P)$ on K, the restriction of $U^{\mu_n}(P)$ to K is continuous and

$$U^{\mu_n}(P_n) > 2^n n f(P_n).$$

We set $\nu_n = \mu_n / 2^n$ and $\nu = \sum_{n=1}^{\infty} \nu_n$. Certainly $S_\nu \subset K$. Suppose that the total mass of ν were infinite. Then there would be a point $P_0 \in K$ such that the ν-value of any neighborhood of P_0 is infinite. Let N_0 be a neighborhood of P_0 such that $\Phi(P, Q) > a > 0$ on $N_0 \times N_0$. Then

$$
\begin{aligned}
\infty = \int_{N_0} \Phi(P, Q)\, d\nu(Q) \leqq U^{\nu}(P) &= \sum_{n=1}^{\infty} \frac{1}{2^n} U^{\mu_n}(P) \\
&\leqq \sum_{n=1}^{\infty} \frac{1}{2^n} f(P) = f(P) < \infty
\end{aligned}
$$

on $N_0 \cap K$. This is impossible and it is proved that the total mass of ν is finite. Since $f(P)$ is bounded on K by assumption, the convergence of $\sum_{n=1}^{\infty} U^{\nu_n}(P)$ is uniform on K and hence the restriction of $U^{\nu}(P)$ to K is continuous. On the other hand

$$U^{\nu}(P_n) \geqq U^{\nu_n}(P_n) > n f(P_n).$$

4) In this paper a positive function is strictly positive; it never vanishes in its domain of definition or on the specified set. Fuglede [1] defined it otherwise; i.e., a kernel is strictly positive if it is nonnegative in $\Omega \times \Omega$ and never vanishes on the diagonal set in $\Omega \times \Omega$.

This contradicts the assumption and the lemma is proved.

Similarly we can prove

LEMMA 1.2. *Let X and K be the same as above, and $f(P)$ be a function defined on $K\cup X$ and $<\infty$ on K. Consider a kernel which is positive on the diagonal set and assume that the potential of any measure of $\mathscr{E}_K$ is nonnegative on $K\cup X$ and that $U^{\nu}(P)\leqq f(P)$ on K for $\nu\in\mathscr{E}_K$ implies $U^{\nu}(P)\leqq cf(P)$ on X with a finite constant $c=c(K, X, \nu)>0$. Then there is a finite constant $c'=c'(K, X)>0$ such that $U^{\mu}(P)\leqq f(P)$ on K implies $U^{\mu}(P)\leqq c'f(P)$ on X for any $\mu\in\mathscr{E}_K$.*

LEMMA 1.2′. *We obtain the same conclusion as in Lemma 1.2, if we add the new assumption that the kernel is positive on $K\times K$ but weaken the condition $f(P)<\infty$ on K, by replacing it with $f(P)\not\equiv\infty$ on K.*

From these lemmas follow easily

$$(1.6)\qquad\qquad (\mathrm{B})\ (\rightarrow)\ (\mathrm{K})\qquad \text{and}\qquad (\mathrm{B}')\ (\rightarrow)\ (\mathrm{K}'),$$

where $(\rightarrow)$ indicates that $\rightarrow$ is true provided that the kernel is positive; the meaning of $(\rightarrow)$ may be different later.

(II) *Principles* (F), (U), (U′), $(\mathrm{U_K})$, (B), (B′), $(\mathrm{B_K})$, $(\mathrm{B'_K})$. We shall show by examples that there is no more $\rightarrow$ relation in (1.4) in the special case that $\mathscr{F}$ consists of all finite constants.

Example for $(\mathrm{U})\not\rightarrow(\mathrm{F})$ (Ohtsuka [4]): Consider $\Omega=[0, 1]\cup\{2\}$ as a subspace of the x-axis, and set $\Phi(x, y)=\Phi(y, x)=-\log|x-y|$ for $x, y\in[0, 1]$, $\Phi(2, 2)=\infty$, $\Phi(2, x)=\Phi(x, 2)=a>\log 4$ for $x\in[0, 1]$. This $\Phi(x, y)$ is symmetric, continuous in the extended sense and finite outside the diagonal set in $\Omega\times\Omega$. Let μ_0 be the unit measure on $[0, 1]$ which gives a constant potential there. Then $U^{\mu_0}(x)=\log 4$ on $S_{\mu_0}=[0, 1]$ but $U^{\mu_0}(2)=a>\log 4$. Thus (F) is not satisfied. On the other hand, for any μ supported by $[0, 1]$, we have

$$\sup_{x\in\Omega} U^{\mu}(x)\leqq a \sup_{x\in S_{\mu}} U^{\mu}(x)$$

and (U) is satisfied.

Later we shall give examples for $(\mathrm{U})\not\rightarrow(\mathrm{FV})$ which apparently serve as examples for $(\mathrm{U})\not\rightarrow(\mathrm{F})$. One of them will be a so-called α-potential.

The following theorem by Kametani [2] and Ugaheri [1; 2] motivated the terminology of Ugaheri's maximum principle:

Let $\varphi(t)\geqq 0$ be a decreasing continuous function such that $\varphi(t)\rightarrow\infty$ as $t\rightarrow 0$. Then there is a constant $c_n\geqq 1$, depending only on the dimension n of the euclidean space E_n, with the property that

$$\sup_{E_n} U^{\mu}(P)\leqq c_n \sup_{S_{\mu}} U^{\mu}(P)$$

for every measure $\mu\neq 0$, where $\varphi(\overline{PQ})$ is taken as the kernel. We refer to

Choquet [2] and Ninomiya [7] for various generalizations of the result of Kametani and Ugaheri.

Example for (U$'$)$\not\rightarrow$(U) (Ohtsuka [4]): Consider $\Omega=\bigcup_{n=0}^{\infty}([2n, 2n+1]\cup\{2n+3/2\})$ as a subspace of the x-axis, and set

$$\begin{aligned}
&\Phi(x, y)=\Phi(y, x)=-\log|x-y| && \text{for } x, y\in[2n, 2n+1],\ n=0, 1, 2,\cdots,\\
&\Phi(2n+3/2, 2n+3/2)=\infty,\\
&\Phi(2n+3/2, x)=\Phi(x, 2n+3/2)=n && \text{if } x\in[2n, 2n+1],\\
&\Phi(x, y)=\Phi(y, x)=1 && \text{if } x\in[2n, 2n+1] \text{ and } y\in[2m, 2m+1],\ n\neq m,\\
& && \text{or if } x=2n+3/2 \text{ and } y\in[2m, 2m+1],\ n\neq m.
\end{aligned}$$

This $\Phi(x, y)$ is symmetric, continuous in the extended sense and finite outside the diagonal set in $\Omega\times\Omega$. Let μ_n be the unit measure on $[2n, 2n+1]$ which gives a constant potential on that interval. Then $U^{\mu_n}(2n+3/2)=n\rightarrow\infty$ while $U^{\mu_n}(x)=\log 4$ for $x\in S_{\mu_n}=[2n, 2n+1]$. Thus (U) is not satisfied. Next let K be any compact set with $V_i(K)<\infty$ in Ω and μ be any unit measure with $S_\mu\subset K$ and $V(\mu)<\infty$. If $K\subset\bigcup_{n=1}^{N}([2n, 2n+1]\cup\{2n+3/2\})$, then $S_\mu\subset\bigcup_{n=1}^{N}[2n, 2n+1]$. Let μ_n denote the restriction of μ to $[2n, 2n+1]$. If $\mu_n\not\equiv 0$, we have

$$\begin{aligned}
U^\mu(x)&=U^{\mu_n}(x)+1-\mu_n(\Omega)\leqq V(\mu_n)+1-\mu_n(\Omega)\\
&=\sup_{y\in S_{\mu_n}} U^\mu(y)\leqq \sup_{y\in S_\mu} U^\mu(y)=V(\mu) \qquad \text{for } x\in[2n, 2n+1]
\end{aligned}$$

and

$$V(\mu)\geqq V(\mu_n)+1-\mu_n(\Omega)\geqq\mu_n(\Omega)\log 4+1-\mu_n(\Omega)>1.$$

If $\mu_n\equiv 0$, then

$$U^\mu(x)=1<V(\mu) \qquad \text{for } x\in[2n, 2n+1].$$

We observe also that

$$U^\mu(2n+3/2)\leqq N\leqq NV(\mu) \qquad \text{for } n=0,\cdots, N.$$

Consequently we can take N for $c(K)$ in (U$'$). Thus (U$'$) is established.

Example for (B)$\not\rightarrow$(U$_K$) (modified form of the example in Ohtsuka [2]): Consider $\Omega=\bigcup_{n=1}^{\infty}[(2n+1)^{-1}, (2n)^{-1}]\cup\{0\}$ as a subspace of the x-axis, and set

$$\begin{aligned}
&\Phi(x, y)=-\log|x-y| && \text{if } x, y\in[(2n+1)^{-1}, (2n)^{-1}],\\
&\Phi(x, y)=\Phi(y, x)=n\log\{8n(2n+1)\}\\
& && \text{if } x\in[(2n+1)^{-1}, (2n)^{-1}],\ y\in[(2m+1)^{-1}, (2m)^{-1}],\ n<m,\\
& && \text{or if } x\in[(2n+1)^{-1}, (2n)^{-1}],\ y=0,\\
&\Phi(0, 0)=\infty.
\end{aligned}$$

This $\Phi(x, y)$ is symmetric, continuous in the extended sense and finite outside the diagonal set in $\Omega\times\Omega$. Let μ_n be the unit measure on $[(2n+1)^{-1}, (2n)^{-1}]$ whose potential is constant there. Then $U^{\mu_n}(x)=\log\{8n(2n+1)\}$ on the interval. We set

$$\mu'_n=\mu_n/\log\{8n(2n+1)\}.$$

Then $U^{\mu'_n}(x)=1$ on $[(2n+1)^{-1}, (2n)^{-1}]$ and $U^{\mu'_n}(0)=n$. This shows that $(\mathrm{U_K})$ is not satisfied.

Next let μ be a measure such that the supremum of $U^{\mu}(x)$ on S_{μ} is equal to 1. We denote by μ_n the restriction of μ on $[(2n+1)^{-1}, (2n)^{-1}]$. Naturally $\mu(\{0\})=0$. On $[(2n+1)^{-1}, (2n)^{-1}]$ we have

$$U^{\mu_n}(x)\leqq \sup_{y\in S_{\mu_n}} U^{\mu_n}(y)\leqq \sup_{y\in S_{\mu}} U^{\mu}(y)\leqq 1.$$

If $\mu_n\not\equiv 0$, we have, for $x\in[(2n+1)^{-1}, (2n)^{-1}]$,

$$\begin{aligned}U^{\mu}(x)&=\sum_{k=1}^{n-1} k\log\{8k(2k+1)\}\cdot\mu_k(\Omega)+U^{\mu_n}(x)+n\log\{8n(2n+1)\}\cdot\sum_{k=n+1}^{\infty}\mu_k(\Omega)\\&\leqq \sup_{x\in S_{\mu_n}} U^{\mu_n}(x)+U^{\mu-\mu_n}(x)=\sup_{x\in S_{\mu_n}} U^{\mu}(x)\leqq 1.\end{aligned}$$

If $\mu_n\equiv 0$, we have, for $x\in[(2n+1)^{-1}, (2n)^{-1}]$,

$$\begin{aligned}U^{\mu}(x)&=\sum_{k=1}^{n-1} k\log\{8k(2k+1)\}\cdot\mu_k(\Omega)+n\log\{8n(2n+1)\}\cdot\sum_{k=n+1}^{\infty}\mu_k(\Omega)\\&\leqq \sum_{k=1}^{\infty} k\log\{8k(2k+1)\}\cdot\mu_k(\Omega)=U^{\mu}(0)<\infty.\end{aligned}$$

Therefore $U^{\mu}(x)$ is uniformly bounded in Ω. Thus (B) is proved.

Example for $(\mathrm{U_K})\nrightarrow(\mathrm{B}')$: $\Omega=E_3$, $\Phi(P, Q)=1+\overline{PQ}$. Obviously this does not satisfy (B'). But if $S_{\mu}\subset K$, then

$$\sup_{P\in K} U^{\mu}(P)\leqq(1+\operatorname{diam} K)\,\mu(\Omega)\leqq(1+\operatorname{diam} K)\sup_{P\in S_{\mu}} U^{\mu}(P)$$

and $(\mathrm{U_K})$ is satisfied.

This example serves to give $(\mathrm{B_K})\nrightarrow(\mathrm{B}')$ and $(\mathrm{B}'_{\mathrm{K}})\nrightarrow(\mathrm{B}')$. As logical consequences we obtain $(\mathrm{B})\nrightarrow(\mathrm{U}')$, $(\mathrm{B}')\nrightarrow(\mathrm{U_K})$, $(\mathrm{B_K})\nrightarrow(\mathrm{U_K})$, $(\mathrm{U_K})\nrightarrow(\mathrm{B})$ and $(\mathrm{U_K})\nrightarrow(\mathrm{U}')$. From $(\mathrm{B_K})\nrightarrow(\mathrm{B}')$ and $(\mathrm{B})\rightarrow(\mathrm{B}')$ it follows that $(\mathrm{B_K})\nrightarrow(\mathrm{B})$.

Next we shall examine if $(\mathrm{B}'_{\mathrm{K}})\rightarrow(\mathrm{B_K})$ as in (1.3) in our general case too. The answer is negative.

Example for $(\mathrm{B}'_{\mathrm{K}})\nrightarrow(\mathrm{B_K})$. Consider $\Omega=\bigcup_{n=1}^{\infty}\{1/n\}\cup\bigcup_{n=1}^{\infty}\{-1/n\}\cup\{0\}$ as a subspace of the x-axis, and set

$$\Phi(1/n, 1/m)=\min(n, m),$$
$$\Phi(0, 1/n)=\Phi(1/n, 0)=n/2,$$

$$\begin{aligned}
&\Phi(0,\ 0)=\infty,\\
&\Phi(0,\ -1/n)=\Phi(-1/n,\ 0)=2^{n-1},\\
&\Phi(1/n,\ -1/m)=\Phi(-1/m,\ 1/n)=\Phi(-1/m,\ -1/n)=2^{\min(n,m)}.
\end{aligned}$$

Let μ_0 be the measure supported by $\bigcup_{n=1}^{\infty}\{1/n\}\cup\{0\}$ such that the mass of the restriction at $1/n$ is equal to $1/2^n$ and the one at 0 is zero. Then

$$U^{\mu_0}\left(\frac{1}{n}\right)=\sum_{k=1}^{n}\frac{k}{2^k}+n\sum_{k=n+1}^{\infty}\frac{1}{2^k}, \qquad U^{\mu_0}(0)=\sum_{k=1}^{\infty}\frac{k}{2^{k+1}}$$

and

$$U^{\mu_0}\left(-\frac{1}{m}\right)=\sum_{k=1}^{m}\frac{2^k}{2^k}+2^m\sum_{k=m+1}^{\infty}\frac{1}{2^k}>m.$$

Thus $U^{\mu_0}(x)$ is bounded on S_μ but not on Ω. Next let μ be any measure on Ω for which the restriction of $U^\mu(x)$ to S_μ is continuous. We denote the mass at $1/n$ by a_n and the mass at $-1/2$ by b_n. We have

$$U^{\mu}\left(\frac{1}{n}\right)=\sum_{k=1}^{n}ka_k+n\sum_{k=n+1}^{\infty}a_k+\sum_{k=1}^{n}2^k b_k+2^n\sum_{k=n+1}^{\infty}b_k$$

and

$$U^{\mu}(0)=\frac{\sum_{k=1}^{\infty}ka_k+\sum_{k=1}^{\infty}2^k b_k}{2}.$$

As $n\to\infty$, $U^\mu(1/n)\to 2U^\mu(0)$. Therefore S_μ contains only a finite number of $\{1/n\}$. Similarly we can see that S_μ contains only a finite number of $\{-1/n\}$. Consequently, there exists N such that

$$S_\mu\subset\bigcup_{n=1}^{N}\left\{\frac{1}{n}\right\}\cup\bigcup_{n=1}^{N}\left\{-\frac{1}{n}\right\}.$$

Then $U^\mu(x)\leqq 2U^\mu(0)<\infty$ in Ω. Thus (B'_K) is satisfied.

This example serves to show $(\mathrm{B}')\nrightarrow(\mathrm{B})$ and $(\mathrm{B}')\nrightarrow(\mathrm{B}_\mathrm{K})$ naturally. Now we know the following diagram:

(1.7)
$$\begin{tikzcd}
 & & & (\mathrm{B}) \arrow[r, shift left] \arrow[d, shift right, "/" marking] \arrow[dr, shift left] \arrow[dll, shift left, "/" marking] & (\mathrm{B}') \arrow[l, shift left, "/" marking] \arrow[d, shift right, "/" marking] \arrow[dr, shift left] \arrow[dl, shift left, "/" marking] & \\
(\mathrm{F}) \arrow[r, shift left] & (\mathrm{U}) \arrow[l, shift left, "/" marking] \arrow[r, shift left] & (\mathrm{U}') \arrow[l, shift left, "/" marking] \arrow[ur, shift left] \arrow[dr, shift left] & & & (\mathrm{B}'_\mathrm{K}) \arrow[ul, shift left, "/" marking] \arrow[dl, shift left, "/" marking] \\
 & & & (\mathrm{U}_\mathrm{K}) \arrow[r, shift left] \arrow[u, shift right, "/" marking] \arrow[ur, shift left, "/" marking] \arrow[ul, shift left, "/" marking] & (\mathrm{B}_\mathrm{K}) \arrow[l, shift left, "/" marking] \arrow[u, shift right, "/" marking] \arrow[ul, shift left, "/" marking] \arrow[ur, shift left] &
\end{tikzcd};$$

there is no $\rightarrow$ relation more than in (1.4).

(III) *Principles* (K), (K′), (K_K) *and* (K'_K). We know by (I) that

$$\begin{array}{ccccc} (\mathrm{U}') \rightarrow & (\mathrm{K}) & \rightarrow & (\mathrm{K}') \\ & \downarrow(\uparrow) & & \downarrow(\uparrow) \\ & (\mathrm{B}) & \substack{\rightarrow \\ \not\leftarrow} & (\mathrm{B}') \end{array} \qquad \text{and} \qquad \begin{array}{ccc} (\mathrm{K}_{\mathrm{K}}) & \rightarrow & (\mathrm{K}'_{\mathrm{K}}) \\ \downarrow(\uparrow) & & \downarrow(\uparrow) \\ (\mathrm{B}_{\mathrm{K}}) & \substack{\rightarrow \\ \not\leftarrow} & (\mathrm{B}'_{\mathrm{K}}) \end{array} ;$$

see (II) for $(\mathrm{B}) \substack{\rightarrow \\ \not\leftarrow} (\mathrm{B}')$ and for $(\mathrm{B}_{\mathrm{K}}) \substack{\rightarrow \\ \not\leftarrow} (\mathrm{B}'_{\mathrm{K}})$.

Example for $(\mathrm{B}) \not\rightarrow (\mathrm{K}'_{\mathrm{K}})$. A simple example is as follows: $\Omega = \{0, 1\}$, $\Phi(0, 0) = 0$ and $\Phi(0, 1) = \Phi(1, 0) = \Phi(1, 1) = 1$. This Φ may be expressed by

$$\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

Take $x = 0$ for K and let μ be the unit measure at $x = 0$. Then

$$\sup_{x \in K} U^{\mu}(x) = U^{\mu}(0) = 0 \qquad \text{and} \qquad U^{\mu}(1) = 1.$$

There is no c which gives $cU^{\mu}(0) \geqq U^{\mu}(1)$ and hence (K′) is not satisfied. It is clear that (B) is satisfied.

However, in this example a constant c, actually $c = 1$, exists such that

$$c\left\{\sup_{y \in K} U^{\mu}(y) + \mu(\Omega)\right\} \geqq U^{\mu}(x) \qquad \text{for } x \in \Omega.$$

We shall give a little later an example in which such kind of inequality is not satisfied.

We know by (1.6) that $(\mathrm{B}) \rightarrow (\mathrm{K})$ and $(\mathrm{B}') \rightarrow (\mathrm{K}')$ if the kernel is positive. Let us suppose that $\Phi(P, Q) > m > -\infty$, $m < 0$, on $\Omega \times \Omega$ and set $\Phi_1(P, Q) = \Phi(P, Q) - m$. If (B′) is true for Φ, it is true for Φ_1 and, by the above result, (K′) is true for Φ_1. Namely, there is $c(K) \geqq 1$ such that

$$U^{\mu}(P) \leqq U^{\mu}(P) - m\mu(\Omega) \leqq c(K)\left\{\sup_{Q \in K} U^{\mu}(Q) - m\mu(\Omega)\right\} \qquad \text{in } \Omega,$$

whenever $S_{\mu} \subset K$ and the restriction of $U^{\mu}(P)$ to K is continuous. We can say that, if $\Phi(P, Q)$ satisfies (B′) and is bounded from below on $\Omega \times \Omega$, there are constants $c_1(K)$ and $c_2(K)$ such that

$$\sup_{P \in \Omega} U^{\mu}(P) \leqq c_1(K) \sup_{P \in K} U^{\mu}(P) + c_2(K) \tag{1.8}$$

for any unit measure μ, provided $S_{\mu} \subset K$ and the restriction of $U^{\mu}(P)$ to K is continuous.

We can state a similar remark in case (B) is satisfied.

We shall give an example of kernel which satisfies (B) but not (1.8). Consider $\Omega = \bigcup_{n=1}^{\infty} \{1/n\} \cup \bigcup_{n=0}^{\infty} \{-n\}$ as a subspace of the x-axis and set

$$\begin{aligned}
&\Phi(1/n,\ 1/m)=\Phi(1/n,\ 0)=\Phi(0,\ 1/n)=\Phi(0,\ 0)=1, \\
&\Phi(-n,\ 1/n)=\Phi(1/n,\ -n)=n, \\
&\Phi(-n,\ 1/m)=\Phi(1/m,\ -n)=-n \qquad \text{if } n\neq m, \\
&\Phi(-n,\ 0)=\Phi(0,\ -n)=-n, \\
&\Phi(-n,\ -m)=1.
\end{aligned}$$

This kernel is symmetric and continuous. We take $\{1, 1/2,\cdots,0\}$ for K. Let μ_n be the unit measure at $x=1/n$. Then $U^{\mu_n}(x)=1$ for $x\in K$ and $U^{\mu_n}(-n)=n$. Thus (1.8) is not satisfied. Next let μ be any unit measure with compact support and let n_0 be a number such that $x=-n_0, -n_0-1,\cdots$ do not belong to S_μ. It is easy to see that $U^\mu(x)$ is bounded for $x>-n_0$. Let μ' be the restriction of μ to $\bigcup_{n=1}^{\infty}\{-n\}$. We have, for $n\geqq n_0$,

$$U^\mu(-n)=\mu'(\Omega)+n\mu\left(\left\{\frac{1}{n}\right\}\right)-n\sum_{k\neq n}\mu\left(\left\{\frac{1}{k}\right\}\right)-n\mu(\{0\}).$$

This is bounded from above if

$$\mu\left(\left\{\frac{1}{n}\right\}\right)\leqq\sum_{k\neq n}\mu\left(\left\{\frac{1}{k}\right\}\right)+\mu(\{0\})$$

except for a finite number of n. Let us examine the inverse inequality:

$$2\mu\left(\left\{\frac{1}{n}\right\}\right)>\sum_{k=1}^{\infty}\mu\left(\left\{\frac{1}{k}\right\}\right)+\mu(\{0\})=1-\mu'(\Omega).$$

If $\mu'(\Omega)<1$, there are only a finite number of n for which this inequality is satisfied. If $\mu'(\Omega)=1$, then $\mu(\{1/n\})=0$ for every n. Therefore it is concluded that $U^\mu(x)$ is bounded from above in Ω.

We consider the examples given in (II) to show $(\mathrm{B}'_{\mathrm{K}})\not\to(\mathrm{B}_{\mathrm{K}})$ and $(\mathrm{B})\not\to(\mathrm{U}_{\mathrm{K}})$. In the first example the space is compact and the kernel is positive. Therefore (K') and $(\mathrm{K}'_{\mathrm{K}})$ are satisfied but (K) and $(\mathrm{K}_{\mathrm{K}})$ are not. Thus $(\mathrm{K}')\not\to(\mathrm{B}_{\mathrm{K}})$, $(\mathrm{K}'_{\mathrm{K}})\not\to(\mathrm{B}_{\mathrm{K}})$, $(\mathrm{K}')\not\to(\mathrm{K})$ and $(\mathrm{K}')\not\to(\mathrm{K}_{\mathrm{K}})$. Secondly, since the kernel is positive and satisfies (B) in the example for $(\mathrm{B})\not\to(\mathrm{U}_{\mathrm{K}})$, it satisfies (K). It does not satisfy (U'), because if it did it would satisfy $(\mathrm{U}_{\mathrm{K}})$ in virtue of $(\mathrm{U}')\to(\mathrm{U}_{\mathrm{K}})$. Thus $(\mathrm{K})\not\to(\mathrm{U}')$ is shown. This example gives also $(\mathrm{K})\not\to(\mathrm{U}_{\mathrm{K}})$. Finally, since $(\mathrm{U}_{\mathrm{K}})\not\to(\mathrm{B}')$, $(\mathrm{U}_{\mathrm{K}})\to(\mathrm{K}_{\mathrm{K}})$, $(\mathrm{U}_{\mathrm{K}})\to(\mathrm{K}'_{\mathrm{K}})$ and $(\mathrm{K}')\to(\mathrm{B}')$, it follows that $(\mathrm{U}_{\mathrm{K}})\not\to(\mathrm{K}')$, $(\mathrm{K}_{\mathrm{K}})\not\to(\mathrm{K}')$ and $(\mathrm{K}'_{\mathrm{K}})\not\to(\mathrm{K}')$. Taking easy logical consequences into consideration, we now have

$$
(1.9)\qquad
\begin{tikzcd}
 & (\mathrm{B}) \arrow[rr, shift left] \arrow[dr] \arrow[dd, shift left, "/" marking] & & (\mathrm{B}') \arrow[ll, shift left, "/" marking] \arrow[dd, shift left, "/" marking] & \\
 & & (\mathrm{B}_{\bar{\mathrm{K}}}) & & \\
(\mathrm{U}') \arrow[r, shift left] & (\mathrm{K}) \arrow[l, shift left, "/" marking] \arrow[uu, shift left] \arrow[rr, shift left] \arrow[d, shift right] & & (\mathrm{K}') \arrow[ll, shift left, "/" marking] \arrow[uu, shift left] \arrow[ul, "/" marking] \arrow[d, shift right] \arrow[dll, shift left, "/" marking] & \\
 & (\mathrm{K}_{\mathrm{K}}) \arrow[u, shift right, "/" marking] \arrow[rr, shift left] \arrow[urr, shift left, "/" marking] \arrow[d, shift right] & & (\mathrm{K}'_{\mathrm{K}}) \arrow[ll, shift left, "/" marking] \arrow[u, shift right, "/" marking] \arrow[r, shift left, "/" marking] \arrow[d, shift right] \arrow[dll, shift left, "/" marking] & (\mathrm{B}) \arrow[l, shift left, "/" marking] \\
 & (\mathrm{B}_{\mathrm{K}}) \arrow[u, shift right, "/" marking] \arrow[rr, shift left] \arrow[urr, shift left, "/" marking] & & (\mathrm{B}'_{\mathrm{K}}) \arrow[ll, shift left, "/" marking] \arrow[u, shift right, "/" marking] &
\end{tikzcd}
,\qquad
\begin{tikzcd}
(\mathrm{K}) \arrow[dr, shift right, "/" marking] & & (\mathrm{K}') \arrow[dl, shift left, "/" marking] \\
 & (\mathrm{U}_{\mathrm{K}}) \arrow[ul, shift right, "/" marking] \arrow[ur, shift left, "/" marking] \arrow[dl, shift left] \arrow[dr, shift left] & \\
(\mathrm{K}_{\mathrm{K}}) \arrow[ur, shift left, "/" marking] & & (\mathrm{K}'_{\mathrm{K}}) \arrow[ul, shift left, "/" marking]
\end{tikzcd}
,
$$

$$
\begin{array}{ll}
(\mathrm{K})\ (\leftarrow)\ (\mathrm{B}), & (\mathrm{K}')\ (\leftarrow)\ (\mathrm{B}'), \\
(\mathrm{K}_{\mathrm{K}})\ (\leftarrow)\ (\mathrm{B}_{\mathrm{K}}), & (\mathrm{K}'_{\mathrm{K}})\ (\leftarrow)\ (\mathrm{B}'_{\mathrm{K}}),
\end{array}
$$

where $(\leftarrow)$ means that $\leftarrow$ is true if the kernel is positive.

(IV) *Continuity principle* (C). First we give

Example for (F)$\not\to$(C). We observe first that, if there are points P_0 and Q_0 in Ω such that $\Phi(Q_0, Q_0)<\infty$ and

$$\varlimsup_{P\to P_0} \Phi(P, Q_0)>\Phi(P_0, Q_0),$$

then (C) is not satisfied. In fact, let μ_0 be the unit point measure at Q_0. Then $U^{\mu_0}(Q_0)=\Phi(Q_0, Q_0)<\infty$ and $U^{\mu_0}(P)$ is certainly continuous as a function on $S_{\mu_0}=\{Q_0\}$, but $U^{\mu_0}(P)=\Phi(P, Q_0)$ does not tend to $\Phi(P_0, Q_0)$ as $P\to P_0$.

In particular, if (C) is satisfied, the kernel must be continuous in the extended sense at every point of the diagonal set.

For later use (in § 1. 4) we shall give an example which is not quite the simplest. Consider

$$\Omega=\{0\}\cup\bigcup_{n=1}^{\infty}\left\{\frac{1}{n}\right\}\cup\bigcup_{n=1}^{\infty}\left\{1+\frac{1}{n}\right\}\cup\bigcup_{n=4}^{\infty}\left\{\frac{1}{2}+\frac{1}{n}\right\}\cup\cdots\bigcup_{n=k^2}^{\infty}\left\{\frac{1}{k}+\frac{1}{n}\right\}\cup\cdots$$

as a subspace of the x-axis and set

$$
\begin{array}{ll}
\Phi(0, 0)=1, & \\
\Phi(x, x)=\infty & \text{for } x\neq 0, \\
\Phi(0, 1/k)=\Phi(1/k, 0)=1-1/k, & \\
\Phi(0, x)=\Phi(x, 0)=1 & \text{for } x\neq 0,\ 1/k;
\end{array}
$$

otherwise we define it so that it is continuous in the extended sense and finite outside the diagonal set. If we note the discontinuity at the points $(0, 1/k)$, we see that (C) is not satisfied by the above reasoning. Now let μ be any unit measure on Ω which gives $\sup\limits_{x\in S_\mu} U^\mu(x)<\infty$. It follows that $S_\mu=\{0\}$. We see that

$$\sup_{x \in S_\mu} U^\mu(x)=1, \qquad \text{and} \qquad U^\mu(x)=1 \qquad\qquad \text{for } x \neq 0.$$

Thus (F) is satisfied.

The example given in (II) for $(\mathrm{U_K}) \nrightarrow (\mathrm{B}')$ serves to show $(\mathrm{C}) \nrightarrow (\mathrm{B}')$. However, we can prove

$(\mathrm{C}) \rightarrow (\mathrm{B_K})$: Let μ be a measure with compact support such that $U^\mu(P) \leqq 1$ on S_μ. We assume that there is a sequence of points $\{P_n\}$ on a compact set $K \supset S_\mu$ with $U^\mu(P_n) > n2^n$. First we consider the case that $\Phi(P, Q) \geqq 0$ on $K \times K$. By Lusin's theorem we choose a compact set $K_n \subset K$ such that the restriction of $U^\mu(P)$ to K_n is continuous on K_n and that $\mu(K-K_n)$ is arbitrarily small. If we denote by μ_n the restriction of μ to K_n, the restriction of $U^{\mu_n}(P)=U^\mu(P)-U^{\mu-\mu_n}(P)$ to K_n is continuous because it is at the same time upper and lower semicontinuous on K_n. By (C) it is continuous in Ω. Since $U^{\mu_n}(P_n) \rightarrow U^\mu(P_n)$ as $\mu(K-K_n) \rightarrow 0$, we could choose K_n such that $U^{\mu_n}(P_n) > n2^n$. We divide μ_n by 2^n and denote the measure thus obtained by ν_n. We set $\sum_n \nu_n = \nu$. Since $U^{\nu_n}(P)$ is continuous and not greater than $1/2^n$ on S_μ, the restriction of $U^\nu(P)$ to $S_\mu \supset S_\nu$ is continuous and by (C) $U^\nu(P)$ is continuous on Ω, particularly on K. Consequently $U^\nu(P)$ is bounded on K. However, $U^\nu(P_n) \geqq U^{\nu_n}(P_n) > n$ and we have a contradiction. Thus our proof is completed in the case $\Phi(P, Q) \geqq 0$ on $K \times K$. In the general case we set $m = \min \Phi(P, Q)$ on $K \times K$ and $\Phi_1(P, Q) = \Phi(P, Q) - m$. If we assume (C) on $\Phi(P, Q)$, then $\Phi_1(P, Q)$ satisfies (C) and

$$\int \Phi_1(P, Q)\, d\mu(Q) = \int \Phi(P, Q)\, d\mu(Q) - m\mu(\Omega)$$

is bounded on K. Therefore $\int \Phi(P, Q)\, d\mu(Q) = U^\mu(P)$ is bounded on K and $(\mathrm{C}) \rightarrow (\mathrm{B_K})$ is completely proved.

In the following lines we shall prove that, on a compact space, (B) $(=(\mathrm{B_K}))$ means (C) if $\Phi(P, Q)$ is continuous in the extended sense and finite outside the diagonal set of $\Omega \times \Omega$. In our example in (II) for $(\mathrm{B}) \nrightarrow (\mathrm{U_K})$ the condition is satisfied and $(\mathrm{C}) \nrightarrow (\mathrm{U_K})$ follows; this was stated also in Choquet [2]. Thus we have now

$$\begin{tikzcd}
(\mathrm{F}) \arrow[r, "/" marking] & (\mathrm{C}) \arrow[r] \arrow[dl, "/" marking] \arrow[dr, "/" marking] & (\mathrm{B_K}) \\
(\mathrm{U_K}) & & (\mathrm{B}')
\end{tikzcd}$$

If we take (1.7) into consideration, we have

$$\begin{tikzcd}
(\mathrm{F}) \arrow[r, shift left, "/" marking] & (\mathrm{C}) \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[dl, shift left, "/" marking] \arrow[dr, shift left, "/" marking] & (\mathrm{B_K}) \arrow[l, shift left, "/" marking] \\
(\mathrm{U_K}) \arrow[ur, shift left, "/" marking] & & (\mathrm{B}') \arrow[ul, shift left, "/" marking]
\end{tikzcd}$$

Next we shall prove

$$(1.10)\qquad (\mathrm{C}) \rightleftarrows (\mathrm{B_K}) \rightleftarrows (\mathrm{B'_K})$$

under the hypothesis that $\Phi(P, Q)$ is continuous in the extended sense and finite outside the diagonal set in $\Omega \times \Omega$. It is sufficient to prove

$(\mathrm{B'_K}) \rightarrow (\mathrm{C})$ (Ohtsuka [2; 4]): Let the restriction of $U^{\mu}(P)$ to the compact support S_{μ} of μ be continuous. It is enough to show that $U^{\mu}(P)$ is continuous at an arbitrary point P_0 of S_{μ}. If $\Phi(P_0, P_0) < \infty$, $U^{\mu}(P)$ is obviously continuous at P_0. Suppose $\Phi(P_0, P_0) = \infty$. We take a neighborhood N_{P_0} of P_0 such that $\Phi(P, Q) > 0$ on $N_{P_0} \times N_{P_0}$. Since the potential of the restriction of μ to the outside of N_{P_0} is continuous at P_0, we may assume from the beginning that Ω is compact and the kernel is positive on $\Omega \times \Omega$. We denote by μ_n the restriction of μ to

$$B_n = \{P;\ \Phi(P_0, P) > n\}.$$

Then the restriction of $U^{\mu_n}(P)$ to S_{μ} is continuous. Since $U^{\mu}(P)$ is finite, $U^{\mu_n}(P)$ decreases to 0 with $1/n$ on S_{μ} and the convergence is uniform by a theorem known as Dini's theorem in the classical case.

If we take S_{μ} as a compact set K in the definition of (K'), there is a constant $c = c(S_{\mu})$ such that

$$\sup_{P \in \Omega} U^{\mu_n}(P) \leqq c \sup_{P \in S_{\mu}} U^{\mu_n}(P)$$

for every n by $(\mathrm{B}') \rightarrow (\mathrm{K}')$; note that $(\mathrm{B'_K}) = (\mathrm{B}')$ in the present case because Ω is supposed compact. Therefore $\sup_{P \in \Omega} U^{\mu_n}(P) \searrow 0$ as $n \rightarrow \infty$. Given $\varepsilon > 0$, we take n_0 such that $U^{\mu_{n_0}}(P) < \varepsilon$ in Ω. Since $U^{\mu - \mu_{n_0}}(P)$ is continuous in B_{n_0}, we have, for P sufficiently near P_0,

$$|U^{\mu - \mu_{n_0}}(P) - U^{\mu - \mu_{n_0}}(P_0)| < \varepsilon$$

and hence

$$|U^{\mu}(P) - U^{\mu}(P_0)| < \varepsilon + U^{\mu_{n_0}}(P) + U^{\mu_{n_0}}(P_0) < 3\varepsilon.$$

Remark. By the aid of Lemma 1.1 we can generalize this result as follows: Let K be a compact set and X be an arbitrary set in Ω. Instead of $(\mathrm{B'_K})$ assume that whenever the potential $U^{\mu}(P)$ of a measure μ with $S_{\mu} \subset K$ is continuous as a function on S_{μ}, it is bounded on X. Then $U^{\mu}(P)$ is continuous as a function on $S_{\mu} \cup X$.

(V) *Supplementary relation between* (B) *and* (B′). First we notice that $(\mathrm{B}') \rightarrow (\mathrm{B})$ provided that $\Phi(P, Q)$ is continuous in the extended sense and finite outside the diagonal set and that $\inf_{\Omega \times \Omega} \Phi(P, Q) > -\infty$. In fact, $(\mathrm{B}') \rightarrow (\mathrm{B'_K}) \rightarrow (\mathrm{C})$ and the proof for $(\mathrm{C}) \rightarrow (\mathrm{B_K})$ applies to show $(\mathrm{C}) \rightarrow (\mathrm{B})$ under the condition that $\inf_{\Omega \times \Omega} \Phi(P, Q) > -\infty$. We have given an example for $(\mathrm{B}') \nrightarrow (\mathrm{B})$ (originally for

$(\mathrm{B}_K') \nrightarrow (\mathrm{B}_K)$) in (II) in which the kernel is positive but discontinuous. Now we show that $(\mathrm{B}') \nrightarrow (\mathrm{B})$ even if $\Phi(P, Q)$ is continuous in the extended sense and finite outside the diagonal set in $\Omega \times \Omega$.

Example for $(\mathrm{B}') \nrightarrow (\mathrm{B})$: Consider $\Omega = \bigcup_{n=1}^{\infty} \{1/n\} \cup \bigcup_{n=2}^{\infty} \{-n\}$ as a subspace of the x-axis and set

$$
\begin{aligned}
&\Phi(0, 0) = \infty, && \\
&\Phi(0, 1/n) = \Phi(1/n, 0) = n, && \\
&\Phi(1/n, 1/n) = n^3, && \\
&\Phi(1/n, 1/m) = \min(n, m) && \text{if } n \neq m, \\
&\Phi(-n, 1/m) = \Phi(1/m, -n) = \begin{cases} -n & \\ 3n^3 & \end{cases} && \begin{array}{l} \text{if } m < n, \\ \text{if } m \geqq n,\ n \neq 0, \end{array} \\
&\Phi(-n, 0) = \Phi(0, -n) = 3n^3 && \text{if } n \neq 0, \\
&\Phi(-n, -n) = \infty && \text{if } n \neq 0, \\
&\Phi(-n, -m) = \Phi(-m, -n) = 1 && \text{if } n \neq m,\ n \neq 0,\ m \neq 0.
\end{aligned}
$$

The kernel is continuous in the extended sense and finite outside the diagonal set. Let μ_0 be the measure on $\bigcup_{n=1}^{\infty} \{1/n\}$ such that each point $1/n$ supports the mass $1/n^3$. We have

$$
\begin{aligned}
U^{\mu_0}(-n) &= -n \sum_{k=1}^{n-1} \frac{1}{k^3} + 3n^3 \sum_{k=n}^{\infty} \frac{1}{k^3} > -n\left(\frac{9}{8} + \int_2^{\infty} \frac{dx}{x^3}\right) + 3n^3 \int_n^{\infty} \frac{dx}{x^3} \\
&= -\frac{5}{4} n + \frac{3}{2} n = \frac{n}{4} \to \infty
\end{aligned}
$$

with n. It holds that

$$
\begin{aligned}
U^{\mu_0}\left(\frac{1}{n}\right) &= \sum_{k=1}^{n-1} \frac{k}{k^3} + 1 + n \sum_{k=n+1}^{\infty} \frac{1}{k^3} < \sum_{k=1}^{\infty} \frac{1}{k^2} + 1 + n \int_n^{\infty} \frac{dx}{x^3} \\
&= \sum_{k=1}^{\infty} \frac{1}{k^2} + 1 + \frac{1}{2n}
\end{aligned}
$$

and

$$
U^{\mu_0}(0) = \sum_{k=1}^{\infty} \frac{1}{k^2}.
$$

Thus $U^{\mu_0}(x)$ is bounded on $S_{\mu_0} = \sum_{n=1}^{\infty} \{1/n\} \cup \{0\}$ and (B) is not satisfied.

Next let μ be a unit measure such that $U^{\mu}(x)$ is continuous as a function on $S_\mu \subset \bigcup_{n=1}^{\infty} \{1/n\} \cup \{0\}$ and $\mu(\{0\}) = 0$. It is easy to discuss the case in which S_μ contains only a finite number of points and hence we assume that $0 \in S_\mu$. We set $\mu(\{1/n\}) = m_n$.

We have

$$U^{\mu}\left(\frac{1}{n}\right)=\sum_{k=1}^{n-1} km_k+n^3 m_n+n\sum_{k=n+1}^{\infty} m_k$$

and

$$U^{\mu}(0)=\sum_{k=1}^{\infty} km_k.$$

Our assumption requires that $U^{\mu}(1/n)\to U^{\mu}(0)$ as $n\to\infty$. Hence $n^3 m_n\to 0$ as $n\to\infty$. Then it follows that $n\sum_{k=n+1}^{\infty} m_k\to 0$. Therefore $U^{\mu}(x)$ is continuous as a function on S_{μ} if and only if $n^3 m_n\to 0$ as $n\to\infty$. We have

$$U^{\mu}(-n)=-n\sum_{k=1}^{n-1} m_k+3n^3\sum_{k=n}^{\infty} m_k.$$

For a large n it holds that

$$\sum_{k=1}^{n-1} m_k>\frac{1}{2} \qquad \text{and} \qquad k^3 m_k<\frac{1}{6} \text{ if } k\geqq n.$$

For such n

$$U^{\mu}(-n)<-\frac{n}{2}+\frac{n^3}{2}\sum_{k=n}^{\infty}\frac{1}{k^3}<-\frac{n}{2}+\frac{n^3}{2}\left(\frac{1}{n^3}+\int_n^{\infty}\frac{dx}{x^3}\right)<\frac{1}{2}-\frac{n}{2}+\frac{n}{4}$$
$$=\frac{1}{2}-\frac{n}{4}<\frac{1}{2}.$$

This shows that $U^{\mu}(x)$ is bounded from above in Ω and it is verified that (B′) is satisfied.

We now know that

(1.11) $\qquad$ (B′) (→) (B),

if $\Phi(P, Q)$ is continuous in the extended sense and finite outside the diagonal set in $\Omega\times\Omega$ and if $\inf_{\Omega\times\Omega}\Phi(P, Q)>-\infty$, and that both the continuity in the extended sense and the lower boundedness of $\Phi(P, Q)$ are necessary for → to be true.

(VI) *Domination principle* (D) *and restricted domination principle* (D*). First we give

Example for (D*)$\nrightarrow$(D): $\Omega=\{0\}\cup\{1\}\cup\{2\}$, $\Phi(0, 0)=\Phi(1, 2)=\Phi(2, 1)=\Phi(0, 2)=\Phi(2, 0)=1$, $\Phi(1, 1)=\Phi(2, 2)=\infty$, $\Phi(1, 0)=\Phi(0, 1)=2$. It is obvious that (D*) is satisfied. Let μ be the unit point measure at 0 and ν be the point measure at 1 with total mass 1/2. Then $U^{\mu}(0)=U^{\nu}(0)$ but $U^{\mu}(2)=1>1/2=U^{\nu}(2)$. Thus (D) is not satisfied.

Examples for (D)$\nrightarrow$(B_K'). First example: Consider $\Omega=\bigcup_{n=1}^{\infty}\{1/n\}\cup\{0\}$ as a subspace of the x-axis, and set $\Phi(1/n, 1/m)=nm$, $\Phi(0, 1/n)=\Phi(1/n, 0)=n$, $\Phi(0, 0)=1$. This does not satisfy (B'_{K}); consider a measure at $x=0$. Next we observe that, for any μ,

$$U^{\mu}(1/n)=n\mu(\{0\})+n\sum_{m} m\mu(\{1/m\})=nU^{\mu}(0).$$

Let μ, ν be measures such that $U^{\mu}(x)\leqq U^{\nu}(x)$ on S_{μ}. Then it follows that $U^{\mu}(0)\leqq U^{\nu}(0)$ and this shows that $U^{\mu}(1/n)\leqq U^{\nu}(1/n)$ for every n. Thus $U^{\mu}(x)\leqq U^{\nu}(x)$ everywhere in Ω. We have the conclusion that (D) is satisfied.

Second example: $\Omega=\{0\}\cup\{1\}$, $\Phi(0, 0)=1$, $\Phi(1, 1)=\Phi(0, 1)=\Phi(1, 0)=\infty$.

In the first example the kernel is finite-valued in Ω but discontinuous at $(0, 0)$, and in the second example it is continuous in the extended sense but infinite even outside the diagonal set. If the kernel is continuous outside the diagonal set and positive on the diagonal set, then we can derive more than (B'_{K}) from (D); Ninomiya [8], Lemma 3, proved (D)$\rightarrow$(B'_{K}) under some additional conditions. We shall prove general results in § 1.5.

Next we give

Example for (D)$\not\rightarrow$(B′), even if the kernel is positive, symmetric, continuous in the extended sense and finite outside the diagonal set. We take for Ω a unit ball B and a sequence of points $\{P_n\}$ outside B in E_3 which tends to the point at infinity.

We set

$$\Phi(P, Q)=\Phi(Q, P)=\begin{cases} \dfrac{1}{\overline{PQ}} & \text{for } P, Q\in B,\\ n & \text{for } P\in B,\ Q=P_n,\\ mn & \text{for } P=P_m,\ Q=P_n (m\neq n),\\ \infty & \text{for } P=Q=P_n. \end{cases}$$

Let us see that $\Phi(P, Q)$ satisfies (D). If $\mu\in\mathscr{E}$, $S_{\mu}\cup S_{\nu}\subset B$ and $U^{\mu}(P)\leqq U^{\nu}(P)$ on S_{μ}, then $U^{\mu}(P)\leqq U^{\nu}(P)$ on B as it is known for Newtonian potentials. Let λ be the unit equilibrium measure on S_{μ}, B being as a space, and $W(S_{\mu})$ be the equilibrium constant. Then $U^{\lambda}(P)\leqq W(S_{\mu})$ on B and $U^{\lambda}(P)=W(S_{\mu})$ p.p.p. on S_{μ}. Therefore

$$(\mu, \lambda)=W(S_{\mu})\,\mu(B)\leqq(\nu, \lambda)\leqq W(S_{\mu})\nu(B),$$

and hence $\mu(B)\leqq\nu(B)$. Consequently

$$U^{\mu}(P_n)=n\mu(B)\leqq n\nu(B)=U^{\nu}(P_n).$$

Thus $U^{\mu}(P)\leqq U^{\nu}(P)$ everywhere in Ω. The next case is when $\mu\in\mathscr{E}$ and $S_{\mu}\subset B$, but S_{ν} is a general set. We denote by ν_B the restriction of ν to B and by ν_n the restriction of ν to P_n. We have

$$U^{\nu}(P)=U^{\nu_B}(P)+\sum_{n=1}^{\infty} n\nu_n(\Omega) \qquad \text{on } B.$$

Suppose that $U^{\mu}(P)\leqq U^{\nu}(P)$ on S_{μ}. Let λ_0 be the uniform unit measure on ∂B, and set $\nu'=\nu_B+\sum_{n=1}^{n} n\nu_n(\Omega)\lambda_0$. We have

$$U^{\mu}(P)\leqq U^{\nu}(P)=U^{\nu_B}(P)+\sum_{n=1}^{\infty} n\nu_n(\Omega)\, U^{\lambda_0}(P)=U^{\nu'}(P) \qquad \text{on } S_{\mu}.$$

This is true on B. Using the same λ as in the first case we find

$$W(S_{\mu})\,\mu(B)\leqq W(S_{\mu})\,\{\nu_B(B)+\sum_{n=1}^{\infty} n\nu_n(\Omega)\}.$$

Therefore

$$\mu(B)\leqq \nu_B(B)+\sum_{n=1}^{\infty} n\nu_n(\Omega).$$

If $\nu_k\neq 0$, then $U^{\nu}(P_k)=\infty$ and naturally $U^{\mu}(P_k)<U^{\nu}(P_k)$. If $\nu_k=0$, then $U^{\mu}(P_k)=k\mu(B)$ and $U^{\nu}(P_k)=k\nu_B(B)+k\sum_{n=1}^{\infty} n\nu_n(\Omega)$. Therefore $U^{\mu}(P_k)\leqq U^{\nu}(P_k)$. Thus $U^{\mu}(P)\leqq U^{\nu}(P)$ everywhere in Ω and (D) is true. However, if we consider λ_0 on ∂B, then $U^{\lambda_0}(P)=1$ on B but $U^{\lambda_0}(P_n)=n$. This shows that (B′) is not satisfied.

Finally we give

Example for (F)$\nrightarrow$(D*): The simplest example of Φ is given by

$$\begin{pmatrix} c & c & c \\ c & c & 1 \\ c & 1 & c \end{pmatrix}$$

with $c>1$. This shows that, given $c>1$, there is a kernel which satisfies (F) but not $(\mathrm{U}_{\mathrm{d}}^{*})_c$.

We refer to the work of Ninomiya [8] for some results in the direction (D)$\rightarrow$(F); this will be discussed in § 2.11 of Chapter II in our paper.

Let us repeat what we have obtained so far:

(1.12) $$(\mathrm{F})\nrightarrow(\mathrm{D}^*)\underset{\nrightarrow}{\leftarrow}(\mathrm{D})\nrightarrow(\mathrm{B}'_{\mathrm{K}}).$$

Even if $\Phi(P, Q)$ is positive, symmetric, continuous in the extended sense and finite outside the diagonal set,

(1.13) $$(\mathrm{D})\nrightarrow(\mathrm{B}').$$

We shall discuss again (D) and (D*) in § 1.5 and in the next chapter.

1.4. Local behavior of potentials.

A point $P_0\in\Omega$ is called by Choquet [2] a *point of c-undulation* $(c>0)$ for kernel Φ if, for any neighborhood N of P_0, there is a measure μ with compact support $S_\mu \subset N$ such that

$$\sup_{P\in N} U^\mu(P) > c \sup_{P\in S_\mu} U^\mu(P);\ ^{5)}$$

he defined it in case the kernel is positive. We denote by $\mathbf{O}_c$ the set of all points of c-undulation and set $\mathbf{O}_\infty = \bigcap_{c>1} \mathbf{O}_c$. These sets are naturally closed. Choquet [2] observed several relations of these sets with some principles. Let us define the *undulation coefficient* $\mathbf{o}(P)$ of a point P by the supremum of c such that $P \in \mathbf{O}_c$.

We shall define different kinds of points and see relations among them. First we define, for a kernel of general sign and a class $\mathscr{F}$ of functions in Ω,

Point of $\mathscr{F}$-relative c-undulation $P_0 (c>0)$. For any neighborhood N of P_0, there are a μ with $S_\mu \subset N$ and an $f \in \mathscr{F}$ such that $U^\mu(P) \leqq f(P)$ on S_μ but $U^\mu(P') > cf(P')$ at some point $P' \in N$.

We denote by $\mathbf{O}_c^{(\mathscr{F})}$ the set of all points of $\mathscr{F}$-relative c-undulation and set $\mathbf{O}_\infty^{(\mathscr{F})} = \bigcap_{c>1} \mathbf{O}_c^{(\mathscr{F})}$. We define the $\mathscr{F}$-relative undulation coefficient $\mathbf{o}_{\mathscr{F}}(P)$ by $\sup \{c;\ P \in \mathbf{O}_c^{(\mathscr{F})}\}$. In case $\mathscr{F}$ consists of all positive potentials, which are defined everywhere, and $\mathscr{F}$ is not empty, a point of $\mathbf{O}_c^{(\mathscr{F})}$ will be called a *point of c-revolution* and denoted by $\mathbf{R}_c$. The notations $\mathbf{R}_\infty$ and $\mathbf{r}(P)$ will be used. In case $\mathscr{F}$ consists of all positive potentials of measures of $\mathscr{E}$ with compact support and $\mathscr{F}$ is not empty, we add the adjective 'restricted' and the symbol *; for example, $\mathbf{R}_c^*$ will mean the set of all points of restricted c-revolution.

Furthermore we define

Point of $\mathscr{F}$-relative c-cliff $P_0 (c>0)$. P_0 is not isolated and, for any neighborhood N of P_0, there are an $f \in \mathscr{F}$, which does not vanish in a neighborhood of P_0, and a measure μ with compact support S_μ, containing P_0 and included in N, such that

$$\varlimsup_{P\to P_0} \frac{U^\mu(P)}{f(P)} > c \varlimsup_{P\in S_\mu, P\to P_0} \frac{U^\mu(P)}{f(P)}; \tag{1.14}$$

the value at P_0 is not considered when we take $\varlimsup$ as $P \to P_0$ but the right side is replaced by $U^\mu(P_0)/f(P_0)$ in case P_0 is isolated on S_μ.

We shall denote the set of all points of $\mathscr{F}$-relative c-cliff by $\mathbf{P}_c^{(\mathscr{F})}$ and set $\mathbf{P}_\infty^{(\mathscr{F})} = \bigcap_{c>1} \mathbf{P}_c^{(\mathscr{F})}$. The $\mathscr{F}$-relative cliff coefficient $\mathbf{p}_{\mathscr{F}}(P)$ is defined by $\sup \{c;\ P \in \mathbf{P}_c^{(\mathscr{F})}\}$. In case $\mathscr{F}$ consists of positive constants we drop the adjective '$\mathscr{F}$-relative' and the superscript $\mathscr{F}$. In case $\mathscr{F}$ consists of all potentials, which are defined everywhere in Ω, we shall call a point of $\mathbf{P}_c^{(\mathscr{F})}$ a point of *c-gap* and write $\mathbf{S}_c$ for $\mathbf{P}_c^{(\mathscr{F})}$. We also write $\mathbf{s}(P)$ for $\mathbf{p}_{\mathscr{F}}(P)$. In case $\mathscr{F}$ consists of all potentials of $\mathscr{E}$ with compact support, we add the adjective 'restricted' and

5) Originally Choquet [2] included the equality with the additional assumption that the right side is finite.

the symbol *.

In (1.14) we can not take $c=\infty$ if the upper limit on the right side is positive. Suppose, however, that there is a point P_0 with the following property: For any neighborhood N of P_0 there is a measure μ with S_μ, containing P_0 and included in N, such that $U^\mu(P)$ is bounded from above on S_μ but $\overline{\lim\limits_{P\to P_0}}\, U^\mu(P)=\infty$. We shall denote the set of such points by $\mathbf{P}'_\infty$. More generally, we can define $\mathbf{P}'^{(\mathscr{F})}_\infty$ in a similar fashion.

Point of $\mathscr{F}$-relative unboundedness P_0. In any neighborhood N of P_0 the $\mathscr{F}_N$-relative upper boundedness principle is not true, where $\mathscr{F}_N$ consists of the restrictions of the functions of $\mathscr{F}$ to N.

We shall denote the set of all points of $\mathscr{F}$-relative unboundedness by $\mathbf{Q}_\mathscr{F}$. In case $\mathscr{F}$ consists of all positive constants we drop the adjective '$\mathscr{F}$-relative' and the subscript $\mathscr{F}$. In case $\mathscr{F}$ consists of all positive potentials, which are defined everywhere in Ω, and $\mathscr{F}$ is not empty, we call a point of $\mathbf{Q}_\mathscr{F}$ a point of relative unboundedness and write $\mathbf{Q}_\mathrm{d}$ for $\mathbf{Q}_\mathscr{F}$. Corresponding change is to be made in the restricted case.

Point of $\mathscr{F}$-relative weak unboundedness P_0. For any neighborhood N of P_0 there are an $f\in\mathscr{F}$ and a measure μ with compact support $S_\mu\subset N$ such that the restriction of U^μ/f to S_μ can be defined and is continuous but U^μ/f is not defined or is not bounded in N.

We shall denote the set of all points of weak unboundedness by $\mathbf{Q}'_\mathscr{F}$. In special cases we shall make changes of terminologies and notations in the same way as for points of ($\mathscr{F}$-relative) unboundedness.

Point of $\mathscr{F}$-relative discontinuity P_0. For any neighborhood N of P_0 there are an $f\in\mathscr{F}$ and a measure μ with compact support $S_\mu\subset N$ such that the restriction of U^μ/f to S_μ can be defined and is continuous but not in N.

We shall denote the set of all points of $\mathscr{F}$-relative discontinuity by $\mathbf{D}$. In special cases we shall make some changes of terminologies and notations.

We shall see relations among these sets. An obvious relation is

$$\mathbf{Q}'_\mathscr{F}\subset\mathbf{Q}_\mathscr{F}\subset\mathbf{O}^{(\mathscr{F})}_\infty.$$

In (IV) of the preceding section § 1.3, we proved that (C)$\to$($\mathrm{B_K}$). From this fact it follows that

$$\mathbf{Q}\subset\mathbf{D}.$$

The example for ($\mathrm{B'_K}$)$\not\to$($\mathrm{B_K}$) given in (II) of § 1.3 shows that $\mathbf{Q}'$ can be empty while $\mathbf{Q}\neq\emptyset$. Taking the inclusion relation into consideration, we shall express this fact by $\mathbf{Q}'\lneqq\mathbf{Q}$. The example for (F)$\not\to$(C) given in (IV) of § 1.3 shows that $\mathbf{Q}\lneqq\mathbf{D}$ and that, for every $c>1$, $\mathbf{O}_c$ can be empty while $\mathbf{D}\neq\emptyset$. We shall express this fact by $\mathbf{D}\not\lneqq\mathbf{O}_c$. The example for (B)$\not\to$($\mathrm{U_K}$) given in (II) of § 1.3 shows that $\mathbf{Q}\lneqq\mathbf{O}_\infty$ even if the kernel is symmetric, continuous in the extended sense and finite outside the diagonal set. This example shows also that (C)$\nrightarrow$(U_K) and hence that $\mathbf{O}_\infty \not\subset \mathbf{D}$. We now have

$$\mathbf{Q}' \subset \mathbf{Q} \subset \mathbf{O}_\infty, \qquad \begin{array}{c}\mathbf{Q} \subset \mathbf{D} \not\subset \mathbf{O}_c \\ \not\cup \\ \mathbf{O}_\infty\end{array} \qquad \text{for any } c>1.$$

If the kernel is continuous in the extended sense and finite outside the diagonal set, we have

$$\mathbf{Q}=\mathbf{Q}'=\mathbf{D}$$

by (1.10) of §1.3.

Next we are particularly concerned with $\mathbf{P}_c$ and $\mathbf{O}_c$ ($c \leqq \infty$).

(i) We shall give two examples as to undulation coefficient.

For any c, $1<c<\infty$, there are a positive kernel, which is continuous in the extended sense and finite outside the diagonal set, and a point P_0 such that $\mathbf{o}(P_0) = c$ and $P_0 \notin \mathbf{O}_c$.

We consider $\Omega = \bigcup_{n=2}^{\infty} ([(2n+1)^{-1}, (2n)^{-1}] \cup \{-1/n\}) \cup \{0\}$ as a subspace of the x-axis, and, denoting $[(2n+1)^{-1}, (2n)^{-1}]$ by I_n, we set

$$\Phi(-1/n, x) = \Phi(x, -1/n) = c \log \{8n(2n+1)\} \qquad \text{if } x \in I_n,$$

$$\Phi(x, y) = \Phi(y, x) = \log \frac{1}{|x-y|} \qquad \text{for any other } (x, y).$$

If we take a unit measure μ_n on I_n which gives a constant potential on I_n, then $U^{\mu_n}(x) = \log\{8n(2n+1)\}$ on I_n and $U^{\mu_n}(-1/n) = c \log\{8n(2n+1)\}$. This shows that $0 \in \mathbf{O}_{c'}$ for every $c'<c$. Let μ be any measure with $\sup_{S_\mu} U^\mu < \infty$. Then S_μ is included in the positive axis. Since $\Phi(x, y)$ is logarithmic for $x, y \geqq 0$, $U^\mu(x) \leqq \sup_{y\in S_\mu} U^\mu(y)$ for $x \geqq 0$. If $\mu(I_n)>0$, we have

$$\begin{aligned} U^\mu\left(-\frac{1}{n}\right) &= \int_{y \notin I_n} \log \frac{1}{\left|y+\dfrac{1}{n}\right|}\, d\mu(y) + c\mu(I_n) \log\{8n(2n+1)\} \\ &\leqq \int_{y\notin I_n} \log \frac{1}{|y-x'|}\, d\mu(y) + c \sup_{x\in S_{\mu_n}} U^{\mu_n}(x) \end{aligned}$$

for every point $x' \in I_n$, where μ_n is the restriction of μ to I_n. Therefore

$$U^\mu\left(-\frac{1}{n}\right) \leqq c \sup_{x\in S_{\mu_n}} U^\mu(x) \leqq c \sup_{x\in S_\mu} U^\mu(x).$$

If $\mu(I_n)=0$, then $\Phi(-1/n, y)$ is logarithmic for every $y \in S_\mu$ and hence $U^\mu(-1/n) \leqq \sup_{x\in S_\mu} U^\mu(x)$. Therefore

$$U^\mu(x) \leqq c \sup_{y\in S_\mu} U^\mu(y) \qquad \text{in } \Omega$$

in any case and it is proved at the same time that $0 \notin \mathbf{O}_c$ and $\mathbf{o}(0)=c$.

Secondly, given any c, $1<c<\infty$, *there are a positive kernel, which is continuous in the extended sense and finite outside the diagonal set, and a point* P_0 *such that* $\mathbf{o}(P_0)=c$ *and* $P_0 \in \mathbf{O}_c$.

We choose $c_1, \cdots, c_n, \cdots, c_n>c$, decreasing to c. We take the same Ω as in the first example and set

$$\Phi(-1/n, x)=\Phi(x, -1/n)=c_n \log \{8n(2n+1)\} \qquad \text{if } x \in I_n,$$

$$\Phi(x, y)=\Phi(y, x)=\log \frac{1}{|x-y|} \qquad \text{for any other } (x, y).$$

Let μ be a measure with $S_\mu \subset \bigcup_{k=n}^{\infty} I_k$ and with finite sup U^μ on S_μ. As in the above example we see that

$$U^\mu(x) \leqq c_n \sup_{y \in S_\mu} U^\mu(y) \qquad \text{in } \Omega.$$

Therefore $\mathbf{o}(0) \leqq c$. For the unit measure μ_n on I_n which gives a constant potential on I_n, $U^{\mu_n}(x)=\log \{8n(2n+1)\}$ on I_n and $U^{\mu_n}(-1/n)=c_n \log \{8n(2n+1\}$. Since $c_n>c$, it follows that $0 \in \mathbf{O}_c$. Hence $\mathbf{o}(0)=c$.

(ii) *Let* $c>1$. *Consider a nonnegative kernel which is continuous outside the diagonal set and assume that* $\Phi(P_0, P_0)=\infty$ *or* $\overline{\lim\limits_{P \to P_0}} \Phi(P, P_0)<c\, \Phi(P_0, P_0)<\infty$ *for a point* P_0. *If, for any neighborhood* N *of* P_0, *there is a measure* μ *with compact support* $S_\mu \ni P_0$, *contained in* N, *such that*

$$\overline{\lim_{P \to P_0}}\, U^\mu(P) \geqq c \overline{\lim_{P \in S_\mu, P \to P_0}}\, U^\mu(P) \; (<\infty), \tag{1.15}$$

then there exists $c'>c$ *for which* $P_0 \in \mathbf{P}_{c'}$.

From our assumption on $\Phi(P_0, P_0)$, it follows that $\Phi(P_0, P_0)>0$. Hence $\Phi(P, Q)>0$ on $N \times N$ for some neighborhood N of P_0. We assume the existence of $\mu \neq 0$ with $S_\mu \subset N$ and satisfying (1. 15). By our assumption on $\Phi(P_0, P_0)$, S_μ does not coincide with P_0. Let μ' denote the restriction of μ to the outside of P_0. If $\mu'(N)=0$, $\Phi(P_0, P_0)<\infty$ and by (1.15) we should have

$$0 \leqq (c-1)\, U^{\mu'}(P_0) \leqq \mu(\{P_0\}) \left\{ \overline{\lim_{P \to P_0}}\, \Phi(P, P_0) - c\, \Phi(P_0, P_0) \right\} < 0.$$

This is a contradiction. Accordingly $\mu'(N)>0$ and hence $U^{\mu'}(P_0)>0$. We denote also the upper limits in (1.15) by α and β respectively. We take a directed set D which provides P_0 with a base of neighborhoods, and denote by μ_V the restriction of μ to $V \in D$. As $V \in D$ converges to P_0, S_{μ_V} tends to P_0. We set

$$\alpha_V = \overline{\lim_{P \to P_0}}\, U^{\mu_V}(P) \qquad \text{and} \qquad \beta_V = \overline{\lim_{P \in S_\mu, P \to P_0}}\, U^{\mu_V}(P).$$

Since $\alpha_V \geqq \alpha_{V'}$, and $\beta_V \geqq \beta_{V'}$ if $V \supset V'$, α_V tends to a certain value $\alpha_0 \geqq 0$ and β_V does to $\beta_0 \geqq 0$ as $V \to P_0$ on D. We see that $\alpha = U^{\mu-\mu_V}(P_0) + \alpha_V$ and that the right side tends to $U^{\mu'}(P_0) + \alpha_0$ as $V \to P_0$. Similarly $\beta = U^{\mu'}(P_0) + \beta_0$. By (1.15)

$$\alpha = U^{\mu'}(P_0) + \alpha_0 \geqq c(U^{\mu'}(P_0) + \beta_0)$$

and hence

$$\alpha_0 - c\beta_0 \geqq (c-1)\, U^{\mu'}(P_0) > 0.$$

We find $V_0 \in D$ such that $V_0 \subset N$ and

$$\beta_V < \beta_0 + \frac{1}{2}\left(1 - \frac{1}{c}\right) U^{\mu'}(P_0) \qquad \text{for every } V \subset V_0,\ V \in D.$$

We denote the second term of the right side by c'' and have

$$\alpha_V \geqq \alpha_0 \geqq c\beta_0 + 2cc'' > c\beta_{V_0} + cc'' = (c + cc''/\beta_{V_0})\,\beta_{V_0} \geqq (c + cc''/\beta_{V_0})\beta_V \quad \text{if } V \subset V_0.$$

This shows that $P_0 \in \mathbf{P}_{c'}$ with $c' = c + cc''/\beta_{V_0}$.

COROLLARY. *Let $c > 1$ and assume that there are a measure μ and a point P_0, not isolated on S_μ, such that*

$$\varlimsup_{P \to P_0} U^\mu(P) = c \varlimsup_{P \in S_\mu,\, P \to P_0} U^\mu(P) < \infty.$$

Then $P_0 \in \mathbf{P}_{c'}$ for some $c' > c$ unless $\Phi(P_0, P_0) < \infty$ and $\varlimsup_{P \to P_0} \Phi(P, P_0) = c\,\Phi(P_0, P_0)$.

REMARK. Let $c \geqq 1$. Consider a nonnegative kernel which is continuous outside the diagonal set. If $P_0 \in \mathbf{P}_c$, then $P_0 \in \mathbf{P}_{c'}$ for some $c' > c$.

Suppose that there is a measure μ satisfying

$$\alpha = \varlimsup_{P \to P_0} U^\mu(P) > c \varlimsup_{P \in S_\mu,\, P \to P_0} U^\mu(P) = c\beta.$$

We choose c' such that $\alpha/\beta > c' > c$. Let V be any neighborhood of P_0, and use the same notations μ_V, α_V and β_V as above. It holds that $\alpha = U^{\mu-\mu_V}(P_0) + \alpha_V$ and $\beta = U^{\mu-\mu_V}(P_0) + \beta_V$. Therefore

$$\alpha_V > c'\beta_V + (c'-1)\, U^{\mu-\mu_V}(P_0) \geqq c'\beta_V.$$

This means that $P_0 \in \mathbf{P}_{c'}$.

(iii) *For a nonnegative kernel which is continuous outside the diagonal set,*

$$\mathbf{P}_c \subset \mathbf{O}_c \qquad \textit{for any } c > 1.$$

We take $P_0 \in \mathbf{P}_c$. If $\varlimsup_{P \to P_0} \Phi(P, P_0) > c\Phi(P_0, P_0)$, we see $P_0 \in \mathbf{O}_c$ immediately. Therefore we assume the existence of a unit measure μ with $S_\mu \ni P_0$, $S_\mu \neq \{P_0\}$, satisfying

$$\overline{\lim_{P\to P_0}}\ U^{\mu}(P)>c\ \overline{\lim_{P\in S_{\mu},\,P\to P_0}}\ U^{\mu}(P);$$

we denote the upper limits by α and β as in (ii). First we suppose that the kernel is positive on $\Omega\times\Omega$. With the same notations as in (ii) we find, for V_0 chosen in (ii), an element $V_1\in D$ included in V_0 with the property that

$$\sup_{V_1\cap S_{\mu}} U^{\mu_{V_0}}<\beta_{V_0}+c''.$$

Therefore

$$\sup_{V_1} U^{\mu_{V_1}}\geqq\alpha_{V_1}\geqq\alpha_0\geqq c\beta_{V_0}+cc''>c\sup_{V_1\cap S_{\mu}} U^{\mu_{V_0}}\geqq c\sup_{S_{\mu_{V_1}}} U^{\mu_{V_1}}. \tag{1.16}$$

This shows that $P_0\in\mathbf{O}_c$.

Next consider $\Phi\geqq 0$. We choose $\varepsilon>0$ such that $\alpha>c\beta+\varepsilon(c-1)$ and denote by $U_{\varepsilon}(P)$ the potential with kernel $\Phi(P,Q)+\varepsilon$ of a measure ν. Since this kernel is positive, (1.16) holds for it and

$$\sup_{V_1} U_{\varepsilon}^{\mu_{V_1}}=\sup_{V_1} U^{\mu_{V_1}}+\varepsilon\mu_{V_1}(\Omega)>c\sup_{S_{\mu_{V_1}}} U_{\varepsilon}^{\mu_{V_1}}=c\sup_{S_{\mu_{V_1}}} U^{\mu_{V_1}}+c\varepsilon\mu_{V_1}(\Omega).$$

Thus $P_0\in\mathbf{O}_c$ in this case too.

COROLLARY. *If the kernel is continuous in the extended sense and finite outside the diagonal set, then*

$$\mathbf{P}_c\subset\mathbf{O}_c \qquad \textit{for any } c<1.$$

Since any potential is continuous at P_0 for which $\Phi(P_0,P_0)<\infty$, $\Phi(P_0,P_0)=\infty$ at P_0 belonging to $\mathbf{P}_c$ with $c>1$. Our problem is local and hence we may assume from the beginning that the kernel is positive. Thus our case reduces to (iii).

(iv) $0\in\mathbf{O}_c$ *but* $\mathbf{P}_c=\emptyset$ *for every* $c\geqq 1$ *in the example for* (B)$\nrightarrow$(U$_{\text{K}}$) *in* (II) *of* § 1.3.

It is seen in (II) that $0\in\mathbf{O}_{\infty}$. We shall denote by I_n the interval $[(2n+1)^{-1}, (2n)^{-1}]$. Let μ be any measure and x_0 be any point of S_{μ}. If $x_0\in I_n$,

$$\overline{\lim_{x\to x_0}}\ U^{\mu}(x)=\overline{\lim_{x\in S_{\mu},\,x\to x_0}}\ U^{\mu}(x)$$

because we are concerned with a logarithmic potential. If $0\in S_{\mu}$ and $\mu(I_n)=0$, then $U^{\mu}(x)\leqq U^{\mu}(0)\leqq \overline{\lim_{y\in S_{\mu},\,y\to 0}}\ U^{\mu}(y)$ for $x\in I_n$ as was shown in (II). If $\mu(I_n)>0$, $\sup_{x\in I_n} U^{\mu}(x)\leqq\sup_{x\in S_{\mu_n}} U^{\mu}(x)$. Therefore

$$\overline{\lim_{x\to 0}}\ U^{\mu}(x)=\overline{\lim_{x\in S_{\mu},\,x\to 0}}\ U^{\mu}(x)$$

in this case too. This shows that $\mathbf{P}_c=\emptyset$ for every $c\geqq 1$.

(v) *Consider a nonnegative kernel which is locally bounded outside the diagonal set. If* c_n, $1 \leqq c_n \leqq \infty$, *tends to* c *and if* $P_n \in \mathbf{O}_{c_n}$ *with* $\Phi(P_n, P_n) = \infty$ *converges to* P_0, *then, for any* $\varepsilon > 0$, *there is a measure* μ *with* $S_\mu \ni P_0$ *such that* $U^\mu(P) \leqq 1$ *on* S_μ *and* $\overline{\lim}\, U^\mu(P) \geqq c - \varepsilon$ *as* $P \to P_0$.

We may assume that all c_n are finite. In a neighborhood N_1 of $P_{n_1} = P_1$ which does not contain P_0 and on whose product $\Phi(P, Q) > 1$, we choose μ_1 and a point P_1' such that

$$\sup_{P \in S_{\mu_1}} U^{\mu_1}(P) \leqq 1, \qquad U^{\mu_1}(P_1') > c_1{}^{-1} \qquad \text{and} \qquad U^{\mu_1}(P_0) < \frac{\varepsilon}{2},$$

where $\varepsilon > 0$ is a given number. Assuming that P_{n_i}, a neighborhood N_i of P_{n_i}, μ_i with $S_{\mu_i} \subset N_i$ and $P_i' \in N_i$ are chosen up to $i = k$ so that

$$\Phi(P, Q) > i \qquad \text{on } N_i \times N_i,$$

$$\sup_{S_{\mu_i}} U^{\mu_1 + \cdots + \mu_{i-1}}(P) < \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2^{i-1}},$$

$$\sup_{S_{\mu_i}} U^{\mu_i}(P) \leqq 1, \qquad U^{\mu_i}(P_i') > c_i - \frac{1}{i},$$

and

$$U^{\mu_i}(P) < \frac{\varepsilon}{2^i} \qquad \text{on } \bigcup_{j=1}^{i-1} S_{\mu_j} \cup \{P_0\},$$

we choose $P_{n_{k+1}}$, a neighborhood N_{k+1} of $P_{n_{k+1}}$, μ_{k+1} and P_{k+1}' so that $S_{\mu_{k+1}} \subset N_{k+1}$, $P_{k+1}' \in N_{k+1}$,

$$\Phi(P, Q) > k+1 \qquad \text{on } N_{k+1} \times N_{k+1},$$

$$\sup_{S_{\mu_{k+1}}} U^{\mu_1 + \cdots + \mu_k}(P) < \frac{\varepsilon}{2} + \cdots + \frac{\varepsilon}{2^k},$$

$$\sup_{S_{\mu_{k+1}}} U^{\mu_{k+1}}(P) \leqq 1, \qquad U^{\mu_{k+1}}(P_{k+1}') > c_{k+1} - \frac{1}{k+1}$$

and

$$U^{\mu_{k+1}}(P) < \frac{\varepsilon}{2^{k+1}} \qquad \text{on } \bigcup_{i=1}^{k} S_{\mu_i} \cup \{P_0\}.$$

If we set

$$\mu = \mu_1 + \mu_2 + \cdots,$$

we have

$$U^{\mu}(P) < 1 + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} + \cdots = 1 + \varepsilon \qquad \text{on each } S_{\mu_k} \text{ and at } P_0.$$

Thus $\sup_{P \in S_\mu} U^{\mu}(P) \leqq 1 + \varepsilon$. On the other hand $U^{\mu}(P'_n) > c_n - 1/n$. Since the kernel is locally bounded outside the diagonal set, since $\Phi(P, Q) > k$ on $N_k \times N_k$ and since $N_k \ni P_{n_k}$ tends to P_0, it is seen that N_k tends to P_0 as a whole. Therefore P'_n tends to P_0 and

$$\overline{\lim_{P \to P_0}}\, U^{\mu}(P) \geqq \frac{c}{1+\varepsilon} \sup_{P \in S_\mu} U^{\mu}(P).$$

Since ε is arbitrarily small, our assertion is seen to be true.

Corollary. *Consider a nonnegative kernel which is locally bounded outside the diagonal set. If c_n tends to c and if $P_n \in \mathbf{O}_c$ with $\Phi(P_n, P_n) = \infty$ converges to P_0, then $P_0 \in \mathbf{P}_{c'}$ for any $c' < c$ if $c < \infty$ and $P_0 \in \mathbf{P}'_\infty$ if $c = \infty$.*

(vi) *We shall show by an example that $\mathbf{P}_c$ is not necessarily closed.*

An outline of the construction is as follows. We shall choose $P_n \in \mathbf{P}_c$ so that $P_n \to P_0$ and that, for every k, we can find a neighborhood of P_0 in which the $(c+1/k)$-dilated maximum principle is true. Then $P_0 \in \mathbf{P}_{c'}$ for every $c' < c$ by (v) and $P_0 \notin \mathbf{O}_{c''}$ for every $c'' > c$. Consequently $\mathbf{p}(P_0) = c$ and $P_0 \notin \mathbf{P}_c$ by Remark in (ii).

We consider $\Omega_0 = \bigcup_{n=1}^{\infty} ([(2n+1)^{-1}, (2n)^{-1}] \cup \{-1/n\}) \cup \{0\}$ as a subspace of the x-axis and denote by $\Omega_0(a, l)$ the set obtained by contracting Ω_0 by the ratio l and then translating it so that a corresponds to the origin:

$$\Omega_0(a, l) = \bigcup_{n=1}^{\infty} \left(\left[a + \frac{l}{2n+1}, a + \frac{l}{2n} \right] \cup \left\{ a - \frac{l}{n} \right\} \right) \cup \{a\}.$$

We set

$$\Phi_{c,a,l}(x, y) = \Phi_{c,a,l}(y, x) = c \log \frac{8n(2n+1)}{l}$$

for x which is the transform of $-1/n$ and for y which belongs to the transform of $I_n = [(2n+1)^{-1}, (2n)^{-1}]$, and set

$$\Phi_{c,a,l}(x, y) = \Phi_{c,a,l}(y, x) = \log \frac{1}{|x-y|} \qquad \text{for any other } (x, y).$$

Setting

$$a_n = \frac{4n+1}{4n(2n+1)},$$

we consider $\Omega_0(a_n, l_n)$ for each n; a_n is equal to the middle point of I_n and $\{l_n\}$ are chosen so small that $\log |x-y| \geqq c \log |x'-y|$ whenever $x, x' \in \Omega_0(a_n, l_n)$

and $y \in \Omega_0(a_m, l_m)$, $m \neq n$. We put

$$\omega = \bigcup_{n=1}^{\infty} \Omega_0(a_n, l_n) \cup \{0\}$$

and define $\Phi_c(x, y)$ on ω by

$$\Phi_c(x, y) = \begin{cases} \Phi_{c, a_n, l_n}(x, y) & \text{for } x, y \in \Omega_0(a_n, l_n), \\ \log \dfrac{1}{|x-y|} & \text{for any other } (x, y). \end{cases}$$

We denote by $\omega(a_k, l_k)$ the transform of ω obtained in the same way as $\Omega_0(a_n, l_n)$ was obtained from Ω_0, and by $\Phi_{c, a_n, l_n}(x, y)$ the function obtained from $\Phi_c(x, y)$ similarly. We set

$$\Omega = \bigcup_{k=1}^{\infty} \omega(a_k, l_k) \cup \{0\}$$

and define $\Phi(x, y)$ on Ω by

$$\Phi(x, y) = \begin{cases} \Phi_{c+1/k, a_k, l_k}(x, y) & \text{for } x, y \in \omega(a_k, l_k), \\ \log \dfrac{1}{|x-y|} & \text{for any other } (x, y). \end{cases}$$

We see that $a_k \in \mathbf{P}_{c+(2k)^{-1}}$ by (v). Let μ be any measure supported by $\bigcup_{j=k}^{\infty} \omega(a_j, l_j) \cup \{0\}$ such that it is bounded on the support. If x is not equal to any transform of any point of Ω_0 of the form $-1/n$, $\Phi(x, y)$ is logarithmic for any $y \in \Omega$. For such x we see that $U^\mu(x) \leqq \sup_{y \in S_\mu} U^\mu(y)$, because S_μ does not contain any transform of any point of the form $-1/n$. We assume that $x \in \omega(a_j, l_j)$, $j \geqq k$, is a transform of $-1/n$ and denote by Ω_0' the transform of Ω_0 which contains x. We assume that $\mu(\Omega_0') \neq 0$; if $\mu(\Omega_0') = 0$ then $U^\mu(x) \leqq \sup_{y \in S_\mu} U^\mu(y)$. We have

$$U^\mu(x) = \int_{\Omega_0'} \Phi_{c+1/j, a_j, l_j}(x, y) d\mu(y) + \int_{\Omega - \Omega_0'} \log \frac{1}{|x-y|} d\mu(y).$$

As we have seen in (i)

$$\int_{\Omega_0'} \Phi_{c+1/j, a_j, l_j}(x, y) d\mu(y) \leqq \left(c + \frac{1}{j}\right) \sup_{z \in S_\mu \cap \Omega_0'} \int_{\Omega_0'} \Phi_{c+1/j, a_j, l_j}(z, y) d\mu(y)$$

for $x \in \Omega_0'$.

Since

$$\log \frac{1}{|x-y|} = \frac{\log \dfrac{1}{|x-y|}}{\log \dfrac{1}{|x'-y|}} \log \frac{1}{|x'-y|} \leqq c \log \frac{1}{|x'-y|}$$

for any $x, x' \in \Omega_0'$ and $y \in \Omega - \Omega_0'$ on account of the choice of $\{l_n\}$, it follows that

$$U^{\mu}(x) \leqq \left(c + \frac{1}{j}\right) \sup_{y \in S_{\mu} \cap \Omega_0'} \int_{\Omega_0'} \log \frac{1}{|y-y'|} d\mu(y') + c \int_{\Omega-\Omega_0'} \log \frac{1}{|x'-y|} d\mu(y)$$

for any $x' \in \Omega_0'$. Consequently

$$U^{\mu}(x) \leqq \left(c + \frac{1}{j}\right) \sup_{y \in S_{\mu} \cap \Omega_0'} U^{\mu}(y) \leqq \left(c + \frac{1}{j}\right) \sup_{y \in S_{\mu}} U^{\mu}(y).$$

The proof is now completed.

COROLLARY 1. *We can not replace c' by c in* (v).

COROLLVRY 2. *Given any c, $1 < c < \infty$, we can find P with $\mathbf{p}(P) = c$.*

(vii) *Consider a kernel which is locally bounded outside the diagonal set and positive on the diagonal set. Assume that there are a sequence $\{P_n\}$ of points tending to a point P_0, a sequence $\{\mu_n\}$ of measures and a sequence $\{V_n\}$ of compact neighborhoods of P_0 with the following property: $V_n \cap S_{\mu_n} = \emptyset$, V_n and S_{μ_n} tend to P_0 as a whole, $U^{\mu_n}(P_n) \to \infty$ and $U^{\mu_n}(P) < 1$ on $S_{\mu_n} \cup V_n$. Then there is μ such that $U^{\mu}(P)$ is bounded on S_{μ} and $\overline{\lim_{n\to\infty}} U^{\mu}(P_n) = \infty$. If, in addition, the kernel is continuous outside the diagonal set, μ can be chosen so that $U^{\mu}(P)$ is continuous as a function on S_{μ}.*

We may assume that the kernel is positive. We choose a measure μ_{n_1} such that $U^{\mu_{n_1}}(P_{n_1}) > 1$. We set

$$m_1 = \inf_{V_1 \times V_1} \Phi(P, Q) \qquad \text{and} \qquad M_1 = \sup_{V_1 \times S_{\mu_{n_1}}} \Phi(P, Q)$$

and choose μ_{n_2} such that $S_{\mu_{n_2}} \subset V_1$ and $U^{\mu_{n_2}}(P_{n_2}) > \max(4, 4M_1 m_1^{-1})$. We see that $m_1 \mu_{n_2}(\Omega) < 1$ and hence

$$U^{\mu_{n_2}}(P) \leqq M_1 \, \mu_{n_2}(\Omega) < \frac{M_1}{m_1} \qquad \text{on } S_{\mu_{n_1}}.$$

We set $\mu_1' = \mu_{n_1}$, $\mu_2' = \mu_{n_2} (\max(2, 2M_1 m_1^{-1}))^{-1}$ and observe that

$$U^{\mu_2'}(P_{n_2}) > 2 \qquad \text{and} \qquad U^{\mu_2'}(P) < \frac{1}{2} \qquad \text{on } S_{\mu_1'} \cup S_{\mu_2'} \cup V_{n_2}.$$

By induction we can find easily $\{n_k\}$ and $\{\mu_k'\}$ such that $S_{\mu_k'} = S_{\mu_{n_k}} \subset V_{n_{k-1}}$, $(\bigcup_{j=1}^{k-1} S_{\mu_j}) \cap S_{\mu_k} = \emptyset$, $U^{\mu_k'}(P_{n_k}) > 2^{k-1}$ and $U^{\mu_k'}(P) < 2^{-(k-1)}$ on $S = \{P_0\} \cup \bigcup_{k=1}^{\infty} S_{\mu_k'}$. Now we set $\mu = \mu_1' + \mu_2' + \cdots$. Since $U^{\mu_k'}(P) < 2^{-(k-1)}$ on S, $U^{\mu} = \sum_{k=1}^{\infty} U^{\mu_k'}$ is bounded on $S = S_{\mu}$. On the other hand $U^{\mu}(P_{n_k}) \geqq U^{\mu_k'}(P_{n_k}) > 2^{k-1}$. If the kernel is continuous outside the diagonal set, each $U^{\mu'_k}(P)$ may be assumed continuous as a function on S. Since the convergence is uniform on S, $U^{\mu}=\sum_{k=1}^{\infty} U^{\mu'_k}$ is continuous as a function on $S=S_{\mu}$.

(viii) *We consider a kernel which is locally bounded outside the diagonal set and positive on the diagonal set in $\Omega\times\Omega$. If P_0 is a point of accumulation of $\mathbf{O}_{\infty}$ and has a countable base of neighborhoods, then there exists μ such that $U^{\mu}(P)$ is bounded on S_{μ} but $\overline{\lim}_{P\to P_0} U^{\mu}(P)=\infty$. If, in addition, the kernel is continuous outside the diagonal set, the restriction of $U^{\mu}(P)$ to S_{μ} may be assumed continuous.*[6]

Let us see that the conditions of (vii) are satisfied. We may assume that the kernel is positive in $\Omega\times\Omega$. Let $\{V_n\}$ be a sequence of relatively compact open neighborhoods of P_0 decreasing to P_0. We choose a point $P_n\in\mathbf{O}_{\infty}$ different from P_0, a neighborhood N_n of P_n and a compact neighborhood V_n of P_0 disjoint from N_n. We set

$$m_n=\inf_{N_n\times N_n}\Phi(P,Q) \qquad \text{and} \qquad M_n=\sup_{V_n\times N_n}\Phi(P,Q);$$

$0<m_n$ and $0<M_n<\infty$ by our assumption on the kernel. We choose a finite number $a_n>\max(1, M_n m_n^{-1})$ and a measure ν_n such that $S_{\nu_n}\subset N_n$, $U^{\nu_n}(P)<1$ on S_{μ_n} and $\sup_{N_n} U^{\mu_n}>na_n$. We see that $m_n\nu_n(\Omega)<1$ and hence

$$U^{\nu_n}(P_0)\leqq M_n\,\nu_n(\Omega)<\frac{M_n}{m_n}.$$

It is easy to verify that $\mu_n=\nu_n/a_n$ fulfils the conditions required in the lemma. In the above we may assume that P_n, N_n and V_n all tend to P_0 as $n\to\infty$.

COROLLARY 1. *We assume that the kernel is locally bounded outside the diagonal set and positive on the diagonal set. If each point of accumulation of $\mathbf{P}_{\infty}$ has a countable base of neighborhoods, then $\mathbf{P}_{\infty}$ is closed.*

COROLLARY 2. (Choquet [2]). *We assume that the kernel is continuous outside the diagonal set and positive on the diagonal set. If $\mathbf{O}_{\infty}$ is not discrete and at least one point of accumulation of $\mathbf{O}_{\infty}$ has a countable base of neighborhoods, then the continuity principle is not satisfied.*

However, the fact $\mathbf{O}_{\infty}\neq\emptyset$ does not mean that the continuity principle is not valid. In fact, an example will be given in Corollary 4 in $(\mathrm{U_K})$, (ii) of the next section.

(ix) *We consider a kernel which is locally bounded outside the diagonal set*

6) Choquet [2] announced the following theorem: If the kernel is continuous outside the diagonal set and positive on $\Omega\times\Omega$, then, for the limiting point P_0 of a sequence of points of $\mathbf{O}_{\infty}$ and for any neighborhood V of P_0, there exists a measure μ with compact $S_{\mu}\subset V$ such that the restriction of $U^{\mu}(P)$ to S_{μ} is continuous and $U^{\mu}(P)$ is discontinuous at P_0. We note that no countability condition is required there. Proof is not given.

and positive on the diagonal set in $\Omega\times\Omega$, and assume that each point of accumulation of $\mathbf{P}_\infty$ has a countable base of neighborhoods. Then we have

$$\mathbf{Q}=\mathbf{P}'_\infty.$$

If, in addition, the kernel is continuous outside the diagonal set,

$$\mathbf{Q}=\mathbf{P}_\infty.$$

We may assume that the kernel is positive. By the definition of $\mathbf{Q}$, given any neighborhood N of $P_0\in\mathbf{Q}$, there exists μ with compact $S_\mu\subset N$ such that $U^\mu(P)$ is bounded on S_μ but unbounded in N. If P_0 is a point of S_μ with the property that $U^\mu(P)$ is unbounded in any neighborhood of P_0, then P_0 belongs to $\mathbf{P}'_\infty$. Otherwise there is a point of $\mathbf{P}_\infty$ in N and it follows that P_0 is a point of accumulation of points of $\mathbf{P}_\infty\subset\mathbf{O}_\infty$. Consequently $P_0\in\mathbf{P}'_\infty$ by (viii). The inclusion $\mathbf{P}'_\infty\subset\mathbf{Q}$ is evident.

The identity $\mathbf{Q}=\mathbf{P}_\infty$ will follow if we can choose $\{P_n\}$, $\{\mu_n\}$ and $\{V_n\}$ as in (vii). Assume $P_0\in\mathbf{P}_\infty$ and that the kernel is continuous outside the diagonal set. Then by (1.16) there is a measure ν_n such that S_{ν_n} contains P_0, it is near to P_0, $U^{\nu_n}(P)<1$ on S_{ν_n} and $\overline{\lim}\ U^{\nu_n}(P)>n$ as $P\to P_0$. Let P_n be a point sufficiently close to P_0 such that $U^{\nu_n}(P_n)>n$. Suppose that we can choose ν_n such that $\nu_n(\{P_0\})=0$. Then the restriction μ_n of ν_n to the outside of some neighborhood of P_0 satisfies $U^{\mu_n}(P_n)>n$. Since $U^{\mu_n}(P)\leqq U^{\nu_n}(P)<1$ on $S_{\nu_n}\supset S_{\mu_n}\cup\{P_0\}$ and, by the continuity of the kernel, there is a compact neighborhood V_n of P_0 disjoint from S_{μ_n} on which $U^{\mu_n}(P)<1$, all the required conditions are satisfied. Let us see therefore that we may assume $\nu_n(\{P_0\})=0$. This is so if $\Phi(P_0, P_0)=\infty$ and hence $\Phi(P_0, P_0)$ is assumed to be finite. If $\overline{\lim\limits_{P\to P_0}}\ \Phi(P, P_0)=\infty$, evidently $P_0\in\mathbf{Q}$. Hence we suppose $\Phi(P, P_0)<M<\infty$ in a neighborhood V of P_0. Let λ_n satisfy $\overline{\lim}\ U^{\lambda_n}(P)>n\ \overline{\lim\limits_{S_{\lambda_n}}}\ U^{\lambda_n}(P)$ as $P\to P_0$. Denote the restriction of λ_n to the outside of P_0 by λ'_n and the mass at P_0 by α. It follows that

$$\overline{\lim_{P\to P_0}}\ U^{\lambda'_n}(P)+\alpha M>n\left\{\overline{\lim_{S_{\lambda'_n}}}\ U^{\lambda'_n}(P)+\alpha\Phi(P_0, P_0)\right\}.$$

Since $M<n\Phi(P_0, P_0)$ for large n, $\overline{\lim\limits_{P\to P_0}}\ U^{\lambda'_n}(P)>n\ \overline{\lim\limits_{S_{\lambda'_n}}}\ U^{\lambda'_n}(P)$. We divide the restriction of λ'_n to a sufficiently small neighborhood of P_0 by its total mass and take it for ν_n; then $\nu_n(\{P_0\})=0$ certainly.

In the proof we have proved more than $\mathbf{Q}=\mathbf{P}_\infty$. Namely

COROLLARY 1. *Under the assumption that the kernel is continuous outside the diagonal set and positive on the diagonal set, a point P_0 belongs to $\mathbf{Q}=\mathbf{P}_\infty$, if and only if there is a measure μ such that $P_0\in S_\mu$, $U^\mu(P)$ is continuous as a* *function on* S_μ *and* $\overline{\lim}_{P\to P_0} U^\mu(P)=\infty$.

COROLLARY 2. *Assume that the kernel is continuous in the extended sense, finite outside the diagonal set and positive on the diagonal set. Then, for any* $P_0\in \mathbf{D}$, *we can find* μ *whose potential is continuous as a function on* S_μ *and satisfies* $\overline{\lim}_{P\to P_0} U^\mu(P)=\infty$. *Namely we can let a discontinuity arise at* P_0; *in the definition of* $\mathbf{D}$ *it is sufficient that a discontinuity exists near* P_0.

This is because $\mathbf{D}=\mathbf{Q}$ and by Corollary 1.

COROLLARY 3. *Assume the same on the kernel. Let* P_0 *be a point with the following property: Any neighborhood of* P_0 *supports a measure* μ *such that* $U^\mu(P)$ *is continuous as a function on* S_μ *and*

$$\overline{\lim_{P\to P_0}}\, U^\mu(P)>c \lim_{P\in S_\mu,\, P\to P_0} U^\mu(P)=cU^\mu(P_0)$$

with $c\geqq 1$. *Then we find* μ *with the same character as in Corollary* 2; *we can have thus the inequality with arbitrarily large* c.

(x) *Consider a nonnegative kernel which is continuous in the extended sense, finite outside the diagonal set and positive on the diagonal set in* $\Omega\times\Omega$. *Then*

$$\mathbf{S}_c^*\supset \mathbf{P}_c \qquad \textit{for every } c\geqq 1.$$

We assume $P_0\notin \mathbf{S}_c^*$ and that P_0 is not isolated in Ω. Then there is a neighborhood N of P_0 such that the kernel is positive on $N\times N$ and

$$\overline{\lim_{P\to P_0}}\, \frac{U^\lambda(P)}{U^\nu(P)} \leqq c \overline{\lim_{P\in S_\lambda,\, P\to P_0}}\, \frac{U^\lambda(P)}{U^\nu(P)}$$

whenever $P_0\in S_\lambda\subset N$, $\nu\in\mathscr{E}$ has a compact support and $U^\nu(P_0)>0$. We take any μ such that $S_\mu\subset N$, $P_0\in S_\mu$ and $\overline{\lim}_{P\in S_\mu,\, P\to P_0} U^\mu(P)<\infty$. If $\Phi(P_0, P_0)<\infty$, $U^\mu(P)$ is continuous at P_0 and $P_0\notin \mathbf{P}_c$. Therefore we assume that $\Phi(P_0, P_0)=\infty$. It follows that P_0 is not isolated on S_μ. We can find a compact set $K\ni P_0$ in N with the property that

$$0<\sup_{S_{\mu_K}} U^{\mu_K}(P)<\infty,$$

where μ_K is the restriction of μ to K. We set $\mu_K/U^{\mu_K}(P_0)=\nu$. This belongs to $\mathscr{E}$ and $\lim_{P\to P_0} U^\nu(P)=U^\nu(P_0)=1$. We have

$$\overline{\lim_{P\to P_0}}\, U^\mu(P)=\overline{\lim_{P\to P_0}}\, \frac{U^\mu(P)}{U^\nu(P)} \leqq c \overline{\lim_{P\in S_\mu,\, P\to P_0}}\, \frac{U^\mu(P)}{U^\nu(P)}=c \overline{\lim_{P\in S_\mu,\, P\to P_0}}\, U^\mu(P).$$

Thus $P_0\notin \mathbf{P}_c$ and $\mathbf{S}_c^*\supset \mathbf{P}_c$.

REMARK 1. Similarly we can show $\mathbf{S}_c \supset \mathbf{P}_c$ for every c under the assumption that the kernel is nonnegative, positive on the diagonal set and continuous outside the diagonal set.

REMARK 2. If, in addition to the condition in Remark 1, $\Phi(P, P)=\infty$ at each point of $\mathbf{P}_c$, then $\mathbf{S}_c^* \supset \mathbf{P}_c$.

REMARK 3. If we do not assume that $\Phi(P, Q)$ is continuous in the extended sense, $\mathbf{S}_c^* \not\supset \mathbf{P}_c$ in general. To show this, we consider $\bigcup_{n=1}^{\infty} \{1/n\} \cup \{0\}$ as a subspace of the x-axis and define $\Phi(0, 0)=1$, $\Phi(0, 1/n)=\Phi(1/n, 0)=n$, $\Phi(1/n, 1/m)=\min(n, m)$ for $n\neq m$, $\Phi(1/n, 1/n)=\infty$. For the unit point measure μ at 0, we have

$$cU^{\mu}(0)=c<\overline{\lim_{n\to\infty}}\, U^{\mu}(1/n)=\lim_{n\to\infty} n=\infty$$

for every c. Thus $0\in\mathbf{P}_\infty$ but $\mathbf{S}_c^*$ is empty for every $c>1$.

(xi) *We assume the same as in* (x) *on the kernel. Then*

$$\mathbf{R}_c^* \supset \bigcup_{c'>c} \mathbf{O}_{c'} \qquad \textit{for every } c\geqq 1.$$

Take $P_0\notin\mathbf{R}_c^*$. Then there is a neighborhood N of P_0 such that $U^{\lambda}(P)\leqq cU^{\nu}(P)$ in N whenever $U^{\lambda}(P)\leqq U^{\nu}(P)$ on S_λ for λ with $S_\lambda\subset\mathscr{E}_N$ and for $\nu\in\mathscr{E}$ with compact support. If $\Phi(P_0, P_0)<\infty$, every potential is continuous at P_0 by assumption. Therefore we assume that $\Phi(P_0, P_0)=\infty$. If there exists a neighborhood of P_0 which does not support any non-vanishing measure of $\mathscr{E}$, there is nothing to prove. Therefore we assume that there exists a measure $\nu\in\mathscr{E}$, with compact support $S_\nu\not\ni P_0$ whose potential is equal to 1 at P_0. Given $\varepsilon>0$, we take a neighborhood $N_1\subset N$ of P_0 such that

$$1-\varepsilon<U^{\nu}(P)<1+\varepsilon \qquad \text{in } N_1.$$

Take μ such that $S_\mu\subset N_1$ and $V(\mu)=\sup_{S_\mu} U^{\mu}<\infty$. Since

$$U^{\mu}(P)\leqq(1-\varepsilon)^{-1}V(\mu)\,U^{\nu}(P) \qquad \text{on } S_\mu,$$

we have

$$U^{\mu}(P)\leqq c(1-\varepsilon)^{-1}V(\mu)\,U^{\nu}(P)<c(1-\varepsilon)^{-1}(1+\varepsilon)\,V(\mu) \qquad \text{in } N_1.$$

This shows that $P_0\notin\mathbf{O}_{c(1+\varepsilon)/(1-\varepsilon)}$ and hence that

$$\mathbf{R}_c^* \supset \mathbf{O}_{c(1+\varepsilon)/(1-\varepsilon)}.$$

Therefore

$$\mathbf{R}_c^* \supset \bigcup_{c'>c} \mathbf{O}_{c'}.$$

COROLLARY. *Consider a kernel which is nonnegative, continuous outside* *the diagonal set and positive on the diagonal set. Then*

$$\mathbf{R}_c \supset \mathbf{P}_c.$$

If, in addition, the kernel is continuous in the extended sense, then

$$\mathbf{R}_c^* \supset \mathbf{P}_c.$$

These facts are proved by making use of Remark in (ii).

REMARK 1. Similarly we can show $\mathbf{R}_c \supset \bigcup_{c'>c} \mathbf{O}_{c'}$ under the assumption that the kernel is positive on the diagonal set in $\Omega \times \Omega$ and continuous outside the diagonal set.

REMARK 2. If, in addition to the assumptions in Remark 1, $\Phi(P, P) = \infty$ at each point of $\mathbf{P}_c$, then $\mathbf{R}_c^* \supset \bigcup_{c'>c} \mathbf{O}_{c'}$.

REMARK 3. In the example in (x), $\mathbf{R}_c^* = \emptyset$ but $0 \in \mathbf{O}_\infty$.

REMARK 4. If a kernel is locally bounded outside the diagonal set and positive on the diagonal set, then $\mathbf{R}_c = \emptyset$ for some $c \geqq 1$ implies $\mathbf{O}_\infty = \emptyset$.

(xii) *Let $c \geqq 1$ be given. There is a positive symmetric kernel which is continuous in the extended sense and finite outside the diagonal set, and for which $\mathbf{R}_c \not\supset \mathbf{O}_c$.*

LEMMA. *Consider a positive symmetric kernel which is continuous in the extended sense and finite outside the diagonal set. In order that* $(\mathrm{U_d})_c$ $(c \geqq 1)$ *be true the following condition is necessary and sufficient:*

Let $\mu \in \mathscr{E}$ be any measure with compact support and P_0 be any point outside S_μ. If

$$U^\mu(P) \leqq \Phi(P, P_0) \qquad \text{on } S_\mu,$$

then $U^\mu(P) \leqq c\Phi(P, P_0)$ everywhere in Ω.

This will be given as Corollary to Theorem 2.44 and we omit the proof here.

We consider

$$\Omega = \bigcup_{n=1}^{\infty} \left(\left\{ \frac{1}{n} \right\} \cup \left\{ -\frac{1}{n} \right\} \right) \cup \{0\}$$

as a subspace of the x-axis and set, with any positive number $d < (2c)^{-1}$,

$$\Phi(0, 0) = \infty,$$

$$\Phi\left(0, \frac{1}{n}\right) = \Phi\left(\frac{1}{n}, 0\right) = n,$$

$$\Phi\left(\frac{1}{n}, \frac{1}{m}\right) = \min(n, m),$$

$$\Phi\left(0, -\frac{1}{n}\right) = \Phi\left(-\frac{1}{n}, 0\right) = n + \frac{d}{n},$$

$$\Phi\left(-\frac{1}{n}, -\frac{1}{m}\right) = \min(n, m) + \frac{d}{\min(n, m)} + \frac{1}{\max(n, m)} \qquad n \neq m,$$

$$\Phi\left(-\frac{1}{n}, -\frac{1}{n}\right) = \infty,$$

$$\Phi\left(-\frac{1}{n}, \frac{1}{m}\right) = \Phi\left(\frac{1}{m}, -\frac{1}{n}\right) = \min(n, m) + \frac{d}{n} \qquad n \neq m,$$

$$\Phi\left(-\frac{1}{n}, \frac{1}{n}\right) = \Phi\left(\frac{1}{n}, -\frac{1}{n}\right) = c\left(n + \frac{d}{n}\right).$$

This is a positive symmetric kernel which is continuous in the extended sense and finite outside the diagonal set.

We consider the unit point measure μ_n at $x = 1/n$. Then

$$\operatorname*{snp}_{S_{\mu_n}} U^{\mu_n} = U^{\mu_n}\left(\frac{1}{n}\right) = n$$

and

$$U^{\mu_n}\left(-\frac{1}{n}\right) = c\left(n + \frac{d}{n}\right).$$

Therefore

$$\sup_{-1/n \leqq x \leqq 1/n} U^{\mu_n}(x) \geqq c\left(n + \frac{d}{n}\right) > c \sup_{S_{\mu_n}} U^{\mu_n}.$$

This shows that $0 \in \mathbf{O}_c$.

Let us see that $0 \notin \mathbf{R}_c$. Let $\mu \neq 0$ be a measure in $\mathscr{E}$, and set $a_n = \mu(\{1/n\})$. For $x_0 \notin S_\mu$ assume that

$$U^\mu(x) \leqq \Phi(x, x_0) \qquad \text{on } S_\mu.$$

First we consider the case $x_0 = 1/n_0 > 0$. The inequality reads

$$U^\mu\left(\frac{1}{n}\right) = \sum_{k=1}^{n} k a_k + n \sum_{k=n+1}^{\infty} a_k \leqq \min(n, n_0) \qquad \text{if } 1/n \in S_\mu.$$

For the largest number $1/n_1$ in S_μ, we have

$$U^\mu\left(\frac{1}{n_1}\right) = n_1 \sum_k a_k \leqq \Phi\left(\frac{1}{n_1}, \frac{1}{n_0}\right) = \min(n_1, n_0) \leqq n_1$$

and hence the total mass $\sum_k a_k \leqq 1$. Therefore, for $n \leqq n_0$,

$$U^\mu\left(\frac{1}{n}\right) \leqq n \sum_k a_k \leqq n.$$

If $S_\mu \subset [1/n_0, 1]$, we have, for $1/n \in [0, 1/n_0]$,

$$U^\mu\left(\frac{1}{n}\right) = \sum_k k a_k = U^\mu\left(\frac{1}{n_0}\right) \leqq n_0.$$

If $S_\mu \not\subset [1/n_0, 1]$, we have, for $1/n \in [0, 1/n_0]$,

$$U^\mu\left(\frac{1}{n}\right) \leqq \sum_k k a_k = U^\mu\left(\frac{1}{n_2}\right) \leqq n_0,$$

where $1/n_2 \geqq 0$ is the smallest number in S_μ. Thus $U^\mu(1/n) \leqq \min\,(n, n_0)$ for any $n \leqq \infty$. Next we observe

$$\begin{aligned} U^\mu\left(-\frac{1}{n}\right) &= \sum_{k=1}^{n-1} k a_k + c\left(n+\frac{d}{n}\right) a_n + n \sum_{k=n+1}^{\infty} a_k + \frac{d}{n} \sum_{k \neq n} a_k \\ &\leqq c\,\Big(\sum_{k=1}^{n} k a_k + n \sum_{k=n+1}^{\infty} a_k\Big) + \frac{cd}{n} \sum_k a_k - (c-1)\Big(\sum_{k=1}^{n-1} k a_k + n \sum_{k=n+1}^{\infty} a_k\Big) \\ &\leqq c\, U^\mu\left(\frac{1}{n}\right) + \frac{cd}{n} \leqq c\, \min\,(n, n_0) + \frac{cd}{n} \leqq c\Phi\left(-\frac{1}{n}, \frac{1}{n_0}\right) \end{aligned}$$

for finite n. Thus it is concluded that

$$U^\mu(x) \leqq c\Phi(x, x_0) \qquad \text{for any } x \in \Omega.$$

Secondly we consider the case $x_0 = 0$. Let $1/n_1$ be the largest number in S_μ. Since

$$U^\mu\left(\frac{1}{n_1}\right) = n_1 \sum_k a_k \leqq \Phi\left(\frac{1}{n_1}, 0\right) = n_1,$$

the total mass $\sum_k a_k \leqq 1$. Therefore, for any $n_0 > 0$,

$$U^\mu\left(\frac{1}{n}\right) = \sum_{k=1}^{n} k a_k + n \sum_{k=n+1}^{\infty} a_k \leqq n \sum_k a_k \leqq n = \Phi\left(\frac{1}{n}, 0\right)$$

and

$$U^\mu\left(-\frac{1}{n}\right) \leqq c U^\mu\left(\frac{1}{n}\right) + \frac{cd}{n} \leqq cn + \frac{cd}{n} = c\Phi\left(-\frac{1}{n}, 0\right).$$

Since $U^\mu(0) \leqq c\Phi(0, 0) = \infty$, $U^\mu(x) \leqq c\Phi(x, 0)$ in all cases.

Finally we consider the case $x_0 = -1/n_0$, and let $1/n_1$ be the largest number in S_μ. We have

$$U^\mu\left(\frac{1}{n_1}\right) = n_1 \sum_k a_k \leqq \Phi\left(\frac{1}{n_1}, -\frac{1}{n_0}\right) \leqq c\left\{\min\,(n_1, n_0) + \frac{d}{n_0}\right\} \leqq c\left(n_1 + \frac{d}{n_0}\right).$$

Hence

$$\sum_k a_k \leqq c\left(1 + \frac{d}{n_0 n_1}\right). \tag{1.17}$$

On the other hand let $1/n_2 \geqq 0$ be the smallest number in S_μ. Then

$$U^\mu\left(\frac{1}{n_2}\right)=\sum_k ka_k \leqq \Phi\left(\frac{1}{n_2}, -\frac{1}{n_0}\right)\leqq c\left\{\min (n_0, n_2)+\frac{d}{n_0}\right\}.$$

If $n_0 < n \leqq \infty$,

$$U^\mu\left(\frac{1}{n}\right)\leqq \sum_k ka_k = U^\mu\left(\frac{1}{n_2}\right)\leqq c\left(n_0+\frac{d}{n_0}\right)=c\Phi\left(\frac{1}{n}, -\frac{1}{n_0}\right).$$

Consequently we shall consider n such that $n \leqq n_0$. If, in addition, $n \leqq n_1$, then by (1.17)

$$U^\mu\left(\frac{1}{n}\right)\leqq n\sum_k a_k \leqq cn\left(1+\frac{d}{n_0 n_1}\right)\leqq cn + c\frac{d}{n_0}\leqq c\Phi\left(\frac{1}{n}, -\frac{1}{n_0}\right).$$

For $n > n_2$, it follows that $n_2 < n_0$ and

$$U^\mu\left(\frac{1}{n}\right)=\sum_k ka_k = U^\mu\left(\frac{1}{n_2}\right)\leqq n_2+\frac{d}{n_0} < n+\frac{d}{n_0}\leqq \Phi\left(\frac{1}{n}, -\frac{1}{n_0}\right).$$

If $n_1 < n < n_2$, we take the nearest greater number n_4 and the nearest smaller number n_3 in S_μ: $n_4 > n > n_3$; $n_3 < n_0$ because $n \leqq n_0$. It holds that

$$U^\mu\left(\frac{1}{n_3}\right)=\sum_{k=1}^{n_3} ka_k + n_3\sum_{k=n_4}^{\infty} a_k \leqq \Phi\left(\frac{1}{n_3}, -\frac{1}{n_0}\right)= n_3+\frac{d}{n_0}$$

and

$$U^\mu\left(\frac{1}{n_4}\right)=\sum_{k=1}^{n_3} ka_k + n_4\sum_{k=n_4}^{\infty} a_k \leqq \Phi\left(\frac{1}{n_4}, -\frac{1}{n_0}\right)\leqq c\left(n_4+\frac{d}{n_0}\right).$$

We consider the following linear expression in x:

$$(c-\sum_{k=n_4}^{\infty} a_k)\, x + \frac{cd}{n_0} - \sum_{k=1}^{n_3} ka_k.$$

This is nonnegative for $x=n_3$ and n_4. Hence it is so for any n in between. Thus

$$U^\mu\left(\frac{1}{n}\right)=\sum_{k=1}^{n_3} ka_k + n\sum_{k=n_4}^{\infty} a_k \leqq c\left(n+\frac{d}{n_0}\right)\leqq c\Phi\left(\frac{1}{n}, -\frac{1}{n_0}\right).$$

Next we shall evaluate $U^\mu(-1/n)$. Naturally

$$U^\mu\left(-\frac{1}{n_0}\right) < c\Phi\left(-\frac{1}{n_0}, -\frac{1}{n_0}\right)=\infty.$$

If $1/n \in S_\mu$, $U^\mu(1/n)\leqq \Phi(1/n, -1/n_0)$. For $n < n_0$, we have as before

$$U^\mu\left(-\frac{1}{n}\right)\leqq cU^\mu\left(\frac{1}{n}\right)+\frac{cd}{n}\sum_k a_k.$$

It follows that

$$U^{\mu}\left(-\frac{1}{n}\right) \leqq c\Phi\left(\frac{1}{n}, -\frac{1}{n_0}\right) + \frac{cd}{n}\sum_k a_k = cn + \frac{cd}{n_0} + \frac{cd}{n}\sum_k a_k$$
$$= c\Phi\left(-\frac{1}{n}, -\frac{1}{n_0}\right) - \frac{cd}{n} + \frac{cd}{n_0} - \frac{c}{n_0} + \frac{cd}{n}\sum_k a_k$$
$$= c\Phi\left(-\frac{1}{n}, -\frac{1}{n_0}\right) - A,$$

where

$$A = \frac{cd}{n} - \frac{cd}{n_0} + \frac{c}{n_0} - \frac{cd}{n}\sum_k a_k.$$

Let us show that $A>0$. First we observe that, if $1/n_1$ is the largest number in S_μ, then $1/n_1 \geqq 1/n > 1/n_0$, and we derive

$$\sum_k a_k \leqq 1 + \frac{d}{n_0 n_1}$$

as before. We see then

$$\frac{A}{c} = \frac{d}{n} - \frac{d}{n_0} + \frac{1}{n_0} - \frac{d}{n}\sum_k a_k \geqq \frac{d}{n} - \frac{d}{n_0} + \frac{1}{n_0} - \frac{d}{n}\left(1 + \frac{d}{n_0 n_1}\right)$$
$$= \frac{1}{n_0}\left(1 - d - \frac{d^2}{nn_1}\right) \geqq \frac{1-2d}{n_0} > 0.$$

Thus

$$U^{\mu}\left(-\frac{1}{n}\right) < c\Phi\left(-\frac{1}{n}, -\frac{1}{n_0}\right).$$

We have, for $n > n_0$,

$$U^{\mu}\left(-\frac{1}{n}\right) \leqq c\Phi\left(\frac{1}{n}, -\frac{1}{n_0}\right) + \frac{cd}{n}\sum_k a_k = cn_0 + c\frac{d}{n_0} + \frac{cd}{n}\sum_k a_k$$
$$= c\Phi\left(-\frac{1}{n}, -\frac{1}{n_0}\right) - \frac{c}{n} + \frac{cd}{n}\sum_k a_k$$
$$\leqq c\Phi\left(-\frac{1}{n}, -\frac{1}{n_0}\right) - \frac{c}{n} + \frac{c^2 d}{n}\left(1 + \frac{d}{n_0 n_1}\right) < c\Phi\left(-\frac{1}{n}, -\frac{1}{n_0}\right),$$

where we used (1.17).

If $n \neq n_0$ and $1/n \notin S_\mu$, then $a_n = 0$ and

$$U^{\mu}\left(-\frac{1}{n}\right) = \sum_{k=1}^{n-1} ka_k + n\sum_{k=n+1}^{\infty} a_k + \frac{d}{n}\sum_k a_k = U^{\mu}\left(\frac{1}{n}\right) + \frac{d}{n}\sum_k a_k.$$

We know that

$$U^{\mu}\Big(\frac{1}{n}\Big) \leqq c\Phi\Big(\frac{1}{n}, -\frac{1}{n_0}\Big) = c\Big\{\min(n, n_0) + \frac{d}{n_0}\Big\} \qquad \text{for } n \neq n_0.$$

By (1.17) we see that

$$U^{\mu}\Big(-\frac{1}{n}\Big) \leqq c\Big\{\min(n, n_0) + \frac{d}{n_0}\Big\} + \frac{cd}{n}\Big(1 + \frac{d}{n_1 n_0}\Big)$$

$$= \begin{cases} c\Big\{n + \dfrac{d}{n_0} + \dfrac{d}{n}\Big(1 + \dfrac{d}{n_0 n_1}\Big)\Big\} < c\Big(n + \dfrac{d}{n} + \dfrac{1}{n_0}\Big) = c\Phi\Big(-\dfrac{1}{n}, -\dfrac{1}{n_0}\Big) & \text{for } n < n_0, \\ c\Big\{n_0 + \dfrac{d}{n_0} + \dfrac{d}{n}\Big(1 + \dfrac{d}{n_1 n_0}\Big)\Big\} < c\Big(n_0 + \dfrac{d}{n_0} + \dfrac{1}{n}\Big) = c\Phi\Big(-\dfrac{1}{n}, -\dfrac{1}{n_0}\Big) & \text{for } n > n_0. \end{cases}$$

In any case

$$U^{\mu}\Big(-\frac{1}{n}\Big) \leqq c\Phi\Big(-\frac{1}{n}, -\frac{1}{n_0}\Big).$$

Consequently

$$U^{\mu}(x) \leqq c\Phi(x, x_0)$$

is concluded for any x from the inequality $U^{\mu}(x) \leqq \Phi(x, x_0)$ assumed on S_{μ}.

Let ν be any measure whose potential is defined everywhere in Ω and satisfies $U^{\mu}(x) \leqq U^{\nu}(x)$ on S_{μ}. By Lemma it follows that

$$U^{\mu}(x) \leqq cU^{\nu}(x) \qquad \text{everywhere in } \Omega.$$

This shows that $0 \notin \mathbf{R}_c$.

We have proved more than $0 \in \mathbf{O}_c - \mathbf{R}_c$ in the above. Actually $0 \in \mathbf{O}_c$ and $(\mathrm{U_d})_c$ is satisfied. In case $c=1$ this example gives (D)$\nrightarrow$(F), although we showed already (D)$\nrightarrow$(B′) in (VI) of § 1.3.

1.5. Global properties of potentials.

Under this title we shall investigate some principles, in particular, (FV), (UV) and $(\mathrm{U_K})$, in connexion with the local properties obtained in § 1.4.

(FV) and (UV). (i) *Let* $c \geqq 1$. *If*

$$\overline{\lim_{P \to P_0}}\, U^{\mu}(P) \leqq c \overline{\lim_{P \in S_{\mu},\, P \to P_0}}\, U^{\mu}(P) \tag{1.18}$$

for every measure μ *with compact support and at every point* P_0 *on* S_{μ}, *then* $(\mathrm{UV})_c$ *is true; in case* P_0 *is isolated on* S_{μ}, *we let the value* $cU^{\mu}(P_0)$ *replace the right side of* (1.18). *Conversely, if the kernel is continuous outside the diagonal set and nonnegative in* $\Omega \times \Omega$ *and if* $(\mathrm{UV})_c$ *is satisfied, then* (1.18) *is true for any* μ *with compact* S_{μ} *and for any* $P_0 \in S_{\mu}$.

PROOF. The first part is easily proved by Heine-Borel's covering theorem. We suppose that (1.18) is not true with $c>1$ for some μ with compact support and for P_0 not isolated on S_μ. Using the same notations as in (ii) of § 1.4, we have

$$\overline{\lim_{P\to S_{\mu_{V_1}}}} U^{\mu_{V_1}}(P) \geqq \alpha_{V_1} > c \sup_{S_{\mu_{V_1}}} U^{\mu_{V_1}}(P)$$

by (1.16), where $\overline{\lim}_{P\to S_{\mu_{V_1}}} U^{\mu_{V_1}}(P)$ is defined by $\inf_G \sup_{P\in G} U^{\mu_{V_1}}(P)$ for open set $G \supset S_{\mu_{V_1}} = V_1 \cap S_\mu$. Thus $(\mathrm{UV})_c$ is denied. If P_0 is isolated on S_μ and $\overline{\lim}_{P\to P_0} U^\mu(P) > cU^\mu(P_0)$, we have, for the restriction μ_0 of μ to P_0,

$$\overline{\lim_{P\to P_0}} U^{\mu_0}(P) > cU^{\mu_0}(P_0) + (c-1)\, U^{\mu-\mu_0}(P_0) \geqq cU^{\mu_0}(P_0)$$

and it is seen that $(\mathrm{UV})_c$ is not satisfied. Finally suppose that (1.18) is not true with $c=1$. We can find $c'>1$ with which still (1.18) is not true. Then we see that $(\mathrm{UV})_{c'}$ is not satisfied. Naturally $(\mathrm{UV})_1 = (\mathrm{FV})$ is not satisfied.

By (ii) of § 1.4 we see that (1.18) is true if $\bigcup_{c'>c} \mathbf{O}_{c'} = \emptyset$. Hence we obtain

COROLLARY 1. *Let* $c \geqq 1$. *Consider a nonnegative kernel which is continuous outside the diagonal set. If* $\mathbf{O}_{c'}$ *is empty for every* $c'>c$, *then* $(\mathrm{UV})_c$ *is satisfied.*

In case $c=1$, this corollary was given by Choquet [2] as Proposition 3.

From (xi) of § 1.4 and Corollary 1 follows

COROLLARY 2. *Let* $c \geqq 1$. *For a nonnegative kernel which is continuous in the extended sense, finite outside the diagonal set and positive on the diagonal set,* $(\mathrm{U}_\mathrm{d}^*)_c \to (\mathrm{UV})_c$. *In particular,* $(\mathrm{D}^*) \to (\mathrm{FV})$.

(ii) *Let* $c \geqq 1$. *If* (1.18) *is always true, then* $\mathbf{P}_c = \emptyset$. *If a nonnegative kernel is continuous outside the diagonal set in* $\Omega \times \Omega$ *and if* $\mathbf{P}_c = \emptyset$, *then* (1.18) *is true.*

PROOF. The first assertion is obviously true. Next we assume the existence of μ and P_0 such that (1.18) is not true. Let N be any compact neighborhood of P_0, and denote by μ_N the restriction of μ to N. Then $S_{\mu_N} \subset N$ and

$$\overline{\lim_{P\to P_0}} U^{\mu_N}(P) > (c-1)\, U^{\mu-\mu_N}(P_0) + c \overline{\lim_{P\in S_\mu, P\to P_0}} U^{\mu_N}(P) \geqq c \overline{\lim_{P\in S_{\mu_N}, P\to P_0}} U^{\mu_N}(P).$$

This shows that $P_0 \in \mathbf{P}_c$.

Combining (ii) with (i) we have

COROLLARY. *Let* $c \geqq 1$. *If a nonnegative kernel is continuous outside the diagonal set, then* $(\mathrm{UV})_c$ *is true if and only if* $\mathbf{P}_c = \emptyset$.

(iii) *Let* $c \geqq 1$. *Consider a nonnegative kernel which is locally bounded* *outside the diagonal set and equal to* ∞ *on the diagonal set. If* $(\mathrm{UV})_c$ *is satisfied, then* $\mathbf{O}_{c'}$ *has no point of accumulation for every* $c'>c$.

This follows immediately from (v) of §1.4. In case $c=1$, this was stated by Choquet under a slightly different condition.[7]

(iv) *Examples to show* $(\mathrm{U}) \not\rightarrow (\mathrm{FV})$ *were given by Kunugui* [1] *and Choquet* [2]. *Here we reproduce the example of Kunugui (p. 77).*

Example for $(\mathrm{U}) \not\rightarrow (\mathrm{FV})$. $\Omega=E_3$ and $\Phi(P, Q)=\overline{PQ}^{-\alpha}$, $0<\alpha<1$. On a half line issuing from the origin 0 we take points $\{P_n\}$ such that $\overline{OP_n}=1/n$. We set

$$k_\alpha = 1 - \frac{1}{2^\alpha\left(1-\dfrac{2}{\alpha}\right)} > 0$$

and choose c, $0<c<k_\alpha/2$, in an arbitrary manner. We set also

$$d_n=\frac{c}{2^{n+2}n(n+1)}.$$

We denote by C_n the spherical surface with P_n as center and $d_n^{1/\alpha}$ as radius, and by μ_n the uniform measure on C_n with total mass d_n. The support of $\mu = \sum_{n=1}^{\infty}\mu_n$ is equal to $\bigcup_{n=1}^{\infty} C_n \cup \{0\}$. We can see by computation that

$$U^{\mu_n}(P_n)=1 \qquad \text{and} \qquad U^{\mu_n}(P)=2^{-\alpha}(1-\alpha/2)^{-1} \qquad \text{on } C_n,$$

and that, for $m \neq n$,

$$U^{\mu_m}(P)<\frac{c}{2^{m+1}} \qquad \text{on } C_n$$

and

$$U^{\mu_n}(0)<\frac{c}{2^{n+1}(n+1)}.$$

Therefore

$$U^{\mu}(P_n)>1$$

and

$$U^{\mu}(P)<2^{-\alpha}(1-\alpha/2)^{-1}+c<1-k_\alpha+\frac{k_\alpha}{2}=1-\frac{k_\alpha}{2}<1 \qquad \text{on } S_\mu.$$

Namely, for any neighborhood V of S_μ,

$$\sup_{P\in V} U^{\mu}(P) > \sup_{P\in S_\mu} U^{\mu}(P) + \frac{k_\alpha}{2},$$

7) Proposition 4 of Choquet [2].

contrary to (FV). It is well known that any α-kernel $\overline{PQ}^{-\alpha}$ satisfies (U).

(v) *We shall complete the relations of* (FV).

Example for $(\mathrm{FV}) \not\rightarrow (\mathrm{B}'_{\mathrm{K}})$: $\Omega = \{0\} \cup \{1\}$ and $\Phi(0, 0) = \Phi(1, 1) = 1$, $\Phi(0, 1) = \Phi(1, 0) = \infty$.

However, if we assume that $\Phi(P, Q)$ is continuous in the extended sense and finite outside the diagonal set, not only $(\mathrm{B}'_{\mathrm{K}})$ but also $(\mathrm{B}_{\mathrm{K}})$ follows obviously from (UV). The fact that $(\mathrm{FV}) \not\rightarrow (\mathrm{B}')$ is seen by the example $\Phi(P, Q) = \overline{PQ}$ in E_3. The example given before for $(\mathrm{B}) \not\rightarrow (\mathrm{U}_{\mathrm{K}})$ provides an example for $(\mathrm{FV}) \not\rightarrow (\mathrm{U}_{\mathrm{K}})$. Actually if we use the same notation $\mu = \sum_{n=1}^{\infty} \mu_n$ as there and if N is the highest subscript with $\mu_N \neq 0$, then we have $U^{\mu}(P) \leqq \sup_{Q \in S_{\mu}} U^{\mu}(Q)$ on $V = \bigcup_{n=1}^{N} [(2n+1)^{-1}, (2n)^{-1}]$. If $0 \in S_{\mu}$, then $U^{\mu}(P) \leqq \sup_{Q \in S_{\mu}} U^{\mu}(Q)$ in the whole Ω.

It is evident that $(\mathrm{F}) \rightarrow (\mathrm{FV})$ and we have now

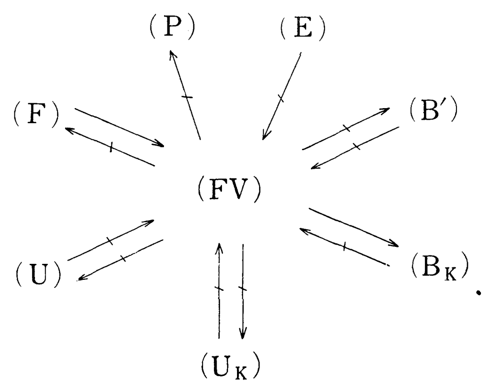
$$(1.19) \qquad \begin{array}{c} (\mathrm{U}) \underset{\not\leftarrow}{\not\rightarrow} (\mathrm{FV}) \underset{\not\leftarrow}{\not\rightarrow} (\mathrm{B}'_{\mathrm{K}}) \\ \uparrow \not\downarrow \\ (\mathrm{F}) \end{array}$$

If $\Phi(P, Q)$ is continuous in the extended sense and finite outside the diagonal set, we have

$$(1.20) \qquad \begin{array}{c} (\mathrm{B}') \underset{\not\leftarrow}{\not\rightarrow} (\mathrm{FV}) \underset{\not\leftarrow}{\rightarrow} (\mathrm{B}_{\mathrm{K}}) \\ \not\uparrow \not\downarrow \\ (\mathrm{U}_{\mathrm{K}}) \end{array}$$

It is easily observed that Example 3 in Ohtsuka [7] satisfies (FV). Hence $(\mathrm{FV}) \not\rightarrow (\mathrm{P})$. Since $(\mathrm{FV}) \rightarrow (\mathrm{B}_{\mathrm{K}}) \rightarrow (\mathrm{C})$ and Example 2 in Ohtsuka [7] shows that $(\mathrm{E}) \not\rightarrow (\mathrm{C})$, it follows that $(\mathrm{E}) \not\rightarrow (\mathrm{FV})$. Taking (1.19) and (1.20) into consideration, we have

(1.21)



(vi) *First we shall give an example for* $(\mathrm{U}_{\mathrm{K}}) \not\rightarrow (\mathrm{UV})$ *and then establish relations of* $(\mathrm{U}_{\mathrm{K}})$ *with other principles.*

We found in (vi) of §1. 4 a compact space, a positive kernel which is continuous in the extended sense and finite outside the diagonal set and which satisfies $(\mathrm{U})_{c+1}$, and a point P_0 of the space with $\mathbf{p}(P_0)=c$. For every n, we take such a space K_n, a kernel $\Phi_n(P, Q)$ satisfying $(\mathrm{U})_{n+1}$ on $K_n\times K_n$ and a point $P_n\in K_n$ with $\mathbf{p}(P_n)=n$. We consider the sumspace Ω of K_n, $n=1, 2,\cdots$, and define the kernel $\Phi(P, Q)$ on $\Omega\times\Omega$ by

$$\Phi(P, Q)=\begin{cases}\Phi_n(P, Q) & \text{if } P, Q\in K_n,\\ 1 & \text{otherwise.}\end{cases}$$

It is easy to see that $(\mathrm{U_K})$ is satisfied. Since $\mathbf{p}(P_n)=n$, $(\mathrm{UV})_c$ is not true for every $c\geqq 1$. Thus $(\mathrm{U_K})\nrightarrow(\mathrm{UV})$.

This example shows also $(\mathrm{B})\nrightarrow(\mathrm{UV})$. Let us still consider positive kernels which are continuous in the extended sense and finite outside the diagonal set. Then we see $(\mathrm{UV})\nrightarrow(\mathrm{U_K})$, $(\mathrm{UV})\nrightarrow(\mathrm{B'})$, $(\mathrm{UV})\nrightarrow(\mathrm{U})$, $(\mathrm{UV})\nrightarrow(\mathrm{P})$ and $(\mathrm{UV})\nrightarrow(\mathrm{FV})$ in view of (1.20). It is easy to see $(\mathrm{UV})\rightarrow(\mathrm{B_K})$, and as in (iv) we observe $(\mathrm{E})\nrightarrow(\mathrm{UV})$ and $(\mathrm{UV})\nrightarrow(\mathrm{P})$. Thus we have

(1.22)
$$\begin{tikzcd}
 & (\mathrm{P}) & (\mathrm{E}) \arrow[dl, "/" marking] & \\
(\mathrm{FV}) \arrow[r, shift left] & (\mathrm{UV}) \arrow[u, "/" marking] \arrow[l, shift left, "/" marking] \arrow[rr, bend left, ""] \arrow[ddrr, "/" marking] \arrow[d, shift right, "/" marking] \arrow[dl, shift left, "/" marking] & & (\mathrm{B_K}) \\
(\mathrm{U}) \arrow[ur, shift left] & (\mathrm{U_K}) \arrow[u, shift right, "/" marking] & & (\mathrm{B}) \arrow[ull, "/" marking] \\
 & & & (\mathrm{B'})
\end{tikzcd}$$

for kernels which are continuous in the extended sense and finite outside the diagonal set.

For general kernels we obtain, in view of (1.19),

(1.23)
$$\begin{tikzcd}
(\mathrm{FV}) \arrow[r, shift left] & (\mathrm{UV}) \arrow[l, shift left, "/" marking] \arrow[r, shift left] \arrow[d, shift left, "/" marking] & (\mathrm{B'_K}) \arrow[l, shift left, "/" marking] \\
 & (\mathrm{U}) \arrow[u, shift left] &
\end{tikzcd}$$

$(\mathrm{U_K})$. (i) *If* $(\mathrm{U_K})$ *is satisfied, then* $\mathbf{O}_\infty=\emptyset$. *Conversely, if* $\mathbf{O}_\infty=\emptyset$ *for a nonnegative kernel which is locally bounded outside the diagonal set and positive on the diagonal set in* $\Omega\times\Omega$, *then the kernel satisfies* $(\mathrm{U_K})$.[8]

Proof. It is obvious that $\mathbf{O}_\infty=\emptyset$ if $(\mathrm{U_K})$ is satisfied. Let us assume that $\mathbf{O}_\infty=\emptyset$ and take a compact set K in Ω. At every point $P\in K$ there are a relatively compact open neighborhood N_P of P and a constant c_P such that

(1.24)
$$\sup_{Q\in N_P} U^\mu(Q)\leqq c_P \sup_{Q\in S_\mu} U^\mu(Q)$$

for any μ with $S_\mu\subset N_P$. In each N_P we take a compact neighborhood N'_P of P.

8) This result is stated in Choquet [2].

We can find $N'_{P_1}, \cdots, N'_{P_n}$ which together cover K. We set $F=\bigcup_{k=1}^{n} N'_{P_k}$. Let μ be any measure with $S_\mu \subset N'_{P_k}$ and with $\sup_{S_\mu} U^\mu < \infty$. By our assumption, $U^\mu(P)$ is bounded on $F-N_{P_k}$ because

$$\{(P, Q);\ P\in F-N_{P_k},\ Q\in S_\mu\}$$

is a compact set disjoint from the diagonal set in $\Omega\times\Omega$. Combining this fact with (1.24), we see that $U^\mu(P)$ is bounded on F. By Lemma 1.2 of (I) in § 1.3 it follows that there is a constant $c_k=c(N_{P_k})\geqq 1$ such that

$$\sup_{P\in F} U^\mu(P) \leqq c_k \sup_{P\in N'_{P_k}} U^\mu(P) \leqq c_k\, c_{P_k} \sup_{Q\in S_\mu} U^\mu(Q)$$

for any μ with $S_\mu \subset N'_{P_k}$.

Now let λ be any measure with $S_\lambda \subset K$ and decompose S_λ in such a way that $S_\lambda=\bigcup_{k=1}^{n} F_k$ and each F_k is a compact set contained in N'_{P_k}. Denoting the restriction of λ to F_k by λ_k, we have

$$\sup_{P\in K} U^\lambda(P) \leqq \sum_{k=1}^{n} \sup_{P\in K} U^{\lambda_k}(P) \leqq \sum_{k=1}^{n} c_k\, c_{P_k} \sup_{P\in S_{\lambda_k}} U^{\lambda_k}(P)$$
$$\leqq \left(\sum_{k=1}^{n} c_k\, c_{P_k}\right) \sup_{P\in S_\lambda} U^\lambda(P).$$

This shows that $(\mathrm{U_K})$ is satisfied.

Since the new $\mathbf{O}_\infty$ is empty if we restrict us to the space $\Omega-\mathbf{O}_\infty$, we have

Corollary 1. *Consider a nonnegative kernel which is locally bounded outside the diagonal set and positive on the diagonal set in $\Omega\times\Omega$. Then it satisfies $(\mathrm{U_K})$ in $\Omega-\mathbf{O}_\infty$, this being considered as a space.*

Corollary 2. *We consider a kernel which is continuous outside the diagonal set and positive on the diagonal set. If $(\mathrm{U_{K,d}})$ is satisfied, $(\mathrm{U_K})$ is true.*

Proof. Let K be any compact subset of Ω. By the relation $\mathbf{R}_c \supset \bigcup_{c'>c} \mathbf{O}_{c'}$ given in Remark 1 in (xi) of § 1.4 it followt from $(\mathrm{U_{K,d}})$ that $\mathbf{O}_c \cap K=\emptyset$ for some $c<\infty$. Therefore $\mathbf{O}_\infty=\emptyset$ in Ω.

If we can show that $U^\mu(P)\geqq 0$ on K for any $\mu\in\mathscr{E}$, then we can apply Lemma 1.2 to show $(\mathrm{U_K})$ in the same way as in (i). So we give

Lemma 1.3. *We consider a kernel which is continuous outside the diagonal set and positive on the diagonal set. If $(\mathrm{U_d^*})$ is satisfied, $U^\mu(P)\geqq 0$ in Ω for any $\mu\in\mathscr{E}$ with compact support.*

Proof. Take any $\mu\in\mathscr{E}$ with compact support and assume that there are points P_0 and Q_0 such that $\Phi(P_0, Q_0)<0$. We take a compact neighborhood N

of Q_0 such that $\Phi(P, Q)>0$ on $N\times N$ and $\Phi(P_0, Q)<0$ for $Q\in N$. Let us assume that $(\mathrm{U}_{\mathrm{d}}^{*})_c$ holds. If $\mu(N)\neq 0$, we denote by μ_N the restriction of μ to N. Then $U^{\mu_N}(P)<U^{k\mu_N}(P)$ on $S_{\mu_N}=S_\mu\cap N$ with $k>\max(1, 1/c)$ and hence $U^{\mu_N}(P)\leqq cU^{k\mu_N}(P)=kcU^{\mu_N}(P)$ in Ω. This shows that $U^{\mu_N}(P)\geqq 0$ in Ω. In particular,

$$U^{\mu_N}(P_0)=\int_N \Phi(P_0, Q)d\mu(Q)\geqq 0.$$

This contradicts the fact that $\Phi(P_0, Q)<0$ for every $Q\in N$. Thus it is proved that, for every $Q\in S_\mu$, $\Phi(P, Q)\geqq 0$ for all $P\in\Omega$. Therefore $U^\mu(P)\geqq 0$ in Ω.

If we use the relation $\mathbf{R}_c^*\supset \bigcup_{c'>c}\mathbf{O}_{c'}$ proved in (xi) of § 1.4, we have

Corollary 3. *We consider a kernel which is continuous in the extended sense, finite outside the diagonal set and positive on the diagonal set. If* $(\mathrm{U}_{\mathrm{K,d}}^{*})$ *is satisfied,* $(\mathrm{U}_{\mathrm{K}})$ *is true.*

Corollary 4. *There exists a positive kernel with the following property: It is continuous in the extended sense and finite outside the diagonal set, it satisfies the continuity principle and* $\mathbf{O}_\infty$ *is not empty.*

Proof. We consider the example for $(\mathrm{B})\nrightarrow(\mathrm{U}_{\mathrm{K}})$ in (II) of § 1.3. By the above result $\mathbf{O}_\infty$ is not empty. Since (B) is equivalent to the continuity principle for our kernel, it fulfills all the requirements.

(ii) *We shall show that the above-stated condition on the sign of the kernel on the diagonal set is really necessary.*

First we give

Example of $\Phi\geqq 0$ for $(\mathrm{D})\nrightarrow(\mathrm{U}_{\mathrm{K}})$: $\Omega=\bigcup_{n=1}^{\infty}\{1/n\}\cup\{0\}$, $\Phi(0, 0)=\Phi(0, 1/n)=\Phi(1/n, 0)=0$, $\Phi(1/n, 1/m)=(nm)^{-1}$. Let $\mu\in\mathscr{E}$, $\mu\neq 0$, and assume that $U^\mu(x)\leqq U^\nu(x)$ on S_μ with some ν. We denote $\mu(\{1/n\})$ and $\nu(\{1/n\})$ by a_n and b_n respectively. We have

$$U^\mu\left(\frac{1}{n}\right)=\sum_{k=1}^{\infty}\frac{a_k}{nk} \qquad \text{and} \qquad U^\nu\left(\frac{1}{n}\right)=\sum_{k=1}^{\infty}\frac{b_k}{nk}.$$

If $1/n_0\in S_\mu$,

$$\sum_{k=1}^{\infty}\frac{a_k}{n_0k}\leqq\sum_{k=1}^{\infty}\frac{b_k}{n_0k} \qquad \text{and hence} \qquad \sum_{k=1}^{\infty}\frac{a_k}{k}\leqq\sum_{k=1}^{\infty}\frac{b_k}{k}.$$

Therefore $U^\mu(1/n)\leqq U^\nu(1/n)$ for every n. Since $U^\mu(0)=U^\nu(0)=0$, $U^\mu(x)\leqq U^\nu(x)$ is true in Ω. If $S_\mu=\{0\}$, $U^\mu(x)=0$ everywhere and $U^\mu(x)\leqq U^\nu(x)$ in Ω. Thus (D) is satisfied. On the other hand, if we give a point measure μ_n at $x=1/n$ with mass n^2, then $U^{\mu_n}(1/n)=1$ but $U^{\mu_n}(1)=n$ and $(\mathrm{U}_{\mathrm{K}})$ is not satisfied.

Example of $\Phi\neq 0$ for $(\mathrm{D})\nrightarrow(\mathrm{U}_{\mathrm{K}})$: $\Omega=\{0\}\cup\{1\}$, $\Phi(0, 0)=-2$, $\Phi(1, 1)=1$, $\Phi(1, 0)=-1$, $\Phi(0, 1)=2$. Let μ be a point measure at 0 and ν be any measure such that $-2\mu(\Omega)=U^\mu(0)\leqq U^\nu(0)$. If we denote the restrictions of ν to the points 0 and 1 by ν_1 and ν_2 respectively, then $U^\nu(0) = -2\nu_1(\Omega) + 2\nu_2(\Omega) \geqq -2\mu(\Omega)$. We have $U^\mu(1) = -\mu(\Omega) \leqq -\nu_1(\Omega) + \nu_2(\Omega) = U^\nu(1)$. If μ is a point measure at 1 and $U^\mu(1) \leqq U^\nu(1)$, we see similarly that $U^\mu(0) \leqq U^\nu(0)$. Thus (D) is satisfied. On the other hand ($\mathrm{U_K}$) is not true because, if $\sup_\Omega U^\mu \leqq c \sup_{S_\mu} U^\mu$ with $c > 0$ for any μ, then we see that $c \geqq 2$ by considering the unit point measure at 1 and also that $c \leqq 1/2$ by considering the unit point measure at 0.

There is a kernel which has general sign on the diagonal set and satisfies both (D) and (F) as the following example shows:

$$\Omega = \{0\} \cup \{1\},\ \Phi(0, 0) = \Phi(0, 1) = -1,\ \Phi(1, 1) = \Phi(1, 0) = 1.$$

However, if we require furthermore that the kernel is symmetric, then there is no kernel which has general sign on the diagonal set outside $\{(P, P);\ P \in G_0\}$ and satisfies both (D*) and ($\mathrm{U_K}$), where G_0 is the set of points each of which has a neighborhood supporting no non-vanishing element of $\mathscr{E}$. To prove this, assume that Q_0 is a point with $\Phi(Q_0, Q_0) < 0$ and $Q_1 \notin G_0$ is a point with $\Phi(Q_1, Q_1) > 0$. We denote by μ_0 the unit point measure at Q_0. Naturally $\mu_0 \in \mathscr{E}$ and $0 \in \mathscr{E}$. Since $U^{\mu_0}(Q_0) = \Phi(Q_0, Q_0) < 0 = U^0(Q_0)$ on $S_{\mu_0} = \{Q_0\}$, it follows by (D*) that $U^{\mu_0}(Q) = \Phi(Q, Q_0) = \Phi(Q_0, Q) \leqq 0$ for every $Q \in \Omega$. Let V be a compact neighborhood of Q_1 such that $\Phi(P, Q) > 0$ on $V \times V$, and $\mu_1 \gneqq 0$ be a measure of $\mathscr{E}_V$. Since $U^{\mu_1}(P) < U^{2\mu_1}(P)$ on S_{μ_1}, $U^{\mu_1}(P) \leqq U^{2\mu_1}(P)$ in Ω. Hence $U^{\mu_1}(P) \geqq 0$ in Ω. In particular $0 \leqq U^{\mu_1}(Q_0) = \int \Phi(Q_0, Q)\, d\mu_1(Q)$. We have seen that $\Phi(Q_0, Q) \leqq 0$ for every $Q \in \Omega$. Hence there is at least one point $Q_1' \in V$ at which $\Phi(Q_0, Q_1') = 0$. Therefore

$$U^{\mu_0}(Q_1') = \Phi(Q_1', Q_0) = 0.$$

On the other hand

$$V(\mu_0) = \sup_{S_{\mu_0}} U^{\mu_0} = U^{\mu_0}(Q_0) = \Phi(Q_0, Q_0) < 0.$$

If we take for K the union of Q_0 and V, there is no $c(K) > 0$ which satisfies $\sup_K U^{\mu_0} \leqq c(K)\, V(\mu_0)$.

Finally we give

Example of $\Phi < 0$ for (D) $\nrightarrow$ ($\mathrm{U_K}$): $\Omega = \bigcup_{n=1}^\infty \{1/n\} \cup \{0\}$, $\Phi(0, 0) = -1$, $\Phi(0, 1/n) = \Phi(1/n, 0) = -1/n$, $\Phi(1/n, 1/m) = -(nm)^{-1}$. It is shown as in the first example that (D) is satisfied. If we denote by μ_0 the unit point measure at 0, then $U^{\mu_0}(0) = -1$ and $U^{\mu_0}(1/n) = -1/n$. There is no $c > 0$ which satisfies

$$U^{\mu_0}\left(\frac{1}{n}\right) \leqq c \sup_{S_{\mu_0}} U^{\mu_0} = -c$$

for each n.

(B_K), (B'_K) and (C). It is easy to verify:

(i) *Consider a kernel which is locally bounded outside the diagonal set. A necessary and sufficient condition for* (B_K) ((B'_K) *resp.*) *to be true is that* $\mathbf{Q} = \emptyset$ ($\mathbf{Q}' = \emptyset$ *resp.*).

(ii) *Consider a kernel which is continuous outside the diagonal set. A necessary and sufficient condition for* (C) *to be true is that* $\mathbf{D} = \emptyset$.

1.6. Topologies.

We shall introduce several kinds of topologies on the class $\mathscr{M}$ of all measures μ in Ω, on the class $\mathscr{M}_0$ of measures, whose potentials are bounded from below on every compact set in Ω and not identically equal to ∞, and on the subclass $\mathscr{E}$ of $\mathscr{M}_0$ of measures with finite energy.

1) *Vague topology.*

Let $\mathscr{C}_0(\Omega)$ be the set of continuous real-valued functions with compact support in Ω. The vague topology is defined on $\mathscr{M}$ by the semi-norms $\mu - \nu \to \left| \int f d\mu - \int f d\nu \right|$, $f \in \mathscr{C}_0(\Omega)$. The space $\mathscr{M}$ is then a Hausdorff space. We call a set $H \subset \mathscr{M}$ vaguely bounded if each semi-norm is bounded on H. This amounts to say that, for every compact set $K \subset \Omega$, $\sup_{\mu \in H} \mu(K) < \infty$.

We state two facts which will be used later; see Bourbaki [1] and Fuglede [1] for them.

Proposition 3. *Any vaguely bounded set H is relatively compact in $\mathscr{M}$ with respect to the vague topology.*

Proposition 4. *Consider a nonnegative kernel or the class $\mathscr{M}_K$ of measures which are supported by a fixed compact set K. Then the mutual energy (μ, ν) is lower semicontinuous on $\mathscr{M} \times \mathscr{M}$ or on $\mathscr{M}_K \times \mathscr{M}_K$ respectively. Also $U^\mu(P)$ is lower semicontinuous as a function on $\mathscr{M} \times \Omega$ or on $\mathscr{M}_K \times K$ respectively.*

We consider the unit point measure at every point of Ω and define a topology of Ω by the vague topology of the corresponding unit point measures. Then this topology of Ω coincides with the original topology of Ω.

2) *Fine topology.*

This was first introduced by H. Cartan [6] for the Newtonian kernel. We shall denote by $\mathscr{L}$ ($\check{\mathscr{L}}$ resp.) the set of measures $\lambda \in \mathscr{M}_0$ with the property that (λ, μ) ((μ, λ) resp.) is defined and finite for every $\mu \in \mathscr{M}_0$. Obviously $\mathscr{L} \subset \mathscr{E}$. The fine (adjoint fine resp.) topology is defined on $\mathscr{M}_0$ by the semi-norms $\mu - \nu \to |(\lambda, \mu) - (\lambda, \nu)|$, $\lambda \in \mathscr{L}$ ($\mu - \nu \to |(\mu, \lambda) - (\nu, \lambda)|$, $\lambda \in \check{\mathscr{L}}$ resp.). The space $\mathscr{M}_0$ with this topology may not be a Hausdorff space.

In case there is no P for which $\Phi(Q, P)$ ($\Phi(P, Q)$ resp.) $\equiv \infty$ as a function of Q, we define the fine (adjoint fine resp.) topology of Ω by the fine (adjoint fine resp.) topology of the corresponding unit point measures; every point measure belongs to $\mathscr{M}_0$ in our case. This is the weakest topology which makes all potentials (adjoint potentials resp.) of measures of $\mathscr{L}$ ($\check{\mathscr{L}}$ resp.) continuous. We shall discuss the fine topology in Ω later (not in this paper).

3) *Weak topology.*

Under the assumption that the kernel is of positive type, the weak topology is defined on $\mathscr{E}$ by the semi-norms $\mu-\nu \to |(\pi, \mu)-(\pi, \nu)|$, $\pi \in \mathscr{E}$, provided that (π, μ) and (π, ν) are defined and finite.[9] The space $\mathscr{E}$ with this topology may not be a Hausdorff space.

4) *Strong topology.*

Under the assumption that the kernel is of positive type, the semi-norm $\|\mu-\nu\| = \sqrt{(\mu, \mu)+(\nu, \nu)-2(\mu, \nu)}$, considered under the condition that (μ, ν) is defined, defines the strong topology on $\mathscr{E}$. This may not give a Hausdorff space again. A Cauchy net in $\mathscr{E}$ with respect to this topology will be called a strong Cauchy net.

H. Cartan [5; 6] proved in the Newtonian case that, in the space $\mathscr{E}$, the strong topology 4) is stronger than the weak topology 3) and 3) is stronger than the fine topology 2), and that, in the space $\mathscr{M}_0$, 2) is stronger than the vague topology 1). He showed also that, if the energy of each element of a sequence in $\mathscr{E}$ is bounded, then the first three convergences are equivalent.

Let us consider a general kernel of positive type. Obviously 4) is stronger than 3), and 3) is stronger than 2) on $\mathscr{E}$. However, we give

EXAMPLE 1 to show that 2) is not stronger than 1). For Ω we take E_3 and two points P_1 and P_2 which do not belong to E_3. We preserve the topology of E_3 and regard P_1 and P_2 as isolated points. We set

$$\Phi(P_1, P_1)=\Phi(P_2, P_2)=\infty,$$

$$\Phi(P_1, P_2)=\Phi(P_2, P_1)=\Phi(P, P_i)=\Phi(P_i, P)=1 \qquad \text{for } P \in E_3 \text{ and } i=1, 2,$$

$$\Phi(P, Q)=\overline{PQ}^{-1} \qquad \text{for } P, Q \in E_3.$$

We observe that $\mathscr{L}$ is a subclass of the $\mathscr{L}$-class in E_3 for the Newtonian kernel and each measure of $\mathscr{L}$ has a finite total mass. We denote the unit measures at P_1 and P_2 by μ_1 and μ_2. Since $(\lambda, \mu_1)=(\lambda, \mu_2)=\lambda(\Omega)$ for every $\lambda \in \mathscr{L}$, μ_1 and μ_2 are not separated in the space $\mathscr{M}_0$ with the fine topology. This shows that 2) is not stronger than 1) because $\mathscr{M}$ with the vague topology is a Hausdorff space.

We note that the energy and continuity principles are satisfied but μ_1, $\mu_2 \notin \mathscr{E}$ in this example.

Next we are concerned with a net T in $\mathscr{E}$ with bounded energy. Without

9) Choosing $\pi_1, \cdots, \pi_n$ in $\mathscr{E}$ such that each (π_k, μ) is defined and finite,

$$\bigcap_{k=1}^{n} \{\nu \in \mathscr{E}\ ;\ (\pi_k, \nu) \text{ is defined and } |(\pi_k, \mu)-(\pi_k, \nu)|<1\}$$

is taken as a neighborhood of μ and all such neighborhoods constitute a base of neighborhoods of μ.

any condition there is no equivalence relation among 1), 2) and 3); the situation is different from the Newtonian case.

EXAMPLE 2. We shall show the existence of a kernel, satisfying both the energy and continuity principles, and a strongly converging sequence of measures which does not at all converge vaguely. We consider a positive continuous strictly positive definite function $\varphi(x)$ defined on the x-axis. We take for Ω the subset $\bigcup_{n=1}^{\infty}\{1/n\}\cup\{-1\}$ of the x-axis and define

$$\Phi\left(\frac{1}{n}, \frac{1}{m}\right)=\varphi\left(\frac{1}{n}-\frac{1}{m}\right),$$
$$\Phi\left(-1, \frac{1}{n}\right)=\Phi\left(\frac{1}{n}, -1\right)=\varphi\left(\frac{1}{n}\right),$$
$$\Phi(-1, -1)=\varphi(0).$$

This kernel satisfies both the energy and continuity principles. We denote by μ_n the unit measure at the point $x=1/n$, and by μ_0 the unit measure at $x=-1$. We see that $(\mu_n, \mu_n)=\varphi(0)$ for every n and that

$$(\mu_n-\mu_0, \mu_n-\mu_0)=\varphi(0)+\varphi(0)-2\varphi\left(\frac{1}{n}\right)$$

tends to 0 as $n\to\infty$. Hence the sequence $\mu_1, \mu_0, \mu_2, \mu_0, \cdots$ converges strongly to μ_0. However, it does not converge vaguely at all.

EXAMPLE 3. We shall show the existence of a sequence of measures, with bounded energy and supported by a fixed compact set, which converges vaguely but not finely.

We modify Example 9 of Ohtsuka [7]. Let K_0 be the segment $0\leqq x\leqq 1$, $y=0$ in the (x, y)-plane E_2. We define $\Phi(P, Q)=\Phi(Q, P)$ in a neighborhood $V\times V$ of $K_0\times K_0$ as in that example and by $\overline{PQ}^{-1}$ for every Q if P lies sufficiently far, say outside a neighborhood $V_1\supset V$ of K_0. For other pair (P, Q) we define $\Phi(P, Q)=\Phi(Q, P)$ arbitrarily so that it is continuous in the extended sense and finite outside the diagonal set in $E_2\times E_2$. The uniform unit measure μ_0 on K_0 belongs to $\mathscr{L}$. In fact, as P approaches K_0, $U^{\mu_0}(P)$ stays bounded as is calculated in the quoted example. Therefore if μ is a measure for which $U^{\mu}(P)\not\equiv\infty$, and if μ_{V_1} denotes the restriction of μ to V_1, then

$$(\mu_{V_1}, \mu_0)=\iint U^{\mu_0}\, d\mu_{V_1}\leqq \sup_{V_1}\ U^{\mu_0}\cdot\mu(V_1)<\infty$$

and

$$(\mu-\mu_{V_1}, \mu_0)=\iint\frac{1}{\overline{PQ}}\, d(\mu-\mu_{V_1})\, d\mu_0=\int_K U^{\mu-\mu_{V_1}}\, d\mu_0,$$

where $U^{\mu-\mu_{V_1}}$ is the potential of $\mu-\mu_{V_1}$ with kernel $1/\overline{PQ}$. Since it is continuous in V_1, it is bounded on K_0. Therefore $(\mu-\mu_{V_1}, \mu_0)<\infty$. Thus $(\mu, \mu_0)<\infty$ for any μ with $U^{\mu}(P)\not\equiv\infty$ and it is concluded that $\mu_0\in\mathscr{L}$. Now we take the uniform unit measure on the segment K_n: $0\leqq x\leqq 1$, $y=1/n$ for μ_n. Obviously it converges vaguely to μ_0. If n is sufficiently large, then $K_n\subset V$ and

$$(\mu_n, \mu_n)=(\mu_0, \mu_0)=\int_0^1\int_0^1 \frac{1}{\sqrt{|x-\xi|}}\, dx\, d\xi=\frac{8}{3}.$$

However, $\underline{\lim}\, U^{\mu_0}(P)\geqq 4$ as P approaches K_0 along K_n. Therefore

$$\underline{\lim}_{n\to\infty}\ (\mu_0, \mu_n)\geqq 4>\frac{8}{3}=(\mu_0, \mu_0).$$

Thus $\{\mu_n\}$ does not converge finely to μ_0.

In this example the kernel neither is of positive type nor satisfies the continuity principle. We shall give later an example in which the energy and continuity principles are satisfied and yet there exists a sequence of measures with bounded energy and converging vaguely but not finely; see Remark 1 to the corollary of Theorem 1.8.

Fuglede [1] denoted the following condition by (CW):

Any vaguely convergent net in $\mathscr{E}$ with bounded energy converges weakly to the vague limit.

Let us denote by (CW)′ the following weaker condition:

Any vaguely convergent net in $\mathscr{E}$ with bounded energy and supported by a fixed compact set converges weakly to the vague limit.

First we give

LEMMA 1.4. *If the kernel is of positive type and satisfies the continuity principle, then every $\nu\in\mathscr{E}$ can be approximated with respect to the strong topology by the restriction ν_K of ν to some compact set K with the property that $U^{\nu_K}(P)$ is continuous in Ω.*

PROOF. Given $\varepsilon>0$, we choose a compact set K_1 such that $\|\nu-\nu_{K_1}\|<\varepsilon$ where ν_{K_1} is the restriction of ν to K_1. Since the subset of K_1, where $U^{\nu_{K_1}}(P)$ is finite, has a vanishing ν-value, we can find by Lusin's theorem a compact set $K\subset K_1$ such that $\nu(K_1-K)$ is arbitrarily small and the restriction of $U^{\nu_{K_1}}(P)$ to K is continuous. We shall denote by ν_K the restriction of ν to K. Since $U^{\nu_{K_1}-\nu_K}(P)$ is lower semicontinuous, the restriction of

$$U^{\nu_K}(P)=U^{\nu_{K_1}}(P)-U^{\nu_{K_1}-\nu_K}(P)$$

to K is upper semicontinuous and hence continuous on K. By the continuity principle, $U^{\nu_K}(P)$ is continuous in Ω. Let us suppose that we have chosen K such that $\|\nu_{K_1}-\nu_K\|<\varepsilon$ is satisfied. This is possible because

$$\|\nu_{K_1}-\nu_K\|^2=\int_{K_1-K}\int_{K_1-K}\Phi(P, Q)\, d\nu(P)\, d\nu(Q)$$

approaches 0 as $\nu(K_1-K)$ tends to 0. We have then

$$\|\nu-\nu_K\| \leqq \|\nu-\nu_{K_1}\|+\|\nu_{K_1}-\nu_K\| \leqq 2\varepsilon.$$

Now we prove

THEOREM 1.1. *If the kernel is of positive type and satisfies the continuity principle, then* (CW)$'$ *is satisfied.*[10]

PROOF. Let $T=\{\mu_\omega;\ \omega\in D\}$ be a vaguely convergent net with $\|\mu\|<M<\infty$ and $S_\mu\subset F$, a compact set, and let μ_0 be the vague limit. By Proposition 4

$$\|\mu_0\| \leqq \varliminf_{\omega} \|\mu_\omega\| \leqq M<\infty.$$

For given $\nu\in\mathscr{E}$ let ν_K be a measure with the property described in Lemma 1.4. If $\|\nu-\nu_K\|<\varepsilon$ we have for $\mu_\omega\in T$

$$|(\nu,\ \mu_\omega)-(\nu_K,\ \mu_\omega)| \leqq \|\nu-\nu_K\|\ \|\mu\|<\varepsilon M.$$

Similarly

$$|(\nu,\ \mu_0)-(\nu_K,\ \mu_0)| \leqq \varepsilon M.$$

We take a continuous function $f_0(P)$ with compact support which is equal to 1 on F. Then

$$\lim_{\omega}\ (\nu_K,\ \mu_\omega)=\lim_{\omega}\int U^{\nu_K} f_0\, d\mu_\omega=\int U^{\nu_K} f_0\, d\mu_0=\int U^{\nu_K} d\mu_0=(\nu_K,\ \mu_0).$$

Since

$$\begin{aligned}|(\nu,\ \mu_\omega)-(\nu,\ \mu_0)| &\leqq |(\nu,\ \mu_\omega)-(\nu_K,\ \mu_\omega)|+|(\nu_K,\ \mu_\omega)-(\nu_K,\ \mu_0)| \\ &+|(\nu_K,\ \mu_0)-(\nu,\ \mu_0)| \leqq 2\varepsilon M+|(\nu_K,\ \mu_\omega)-(\nu_K,\ \mu_0)|,\end{aligned}$$

we see that

$$\lim_{\omega}\ (\nu,\ \mu_\omega)=(\nu,\ \mu_0).$$

Thus T converges weakly to μ_0.

EXAMPLE 4. We shall show that the energy principle alone is not sufficient in Theorem 1.1. We take a positive continuous strictly positive definite function $\varphi(x)$ on the x-axis, and a positive bounded lower semicontinuous function $\psi(x)$ which has a discontinuity at $x=0$. We take the interval $|x|\leqq 1$ for Ω and shall show that

$$\Phi(x,\ y)=\varphi(x-y)\,\psi(x)\,\psi(y)$$

10) This was essentially proved in Ohtsuka [1].

satisfies the energy principle. Let μ, ν be different measures in Ω. Then $\int \psi d\mu$ and $\int \psi d\nu$ can be regarded as different measures in Ω. It follows that

$$\iint \Phi(x, y)\, d(\mu-\nu)(x)\, d(\mu-\nu)(y) = \iint \varphi(x-y)\, (\psi d\mu - \psi d\nu)(x)(\psi d\mu - \psi d\nu)(y) > 0.$$

We denote the unit measure at $x=1/n$ by μ_n and the one at $x=0$ by μ_0. Evidently μ_n converges vaguely to μ_0. We observe that

$$(\mu_n, \mu_n) = \varphi(0)\psi^2\left(\frac{1}{n}\right)$$

and

$$\lim_{n\to\infty} (\mu_n, \mu_0) = \lim_{n\to\infty} \varphi\left(\frac{1}{n}\right)\psi\left(\frac{1}{n}\right)\psi(0) = \varphi(0)\,\psi(0) \lim_{n\to\infty} \psi\left(\frac{1}{n}\right)$$

$$\neq \varphi(0)\psi^2(0) = (\mu_0, \mu_0),$$

which shows that μ_n does not converge weakly to μ_0. Thus (CW)$'$ is not true.

Fuglede gave a similar example, i.e. Example 5 in [1].

This kernel is not continuous even in the extended sense. So we raise

QUESTION 1. *How about if we require the continuity in the extended sense?*

We have only one example, Example 2 of Ohtsuka [7], of kernel with the following property: It is positive, symmetric, continuous in the extended sense, equal to ∞ only on the diagonal set and satisfies the energy principle but not the continuity principle. It is easy to see that this example does not answer Question 1.

The next problem is as to when the weak (fine resp.) convergence implies the vague convergence. By Bourbaki [1] a family of nonnegative functions is called *positively rich* if, for every compact set K in Ω, we can find a relatively compact neighborhood $N \supset K$ such that every nonnegative continuous function with support contained in K can be approximated arbitrarily close by functions, with support contained in N, of the family.

We have as in the Newtonian case (see H. Cartan [5; 6])

THEOREM 1.2. *If the family of nonnegative functions with compact support of the form*

$$\sum_{k=1}^{n} c_k\, U^{\mu_k};\ c_k \gtreqless 0,\ \mu_K \in \mathscr{E} (\in \mathscr{L} \text{ resp.})$$

is positively rich, then the weak (fine resp.) convergence implies the vague convergence.

Next we establish

THEOREM 1.3.[11] *Suppose that the kernel satisfies the energy principle and*

11) This was proved essentially in Kishi [1].

(CW)$'$. *Then every weakly convergent net $T=\{\mu_\omega;\ \omega\in D\}$ in $\mathscr{E}$ with bounded energy and supported by a compact set K converges vaguely to the weak limit.*

PROOF. First we observe that min $\|\mu\|$ for unit measure $\mu\in\mathscr{E}_K$ is attained by some measure μ^*. In fact, we choose a sequence $\{\mu_n\}$ of unit measures of $\mathscr{E}_K$ such that $\|\mu_n\|\to\inf\|\mu\|$. By Proposition 3 there is a subnet $\{\mu_{\hat\omega};\ \hat\omega\in\hat D\}$ of $\{\mu_n\}$ which converges vaguely to some unit measure μ^*. By Proposition 4 it holds that

$$\varliminf_{\hat\omega}\ (\mu_{\hat\omega},\mu_{\hat\omega})\geqq(\mu^*,\mu^*)>0.$$

It is easy to see that $\mu^*\in\mathscr{E}_K$ and our observation is justified. The assumption $\|\mu_\omega\|<M$, $\omega\in D$, implies that $\mu_\omega(K)$ is bounded for $\omega\in D$, because

$$0<\|\mu^*\|\leqq\left\|\frac{\mu_\omega}{\mu_\omega(K)}\right\|\leqq\frac{M}{\mu_\omega(K)}.$$

Now suppose that T does not converge vaguely to the weak limit μ_0. Then there exists $f\in\mathscr{C}_0(\Omega)$ such that $\int fd\mu_\omega\not\to\int fd\mu_0$, $\omega\in D$. Let $T'=\{\mu_{\omega'};\ \omega'\in D'\}$ be a subnet of T such that $\lim_{\omega'}\int fd\mu_{\omega'}\neq\int fd\mu_0$ exists.

By Proposition 3, there is a subnet $T''=\{\mu_{\omega''};\ \omega''\in D''\}$ of T' which converges vaguely to some measure μ_0'. This μ_0' is different from μ_0 because $\lim_{\omega''}\int fd\mu_{\omega''}=\int fd\mu_0'\neq\int fd\mu_0$. By condition (CW)$'$, T'' converges weakly to μ_0'. Since $(\nu,\mu_0)=\lim_{\omega}(\nu,\mu_\omega)=\lim_{\omega''}(\nu,\mu_{\omega''})=(\nu,\mu_0')$ for any $\nu\in\mathscr{E}$, it follows that $(\nu,\mu_0-\mu_0')=0$. Taking μ_0 and μ_0' for ν, we have $\|\mu_0-\mu_0'\|=0$. Consequently $\mu_0=\mu_0'$ by the energy principle. This is a contradiction. Thus T converges vaguely to μ_0.

Combining this theorem with Theorem 1.1, we have

COROLLARY. *If the kernel satisfies the energy and continuity principles, then we have the same conclusion as in Theorem* 1.3.

Les us consider Example 2 of Ohtsuka [7]. The kernel is given in $E_2\times E_2$. It is equal to $\overline{PQ}^{-1/2}$ on $K\times K$, where K is the interval $[0,1]$ on the x-axis, and the support of any measure of $\mathscr{E}$ is contained in K. Hence it satisfies the energy principle and (CW)$'$. However, no function with non-empty support disjoint from K can be approximated by a linear combination of potentials of $\mathscr{E}$. This shows that Theorem 1.3 cannot be derived from Theorem 1.2.

We can not replace the energy principle by the positivity of type in the theorem. This is shown by $\Phi(P,Q)\equiv1$ in $E_3\times E_3$.

In the case that the measures of a net are not necessarily contained in a fixed compact set, we have to assume stronger conditions in order to conclude

(CW); the necessity is actually seen by Example 2 and Theorem 1.5.

THEOREM 1.4. *Suppose that the kernel is nonnegative and of positive type and satisfies the continuity principle, and that, given any measure* ν *with compact support and* $\varepsilon>0$, *we can find a measure* λ *such that* $\|\lambda\|<\varepsilon$ *and*

$$U^{\nu}(P)\leqq U^{\lambda}(P)$$

outside some compact set. Then (CW) *is satisfied.*

PROOF. Let $T=\{\mu_\omega;\ \omega\in D\}$ be a vaguely convergent net with $\|\mu_\omega\|<M<\infty$. Let μ_0 be the vague limit. By Proposition 4

$$\|\mu_0\|\leqq \varliminf_{\omega}\|\mu_\omega\|\leqq M<\infty.$$

Given $\nu\in\mathscr{E}$ and $\varepsilon>0$, we can find by Lemma 1.4 a compact set K such that the potential of the restriction ν_K to K of ν is continuous in Ω and $\|\nu-\nu_K\|<\varepsilon$. By our assumption there are a compact set F and a measure λ such that $\|\lambda\|<\varepsilon$ and $U^{\nu_K}(P)\leqq U^{\lambda}(P)$ on $\Omega-F$. Take $f(P)$ of $\mathscr{C}_0(\Omega)$ such that $0\leqq f(P)\leqq 1$ in Ω and $f(P)=1$ on F. We have

$$\lim_{\omega}\int U^{\nu_K}fd\mu_\omega=\int U^{\nu_K}fd\mu_0,$$

$$\int U^{\nu_K}(1-f)\,d\mu_\omega\leqq\int_{\Omega-F}U^{\nu_K}d\mu_\omega\leqq\int_{\Omega-F}U^{\lambda}\,d\mu_\omega\leqq(\lambda,\ \mu_\omega)\leqq\|\lambda\|\|\mu_\omega\|\leqq\varepsilon M$$

and

$$\int U^{\nu_K}(1-f)\,d\mu_0\leqq\varepsilon M.$$

It follows that $(\nu_K,\ \mu_\omega)$ tends to $(\nu_K,\ \mu_0)$. From

$$\begin{aligned}|(\nu,\ \mu_\omega)-(\nu,\ \mu_0)|&\leqq|(\nu,\ \mu_\omega)-(\nu_K,\ \mu_\omega)|+|(\nu_K,\ \mu_\omega)-(\nu_K,\ \mu_0)|+|(\nu_K,\ \mu_0)-(\nu,\ \mu_0)|\\&\leqq|(\nu_K,\ \mu_\omega)-(\nu_K,\ \mu_0)|+2\varepsilon M,\end{aligned}$$

we can conclude that $(\nu,\ \mu_\omega)\to(\nu,\ \mu_0)$.

QUESTION 2. *Is the following condition sufficient for* (CW)? *The kernel* $\Phi(P,\ Q)$ *is nonnegative and of positive type and satisfies the continuity principle, and* $\Phi(P,\ Q)$ *tends to* 0 *as* P *tends to the point at infinity while* Q *stays on a compact set.*

Corresponding to Theorem 1.3 we have

THEOREM 1.5. *Suppose that the kernel satisfies the energy principle and* (CW) *and is nonnegative. Then every weakly convergent net* $T=\{\mu_\omega;\ \omega\in D\}$ *in* $\mathscr{E}$ *with bounded energy converges vaguely to the weak limit.*

PROOF. Let K be any compact set, and $(\mu_\omega)_K$ be the restriction of μ_ω to K.

Since the kernel is nonnegative, $\|(\mu_\omega)_K\| \leqq \|\mu_\omega\| < M < \infty$ for $\omega \in D$. As in Theorem 1.3 we see that $\mu_\omega(K)$ is bounded on D. Therefore we can apply Proposition 3. The rest of the proof is the same as in Theorem 1.3.

We prove also

THEOREM 1.6. *Suppose that the continuity principle is satisfied. If a net T of measures supported by a fixed compact set converges vaguely to μ_0 and if $|\check{U}^{\mu_\omega}(P)| < M_F < \infty$ ($|U^{\mu_\omega}(P)| < M_F < \infty$ resp.) on any compact set F in Ω for every $\mu_\omega \in T$ and μ_0,[12] then (ν, μ_ω) ((μ_ω, ν) resp.) tends to (ν, μ_0) ((μ_0, ν) resp.) for any $\nu \in \mathscr{E}$ with compact support.*

PROOF. Given $\nu \in \mathscr{E}$ with compact support and $\varepsilon > 0$, we can find a compact set $K \subset S_\nu$ such that $\nu(S_\nu - K) < \varepsilon$ and the potential of the restriction ν_K of ν to K is continuous in Ω. We have

$$\lim_\omega (\nu_K, \mu_\omega) = (\nu_K, \mu_0)$$

and

$$|(\nu, \mu_\omega) - (\nu_K, \mu_\omega)| \leqq \int |\check{U}^{\mu_\omega}| d(\nu - \nu_K) < M_{S_\nu} \nu(S_\nu - K) < \varepsilon M_{S_\nu} \qquad \text{for } \omega \in D.$$

The same is true for μ_0 and $\lim_\omega (\nu, \mu_\omega) = (\nu, \mu_0)$ follows. The left case is similar.

1.7. Strong completeness.

A class of measures is called *strongly complete* if any strong Cauchy net in the class converges strongly to an element of the class.

Before stating theorems concerning strong completeness we shall introduce several new terminologies. Fuglede [1] called a kernel *consistent* if it is of positive type and any strong Cauchy net converging vaguely to a measure converges strongly to the same measure. Likewise he called a kernel *K-consistent* provided that it is of positive type and that, if any strong Cauchy net in measures supported by a fixed compact set converges vaguely to a measure, then it converges strongly to the same measure.

A kernel is called by Fuglede [1] *pseudo-positive* (*strictly pseudo-positive* resp.) if $(\mu, \mu) \geqq 0$ ($(\mu, \mu) > 0$ resp.) for every μ ($\mu \neq 0$ resp.) with compact support. It will be called strictly pseudo-positive *in the strong sense* if $(\mu, \mu) > 0$ for every $\mu \neq 0$ for which (μ, μ) is defined. We shall give an example of a kernel which is strictly pseudo-positive but not in the strong sense. Consider the logarithmic kernel on L_0: $|x| \leqq 2$ on the x-axis, and the unit equilibrium measure μ_0 on L_0, and take $|x| < 2$ for Ω. Then $(\mu, \mu) > 0$ for every $\mu \neq 0$ with

12) It will follow from Theorem 1.15 that $|\check{U}^{\mu_0}(P)| \leqq M_F$ ($|U^{\mu_0}(P)| \leqq M_F$ resp.) if this inequality is assumed for every $\mu_\omega \in T$.

compact support in Ω but $(\mu_0, \mu_0)=0$, μ_0 being considered as a measure in Ω.
First we prove

LEMMA 1.5. (Fuglede [1]). *A kernel of positive type which satisfies* (CW) ((CW)$'$ *resp.*) *is consistent* (*K-consistent resp.*).

PROOF. We consider a kernel of positive type which satisfies (CW). Let $T=\{\mu_\omega;\ \omega\in D\}$ be a Cauchy net which converges vaguely to a measure μ_0. We may assume that $\|\mu_\omega\|$ is bounded. By (CW) it converges weakly to μ_0. We take any $\nu\in\mathscr{E}$ and see that

$$\|\mu_0-\nu\|\leqq\varliminf_{\omega}\|\mu_\omega-\nu\|,$$

because

$$\|\mu_0-\nu\|^2=\lim_{\omega}(\mu_0-\nu,\ \mu_\omega-\nu)\leqq\varliminf_{\omega}\|\mu_0-\nu\|\,\|\mu_\omega-\nu\|.$$

Given $\varepsilon>0$, if we take a suitable ω_0, then

$$\|\mu_{\omega'}-\mu_0\|\leqq\varliminf_{\omega}\|\mu_{\omega'}-\mu_\omega\|<\varepsilon \qquad \text{for every } \omega'\geqq\omega_0.$$

This shows that T converges strongly to μ_0. The case when (CW)$'$ is satisfied is similar.

THEOREM 1.7. *Let K be a fixed compact set in Ω, and assume that the kernel is strictly pseudo-positive and K-consistent. Then $\mathscr{E}_K$ is strongly complete.*

PROOF. Let $T=\{\mu_\omega;\ \omega\in D\}$ be a Cauchy net in $\mathscr{E}_K$. We may assume that $\|\mu_\omega\|$ is bounded. As we have shown in the proof of Theorem 1.3, there exists a subnet T' of T which converges vaguely to some limit $\mu_0\in\mathscr{E}_K$. By Lemma 1.5 it converges strongly to μ_0, and hence T converges strongly to μ_0.

COROLLARY. *If the kernel satisfies the energy principle and the continuity principle, then $\mathscr{E}_K$ is strongly complete.*

REMARK 1. If we consider the class of measures of general sign of the form $\mu-\nu$, μ, $\nu\in\mathscr{E}_K$, it is not necessarily strongly complete as an example in the Newtonian case was given in H. Cartan [5], footnote 13.

REMARK 2. Even if the kernel satisfies both the energy and continuity principles, $\mathscr{E}$ is not necessarily strongly complete. We shall present an example.[13)]

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13) An example was first communicated orally to the author by Aronszajn in 1959 at Lawrence. The present example was proposed by Ogasawara at Hiroshima in 1960. Fuglede (Example 4 of [1]) showed that $\mathscr{E}$ is not complete for the kernel $1/\overline{PQ}+1$ considered in E_3. If we compactify Ω with the Alexandroff point ∞ and define the kernel by 1 when at least one of the variables is ∞, then $\mathscr{E}$ defined with respect to this extended kernel is complete.

We exclude the closed unit ball with center at the origin from E_3 and take the rest for Ω. We consider the Newtonian kernel $\Phi(P, Q)=1/\overline{PQ}$. It satisfies the energy principle and the continuity principle in Ω naturally. Let μ_n be the uniform unit measure on the spherical surface with center at the origin and with radius equal to $1+1/n$. The sequence $\{\mu_n\}$ is a Cauchy sequence but there is no strong limit; the unique strong limit in E_3 is the uniform unit measure on the surface of the unit ball excluded from E_3.

In the example, however, $\mathscr{E}$ becomes complete if we add the surface of the unit ball to Ω. So the following question was raised orally by Kishi:

QUESTION 3. *Are there a space Ω and a kernel Φ satisfying the energy and continuity principles such that, for any extension $\Omega'\supset\Omega$ and any extension Φ' satisfying the energy principle, $\mathscr{E}$ defined with respect to Ω' and Φ' is not strongly complete?*

If we assume more, we have

THEOREM 1.8. *If the kernel is nonnegative, strictly pseudo-positive*[14] *and consistent, or if* inf (μ, μ) *for unit measures μ with compact support is positive and the kernel is consistent, then $\mathscr{E}$ is strongly complete.*

PROOF. Let $T=\{\mu_\omega;\ \omega\in D\}$ be a Cauchy net. We may assume that $\|\mu_\omega\|$ is bounded by $M<\infty$. If the kernel is nonnegative, $\|(\mu_\omega)_K\|\leqq M$ is true for the restrictions $(\mu_\omega)_K$ to any compact set K and we see that $\{\mu_\omega\}$ is vaguely bounded in $\mathscr{M}$. Under the alternative condition we have the same conclusion. In fact, observing that $W=\inf(\mu, \mu)$ is the same for unit measures μ with or without compact support, we obtain

$$\sqrt{W}\mu_\omega(\Omega)\leqq\|\mu_\omega\|\leqq M.$$

The rest is the same as for Theorem 1.7.

COROLLARY. *In the example in Remark* 2 *to Theorem* 1.7, (CW) *is not satisfied.*

REMARK 1. Actually μ_n in Remark 2 has bounded energy and converges vaguely to 0 but not finely to any measure.

We may need a proof for the last statement. Suppose that the sequence converges finely to μ_0. Every μ_m belongs to $\mathscr{L}$ and $(\mu_m, \mu_n)=(1+1/m)^{-1}$ for every $n\leqq m$. Therefore $\mu_0\neq 0$ but this contradicts the following Remark 2.

Remark 2. In the same example, every finely convergent sequence of measures converges vaguely to the same limit, thus showing that condition (CW) is not always necessary to have the conclusion in Theorem 1.5.

PROOF. It is known (Cartan [5]) that every nonnegative difference μ of uniform measures on spherical surfaces belongs to $\mathscr{L}$ and that the family of

14) As is remarked in § 2.1 of Fuglede [1], a kernel is nonnegative and strictly pseudo-positive if and only if $\Phi(P, Q)\geqq 0$ and $\Phi(P, P)>0$ for any $P, Q\in\Omega$.

all linear combinations with positive coefficients of potentials of above measures μ is positively rich; see the preceding section for the notion of positive richness. This proves the assertion.

We take this occasion to discuss the question raised in footnote at p. 166 of Fuglede [1]. The question is as follows:

Consider a kernel which satisfies the energy principle and assume that $\mathcal{E}$ is complete. Does then any strongly convergent net converge vaguely?

Let us see that Example 2 in the preceding section gives a negative answer. We consider $\Omega_0=\bigcup_{n=1}^{\infty}\{1/n\}\cup\{0\}$ as a subspace of the x-axis and set $\Phi_0(x, y)=\varphi(x-y)$. The space $\mathcal{E}$ defined with respect to Ω_0 and Φ_0 is certainly complete and the completeness of $\mathcal{E}$ defined with respect to Ω and Φ follows. However, as shown in Example 2, there is a strongly converging sequence which does not at all converge vaguely.

APPLICATION 1. *For the kernel* $\overline{PQ}^{-\alpha}$, $0<\alpha<n$, *in* $E_n (n\geqq 3)$, $\mathcal{E}$ *is complete.*[15]

Since it is well known that both the energy and continuity principles are satisfied, it will be sufficient to show (CW). One needs to examine the condition on the existence of λ required in Theorem 1.4. Let ν be a measure with compact support and $\varepsilon>0$ be given. We may assume that ν is a unit measure. Let B_r be the ball with origin as center and with radius r, which contains S_ν in its inside. Let λ_r be the uniform unit measure on the surface ∂B_r, and ω_r be the surface area of ∂B_r. We have

$$\|\lambda_r\|^2=\frac{1}{\omega_r^2}\int_{\partial B_r}\int_{\partial B_r}\frac{1}{\overline{PQ}^{\alpha}}\,dS(P)dS(Q)=\frac{r^{2n-2-\alpha}}{\omega_r^2}\int_{\partial B_1}\int_{\partial B_1}\frac{1}{\overline{PQ}^{\alpha}}\,dS(P)dS(Q)$$
$$=\frac{\|\lambda\|^2}{r^{\alpha}\,\omega_1^2}.$$

Therefore if r is sufficiently large, $\|\lambda_r\|<\varepsilon$. As P tends to the point at infinity, $U^{\nu}(P)$ divided by $2U^{\lambda_r}(P)$ is approximately equal to $\overline{PO}^{\alpha}/(2\overline{PO}^{\alpha})=1/2 <1$, and hence $2\lambda_r$ is a required measure in the theorem.

APPLICATION 2. *Let* Ω *be an n-dimensional Greenian space* $(n\geqq 2)$[16] *and* $G(P, Q)$ *be the Green's function with pole at* Q *on* Ω. *Then* (CW) *is satisfied for the kernel* $G(P, Q)$.[17]

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15) By Theorem 1.8, it is seen that $\mathcal{E}$ is complete. This was first proved in Deny [1]. The authors Aronszajn and Smith of [1] assert at footnote that they have a different proof. It is also proved by Fuglede [1], applying a result concerning the completeness of $\mathcal{E}$ for some convolution kernels (Theorem 7.4).

16) See Brelot and Choquet [1] for Greenian spaces.

17) By Theorem 1. 8, again $\mathcal{E}$ is complete. The possibility of extending Cartan's theory to hyperbolic Riemann surfaces was already asserted in Bader [1] and Parreau [1]. An explicit proof of the completeness of $\mathcal{E}$ in the case of a hyperbolic Riemann surface was first given in Edwards [1]. We believe that our proof is more direct (see § 6 of his paper). Furthermore, we observe that only the fact that the kernel is of positive type and not the energy principle is proved there.

We begin with an outlining of the proof of two principles.[18] We use the notations $x, y, \cdots$ for points in E_n and recall the following well known property of a potential with kernel $\log \dfrac{1}{|x-y|}$ $(n=2)$ or $\dfrac{1}{|x-y|^{n-2}}$ $(n\geqq 3)$ in E_n:

$$\overline{\lim_{x\to x_0}}\ U^\mu(x)=\overline{\lim_{x\in S_\mu,\ x\to x_0}}\ U^\mu(x)$$

if the right side has a meaning. In order to have such an equality in the case of a Greenian potential, it is sufficient to prove it for μ whose support is contained in an image in E_n of a neighborhood of a point of Ω. In the image in E_n we can write

$$G(P, Q)=\log \frac{1}{|x-y|}+h(x, y) \qquad \text{if } n=2$$

and

$$G(P, Q)=\frac{1}{|x-y|^{n-2}}+h(x, y) \qquad \text{if } n\geqq 3$$

with a function h which is continuous in (x, y). On account of the equality holding for potentials in E_n, we obtain

$$\overline{\lim_{P\to P_0}}\ U^\mu(P)=\overline{\lim_{P\in S_\mu,\ P\to P_0}}\ U^\mu(P)$$

for any Greenian potential of a measure μ with compact support in Ω and for P_0 not isolated on S_μ.

Let μ be a measure with compact support in Ω such that its potential is bounded on S_μ: $U^\mu(P)\leqq M$ on S_μ. Let $\{\Omega_n\}$, $\Omega_n\supset S_\mu$, be an exhaustion of Ω with smooth boundaries, and $G_n(P, Q)$ be the Green's function on Ω_n. Observing that G_n vanishes on the boundary $\partial\Omega_n$ and applying the above equality at each point of S_μ, we see that

$$\int G_n(P, Q)d\mu(Q)\leqq M \qquad \text{on } \Omega_n$$

on account of the maximum principle for harmonic functions. As $n\to\infty$ the left side tends to $U^\mu(P)=\int G(P, Q)d\mu(Q)$ and it is concluded that the first maximum principle ((F) of § 1.2) is satisfied; the continuity principle follows from this by (1.10). By Ninomiya's theorem the kernel is of positive type; see Corollary of Theorem 2.42 in the next chapter. The fact that $\|\mu-\nu\|=0$ only if $\mu\equiv\nu$ can be proved as follows: We infer that $U^\mu\equiv U^\nu$ from the fact $(\mu-\nu, \lambda)=0$ for any measure λ which is equal to the uniform measure on an

18) When these facts were stated to be true in Ohtsuka [1], in the case of a hyperbolic Riemann surface, the author had in his mind the proof which is given here.

image in E_n of a neighborhood in Ω. On account of the following lemma, we can conclude that $\mu\equiv\nu$.

LEMMA 1.6. *Let μ, ν be measures with compact support in E_n $(n\geqq 2)$, and $h(x)$ be a harmonic function in an open set $G\subset E_n$. If $U^{\mu}(x)$ and $U^{\nu}(x)$ mean the potentials, with kernel $\Phi(x, y)=\log\dfrac{1}{|x-y|}$ or $\dfrac{1}{|x-y|^{n-2}}$, of μ and ν respectively and*

$$U^{\mu}(x)=U^{\nu}(x)+h(x)$$

in G (except for a set of Lebesgue measure zero), then $\mu-\nu$ vanishes for every Borel set in G.

A proof will be given at the end of the present section.

Let us prove the existence of λ in Theorem 1.4. Let ν be a unit measure with compact support. We take an exhaustion $\{\Omega_n\}$, with smooth boundary, of Ω such that $\Omega_n\cup\partial\Omega_n\subset\Omega_{n+1}$ and Ω_1 contains S_ν in its inside. Let Q_0 be an arbitrary point of S_ν. By Harnack's theorem, there is a constant $c>1$ which depends on Q_0 and S_ν such that

$$\frac{1}{c}G(P, Q_0)<G(P, Q)<cG(P, Q_0) \qquad \text{for any } P\notin\Omega_1 \text{ and } Q\in S_\nu.$$

Therefore for $P\notin\Omega_1$ we have

$$U^{\nu}(P)\leqq cG(P, Q_0). \tag{1.25}$$

Let μ_n^P be the harmonic measure at $P\in\Omega_n$ with respect to the domain Ω_n. Then $\int f d\mu_n^P$ gives the value at P of the Dirichlet solution in Ω_n for the given continuous boundary function $f(Q)$ on $\partial\Omega_n$. Particularly for $f(Q)=G(Q, Q_0)$, $\int G(Q, Q_0)d\mu_n^P(Q)=U^{\mu_n^P}(Q_0)$ is equal to $G(P, Q_0)$ on $\partial\Omega_n$. Therefore $G(P, Q_0)-U^{\mu_n^P}(Q_0)$ is equal to the Green's function $G_n(P, Q_0)$ in Ω_n with pole at Q_0. On account of its symmetry, it is equal to $G(Q_0, P)-U^{\mu_n^{Q_0}}(P)$. Thus $U^{\mu_n^{Q_0}}(P)=G(P, Q_0)$ on $\partial\Omega_n$. We shall write simply μ_n for $\mu_n^{Q_0}$. If $m>n$, then

$$U^{\mu_n}(P)\geqq G_m(P, Q_0) \qquad \text{on } \partial\Omega_n\cup\partial\Omega_m.$$

By the maximum principle for harmonic functions, this is true on $\Omega_m-\Omega_n$. By letting $m\to\infty$ it follows

$$U^{\mu_n}(P)\geqq G(P, Q_0) \qquad \text{in } \Omega-\Omega_n. \tag{1.26}$$

Although we need only this inequality, a similar reasoning leads us to the inverse inequality and hence to the equality in $\Omega-\Omega_n$.

We can compute $\|\mu_n\|^2$ explicitly:

$$(\mu_n, \mu_n) = \int_{\partial\Omega_n} U^{\mu_n}(P)\, d\mu_n(P) = \int_{\partial\Omega_n} G(P, Q_0)\, d\mu_n(P) = U^{\mu_n}(Q_0).$$

This is the value at Q_0 of the Dirichlet solution in Ω_n for the boundary value $G(P, Q_0)$. We see that $\lim_{n\to\infty} U^{\mu_n}(P) = h(P)$ is harmonic on Ω. If it were positive, then $G(P, Q_0) - h(P)$ would be a positive harmonic function in Ω except for its singularity at Q_0, which is really smaller than $G(P, Q_0)$. This is impossible and hence $U^{\mu_n}(P)$ decreases to 0 as $n\to\infty$. Consequently we can take $c\mu_n$ with large n for λ on account of (1.25) and (1.26).

Proof of Lemma 1.6. It will be sufficient to prove $\mu(K) = \nu(K)$ for every compact set $K \subset G$. We take a sequence $\{f_n(x)\}$ of three times continuously differentiable functions decreasing to the characteristic function χ_K of K. We assume also that the support of each $f_n(x)$ is contained in G. We know that

$$f_n(x) = \frac{1}{\omega_n} \int \Phi(x, y)\, \Delta f_n(y)\, dy$$

in the classical potential theory, where ω_n is the area of the surface of a unit sphere. We have

$$\int U^{\mu}\, \Delta f_n\, dy = \int d\mu(x) \int \Phi(x, y) \Delta f_n(y)\, dy = \omega_n \int f_n\, d\mu$$

and similarly $\int U^{\nu} \Delta f_n\, dy = \omega_n \int f_n\, d\nu$. Since $\int h \Delta f_n\, dy = 0$, it follows that

$$\int f_n\, d\mu = \int f_n\, d\nu.$$

By letting $n\to\infty$ we obtain

$$\mu(K) = \int \chi_K\, d\mu = \int \chi_K\, d\nu = \nu(K).$$

1.8. Capacity.

There are many ways to define capacities for a positive kernel. One way is to define an inner capacity by

$$\sup\ \{\mu(\Omega);\ \text{compact } S_\mu \subset X,\ U^\mu(P) \leqq 1 \ \text{ in } \ \Omega\}$$

for any set X. In case the kernel has general sign, it is difficult to define capacity in this manner. We shall, instead, consider $V_i(X)$ and $V_e(X)$ defined in § 1.1 or

$$V_i^*(X) = \inf_{\mu} \sup_{P\in\Omega} U^\mu(P)$$

for any set $X \neq \emptyset$, where the infimum is taken with respect to unit measures

μ with compact $S_\mu \subset X$. We set $V_i^*(\emptyset)=\infty$ and

$$V_e^*(X)=\sup_{G\supset X} V_i^*(G),$$

where G is an open set. Obviously $V_i(X)\leqq V_i^*(X)$ and $V_e(X)\leqq V_e^*(X)$ but, as Example 5 of Ohtsuka [7] shows, there is a case in which $V_i(K)$ and $V_i^*(K)$ are essentially different for a compact set K.

Choquet [3; 5] obtained several results on $C_i^*(X)=1/V_i^*(X)$ and $C_e^*(X)=1/V_e^*(X)$, in case kernels are positive. We shall aim at giving similar results on $V_i(X)$ and $V_e(X)$ in our section. Our intention, however, lies primarily in preparing for the next section and a full account on set functions related to capacity will be given on another occasion.

We begin with studying sets on which potentials are equal to ∞.

THEOREM 1.9. *For a kernel whose adjoint kernel satisfies* (B_K') *the potential of any measure* μ *with compact support is finite p.p.p. in* Ω.

PROOF. Suppose that there is a compact set K with $V_i(K)<\infty$ on which $U^\mu(P)=\infty$. We take a unit measure $\nu\in\mathscr{E}_K$, whose potential is bounded on S_ν, and find a compact subset K' of K such that $\nu(K')>0$ and $\check{U}^\nu(P)$ is continuous as a function on K', by means of Lusin's theorem. By our assumption $\check{U}^\nu(P)$ is bounded on S_μ. Hence $\int \check{U}^\nu\, d\mu<\infty$. But this is impossible because

$$\int \check{U}^\nu\, d\mu=\int U^\mu\, d\nu=\infty.$$

Thus $U^\mu(P)<\infty$ p.p.p. in Ω.

THEOREM 1.10. *If Ugaheri's maximum principle is satisfied, then the adjoint potential* $\check{U}^\mu(P)$ *of any measure* μ *with compact support is finite q.p. in* Ω.

PROOF. The set

$$G_n=\{P;\ \check{U}^\mu(P)>n\}$$

is open in Ω. We assume that $G_n\neq\emptyset$ for each n. Let ν be any unit measure with compact support $S_\nu\subset G_n$. If $\sup_{S_\nu} U^\nu(P)<\infty$, by assumption there is a constant $c>0$ such that

$$\sup_{\Omega} U^\nu(P)\leqq c\sup_{S_\nu} U^\nu(P).$$

We have

$$n<\int \check{U}^\mu d\nu=\int U^\nu\, d\mu\leqq c\sup_{S_\nu} U^\nu(P)\,\mu(\Omega).$$

It follows from this that $n\leqq c\mu(\Omega)V_i(G_n)$. Since the set $\{P;\ \check{U}^\mu(P)=\infty\}$ is equal to $\bigcap_n G_n$, its V_e-value is infinite.

If we assume the continuity principle, we have

THEOREM 1.11. *Consider a kernel which is locally bounded outside the diagonal set and which satisfies the continuity principle. Assume that $\Phi(P, P) = \infty$ at each $P \in \mathbf{O}_\infty$. Then $\check{U}^\mu(P)$ is finite q.p. in Ω for every μ with compact support.*

PROOF. Since the potential of any measure with compact support is locally bounded outside the support, we may assume that the space Ω is compact and, by adding a positive constant if necessary, that the kernel has a positive lower bound; $\mathbf{O}_\infty$ for the new kernel is included in $\mathbf{O}_\infty$ for the old kernel. We set

$$G_n = \{P;\ \sup_{Q \in \mathbf{O}_\infty} \Phi(P, Q) > n\}.$$

Then G_n is an open set and decreases to $\mathbf{O}_\infty$ as $n \to \infty$. Since the kernel is locally bounded outside the diagonal set, there is, for given m, a number n such that the closure of G_n is contained in G_m. According to Corollary 1 of $(\mathrm{U_K})$ (i) of § 1.5, Ugaheri's maximum principle (U) is true on $\Omega - G_n$, this being considered as a space. Let us denote the restriction of a measure μ to G_n by μ_n. The potential $\check{U}^{\mu - \mu_n}(P)$ is finite q.p. in $\Omega - G_n$ by the preceding theorem. Since $\check{\Phi}(P, Q)$ is locally bounded outside the diagonal set, $\check{U}^{\mu_n}(P)$ is locally bounded in $\Omega - G_m$. Therefore $\check{U}^\mu(P)$ is finite q. p. in $\Omega - G_m$. It follows that $\check{U}^\mu(P)$ is finite q.p. in $\bigcup_m (\Omega - G_m) = \Omega - \mathbf{O}_\infty$ by proposition 2 of § 1.6. Corollary 2 of (viii) in § 1.4 shows that $\mathbf{O}_\infty$ consists of at most a finite number of points. Since $\Phi(P, P) = \infty$ at each of them, $V_e(\mathbf{O}_\infty) = \infty$. Consequently $\check{U}^\mu(P)$ is finite q.p. in Ω.

LEMMA 1.7.[19] *If $\Phi(P, Q) > m > -\infty$ on the product $K \times K$ of a compact set K, then*

$$V_i(K) - m \leqq 4(\check{V}_i(K) - m).$$

PROOF. We may assume that $\check{V}_i(K) < \infty$. Given $\varepsilon > 0$, let μ be a unit measure supported by K such that

$$\sup_{P \in S_\mu} \check{U}^\mu(P) \leqq \check{V}_i(K) + \varepsilon.$$

For $t > \check{V}_i(K) + \varepsilon$, we set

$$E_1 = \{P \in K;\ U^\mu(P) > t\} \quad \text{and} \quad E_2 = \{P \in K;\ U^\mu(P) \leqq t\},$$

and denote by μ_1 and μ_2 the restrictions of μ to E_1 and E_2 respectively. We

19) See Choquet [3; 5].

have

$$\check{V}_i(K)+\varepsilon-m \geqq \int (\check{U}^{\mu}-m)\,d\mu = \int (U^{\mu}-m)\,d\mu = \iint (\Phi-m)\,d\mu d\mu$$

$$\geqq \iint (\Phi-m)\,d\mu d\mu_1 = \int (U^{\mu}-m)\,d\mu_1 \geqq (t-m)\,\mu_1(K).$$

Therefore

$$\mu_2(K)=1-\mu_1(K)\geqq 1-\frac{\check{V}_i(K)+\varepsilon-m}{t-m}>0.$$

We observe that

$$V_i(K)\mu_2(K)\leqq V(\mu_2)=\sup_{S_{\mu_2}}\int (\Phi-m)\,d\mu_2+m\mu_2(K)$$

$$\leqq \sup_{S_{\mu_2}}\int (\Phi-m)\,d\mu+m\mu_2(K)\leqq t-m\mu_1(K)\leqq t-m+m\mu_2(K)$$

and

$$V_i(K)-m\leqq \frac{t-m}{\mu_2(K)}\leqq \frac{(t-m)^2}{t-\check{V}_i(K)-\varepsilon}.$$

Now follows

$$V_i(K)-m\leqq \frac{(t-m)^2}{t-\check{V}_i(K)}$$

for any $t>\check{V}_i(K)$. The minimum value of the right side with respect to t is equal to $4(\check{V}_i(K)-m)$ and our lemma is proved.

REMARK. It may deserve attention that this lemma holds without any additional assumption, contrary to the result by Choquet in which the c-dilated maximum principle is assumed and the number c appears in the inequality. The coefficient 4 will be replaced by 2 in the next chapter.

From this lemma follows easily

THEOREM 1.12. *If $\Phi(P, Q)>m>-\infty$ on $X\times X$, then*

$$V_i(X)-m\leqq 4(\check{V}_i(X)-m).$$

If G_0 is an open set such that $\Phi(P, Q)>m>-\infty$ on $G_0\times G_0$, then for any $X\subset G_0$,

$$V_e(X)-m\leqq 4(\check{V}_e(X)-m).$$

COROLLARY. *The notion of p.p.p. is the same for Φ and $\check{\Phi}$. The similar fact is true for q.p. on any compact set in Ω.*

We shall say that a function $f(P)$ in Ω is *quasicontinuous* if, for any $\varepsilon>0$, there is an open set G with $V_i(G) \geqq 1/\varepsilon$ such that the restriction of $f(P)$ to $\Omega - G$ is continuous.

First we give two lemmas which are essentially due to Choquet [3; 5].

LEMMA 1.8. *Consider a positive kernel which is continuous outside the diagonal set, and assume that the kernel satisfies the continuity principle and the adjoint kernel satisfies the c-dilated maximum principle with* $c \geqq 1$. *Then, for any measure* μ *with compact support, any* $\varepsilon > 0$ *and any* $\eta > 0$, *there is an open set* G *with* $V_i(G) \geqq 1/\varepsilon$ *and a decomposition* $\mu = \nu + \pi$ *such that* $\pi(\Omega) < \eta$ *and the restriction of* $U^{\nu}(P)$ *to* $\Omega - G$ *is continuous and not greater than* $4c\mu(\Omega)/\varepsilon$.

PROOF. We set $G = \{P;\ U^{\mu}(P) > 4c\mu(\Omega)/\varepsilon\}$ and see $\check{V}_i(G) \geqq 4/\varepsilon$ as in the proof of Theorem 1.10. By Theorem 1.12, $V_i(G) \geqq \check{V}_i(G)/4 \geqq 1/\varepsilon$. We denote the restrictions of μ to G and $\Omega - G$ by μ_G and μ'_G respectively. There is a compact set $K \subset G$ such that $\mu(G-K) < \eta/2$. We shall denote by μ_K the restriction of μ to K. Since $U^{\mu'_G}(P)$ is finite on $S_{\mu'_G} = S_{\mu} - G$, there exists a compact subset $K_1 \subset S_{\mu} - G$ such that $\mu(S_{\mu} - G - K_1) < \eta/2$ and the potential $U^{\mu_{K_1}}(P)$ of the restriction μ_{K_1} of μ to K_1 is continuous in Ω on account of the continuity principle. The measures $\nu = \mu_K + \mu_{K_1}$ and $\pi = \mu - \nu$ have the required properties respectively.

LEMMA 1.9. *Consider a positive kernel which is continuous outside the diagonal set and assume that the kernel satisfies the continuity principle and the adjoint kernel satisfies Ugaheri's maximum principle. Then the potential of any measure* μ *with compact support is quasicontinuous in* Ω.

PROOF. We denote by G_1, ν_1, π_1 an open set and measures obtained in Lemma 1.8, corresponding to μ, $\varepsilon = \delta/2$, $\eta = (\delta/2)^2$. We shall define G_n, ν_n, π_n by induction. Corresponding to π_{n-1}, $\varepsilon = \delta/2^n$, $\eta = (\delta/2^n)^2$, we obtain G_n, ν_n, π_n as in Lemma 1.8. We set $G = \bigcup_{n=1}^{\infty} G_n$ and, by Proposition 1 in § 1.1, we have

$$\frac{1}{V_i(G)} \leqq \sum_{n=1}^{\infty} \frac{1}{V_i(G_n)} \leqq \sum_{n=1}^{\infty} \frac{\delta}{2^n} = \delta.$$

We see that $\mu = \sum_{n=1}^{\infty} \nu_n$ and hence that $U^{\mu}(P) = \sum_{n=1}^{\infty} U^{\nu_n}(P)$. By definition the restriction of $U^{\nu_n}(P)$ to $\Omega - G_n$ and hence to $\Omega - G$ is continuous. On $\Omega - G_n$ we have

$$0 \leqq U^{\nu_n}(P) \leqq 4c\pi_{n-1}(\Omega)\left(\frac{\delta}{2^n}\right)^{-1} \leqq 4c\left(\frac{\delta}{2^{n-1}}\right)^2 \frac{2^n}{\delta} = 4c\frac{\delta}{2^{n-2}} \qquad \text{for } n \geqq 2,$$

where $c \geqq 1$ is a constant such that the c-dilated maximum principle is true on Ω for the adjoint kernel. Therefore the convergence of $\sum_n U^{\nu_n}(P)$ on $\Omega - G$ is uniform, and hence the restriction of $U^{\mu}(P) = \sum_n U^{\nu_n}(P)$ to $\Omega - G$ is continuous.

Now we prove

THEOREM 1.13. (cf. Choquet [3; 5], Kishi [2; 3]). *Assume that the kernel is continuous outside the diagonal set and that $\Phi(P, P)=\infty$ at each point P of $\mathbf{O}_\infty$. If both the kernel and the adjoint kernel satisfy the continuity principle, then any potential of a measure μ with compact support is quasicontinuous.*

PROOF. Since the potential is continuous outside S_μ, we may assume a relatively compact open set containing S_μ to be the space. So we suppose from the beginning that Ω is compact and that the kernel is positive. By Corollary 2 of (viii) of § 1.4, there is only a finite number of points $\{P_k\}$ of $\mathbf{O}_\infty$ for the adjoint kernel. Given $\varepsilon>0$ we can enclose each P_k by an open neighborhood N_k such that $V_i(\bigcup_k N_k)>1/\varepsilon$. In each N_k we choose a neighborhood N'_k of P_k which is relatively compact in N_k, and denote the restriction of μ to $\Omega-\bigcup_k N'_k$ by μ'. By Corollary 1 of $(\mathrm{U_K})$ (i) of § 1.5, Ugaheri's maximum principle is true on $\Omega-\bigcup_k N'_k$ for the adjoint kernel. There exists an open set $G\subset\Omega-\bigcup_k N'_k$ such that $V_i(G)\geqq 1/\varepsilon$ and the restriction of $U^{\mu'}(P)$ to $\Omega-\bigcup_k N'_k-G$ is continuous, in virtue of Lemma 1.9. Then the restriction of $U^{\mu}(P)$ to $\Omega-\bigcup_k N_k-G$ is continuous, because $U^{\mu-\mu'}(P)$ is continuous in $\Omega-\bigcup_k N_k$. We see that

$$\frac{1}{V_i(\bigcup_k N_k\cup G)}\leqq\frac{1}{V_i(\bigcup_k N_k)}+\frac{1}{V_i(G)}\leqq 2\varepsilon.$$

Thus $U^{\mu}(P)$ is quasicontinuous.

We shall discuss the so-called problem of capacitability in § 3.6. Here we shall prove the coincidence of the V_i-value and the V_e-value of a compact set. This was first proved in Fuglede [1] under our general circumstances.

We shall use the following lemma in a special case in this section; it will be used in Chapter III in full generality.

LEMMA 1.10. *Let $f(P)<\infty$ be an upper semicontinuous function and $g(P)$ be a continuous function, both defined on a set $X\subset\Omega$. Let D be a directed set and $T=\{\mu_\omega;\ \omega\in D\}$ be a net of measures, converging vaguely to μ_0, and $\{a_\omega;\ \omega\in D\}$ be a net of real numbers converging to a finite number a_0. Let $\{Y_\omega;\ \omega\in D\}$ be a net of subsets of X and Y_0 be a subset of X with the property that every neighborhood of any point of Y_0 intersects all Y_ω, $\omega\geqq\omega_0$; this ω_0 depends on the point and the neighborhood in general. Then*

$$(1.27)\qquad \varliminf_{\omega}\ \sup_{Y_\omega}\ \{U^{\mu_\omega}(P)-f(P)-a_\omega g(P)\}\geqq\sup_{Y_0}\ \{U^{\mu_0}(P)-f(P)-a_0 g(P)\}.$$

PROOF. We set

$$V_f(\mu_\omega) = \sup_{Y_\omega} \{U^{\mu_\omega}(P) - f(P) - a_\omega g(P)\},\ \omega \in D \text{ or } \omega = 0,$$

and assume $V_f(\mu_0) > -\infty$. Given $t < V_f(\mu_0)$, let P_0 be a point of Y_0 such that $U^{\mu_0}(P_0) - f(P_0) - a_0 g(P_0) > t$. In case the left side is finite, we take a number t_1 in between. In view of Proposition 4 in § 1.6, there are a compact neighborhood N of P_0 and a neighborhood M of μ_0 with respect to the vague topology such that $U^{\mu}(P) - U^{\mu_0}(P_0) > -(t_1 - t)/4$, $-f(P) + f(P_0) > -(t_1 - t)/4$ and $|g(P) - g(P_0)| < \min \{(t_1 - t)(4|a_0|)^{-1}, 1\}$ for any $P \in N \cap X$ and any $\mu \in M$. There is $\omega_0 \in D$ such that $\mu_\omega \in M$, $Y_\omega \cap N \neq \emptyset$ for every $\omega \geqq \omega_0$ and $-a_\omega g(P) + a_0 g(P) > -(t_1 - t)/4$ for every $P \in N \cap X$ and $\omega \geqq \omega_0$. We take an arbitrary point P_ω of $Y_\omega \cap N$ for each $\omega \geqq \omega_0$. It holds that

$$t < U^{\mu_\omega}(P_\omega) - f(P_\omega) - a_\omega g(P_\omega) \leqq V_f(\mu_\omega)$$

for every $\omega \geqq \omega_0$. Therefore

$$t \leqq \varliminf_{\omega} V_f(\mu_\omega).$$

By the arbitrariness of t, we obtain (1.27). The case $U^{\mu_0}(P_0) - f(P_0) - a_0 g(P_0) = \infty$ can be treated in a similar fashion.

THEOREM 1.14. *For any compact set $K \subset \Omega$*

$$V_i(K) = V_e(K).$$

PROOF. We may assume that Ω is a compact space. The set D of all open sets G containing K is directed by $\subset$. We shall write $G_1 \geqq G_2$ if and only if $G_1 \subset G_2$. We assume $V_e(K) < V_i(K)$, and take a number α in between. For every $G \in D$ there is a unit measure μ_G supported by G such that

$$V(\mu_G) < \alpha. \tag{1.28}$$

The set $\{\mu_G;\ G \in D\}$ is a net and bounded in $\mathscr{M}$. Hence a subnet $\{\mu^{(\omega)};\ \omega \in D'\}$ converges to some measure μ_0 vaguely by Proposition 3. We observe that S_{μ_0} is contained in K because $\bigcap_{G \in D} G = K$. Therefore $V(\mu_0) \geqq V_i(K)$. On the other hand, by (1.27) and (1.28) we have

$$\alpha \geqq \varliminf_{\omega} V(\mu^{(\omega)}) \geqq V(\mu_0).$$

This contradicts the assumption $\alpha < V_i(K)$. Thus $V_i(K) \leqq V_e(K)$. The inverse inequality being evident, the equality follows.

1.9. Sequence of potentials.

The next topic is concerning the convergence of potentials as measures converge vaguely or strongly. First we give without proof

THEOREM 1.15. *Let D be a directed set and $T=\{\mu_\omega;\ \omega\in D\}$ be a net converging vaguely to μ_0. If the kernel is nonnegative or if all measures of T are supported by a fixed compact set, then*

$$\varliminf_{\omega} U^{\mu_\omega}(P) \geqq U^{\mu_0}(P)$$

in Ω. In case every measure of T is supported by a fixed compact set K, if the kernel is continuous outside the diagonal set, then

$$\varliminf_{\omega} U^{\mu_\omega}(P) = U^{\mu_0}(P)$$

outside K.

REMARK. There is an example which shows that the condition that every measure of T be supported by a fixed compact set is necessary in order to have the last equality outside a closed set containing $\bigcup_{\omega\in D} S_{\mu_\omega}$ even if $\mu_\omega(\Omega)$, $\omega\in D$, is bounded; in E_3 we take for μ_n a unit measure at $P_n \neq P_0$ which tends to the point at infinity and set $\Phi(P_0, P_n)=n$. However, if for any given $\varepsilon>0$ and for a point P_0 lying outside the closure of $\bigcup_{\omega\in D} S_{\mu_\omega}$ we can find a compact set K such that $|\Phi(P_0, Q)|<\varepsilon$ whenever $Q\in\Omega-K$, and if $\mu_\omega(\Omega)$, $\omega\in D$, is bounded, then the equality is true at P_0.

Next we prove

THEOREM 1.16. (cf. Brelot and Choquet [2], Ohtsuka [5], p. 62). *Assume that the adjoint kernel satisfies the continuity principle. If a subnet $T=\{\mu_\omega;\ \omega\in D\}$ of a sequence of measures, supported by a fixed compact set, converges vaguely to a measure μ_0, then*

$$\varliminf_{\omega} U^{\mu_\omega}(P) = U^{\mu_0}(P) \qquad \textit{p.p.p. in } \Omega.$$

PROOF. On account of Theorem 1.15 it is sufficient to prove that $\varliminf_{\omega} U^{\mu_\omega}(P) \leqq U^{\mu_0}(P)$ p.p.p. in Ω. If we deny this, we can find, by the condition of the continuity principle, a unit measure ν with compact support such that its adjoint potential is continuous in Ω and

$$\varliminf_{\omega} U^{\mu_\omega}(P) > U^{\mu_0}(P) \qquad \text{on } S_\nu.$$

Since $\{\mu_\omega\}$ is a countable class of measures, we can apply Fatou's lemma and see

$$(\mu_0, \nu) < \int \varliminf_{\omega} U^{\mu_\omega}(P)d\nu(P) \leqq \varliminf_{\omega} \int U^{\mu_\omega}(P)d\nu(P) = \lim_{\omega} \int \check{U}^{\nu}(P)d\mu_\omega(P) = (\mu_0, \nu).$$

This is a contradiction and the theorem is concluded.

It is known (Choquet [3; 5], Kishi [2; 3]) that, if a sequenc $\{\mu_n\}$ supported by a fixed compact set converges vaguely to μ_0, then

$$\varliminf_{n\to\infty} U^{\mu_n}(P)=U^{\mu_0}(P)$$

except on a set X with $V_e^*(X)=\infty$ in Ω under some additional condition. In the second chapter we shall need a similar theorem in a slightly general case, and so we prove

THEOREM 1.17. *Assume that the kernel is continuous outside the diagonal set, that $\Phi(P, P)=\infty$ at each $P\in \mathbf{O}_\infty$ and that both the kernel and the adjoint kernel satisfy the continuity principle. If a subnet $T=\{\mu_\omega;\ \omega\in D\}$ of a sequence of measures, supported by a fixed compact set K, converges vaguely to μ_0, then*

$$\varliminf_{\omega} U^{\mu_\omega}(P)=U^{\mu_0}(P) \qquad \textit{q.p. in } \Omega.$$

PROOF. It is sufficient to prove that

$$\varliminf_{\omega} U^{\mu_\omega}(P)\leqq U^{\mu_0}(P) \qquad \text{q.p. in } \Omega,$$

on account of Theorem 1.15; we may suppose that Ω is compact in view of the same theorem. If we use the notation $\leq$ for the order in D, then

$$V_\omega(P)=\inf_{\omega\leqq\omega'} U^{\mu_{\omega'}}(P)$$

increases to $\varliminf_{\omega} U^{\mu_\omega}(P)$. Given $\varepsilon>0$, we can find by Theorem 1.13 and Proposition 2 in § 1.1 an open set G_ε such that $V_i(G_\varepsilon)>1/\varepsilon$ and the restriction of $U^{\mu_\omega}(P)$ for each $\omega\in D$ to $\Omega-G_\varepsilon$ and that of $U^{\mu_0}(P)$ are continuous. For $\eta>0$ we set

$$E_\omega(\eta)=\{P;\ V_\omega(P)-U^{\mu_0}(P)\geqq\eta\}$$

and

$$E_\omega(\varepsilon, \eta)=\{P\in\Omega-G_\varepsilon;\ V_\omega(P)-U^{\mu_0}(P)\geqq\eta\}.$$

Since the restriction of $V_\omega(P)$ to $\Omega-G_\varepsilon$ is upper semicontinuous, $E_\omega(\varepsilon, \eta)$ is a compact set in Ω and hence $V_i(E_\omega(\varepsilon, \eta))=V_e(E_\omega(\varepsilon, \eta))$ by Theorem 1.14.

If $V_i(E_\omega(\varepsilon, \eta))$ were finite, we could find a unit measure ν with $S_\nu\subset E_\omega(\varepsilon, \eta)$ such that $\check{U}^\nu(P)$ is continuous in Ω on account of the continuity principle. It would follow that

$$\eta\leqq\int (V_\omega(P)-U^{\mu_0}(P))\,d\nu(P)\leqq\int (U^{\mu_{\omega'}}(P)-U^{\mu_0}(P))\,d\nu(P)$$

for any $\omega'\in D$ such that $\omega'\geqq\omega$. As $\mu_{\omega'}\to\mu_0$, the right side tends to 0 and we should arrive at a contradiction. Therefore $V_e(E_\omega(\varepsilon, \eta))=V_i(E_\omega(\varepsilon, \eta))=\infty$ for each $\omega \in D$. By Proposition 1 in § 1.1 we have, with $m < \inf \Phi(P, Q)$ taken on $\Omega \times \Omega$,

$$\frac{1}{V_e(E_\omega(\eta))-m} \leq \frac{1}{V_e(E_\omega(\varepsilon, \eta) \cup G_\varepsilon)-m}$$
$$\leq \frac{1}{V_e(E_\omega(\varepsilon, \eta))-m} + \frac{1}{V_e(G_\varepsilon)-m} = \frac{1}{V_i(G_\varepsilon)-m} < \frac{1}{1/\varepsilon - m}.$$

Since ε is arbitrary, we have $V_e(E_\omega(\eta))=\infty$ for every $\eta>0$ and $\omega \in D$. On account of the relation

$$\{P; \varliminf_{\omega} U^{\mu_\omega}(P)-U^{\mu_0}(P)>0\} = \bigcup_{k=1}^{\infty} \bigcup_{\omega \in D} E_\omega\left(\frac{1}{k}\right),$$

the V_e-value of the left set is infinite by Proposition 2 of § 1.1. Namely,

$$\varliminf_{\omega} U^{\mu_\omega}(P) \leq U^{\mu_0}(P) \qquad \text{q.p. in } \Omega.$$

Next we consider a weakly convergent sequence of measures.

THEOREM 1.18. *Assume that the kernel is of positive type and let $\{\mu_n\}$ be a sequence of measures in $\mathscr{E}$ converging weakly to μ_0. Then*

$$\varliminf_{n\to\infty} U^{\mu_n}(P) \leq U^{\mu_0}(P) \qquad \textit{in } G_0$$

except on a set H whose any compact subset vanishes for every measure with finite energy.

PROOF. Suppose that $\varliminf_{n\to\infty} U^{\mu_n}(P) > U^{\mu_0}(P)$ on a compact set $K \subset G_0$ with $V_i(K)<\infty$. We can find by Egoroff's theorem a compact subset K_1 of K with $V_i(K_1)<\infty$ such that $\inf_{m \leqq k} U^{\mu_k}(P)$ tends uniformly to $\varliminf_{n\to\infty} U^{\mu_n}(P)$ on K_1 as $m\to\infty$. Hence $U^{\mu_n}(P)$ is uniformly bounded from below on K_1. We take any $\nu \in \mathscr{E}_{K_1}(1, 1)$ and have by Fatou's lemma

$$(\mu_0, \nu) = \lim_{n\to\infty} (\mu_n, \nu) \geqq \int (\varliminf_{n\to\infty} U^{\mu_n}) d\nu > (\mu_0, \nu).$$

This is a contradiction.

It will be shown in § 2.2 of the next chapter that the above stated character of the exceptional set H is equivalent to $V_i(H)=\infty$. Therefore the inequality is valid in fact p.p.p. in G_0. It will also be proved that the inequality holds q.p. for a strongly convergent sequence under some additional condition (Lemma 3. 8).

1.10. Notes and questions.

One of important tools in classical potential theory was the selection theorem concerning measures; see Frostman [1] for instance. This was generalized to the case of locally compact spaces by Dieudonné [1; 2] (placed in Bourbaki [1] and our Proposition 3 in § 1.6) and it made the discussion of potentials in locally compact spaces possible. This and the importance of kernels which satisfy the continuity principle were recognized and emphasized independently and simultaneously by Choquet and the present author. The idea of applying Egoroff's or Lusin's theorem goes back to Y. Yosida [1]. I recall the following comment on [1] by Kametani stated in Kagaku, 12 (1942), p. 230, in Japanese:…… The above results are rather topological. It seems that the another aspect of the theory of functions, namely the part using the theory of functions of real variables or the metrical part, has been developing centering around potential theory. Recently Mr. Yôiti Yosida of Hokkaido University proved nicely the main theorem in potential theory, i.e. the maximum principle of Frostman, by the aid of the fundamental results in the theory of functions of real variables, particularly using Egoroff's theorem, without applying the theorem on the existence of equilibrium mass-distribution. Aren't there many other theorems and proofs which can be improved after his excellent idea?

We shall state open questions; some other questions as to principles were raised in Ohtsuka [7].

1.1. How are principles related to each other if we restrict ourselves to convolution kernels?

1.2. Question 1 in § 1.6.

1.3. Question 2 in § 1.6.

1.4. Question 3 in § 1.7.

1.5. Under the assumption of Theorem 1.18, is the inequality $\varliminf_{n\to\infty} U^{\mu_n}(P) \geqq U^{\mu_0}(P)$ true in G_0 (with some exception)?

Remarks by B. Fuglede through a letter dated Dec. 20, 1960.

………I take the liberty of making a few comments, in particular concerning the three questions raised on p. 192, p. 194 and p. 197. In this connection I shall refer to my examples 3 and 4, p. 208 ff in my Acta paper. (In example 3 I have forgotten to add the hypothesis that the set of points x where $f(x)=+\infty$ should not be an open set unless it is empty). I use your notations in the sequel.

Ad Question 1. Let $0<f(P)\leqq+\infty$, and suppose f is continuous in the extended sense (i.e. from Ω to $\overline{R}^+$) and that the set $S\subset\Omega$ of points where $f(P)=+\infty$ is neither void nor open. (Example: $\Omega=R=$ the real line; $f(x)=1/x^2$, interpreted as $+\infty$ for $x=0$). Then the kernel

$$\Phi(P, Q)=f(P)f(Q)$$

is of positive type, but not K-consistent (and hence does not fulfill (CW)$'$). In fact, let $P_0\in S$ denote a non-interior point of S, and let $\{Q_\omega;\ \omega\in D\}$ denote a net converging to P_0 and such that $Q_\omega \notin S$. If we put $\mu_\omega =$ the mass $1/f(Q_\omega)$ placed at the point Q_ω, then these measures μ_ω constitute a strong Cauchy net which converges vaguely to 0 (because $f(Q_\omega) \to f(P_0) = +\infty$) but not strongly (because the energy of each μ_ω is 1).— If we want an example in which the kernel moreover satisfies the energy principle, we merely have to add $f(P)$ $f(Q)$ any finite, continuous kernel satisfying the energy principle. I have no example in which $\Phi(P, Q)$ is, in addition, finite for $P \neq Q$. Perhaps this was what you meant?

Ad Question 2. The answer is *no*. Example: Again $\Phi(P, Q) = f(P)f(Q)$, where now $0 < f(P) < +\infty$, f being continuous, and where $f(P) \to 0$ as P approaches infinity. In fact, the measures $\mu_P =$ the mass $1/f(P)$ placed at P determine a strong Cauchy net which approches 0 vaguely, but not strongly, as $P \to$ infinity.— If we want an example in which, in addition, Φ satisfies the energy principle, we may take, e.g.,

$$\Phi(P, Q) = f(P)f(Q)\,(1 + \overline{PQ}^{-1})$$

in $\Omega = R^3$ (with f as above). In fact, this kernel is equivalent in the sense of § 5.1 in my Acta paper to the kernel studied in my Example 4, and hence inconsistent.

As to Question 3, I have no answer, but it is easy to answer the corresponding question concerning *perfect* kernels. In fact, the kernel on R^3 just mentioned has no perfect extension. For let Φ' denote any kernel on a space $\Omega' \supset \Omega$ such that $\Phi' = \Phi$ on $\Omega \times \Omega$. If P' denotes a point of the closure $\bar{\Omega}$ of Ω in Ω', and if $P' \notin \Omega$, then for any $Q \in \Omega$

$$\Phi'(P', Q) \leqq \liminf_{P \to P',\, P \in \Omega} \Phi(P, Q) = 0$$

and for $Q' \in \bar{\Omega} - \Omega$

$$\Phi'(P', Q') \leqq \liminf_{Q \to Q',\, Q \in \Omega} \Phi'(P', Q) = 0.$$

In particular, $\Phi'(P', P') \leqq 0$ for any $P' \in \bar{\Omega} - \Omega$. Consequently, $\bar{\Omega} = \Omega$ (i.e. Ω is closed in Ω') if Φ' satisfies the energy principle, or just if Φ' is strictly pseudo-positive. It follows now that Φ' cannot be perfect, for the restriction of Φ to the *closed* set Ω would then likewise be perfect.

According to another letter he is writing a paper on perfect kernels, consistent kernels, strong completeness, etc.

Chapter II. Gauss variation.

2.1. Potential of an extremal measure.

Let $\mathfrak{A}$ denote the class of all sets which are measurable with respect to every measure in Ω. Let $A \in \mathfrak{A}$ be a set with $\mathscr{E}_A \neq \{0\}$, and $f(P)$ be a function on A which is $\mathfrak{A}$-measurable. We are interested in the problem of minimizing the expression

$$(2.1) \qquad I(\mu)=(\mu, \mu)-2\int f(P)d\mu(P)$$

for $\mu \in \mathscr{E}$ with the property that $\mu(\Omega-A)=0$ and $\int fd\mu$ is defined. This problem will be called *Gauss variational problem.* We shall write at times

$$\int f(P)d\mu(P)=\langle f, \mu\rangle$$

for simplicity. A mutual energy can be written as $\langle U^{\mu}, \nu\rangle$.

If we assume no further condition on μ then the problem will be called *unconditional*, and if we assume some additional condition on μ the problem will be called *conditional.* We begin with a conditional problem.

For measures μ and μ' we shall use the notation $\mu \geqq \mu'$ if $\mu(A) \geqq \mu'(A)$ for every $A \in \mathfrak{A}$. Let a set A of $\mathfrak{A}$ be given and a measure μ with the property that $\mu(\Omega-A)=0$. A family of $\mathfrak{A}$-measurable functions $\{g_k(P)\}$, $k=1, \cdots, n$, defined on A will be called *μ-independent* if there are $\{\mu_k\}$, $k=1, 2, \cdots, n$, such that each $\mu_k \leqq \mu$ and

$$(2.2) \qquad \|\langle g_j, \mu_k\rangle\| \neq 0,$$

where $\| \ \|$ means a determinant. Given $f(P)$, $\{g_k(P)\}$ and finite numbers $\{x_k\}$, we shall consider the following classes of measures:

$$\mathscr{E}_A(\{g_k\}, \{x_k\}, f)=\{\mu \in \mathscr{E}_A;\ \langle g_k, \mu\rangle = x_k \text{ for each } k \text{ and } \langle f, \mu\rangle \text{ is defined}\},$$
$$\mathscr{E}'_A=\{\mu \in \mathscr{E};\ \mu(\Omega-A)=0\}$$

and

$$\mathscr{E}'_A(\{g_k\}, \{x_k\}, f)=\{\mu \in \mathscr{E}'_A;\ \langle g_k, \mu\rangle = x_k \text{ for each } k \text{ and } \langle f, \mu\rangle \text{ is defined}\}.$$

Certainly $\mathscr{E}_K(\{g_k\}, \{x_k\}, f)=\mathscr{E}'_K(\{g_k\}, \{x_k\}, f)$ for any compact set K. For a general set X we set

$$\mathscr{E}_X(\{g_k\}, \{x_k\})=\{\mu \in \mathscr{E}_X;\ S_{\mu} \text{ is compact and } \langle g_k, \mu\rangle = x_k \text{ for each } k\}.$$

In case f is upper semicontinuous and $<\infty$ on K, $\mathscr{E}_K(\{g_k\}, \{x_k\}, f)=\mathscr{E}_K(\{g_k\}, \{x_k\})$ because $\langle f, \mu\rangle$ is always defined for any $\mu \in \mathscr{E}_K$.

First we prove

Theorem 2.1. *Let A be a set of $\mathfrak{A}$ with $\mathscr{E}_A \neq \{0\}$ such that (μ, ν) and (ν, μ) are well-defined for $\mu \in \mathscr{E}'_A$ and any $\nu \in \mathscr{E}_A$, $f(P)$ be an $\mathfrak{A}$-measurable function on A such that $\langle f, \nu\rangle$ is defined for any $\nu \in \mathscr{E}_A$ and $\{g_k(P)\}$ be $\mathfrak{A}$-measurable functions on A such that $\langle g_k, \nu\rangle$ is defined and finite for each k and for any $\nu \in \mathscr{E}_A$. If there exists an extremal measure $\mu^* \in \mathscr{E}'_A(\{g_k\}, \{x_k\}, f)$ such that*

(2.3) $$\infty > I(\mu^*) = \min_{\mu \in \mathcal{E}'_A(\{g_k\}, \{x_k\}, f)} I(\mu) > -\infty$$

and (2.2) *is satisfied with some measures* $\mu_k \leqq \mu^*$,[20] *then,* $\{\gamma_k\}$ *being the solutions of*

(2.4) $$\sum_{j=1}^{n} \langle g_j, \mu_k \rangle \gamma_j = \langle \hat{U}^{\mu^*}, \mu_k \rangle - \langle f, \mu_k \rangle,^{21)}$$

it holds that

(2.5) $$\hat{U}^{\mu^*}(P) \geqq f(P) + \sum_{k=1}^{n} \gamma_k g_k(P)$$

on A *except* H *with* $\mathcal{E}_H \equiv \{0\}$ *and the equality holds* μ_k*-a. e. for each* k.

If, in addition, $f(P)$ *is upper semicontinuous and each* $g_k(P)$ *is continuous on* A, *then*

(2.6) $$\hat{U}^{\mu^*}(P) \leqq f(P) + \sum_{k=1}^{n} \gamma_k g_k(P) \qquad \text{on } \bigcup_{k=1}^{n} S_{\mu_k} \cap A$$

and the above exceptional set H *is the intersection of* A *with an* F_σ*-set*[22] *in* Ω.

PROOF. Our proof will follow a pattern in the calculus of variation; we owe the technique to Nagumo [1]. Let ν be any measure of $\mathcal{E}_A$ and $\{t_k\}$, $k=1, \cdots, n$, be the solutions of the equations

(2.7) $$\sum_{k=1}^{n} \langle g_j, \mu_k \rangle t_k = \langle g_j, \nu \rangle \qquad j=1, \cdots, n.$$

With a positive parameter t we set

$$\mu(t) = \mu^* - t \sum_{k=1}^{n} t_k \mu_k + t\nu.$$

Since

$$\mu^* - t \sum_{k=1}^{n} t_k \mu_k \geqq (1 - t \sum_{k=1}^{n} |t_k|) \mu^*,$$

$\mu(t)$ is nonnegative for sufficiently small t. We have $\langle g_j, \mu(t) \rangle = x_j$ for each j in view of (2.7). If the coefficients of t and t^2 in the polynomial $I(\mu(t))$ are finite, $I(\mu(t)) \geqq I(\mu^*)$ for sufficiently small $t \geqq 0$ and

$$0 \leqq \left. \frac{\partial I(\mu(t))}{\partial t} \right|_{t=0} = 2 \langle \hat{U}^{\mu^*}, \nu - \sum_{k=1}^{n} t_k \mu_k \rangle - 2 \langle f, \nu - \sum_{k=1}^{n} t_k \mu_k \rangle.$$

Substituting (2.4) and (2. 7), we obtain

20) $\{g_k\}$ are then μ^*-independent.

21) We recall that $\hat{U}^{\mu^*} = \int \hat{\Phi} \, d\mu^* = \frac{1}{2} (U^{\mu^*} + \check{U}^{\mu^*})$.

22) An F_σ-set is a countable union of closed sets.

$$\langle \hat{U}^{\mu*}, \nu\rangle - \langle f, \nu\rangle \geqq \sum_{k=1}^{n} t_k \{\langle \hat{U}^{\mu*}, \mu_k\rangle - \langle f, \mu_k\rangle\}$$

$$= \sum_{k=1}^{n} t_k \sum_{j=1}^{n} \langle g_j, \mu_k\rangle \gamma_j = \sum_{j=1}^{n} \gamma_j \sum_{k=1}^{n} \langle g_j, \mu_k\rangle t_k = \sum_{j=1}^{n} \langle g_j, \nu\rangle \gamma_j.$$

Namely

$$\langle \hat{U}^{\mu*}, \nu\rangle \geqq \langle f + \sum_{j=1}^{n} \gamma_j g_j, \nu\rangle. \tag{2.8}$$

The coefficients of t and t^2 in $I(\mu(t))$ are respectively

$$\langle f - \hat{U}^{\mu*}, \sum_k t_k \mu_k - \nu\rangle$$

and

$$(\sum_k t_k \mu_k - \nu, \sum_k t_k \mu_k - \nu).$$

If $\langle \hat{U}^{\mu*}, \nu\rangle = \infty$ or $\langle f, \nu\rangle = -\infty$, (2.8) is true. Hence we assume that $\langle \hat{U}^{\mu*}, \nu\rangle < \infty$ and $\langle f, \nu\rangle > -\infty$. Then the second coefficient is finite. The first coefficient is $< \infty$ and can be equal to $-\infty$ only if $\langle f, \nu\rangle = \infty$. However, if so, $I(\mu^*) \leqq I(\mu(t)) = -\infty$; this is impossible because we assumed $I(\mu^*)$ to be finite. Thus in any case (2.8) holds good. Since $\nu \in \mathscr{E}_A$ is arbitrary, (2.5) follows.

Next we integrate (2.5) with respect to μ_k and obtain

$$\langle \hat{U}^{\mu*}, \mu_k\rangle \geqq \langle f, \mu_k\rangle + \sum_{j=1}^{n} \gamma_j \langle g_j, \mu_k\rangle.$$

We should have the strict inequality here if the strict inequality were true on a set of positive μ_k-value in (2.5). It is impossible on account of (2.4) and the equality is true in (2.5) μ_k-a. e. for each k.

We assume now that $f(P)$ is upper semicontinuous and each $g_k(P)$ is continuous on A. If there were a point $P_0 \in S_{\mu_k} \cap A$ at which

$$\hat{U}^{\mu*}(P_0) > f(P_0) + \sum_{k=1}^{n} \gamma_k g_k(P_0),$$

then the inequality would be true on A in a neighborhood N_{P_0} of P_0 by the lower semicontinuity of $\hat{U}^{\mu*}(P)$ and by the upper semicontinuity of $f(P) + \sum_{k=1}^{n} \gamma_k g_k(P)$. On account of (2.5)

$$\langle \hat{U}^{\mu*}, \mu_k\rangle > \int (f + \sum_{j=1}^{n} \gamma_j g_j)\, d\mu_k = \langle f, \mu_k\rangle + \sum_{j=1}^{n} \langle g_j, \mu_k\rangle \gamma_j.$$

This contradicts (2.4), and (2.6) is proved. Since

$$H_p = \left\{ P \in A;\ \hat{U}^{\mu*}(P) \leqq f(P) + \sum_{k=1}^{n} \gamma_k g_k(P) - \frac{1}{p} \right\}$$

is closed relatively to A, the exceptional set $H=\bigcup_p H_p$ is the intersection of an F_σ-set with A.

COROLLARY. *Assume that $f(P)<\infty$ is upper semicontinuous and each $g_k(P)$ is continuous on A. If μ^* is an extremal measure giving finite $I(\mu^*)$ and if $\{\mu_k\}$ and $\{\gamma_k\}$ are respectively measures and constants defined in the theorem, then*

$$\hat{U}^{\mu^*}(P)-f(P)=\sum_{k=1}^{n}\gamma_k\, g_k(P)$$

on $\bigcup_{k=1}^{n} S_{\mu_k}\cap A$ except a set supporting no nonvanishing measure with finite energy.

REMARK. Let A be an $\mathfrak{A}$-measurable set with $\mathscr{E}_A\neq\{0\}$ and $f(P)$, $g_1(P),\cdots,g_n(P)$ be $\mathfrak{A}$-measurable functions on A. Assume that the following relations are true for $\{\nu_k\}$ and constants $\{c_k\}$, $k=1,\cdots,n$; we do not require any other properties:

$$\nu=\sum_{k=1}^{n}\nu_k\in\mathscr{E}'_A(\{g_k\},\ \{x_k\},f)$$

and, for each k,

$$\sum_{j=1}^{n}\langle g_j,\nu_k\rangle c_j=\langle\hat{U}^\nu,\nu_k\rangle-\langle f,\nu_k\rangle.$$

Then by adding these equalities for $k=1,\cdots,n$, we obtain

$$I(\nu)=\sum_{k=1}^{n}x_k\,c_k-\langle f,\nu\rangle=2\sum_{k=1}^{n}x_k\,c_k-(\nu,\nu). \tag{2.9}$$

This is particularly true for our extremal measure μ^* if $\mu^*=\sum_{k=1}^{n}\mu_k$.

We shall denote by $\mathscr{M}^*$ the class of all extremal measures in $\mathscr{E}'_A(\{g_k\},\{x_k\},f)$ which make $I(\mu)$ minimum. For two measures μ_1 and μ_2, we shall call $a\mu_1+b\mu_2$, with varying $a\geqq 0$ and $b\geqq 0$ such that $a+b=1$, a *segment* and denote it by $\overline{\mu_1\,\mu_2}$.

THEOREM 2.2. *Let $\mu^*,\ \mu^{**}\in\mathscr{M}^*$ and assume that $I(\mu^*)=I(\mu^{**})$ is finite. Then*

$$(\mu^*-\mu^{**},\ \mu^*-\mu^{**})\leqq 0. \tag{2.10}$$

If the strict inequality holds then no inner point of $\overline{\mu^*\mu^{**}}$ belongs to $\mathscr{M}^*$, but if the equality holds then $\overline{\mu^*\mu^{**}}\subset\mathscr{M}^*$.

PROOF. Let $a>0$, $b>0$ and $a+b=1$. Obviously

$$a\mu^* + b\mu^{**} \in \mathscr{E}'_A(\{g_k\}, \{x_k\}, f).$$

A simple calculation shows that

$$I(\mu^*) = I(\mu^{**}) \leqq I(a\mu^* + b\mu^{**}) = I(\mu^*) - ab(\mu^* - \mu^{**}, \mu^* - \mu^{**}).$$

Thus (2.10) follows. If $(\mu^* - \mu^{**}, \mu^* - \mu^{**}) < 0$, then $I(a\mu^* + b\mu^{**}) > I(\mu^*)$ and $a\mu^* + b\mu^{**}$ is not an extremal measure. If $(\mu^* - \mu^{**}, \mu^* - \mu^{**}) = 0$, $a\mu^* + b\mu^{**} \in \mathscr{M}^{**}$ and hence $\overline{\mu^* \mu^{**}}$ is contained in $\mathscr{M}^*$.

COROLLARY 1. *Under the assumption that* $\mathscr{M}^* \neq \emptyset$ *and* $I(\mu^*)$ *is finite on* $\mathscr{M}^*$, $\mathscr{M}^*$ *is a convex set if and only if*

$$(\mu^* - \mu^{**}, \mu^* - \mu^{**}) \geqq 0$$

for any $\mu^*, \mu^{**} \in \mathscr{M}^*$. *This is so in particular if the kernel is of positive type; then* $(\mu^* - \mu^{**}, \mu^* - \mu^{**}) = 0$.

COROLLARY 2. *If* inf $I(\mu)$ *for* $\mu \in \mathscr{E}'_A(\{g_k\}, \{x_k\}, f)$ *is finite and the energy principle is satisfied,* $\mathscr{M}^*$ *consists of at most one measure.*

We shall give in § 2.7 an example (Example 1) in which $\mathscr{M}^*$ consists of just two measures and A is a compact set.

Next we shall examine what follows from (2.4) and (2.5). Let us assume them. We assume also that we could choose $\{\mu_k\}$, $k=1, \dots, n$, such that $\sum_{k=1}^{n} \mu_k = \mu^*$ and (2.2) is satisfied. Let ν be any measure of $\mathscr{E}'_A(\{g_k\}, \{x_k\}, f)$ and integrate both sides of (2.5) with respect to ν. It follows by (2.9) that

$$2\langle \hat{U}^{\mu^*}, \nu \rangle \geqq 2\langle f, \nu \rangle + 2\sum_{k=1}^{n} x_k \gamma_k = 2\langle f, \nu \rangle + I(\mu^*) + (\mu^*, \mu^*).$$

This gives

$$I(\mu^*) + (\mu^* - \nu, \mu^* - \nu) \leqq I(\nu).$$

Thus we have

THEOREM 2.3. *Consider a kernel of positive type. Assume that there are* $\{\mu_k\}$ *satisfying* (2.2) *such that* μ^*, *set equal to* $\sum_{k=1}^{n} \mu_k$, *belongs to* $\mathscr{E}'_A(\{g_k\}, \{x_k\}, f)$ *and* (2.5) *is true with* $\{\gamma_k\}$, *defined by* (2.4), *on* A *except* H *with* $\mathscr{E}_H \equiv \{0\}$. *Then this* μ^* *is an extremal measure.*

For a kernel of positive type we can prove also

THEOREM 2.4. *Consider a kernel of positive type and let* μ^*, ν^* *be extremal measures. Assume that* $I(\mu^*) = I(\nu^*)$ *is finite and that there are* $\{\mu_k\}$ *and* $\{\nu_k\}$ *such that* $\sum_{k=1}^{n} \mu_k = \mu^*$, $\sum_{k=1}^{n} \nu_k = \nu^*$, $\langle g_j, \mu_k \rangle = \langle g_j, \nu_k \rangle$ *for each* j *and* k *and* (2.2) *is true. Then the solutions* $\{\gamma_j\}$ *of* (2.4) *are identical for* μ^*, $\{\mu_k\}$ *and* ν^*, $\{\nu_k\}$.

PROOF. Let $\{\gamma'_j\}$ be the solutions of

$$\sum_{j=1}^{n} \langle g_j, \nu_k \rangle \gamma'_j = (\nu^*, \nu_k) - \langle f, \nu_k \rangle.$$

By integrating (2.5) we have also

$$(\mu^*, \nu_k) - \langle f, \nu_k \rangle \geqq \sum_{j=1}^{n} \langle g_j, \nu_k \rangle \gamma_j.$$

From these relations it follows that

$$(\mu^* - \nu^*, \nu_k) \geqq \sum_{j=1}^{n} \langle g_j, \nu_k \rangle (\gamma_j - \gamma'_j).$$

The left side is zero because

$$(\mu^* - \nu^*, \nu_k)^2 \leqq (\mu^* - \nu^*, \mu^* - \nu^*)\,(\nu_k, \nu_k) = 0$$

on account of Corollary 1 of Theorem 2.2. Consequently

$$\sum_{j=1}^{n} \langle g_j, \nu_k \rangle (\gamma_j - \gamma'_j) \leqq 0.$$

Similarly it holds that

$$\sum_{j=1}^{n} \langle g_j, \mu_k \rangle (\gamma'_j - \gamma_j) \leqq 0.$$

By assumption $\langle g_j, \mu_k \rangle = \langle g_j, \nu_k \rangle$ for each j and k and hence

$$\sum_{j=1}^{n} \langle g_j, \mu_k \rangle (\gamma_j - \gamma'_j) = 0.$$

Since $\|\langle g_j, \mu_k \rangle\| \neq 0$, $\gamma_j = \gamma'_j$ for each j.

We shall find cases in which (2.2) is satisfied.

Theorem 2.5. *Let A and $f(P)$ be the same as in Theorem 2.1, none of $\{x_k\}$ be zero, and $\{g_k(P)\}$ be $\mathfrak{A}$-measurable functions defined on A such that $g_j(P) \equiv 0$ on A_k for any different j and k, where*

$$A_k = \{P \in A;\ x_k\, g_k(P) > 0\},$$

and such that $\langle g_k, \nu \rangle$ is finite for each k and for any $\nu \in \mathscr{E}_A$. Then for any $\mu \in \mathscr{E}'_A(\{g_k\}, \{x_k\}, f)$, the restrictions μ_k of μ to A_k satisfy (2.2).

PROOF. By our assumption $\langle g_j, \mu_k \rangle = 0$ for any different j and k. It will be sufficient to show that $\langle g_k, \mu_k \rangle \neq 0$ for each k. Since $\mu \in \mathscr{E}'_A(\{g_k\}, \{x_k\}, f)$ and none of $\{x_k\}$ is zero,

$$\int g_k\, d\mu = x_k \neq 0.$$

If

$$\langle g_k, \mu_k \rangle = \int_{A_k} g_k \, d\mu = 0,$$

then

$$x_k^2 = x_k \int_{A-A_k} g_k \, d\mu = \int_{A-A_k} (x_k g_k) \, d\mu \leqq 0.$$

This is impossible and $\langle g_k, \mu_k \rangle$ does not vanish for any k.

REMARK. If, in addition, $x_k g_k(P) \geqq 0$ on A for each k, then we define $A_1, \cdots, A_{n-1}$ as above and set $A'_n = A - \bigcup_{k=1}^{n-1} A_k$. We change the definition of μ_n to the restriction on A'_n instead of the restriction on A_n. These $\{\mu_k\}$ satisfy (2.2) and $\sum_{k=1}^{n} \mu_k = \mu$. This is so in particular if $\bigcup_{k=1}^{n} A_k = A$.

2.2. Problem on compact sets.

We shall discuss the existence of extremal measures. It is rather difficult to find conditions which ensure the existence under general circumstances and we shall limit ourselves to the special case in which $A = K$ consists of a finite number of mutually disjoint compact sets $\{K_k\}$, $k = 1, \cdots, n$, $f(P)$ is upper semicontinuous and $< \infty$ on K and $g(P)$ is positive on K; $\{x_k\}$ must be nonnegative then. We shall write simply $\mathscr{E}_K(g, x)$ for $\mathscr{E}_K(\{g_k\}, \{x_k\})$, where $g_k = g$ on K_k and $= 0$ on $K - K_k$ for each k and $x = (x_1, \cdots, x_n)$. The problem which is concerned in

$$\min_{\mu \in \mathscr{E}_K(g, x)} I(\mu)$$

is called n-dimensional. In case $n = 1$ K is not divided into compact subsets and x itself is a number. We shall use the same notation $\mathscr{E}_K(g, x)$ in this case too.

First we give

THEOREM 2.6. *Let K consist of mutually disjoint compact sets $\{K_k\}$, $k = 1, \cdots, n$ such that $\mathscr{E}_{K_k} \neq \{0\}$ for each k, and $\Phi(P, Q)$ be a kernel which is bounded on every $K_j \times K_k$, $j \neq k$. Let $f(P) < \infty$ be an upper semicontinuous function on K and $g(P)$ be a positive continuous function on K. Assume that it is not true that $f(P) = -\infty$ p.p.p. on any K_k. Then, for any nonnegative finite numbers $\{x_k\}$, $k = 1, \cdots, n$, there exists at least one $\mu_x \in \mathscr{E}_K(g, x)$ which gives finite*

$$I(\mu_x) = \min_{\mu \in \mathscr{E}_K(g, x)} I(\mu).$$

PROOF. First we shall prove that there is a measure $\mu_1 \in \mathscr{E}_{K_1}$ such that

$\langle g, \mu_1\rangle = x_1$ and

$$(\mu_1, \mu_1) - 2\langle f, \mu_1\rangle < \infty.$$

We assume $x_1 > 0$ and set

$$B_n = \{P \in K_1;\, f(P) > -n\}.$$

By assumption there is $\mu \in \mathscr{E}_{\underset{n}{\cup} B_n}(g, x_1)$. For some n, say n_0, $\mu(B_{n_0}) > 0$. We denote the restriction of μ to B_{n_0} by μ'. Then the measure $x_1\mu'/\langle g, \mu'\rangle$ has the required properties. We take a similar measure μ_k for each k. It follows that

$$\sum_{k=1}^{n} \mu_k \in \mathscr{E}_K(g, x) \qquad \text{and} \qquad \inf_{\mu\in\mathscr{E}_K(g,x)} I(\mu) \leqq I(\sum_{k=1}^{n} \mu_k) < \infty.$$

We choose $\mu^{(m)} \in \mathscr{E}_K(g, x)$ such that $I(\mu^{(m)})$ tends to the infimum. Since

$$\sum_{k=1}^{n} x_k = \int g d\mu^{(m)} \geqq \min_{P\in K} g(P)\mu^{(m)}(K),$$

$\mu^{(m)}(K)$ is uniformly bounded. By Proposition 3 in § 1.6, $\{\mu^{(m)}\}$ is vaguely bounded as a class of measures. Let $T = \{\nu_\omega;\, \omega \in D\}$ be a subnet of $\{\mu^{(m)}\}$ which converges vaguely to some measure μ_x. It follows that

$$x_k = \lim_{\omega} \int g_k\, d\nu_\omega = \int g_k\, d\mu_x \qquad \text{and} \qquad \overline{\lim_{\omega}} \int f d\nu_\omega \leqq \int f d\mu_x.$$

By Proposition 4 in § 1.6 we have

$$\underline{\lim_{\omega}}\ (\nu_\omega, \nu_\omega) \geqq (\mu_x, \mu_x).$$

Since

$$(\nu_\omega, \nu_\omega) \leqq I(\nu_\omega) + 2\ \max_K\ f \cdot \nu_\omega(K)$$

is bounded from above, we see that $\mu_x \in \mathscr{E}_K(g, x)$. Therefore

$$\inf_{\mu\in\mathscr{E}_K(g,x)} I(\mu) = \lim_{\omega}\ \{(\nu_\omega, \nu_\omega) - 2\langle f, \nu_\omega\rangle\}$$
$$\geqq (\mu_x, \mu_x) - 2\langle f, \mu_x\rangle = I(\mu_x) \geqq \inf_{\mu\in\mathscr{E}_K(g,x)} I(\mu).$$

Thus $I(\mu_x) =$ inf $I(\mu)$ for $\mu \in \mathscr{E}_K(g, x)$ and the existence is shown. The infimum is finite because $\mu_x(K)$ is finite and

$$I(\mu_x) = (\mu_x, \mu_x) - 2\langle f, \mu_x\rangle$$
$$\geqq \min_{P,Q\in K} \Phi(P, Q)\, \mu_x^2(K) - 2\max_{P\in K} f(P)\, \mu_x(K) > -\infty.$$

REMARK. Let $\sum_{k=1}^{n} x_k > 0$ and consider

$$\Psi(P, Q)=\Phi(P, Q)-2f(P)g(Q)/\sum_{k=1}^{n} x_k$$

on $K\times K$. This is lower semicontinuous, and does not take the value $-\infty$. Therefore this can be taken as a kernel on the space K. If we denote by $(\mu, \mu)_\Psi$ the energy of $\mu\in\mathscr{E}_K(g, x)$ with respect to this kernel, then

$$(\mu, \mu)_\Psi=(\mu, \mu)-2\langle f, \mu\rangle=I(\mu).$$

Consequently it is sufficient to prove Theorem 2.6 in the special case that the space is compact, $f(P)\equiv 0$ and $g(P)$ is defined in the whole space. We note also that $(\mu-\nu, \mu-\nu)_\Psi=(\mu-\nu, \mu-\nu)$ if $\mu, \nu\in\mathscr{E}_K(g, x)$. However, $\Psi(P, Q)$ depends on $\sum_{k=1}^{n} x_k$ and hence is not suitable in case $\{x_k\}$ change.

Theorem 2.7. *For an extremal measure μ_x obtained in Theorem 2.6, it holds that, if $x_k>0$,*

$$\hat{U}^{\mu_x}(P)\geqq f(P)+\gamma_k g(P) \tag{2.11}$$

on K_k except H_k with $\mathscr{E}_{H_k}\equiv\{0\}$ and

$$\hat{U}^{\mu_x}(P)\leqq f(P)+\gamma_k g(P) \qquad \text{on } S_{\mu_x}\cap K_k, \tag{2.12}$$

where

$$x_k\,\gamma_k=\int_{K_k}(\hat{U}^{\mu_x}-f)\,d\mu_x. \tag{2.13}$$

It follows that

$$I(\mu_x)=\sum_{k=1}^{n} x_k\,\gamma_k-\langle f, \mu_k\rangle=2\sum_{k=1}^{n} x_k\,\gamma_k-(\mu_x, \mu_x). \tag{2.14}$$

Assume that the kernel is of positive type. Let $\mu\in\mathscr{E}_K(g, x)$ and denote the restriction of μ to K_k by μ_k. If, for each k with $x_k>0$, μ and $\gamma_k=\langle U^\mu-f, \mu_k\rangle/x_k$ satisfy (2.11), *then μ is an extremal measure.*

Proof. If none of $\{x_k\}$ vanishes, this theorem follows from Theorems 2.1, 2.5 (see its Remark) and (2.9). If some x_k but not all of them vanish, say if $x_1>0,\cdots, x_m>0$ and $x_{m+1}=\cdots=x_n=0$, the problem reduces to the "m-dimensional case". Namely, the problem is to minimize $I(\mu)$ for $\mu\in\mathscr{E}_{\overset{m}{\underset{k=1}{\cup}}K_k}(g, x')$ where $x'=(x_1,\cdots, x_m)$. Our theorem is then readily established. The last statement is an immediate consequence of Theorem 2.3.

Corollary. 1. *For any compact set $K=\bigcup_{k=1}^{n} K_k$ such that no K_k is empty,*

$$I(\mu_x)=\inf\sum_{k=1}^{n}\left\{x_k\sup_{P\in S_\mu\cap K_k}\frac{\hat{U}^\mu(P)-f(P)}{g(P)}-\langle f, \mu_k\rangle\right\},$$

where μ is supported by K, μ_k is the restriction of μ to K_k and $\langle g, \mu_k\rangle = x_k$ for each k. In particular, $V_i(K)$ is equal to the infimum of (μ, μ) for unit measure μ supported by K.

PROOF. We may assume that all $x_k > 0$. Let us denote the quantity inside $\{\ \}$ by $W_k(\mu)$. We integrate the inequality $W_k(\mu)g(P) \geqq x_k \hat{U}^\mu(P) - x_k f(P) - \langle f, \mu_k\rangle g(P)$ with respect to μ_k and obtain

$$x_k W_k(\mu) \geqq x_k \int \hat{U}^\mu d\mu_k - 2x_k \langle f, \mu_k\rangle.$$

It follows that $\sum_{k=1}^n W_k(\mu) \geqq (\mu, \mu) - 2\langle f, \mu\rangle = I(\mu)$. If $\mathscr{E}_K(g, x) \neq \emptyset$, $I(\mu) \geqq I(\mu_x) = \sum_{k=1}^n W_k(\mu_k)$ by our theorem. If $\mathscr{E}_K(g, x) = \emptyset$, $I(\mu) = \infty$ and again the equality in the corollary holds. This establishes the corollary.

We shall call μ_1 giving $(\mu_1, \mu_1) = V_i(K)$ a *weak equilibrium measure* on K and $U^{\mu_1}(P)$ a weak equilibrium potential. If we consider $g(P)$ on K, a measure which gives min (μ, μ) among $\mu \in \mathscr{E}_K(g, 1)$ will be called a *weak g-equilibrium measure* on K and its potential a weak potential.

COROLLARY 2. *$V_i(X) \geqq \hat{V}_i(X)$, $\check{V}_i(X) \geqq \hat{V}_i(X)$. If $\Phi(P, Q) \geqq m > -\infty$ on $X \times X$, then*

$$2\hat{V}_i(X) \geqq \max(V_i(X), \check{V}_i(X)) + m$$

and

$$V_i(X) - m \leqq 2(\check{V}_i(X) - m).^{23)}$$

PROOF. Since

$$\hat{V}_i(X) \leqq \hat{V}_i(K) = W(K) \leqq (\mu, \mu) \leqq \sup_{Q \in S_\mu} U^\mu(Q)$$

for any compact set $K \subset X$ and any unit measure μ with $S_\mu \subset K$, we have $\hat{V}_i(X) \leqq V_i(X)$. Similarly $\hat{V}_i(X) \leqq \check{V}_i(X)$. If $\Phi(P, Q) \geqq m > -\infty$ on $X \times X$, then

$$2 \sup_{P \in S_\mu} \hat{U}^\mu(P) \geqq \sup_{P \in S_\mu} U^\mu(P) + m \geqq V_i(X) + m$$

for a unit measure μ with compact $S_\mu \subset X$ and hence $2\hat{V}_i(X) \geqq V_i(X) + m$. It is the same with $\check{V}_i(X)$. Combined with $\check{V}_i(X) \geqq \hat{V}_i(X)$, it gives the last inequality in the theorem.

As a consequence of these corollaries the following two propositions are equivalent:

A property holds p.p.p. on a set X.

A property holds on X except H with $\mathscr{E}_H \equiv \{0\}$.

23) This is an improvement of the evaluation in Theorem 1.12.

Consequently (2.5) and (2.11) hold p.p.p. on A and K_k respectively.

We shall denote by $\mathscr{M}_x^*$ the class of all extremal measures μ_x and set, if $x_k>0$,

$$\gamma_k(\mu_x)=\frac{1}{x_k}\left(\int_{K_k}\hat{U}^{\mu_x}d\mu_x-\int_{K_k}fd\mu_x\right). \tag{2.15}$$

If $x_k=0$, we set $x_k\,\gamma_k(\mu_x)=0$.

Let us prove

THEOREM 2.8. *Under the same condition as in Theorem 2.6, the class $\mathscr{M}_x^*$ is closed under the vague topology. If x_k is positive, $\gamma_k(\mu_x)$ is continuous as a function on the space $\mathscr{M}_x^*$ with the vague topology. If the kernel is of positive type, $x_k\,\gamma_k(\mu_x)$ is uniquely determined for each k.*

PROOF. Let $T=\{\mu^{(\omega)};\ \omega\in D\}$ be a net consisting of extremal measures which converges vaguely to μ^*. As in the proof of Theorem 2.6, we can see that $\mu^*\in\mathscr{E}_K(g,x)$ and we have

$$I(\mu_x)=\lim_{\omega} I(\mu^{(\omega)})\geqq(\mu^*,\mu^*)-2\langle f,\mu^*\rangle=I(\mu^*)\geqq I(\mu_x),$$

where μ_x is any one of extremal measures. Thus $I(\mu^*)=\inf_{\mu\in\mathscr{E}_K(g,x)} I(\mu)$ and μ^* is one of extremal measures.

To prove the last statement of theorem, we assume $x_1>0,\cdots,x_m>0$ and $x_{m+1}=\cdots=x_n=0$. We have that

$$\sum_{k=1}^{m} x_k\,\gamma_k(\mu^{(\omega)})=I(\mu^{(\omega)})+\langle f,\mu^{(\omega)}\rangle$$

for any $\mu^{(\omega)}\in T$ by (2.14). Therefore

$$\overline{\lim_{\omega}}\sum_{k=1}^{m} x_k\,\gamma_k(\mu^{(\omega)})\leqq I(\mu^*)+\langle f,\mu^*\rangle=\sum_{k=1}^{m} x_k\,\gamma_k(\mu^*).$$

On the other hand, by (2.4) and Proposition 3 in § 1.6,

$$\underline{\lim_{\omega}}\ x_k\,\gamma_k(\mu^{(\omega)})=\underline{\lim_{\omega}}\left(\int_{K_k}\hat{U}^{\mu^{(\omega)}}d\mu^{(\omega)}-\int_{K_k}fd\mu^{(\omega)}\right)\geqq\int_{K_k}\hat{U}^{\mu^*}d\mu^*-\int_{K_k}fd\mu^*=x_k\,\gamma_k(\mu^*)$$

for $k=1,\cdots,m$, where we use the fact that the restriction $\mu_k^{(\omega)}$ of $\mu^{(\omega)}$ on K_k converges vaguely to the restriction μ_k^* of μ^* on K_k. We can conclude that each $\lim_{\omega}\gamma_k(\mu^{(\omega)})$ exists and equals $\gamma_k(\mu^*)$ for $k=1,\cdots,m$.

The last statement in the theorem is an immediate consequence of Theorem 2.4.

In case each $x_k>0$, let us consider $(\gamma_1(\mu_x),\cdots,\gamma_n(\mu_x))$ as a point of the euclidean space E_n and denote it by $\gamma(\mu_x)$. We set

$$\Gamma_x=\{\gamma(\mu_x);\ \mu_x\in\mathscr{M}_x^*\}.$$

We know that, if the energy principle is satisfied, $\mathscr{M}_x^*$ consists of a single point. Then Γ_x does too.

Theorem 2.9. *Under the same condition as in Theorem* 2.6 *and the assumption that every* $x_k>0$, Γ_x *is a compact set. If, in addition,* $\mathscr{M}_x^*$ *is a convex set,* Γ_x *is connected.*

PROOF. First we observe that each $\gamma_k(\mu_x)$ is bounded from below because of its definition (2.15) and then that it is bounded from above in virtue of (2.14). Therefore Γ_x is bounded in E_n. In order to see that it is closed, we take a sequence $\{\mu^{(m)}\}\subset\mathscr{M}_x^*$ such that each $\gamma_k(\mu^{(m)})$ tends to some number γ_k. We can find a subnet $T=\{\nu_\omega;\ \omega\in D\}$ of $\{\mu^{(m)}\}$ which vaguely converges to a certain measure μ'. By Theorem 2.8 μ' is a measure of $\mathscr{M}_x^*$ and

$$\gamma_k=\lim_{\omega}\ \gamma_k(\nu_\omega)=\gamma_k(\mu').$$

This shows that the point $(\gamma_1,\cdots,\gamma_n)$ is equal to $\gamma(\mu')$ and hence Γ_x is closed.

Next we assume furthermore that $\mathscr{M}_x^*$ is convex. Hence, for any μ_x, $\mu'_x\in\mathscr{M}_x^*$ and $a\geqq 0$, $b\geqq 0$ such that $a+b=1$, $a\mu_x+b\mu'_x\in\mathscr{M}_x^*$. By a computation we obtain

$$\gamma_k(a\mu_x+b\mu'_x)=a\gamma_k(\mu_x)+b\gamma_k(\mu'_x)-\frac{ab}{x_k}\langle\hat{U}^{\mu_x}-\hat{U}^{\mu'_x},\ \mu_x^{(k)}-\mu_x'^{(k)}\rangle,$$

where $\mu_x^{(k)}$ and $\mu_x'^{(k)}$ are the respective restrictions of μ_x and μ'_x to K_k. If we set

$$\delta_k=\frac{1}{x_k}\langle\hat{U}^{\mu_x}-\hat{U}^{\mu'_x},\ \mu_x^{(k)}-\mu_x'^{(k)}\rangle\qquad\text{and}\qquad\delta=(\delta_1,\cdots,\delta_n),$$

then for a, $0\leqq a\leqq 1$, we have

$$\Gamma_x\ni\gamma(a\mu_x+(1-a)\mu'_x)=a\gamma(\mu_x)+(1-a)\gamma(\mu'_x)-a(1-a)\delta,$$

where $+$ means a vector summation. The right side represents a curve connecting $\gamma(\mu_x)$ and $\gamma(\mu'_x)$. Consequently Γ_x is connected.

2.3. Some general cases.

We shall discuss the existence of extremal measures under different conditions. If the kernel is of positive type, we can have various types of existence theorems. One example is as follows:

Let K be a compact set with $\mathscr{E}_K\equiv\{0\}$ *and* ν, $\lambda\in\mathscr{E}$ *be measures with compact support. Assume that every Cauchy sequence of measures of* $\mathscr{E}_K$ *converges strongly to some measure. Then for any* $x>0$, *there is a measure* $\mu^*\in\mathscr{E}_K(U^\lambda, x)$ *which gives*

$$\|\mu^*-\nu\|=\min_{\mu\in\mathscr{E}_K(U^\lambda,\,x)}\|\mu-\nu\|.$$

The proof is given in a customary way by using the identity

$$\left\|\frac{\mu_1+\mu_2}{2}-\nu\right\|^2+\frac{1}{4}\|\mu_1-\mu_2\|^2=\frac{1}{2}\|\mu_1-\nu\|^2+\frac{1}{2}\|\mu_2-\nu\|^2.$$

We can consider other similar problems. However, we shall take another occasion to discuss such problems in which potentials are taken for f or g or both.

In this section we shall seek a possibility of generalizing f or g in another direction. It does not effect the discussions in the subsequent sections and we can skip over this section. We begin with

THEOROM 2.10. *Let K be the union of mutually disjoint compact sets K_1, $\cdots$, K_n such that $\mathscr{E}_{K_k}\neq\{0\}$ for each k, and $g(P)$ be an upper semicontinuous finite-valued positive function on K. For each k, assume that $(\mu,\mu)>0$ for any $\mu\neq 0$ with $S_\mu\subset K_k$ and that $(\mu,\mu)\geqq(\mu',\mu')$ if $\mu\geqq\mu'$. Then for any positive x_1, $\cdots$, x_n there exists at least one $\mu_x\in\mathscr{E}_K(g,x)$ which gives finite*

$$(\mu_x,\mu_x)=\inf_{\mu\in\mathscr{E}_K(g,x)}(\mu,\mu).$$

PROOF. We consider the special case that $f\equiv 0$, $g\equiv 1$ and $x=1$ in Theorem 2.6. We obtain at least one extremal unit measure $\mu_k\in\mathscr{E}_{K_k}$ which minimizes (μ,μ) for each k. By our assumption we have that $(\mu_k,\mu_k)>0$. We choose $\mu^{(m)}\in\mathscr{E}_K(g,x)$ such that

$$\lim_{m\to\infty}(\mu^{(m)},\mu^{(m)})=\inf_{\mu\in\mathscr{E}_K(g,x)}(\mu,\mu).$$

Let $\mu_k^{(m)}$ be the restriction of $\mu^{(m)}$ to K_k. Since

$$(\mu_k^{(m)},\mu_k^{(m)})\geqq\{\mu_k^{(m)}(K)\}^2(\mu_k,\mu_k),$$

$\mu_k^{(m)}(K)$ is bounded. We can find a subnet $T=\{\nu_\omega;\ \omega\in D\}$ of $\{\mu^{(m)}\}$ which converges vaguely to some μ_x. It holds that

$$x_k=\lim_\omega\int_{K_k}gd\nu_\omega\leqq\int_{K_k}gd\mu_x=\langle g,\mu_x^{(k)}\rangle,$$

where $\mu_x^{(k)}$ denotes the restriction of μ_x to K_k. If we set $\mu'=x_k\,\mu_x^{(k)}/\langle g,\mu_x^{(k)}\rangle$ on K_k for each k, $\mu'\in\mathscr{E}_K(g,x)$ and $\mu'\leqq\mu_x$. By our assumption $(\mu',\mu')\leqq(\mu_x,\mu_x)$. It follows that

$$\inf_{\mu\in\mathscr{E}_K(g,x)}(\mu,\mu)=\lim_\omega(\nu_\omega,\nu_\omega)\geqq(\mu_x,\mu_x)\geqq(\mu',\mu')\geqq\inf_{\mu\in\mathscr{E}_K(g,x)}(\mu,\mu)$$

and all the equalities follow. Thus μ_x is an extremal measure.

We may raise questions as to whether this theorem is true if we consider $I(\mu)=(\mu,\mu)-2\langle f,\mu\rangle$ instead of simple (μ,μ) or if $g(P)$ is lower semicontinuous, and as to whether the condition $(\mu,\mu)>0$ for $\mu\in\mathscr{E}_K$, $\mu\neq 0$, or the condition $(\mu,\mu)\geqq(\mu',\mu')$ for $\mu\geqq\mu'$ can be dropped. Answers are all negative as will be shown below.

EXAMPLE 1. $\Omega=K=\{t;\ 0\leqq t\leqq 1\}$, $\Phi(t, s)\equiv 1$, $f(t)\equiv 1$, $g(t)=2-t$ for $0\leqq t<1$ and $=2$ at $t=1$, and $x\leqq 1$. Observing that

$$\mu(K)\leqq\int g d\mu=x\leqq 1$$

for $\mu\in\mathscr{E}_K(g, x)$ and that

$$I(\mu)=\mu^2(K)-2\mu(K)=(1-\mu(K))^2-1,$$

we see that the problem is to maximize the total mass of a measure of $\mathscr{E}_K(g, x)$. The total mass becomes larger as a measure is distributed nearer the point $t=1$ as a whole, and the infimum of $I(\mu)$ is x^2-2x. However, the I-value of the point measure at $t=1$ is $x^2/4-x$. Therefore there is no extremal measure.

EXAMPLE 2. $\Omega=K=\{t;\ 0\leqq t\leqq 1\}$, $\Phi(t, s)=t+s+1$, $f(t)\equiv 0$, $g(0)=1$, $g(t)=2$ for $0<t\leqq 1$. For $\mu\in\mathscr{E}_K(g, 1)$ we have

$$I(\mu)=(\mu, \mu)=\iint\Phi(t+s+1)d\mu(t)d\mu(s).$$

This becomes smaller as the measure μ is distributed nearer the point $t=0$ as a whole and the infimum is $1/4$. However, if μ is the point measure at $t=0$, then $I(\mu)$ is equal to 1. Hence there is no extremal measure.

Example 3. $\Omega=K=\{t;\ 0\leqq t\leqq 1\}$, $\Phi(t, s)\equiv -1$, $f(t)\equiv 0$, $g(t)=$ the same as in Example 1. In the same way as above the problem is to maximize the total mass of a measure of $\mathscr{E}_K(g, x)$ but there is no extremal measure.

EXAMPLE 4. $K_1=\{t;\ 0\leqq t\leqq 1\}$, $K_2=\{t;\ 2\leqq t\leqq 3\}$, $\Phi(t, s)=1$ on $K_1\times K_1$ and $K_2\times K_2$, $=-4$ on $K_1\times K_2$ and $K_2\times K_1$, $f(t)\equiv 0$, $g(t)$ and $g(t-2)$ are the same as in Example 1 on K_1 and on K_2 respectively, and $2x_1=x_2$. Let $x=(x_1, x_2)$ and $\mu\in\mathscr{E}_{K_1\cup K_2}(g, x)$. We set $\mu(K_1)=m_1$ and $\mu(K_2)=m_2$, and observe that, for a fixed m_1,

$$(\mu, \mu)=m_1^2+m_2^2-4m_1 m_2$$

takes its minimum $-3m_1^2$ when $m_2=2m_1$. Hence the problem is to maximize the total mass of a measure of $\mathscr{E}_{K_1}(g, x_1)$. This again shows that there is no extremal measure. Actually the condition $(\mu, \mu)\geqq(\mu', \mu')$ for $\mu\geqq\mu'$ is not satisfied as is seen by $\mu=$ the unit measure on K_1 plus the unit measure on K_2, $\mu'=$the unit measure on K_1.

Next we shall examine if $f(P)$ can be lower semicontinuous.

EXAMPLE 5. $\Omega=E_3$, $K=\{P;\ \overline{OP}\leqq 1\}$, $\Phi(P, Q)=1/\overline{PQ}$, $f(P)=1+\overline{OP}$ for $P\in K-\partial K$, $=1$ for $P\in\partial K$, $g(t)\equiv 1$. For $\mu\in\mathscr{E}_K(1, x)$, we have

$$I(\mu)=(\mu, \mu)-2\int f(P)d\mu(P)=(\mu, \mu)-2x-\int_{\overline{OP}<1}\overline{OP}d\mu(P).$$

This value becomes smaller as a measure μ is distributed uniformly and nearer ∂K as a whole. The infimum is equal to x^2-3x but the uniform unit measure on ∂K gives the value x^2-2x. Thus there is no extremal measure.

As an application of this example we would point out that Lemma 3 of Cartan and Deny [1] is not valid as it stands. In fact, according to the lemma the following fact should be true:

Consider the Newtonian kernel, and let K be a compact set in E_3 and $f(P)$ be a positive bounded measurable function in E_3. Then there is a unique measure μ^* which gives

$$I(\mu^*)=\min_{\mu\in\mathscr{E}_K} I(\mu).$$

Let us take K and $f(P)$ as in Example 5. If there were an extremal measure μ^* as asserted above, $\mu^*\neq 0$ and μ^* would give the smallest value to $I(\mu)$ among measures of $\mathscr{E}_K(1, x)$ where $x=\mu^*(K)$. This is impossible as was observed in Example 5.

Finally we consider the class

$$\mathscr{G}_K(g, x)=\left\{ \mu\in\mathscr{E}_K; \int_{K_k} gd\mu\geqq x_k \quad \text{for each } k\right\}.$$

THEOREM 2.6′. *Let K be the union of compact sets $K_1, \cdots, K_n$ with $\mathscr{E}_{K_k}\neq\{0\}$ for each k, $\Phi(P, Q)$ be a kernel which is bounded on every $K_j\times K_k$, $j\neq k$, $f(P)<\infty$ be an upper semicontinuous function on K and $g(P)$ be an upper semicontinuous finite-valued positive function on K. Assume that it is not true that $f(P)=-\infty$ p.p.p. on any K_k and suppose that $(\mu, \mu)>0$ for any $\mu\neq 0$ supported by K_k for any k. Then, for any $x>0$, there exists at least one $\mu^*\in\mathscr{G}_K(g, x)$ which gives finite*

$$I(\mu^*)=\min_{\mu\in\mathscr{G}_K(g, x)} I(\mu).$$

PROOF. We choose $\mu^{(m)}\in\mathscr{G}_K(g, x)$ such that $I(\mu^{(m)})$ $(<\infty)$ tends to the infimum as $m\to\infty$. Let μ_k be the unit measure which gives min (μ, μ) among unit measures supported by K_k. It follows for the restriction $\mu_k^{(m)}$ of $\mu^{(m)}$ to K_k that

$$I(\mu_k^{(m)})\geqq(\mu^{(m)}(K_k))^2(\mu_k, \mu_k)-\sup_{P\in K} f(P)\cdot\mu^{(m)}(K_k).$$

This shows that $\mu^{(m)}(K_k)$ is bounded for each k. If $T=\{\nu_\omega; \omega\in D\}$ is a subnet of $\{\mu^{(m)}\}$ which converges vaguely to some μ_0, then

$$\overline{\lim_{\omega}}\int_{K_k} gd\nu_\omega\leqq\int_{K_k} gd\mu_0$$

and hence $\mu_0 \in \mathscr{G}_K(g, x)$. Therefore

$$\inf_{\mu\in\mathscr{G}_K(g,x)} I(\mu)=\lim_{\omega} I(\nu_\omega)\geqq I(\mu_0)\geqq \inf_{\mu\in\mathscr{G}_K(g,x)} I(\mu)$$

and the equality follows.

Hereafter *we shall assume that* $g(P)$ *is positive continuous* on the set where it is defined except in § 2.9

2.4. Change of extremal values.

We raise the question how $I(\mu_x)$ and $\gamma_k(\mu_x)$ change as $f(P)$ or $g(P)$ or both change.

When we specify the function f in expression (2.1) of $I(\mu)$, we write $I_f(\mu)$.

THEOREM 2.11. *Let* $K_1, \cdots, K_n$ *be mutually disjoint compact sets such that* $\mathscr{E}_{K_k}\neq\{0\}$ *for each* k, *and* $\Phi(P, Q)$ *be a kernel which is bounded on every* $K_j\times K_k$, $j\neq k$. *Let* $f(P)$ *be a finite-valued upper semicontinuous function defined on* $K=\bigcup_{k=1}^{n} K_k$, *and* $g(P)$ *be a positive continuous function on* K. *Let* $\{f_p(P)\}$ *be a sequence of upper semicontinuous functions on* K *which tends uniformly to* $f(P)$, *and* $\{g_p(P)\}$ *be a sequence of positive continuous functions on* K *which tends uniformly to* $g(P)$. *Then, for any point* $x=(x_1, \cdots, x_n)$ *in* $x_1\geqq 0, \cdots, x_n\geqq 0$, *the minimum value of* $I_{f_p}(\mu)$ *for* $\mu\in\mathscr{E}_K(g_p, x)$ *tends to that of* $I_f(\mu)$ *for* $\mu\in\mathscr{E}_K(g, x)$ *as* $p\to\infty$. *If there is the unique extremal measure* μ *in* $\mathscr{E}_K(g, x)$, *the sequence* $\{\mu^{(p)}\}$ *consisting of the extremal measures, respectively in* $\mathscr{E}_K(g_p, x)$, *converges vaguely to* μ *and each* $x_k\gamma_k(\mu^{(p)})$ *tends to* $x_k\gamma_k(\mu)$.

PROOF. Let $\mu^{(p)}$ be any one of extremal measures in $\mathscr{E}_K(g_p, x)$ and denote $I_{f_p}(\mu^{(p)})$ simply by I_p. We denote also the minimum value of $I_f(\mu)$ for $\mu\in\mathscr{E}_K(g, x)$ by I. Since the total mass $\mu^{(p)}(K)$ is bounded, we can extract a vaguely convergent subnet $T=\{\nu_\omega;\ \omega\in D\}$ of $\{\mu^{(p)}\}$ and denote by μ^* the vague limit. It is easily seen that $\lim_\omega \int_{K_k} g^{(\omega)}d\nu_\omega=\int_{K_k} gd\mu^*$ and $\lim_\omega \int (f^{(\omega)}-f)\, d\nu_\omega=0$, where $g^{(\omega)}$ is the one of $\{g_p\}$ corresponding to ν_ω and $f^{(\omega)}$ is the corresponding one of $\{f_p\}$. It follows that $\mu^*\in\mathscr{E}_K(g, x)$ and also that

$$\overline{\lim_\omega}\,\langle f^{(\omega)}, \nu_\omega\rangle\leqq \overline{\lim_\omega}\,\langle f^{(\omega)}-f, \nu_\omega\rangle+\overline{\lim_\omega}\,\langle f, \nu_\omega\rangle\leqq\langle f, \mu^*\rangle.$$

It holds that

$$\underline{\lim_\omega}\, I_{f^{(\omega)}}(\nu_\omega)=\underline{\lim_\omega}\,\{(\nu_\omega, \nu_\omega)-2\,\langle f^{(\omega)}, \nu_\omega\rangle\}\geqq \underline{\lim_\omega}\,(\nu_\omega, \nu_\omega)-2\,\overline{\lim_\omega}\,\langle f^{(\omega)}, \nu_\omega\rangle$$
$$\geqq(\mu^*, \mu^*)-2\,\langle f, \mu^*\rangle\geqq I.$$

Let μ_x be an extremal measure in $\mathscr{E}_K(g, x)$ and define λ_ω by setting it equal to $x_k\,\mu_x^{(k)}\langle g^{(\omega)}, \mu_x^{(k)}\rangle^{-1}$ on K_k, where $\mu_x^{(k)}$ is the restriction of μ_x to K_k; if $\mu_x^{(k)}\equiv 0$, we set $\lambda_\omega(K_k)=0$. It belongs to $\mathscr{E}_K(g^{(\omega)}, x)$ and

$$\overline{\lim_{\omega}}\ I_{f^{(\omega)}}(\nu_\omega) \leqq \lim_{\omega}\ \{(\lambda_\omega, \lambda_\omega) - 2\langle f^{(\omega)}, \lambda_\omega\rangle\} = (\mu_x, \mu_x) - 2\langle f, \mu_x\rangle = I.$$

Consequently $\lim_{\omega} I_{f^{(\omega)}}(\nu_\omega)$ exists and equals I. It follows also that μ^* is an extremal measure in $\mathscr{E}_K(g, x)$. Because of the arbitrariness in choosing T, it is concluded that $\lim_{p\to\infty} I_p = I$. If there is only one extremal measuse μ in $\mathscr{E}_K(g, x)$, then $\mu^{(p)}$ converges vaguely to μ and by (2.14)

$$\overline{\lim_{p\to\infty}} \sum_{k=1}^{n} x_k\,\gamma_k(\mu^{(p)}) = \lim_{p\to\infty} I_p + \overline{\lim_{p\to\infty}}\ \langle f, \mu^{(p)}\rangle \leqq I + \langle f, \mu\rangle = \sum_{k=1}^{n} x_k\,\gamma_k(\mu).$$

On the other hand, for each k,

$$\begin{aligned}\underline{\lim}_{p\to\infty}\ x_k\,\gamma_k(\mu^{(p)}) &\geqq \underline{\lim}_{p\to\infty}\ (\mu^{(p)}, \mu_k^{(p)}) - \overline{\lim_{p\to\infty}}\ \langle f, \mu_k^{(p)}\rangle \\ &\geqq (\mu, \mu_k) - \langle f, \mu_k\rangle = x_k\,\gamma_k(\mu),\end{aligned}$$

where the subscript k indicates the restriction of measure to K_k. From these relations it follows thet $x_k\,\gamma_k(\mu^{(p)})$ tends to $x_k\,\gamma_k(\mu)$ for each k.

Remark. If all $f_p = f$, f may take $-\infty$ (not ∞) but it is required that $f(P) > -\infty$ on some subset $X_k \subset K_k$ with $\mathscr{E}_{X_k} \neq \{0\}$ for every k.

We shall assume in §§ 2.4–2.7 that $K_1, \cdots, K_n$ are mutually disjoint compact sets such that $\mathscr{E}_{K_k} \neq \{0\}$ for each k, that $\Phi(P, Q)$ is a kernel which is bounded on every $K_j \times K_k$, $j \neq k$, that $f(P) < \infty$ is an upper semicontinuous function defined on $K = \bigcup_{k=1}^{n} K_k$ such that $f(P) > -\infty$ on some subset $X_k \subset K_k$ with $\mathscr{E}_{X_k} \neq \{0\}$ for every k, and that $g(P)$ is a positive continuous function on K. Also the assumption that kernels are symmetric will not affect any generality in the following discussions and hence it will be assumed hereafter in this chapter unless otherwise stated.

We shall study the change of $I(\mu_x)$ in details when $f(P)$ and $g(P)$ are fixed but $x = (x_1, \cdots, x_n)$ changes. When $x_1 > 0, \cdots, x_n > 0$, we define $g_x(P)$ by $g(P)/x_k$ on K_k, $k = 1, \cdots, n$; then $\int_{K_k} g_x d\mu = 1$ for any $\mu \in \mathscr{E}_K(g, x)$. We can apply the preceding theorem (see its remark) and conclude that $I(\mu_x)$ is a continuous function in $x_1 > 0, \cdots, x_n > 0$. However, as any one of x_k's approaches zero, $g_x(P)$ becomes unbounded and the continuity of $I(\mu_x)$ in $x_1 \geqq 0, \cdots, x_n \geqq 0$ cannot be seen in this manner.

Before proving the continuity in $x_1 \geqq 0, \cdots, x_n \geqq 0$, we shall show that (μ_x, μ_x) and each $x_k\,\gamma_k(\mu_x)$ are bounded if x is bounded in E_n. We denote $(1, \cdots, 1)$ by e and take $\nu \in \mathscr{E}_K(g, e)$ such that $f(P)$ is bounded on S_ν. We shall write in general $x\nu$ for the measure which is equal to $x_k\,\nu_k$ on K_k, where ν_k is the restriction of ν to K_k. It holds that

$$I(\mu_x) = (\mu_x, \mu_x) - 2\langle f, \mu_x\rangle \leqq I(x\nu) = \sum_{j,k=1}^{n} x_j\,x_k(\nu_j, \nu_k) - 2\sum_{k=1}^{n} x_k\,\langle f, \nu_k\rangle.$$

The right side is a continuous function of x and hence $I(\mu_x)$ is bounded from above if $|x|=\sqrt{x_1^2+\cdots+x_n^2}<r_0$. On the other hand $\mu_x(K)\leqq\sum_{k=1}^{n} x_k\,(\min g)^{-1}$ and

$$I(\mu_x)\geqq \min_{K\times K}\,\Phi\cdot\mu_x^2(K)-2\,\max_K f\cdot\mu_x(K)$$

is bounded from below if $|x|<r_0$. Therefore

$$(\mu_x,\,\mu_x)=I(\mu_x)+2\,\langle f,\,\mu_x\rangle\leqq I(\mu_x)+2\,\max_K\,f\cdot\mu_x(K)$$

is bounded if $|x|<r_0$. We see also that $\langle f,\,\mu_x\rangle$ and each $(\mu_x^{(j)},\,\mu_x^{(k)})$ are bounded, where $\mu_x^{(k)}$ is the restriction of μ_x to K_k. It follows that $\sum_{k=1}^{n} x_k\,\gamma_k(\mu_x)$ $=(\mu_x,\,\mu_x)+I(\mu_x)$ is bounded and that each $x_k\,\gamma_k(\mu_x)$ is bounded because $x_k\,\gamma_k(\mu_x)$ $=(\mu_x,\,\mu_x^{(k)})-\langle f,\,\mu_x^{(k)}\rangle$ is bounded from below. One sees that each $\langle f,\,\mu_x^{(k)}\rangle$ is bounded too.

Let $\nu\in\mathscr{E}_K(g,\,e)$, $x_1\geqq 0,\,\cdots,\,x_n\geqq 0$, and $x=(x_1,\,\cdots,\,x_n)$. Then $x\nu\in\mathscr{E}_K(g,\,x)$, and we have

$$I(\mu_x)\leqq I(x\nu)=(x\nu,\,x\nu)-2\int fd(x\nu)=\sum_{j,\,k=1}^{n} x_j\,x_k(\nu_j,\,\nu_k)-2\sum_{k=1}^{n} x_k\,\langle f,\,\nu_k\rangle. \tag{2.16}$$

Let us denote by $P(\nu)$ the branch of the parabolic quadratic surface in $x_1\geqq 0$, $\cdots,\,x_n\geqq 0$, expressed by the right side. For $\xi=(\xi_1,\,\cdots,\,\xi_n)$, $\xi_1>0,\,\cdots,\,\xi_n>0$, and an extremal measure μ_ξ, we define $\nu_\xi\in\mathscr{E}_K(g,\,e)$ by setting its restriction to K_k equal to the restriction of μ_ξ/ξ_k to K_k. The surface $P(\nu_\xi)$ touches the surface $I(\mu_x)$ at $x=\xi$ and $I(\mu_x)\leqq P(\nu_\xi)$ in $x_1\geqq 0,\,\cdots,\,x_n\geqq 0$. If $\xi_k>0$ we define ν_ξ by $\mu_\xi^{(k)}/\xi_k$ on K_k, and by any measure $\nu_\xi^{(k)}\in\mathscr{E}_{K_k}(g,\,1)$ such that $\langle f,\,\nu_\xi^{(k)}\rangle$ is finite if $\xi_k=0$. Similar fact is true for $P(\nu_\xi)$ in this case. We denote by Π the family $\{P(\nu_\xi);\,0\leqq\xi_1<\infty,\,\cdots,\,0\leqq\xi_n<\infty\}$; we note that $P(\nu_\xi)$ is not uniquely determined by μ_ξ if some of $\{\xi_k\}$ vanish. We can state

THEOREM 2.12. *$I(\mu_x)$ is the lower envelope*[24] *of Π on $x_1\geqq 0,\,\cdots,\,x_n\geqq 0$.*

Let us prove the continuity of $I(\mu_x)$ in $x_1\geqq 0,\,\cdots,\,x_n\geqq 0$. As a lower envelope of continuous functions, $I(\mu_x)$ is an upper semicontinuous function there. Let $x_0=(x_1^{(0)},\,\cdots,\,x_n^{(0)})$ be any point with nonnegative coordinates and $\{x^{(p)}\}=\{(x_1^{(p)},\,\cdots,\,x_n^{(p)})\}$ be a sequence of points with nonnegative coordinates tending to x_0. Since $(\mu_x^{(j)},\,\mu_x^{(k)})$ and $\langle f,\,\mu_x^{(k)}\rangle$ are bounded for bounded x,

$$I(\mu_{x_0})\leqq \varliminf_{p\to\infty}\,I(x_0\,\nu_x^{(p)})=\varliminf_{p\to\infty}\left\{\sum_{j,\,k}{}'\,\frac{x_j^{(0)}\,x_k^{(0)}}{x_j^{(p)}\,x_k^{(p)}}\,(\mu_x^{(j)},\,\mu_{x^{(p)}}^{(k)})-2\sum{}''\,\frac{x_k^{(0)}}{x_k^{(p)}}\langle f,\,\mu_{x^{(p)}}^{(k)}\rangle\right\},$$

where the superscript k indicates a restriction to K_k and the summations $\sum'$, $\sum''$ are taken over j, k for which $x_j^{(0)},\,x_k^{(0)}>0$. The last side is equal to $\varliminf_{p\to\infty}$

24) This is defined at each point by the infimum of the values of the functions of Π.

$I(\mu_{x^{(p)}})$ if every $x_k^{(0)}>0$, and the theorem is proved. We shall show that in general the last side is not larger than $\underline{\lim}_{p\to\infty} I(\mu_{x^{(p)}})$; this will complete the proof. For that purpose it is sufficient to show that

$$0\leqq \underline{\lim}_{p\to\infty}\left\{I(\mu_{x^{(p)}})-\sum_{j,k}{}' \frac{x_j^{(0)}x_k^{(0)}}{x_j^{(p)}x_k^{(p)}}(\mu_{x^{(p)}}^{(j)},\mu_{x^{(p)}}^{(k)})+2\sum_k{}'' \frac{x_k^{(0)}}{x_k^{(p)}}\langle f,\mu_{x^{(p)}}^{(k)}\rangle\right\}$$

$$=\underline{\lim}_{p\to\infty}\left\{\sum_k{}''' (\mu_{x^{(p)}},\mu_{x^{(p)}}^{(k)})-2\sum_k{}''' \langle f,\mu_{x^{(p)}}^{(k)}\rangle\right\},$$

where $\sum'''$ is taken over k for which $x_k^{(0)}=0$. Since the total mass of $\mu_{x^{(p)}}^{(k)}$ tends to zero it converges vaguely to zero. Hence

$$\overline{\lim_{p\to\infty}}\langle f,\mu_{x^{(p)}}^{(k)}\rangle\leqq\langle f,0\rangle=0$$

and

$$\underline{\lim}_{p\to\infty}(\mu_{x^{(p)}}^{(k)},\mu_{x^{(p)}}^{(k)})\geqq 0.$$

Since the kernel is bounded on any $K_j\times K_k$, $j\neq k$, $\lim_{p\to\infty}(\mu_{x^{(p)}}^{(j)},\mu_{x^{(p)}}^{(k)})=0$ for any $j\neq k$. Hence

$$\underline{\lim}_{p\to\infty}\left\{\sum_k{}'''\sum_{j=1}^{n}(\mu_{x^{(p)}}^{(j)},\mu_{x^{(p)}}^{(k)})-2\sum_k{}'''\langle f,\mu_{x^{(p)}}^{(k)}\rangle\right\}\geqq 0.$$

THEOREM 2.13. *$I(\mu_x)$ is continuous on $x_1\geqq 0,\ \cdots,\ x_n\geqq 0$.*

We shall use Theorem 2.12 to obtain further properties of the graph of $I(\mu_x)$. Let A be a compact set in $x_1>0,\ \cdots,\ x_n>0$. We have seen that $(\mu_\xi^{(k)},\mu_\xi^{(k)})$ and $\langle f,\mu_\xi^{(k)}\rangle$ are bounded if $|\xi|$ is bounded. Hence $(\nu_\xi^{(k)},\nu_\xi^{(k)})$ and $\langle f,\nu_\xi^{(k)}\rangle$ are bounded on A. Therefore for a large c_A and for any $\xi\in A$,

$$P(\nu_\xi)-c_A(x_1^2+\cdots+x_n^2)$$

is concave as a function of $(x_1,\ \cdots,\ x_n)\in A$.[25] Its lower envelope $I(\mu_x)-c_A(x_1^2+\cdots+x_n^2)$ is concave there. It has a directional derivative at each inner point of A and it is totally differentiable a.e. in A. It follows that these facts are true for $I(\mu_x)$ everywhere in $x_1>0,\cdots,\ x_n>0$. We shall compute a directional derivative explicitly in terms of $\{\gamma_k(\mu)\}$.

First we prove

LEMMA 2.1. *Let $T=\{\mu^{(\omega)};\ \omega\in D\}$ be a net consisting of extremal measures at some points and converging vaguely to μ^*. If $x_k^{(\omega)}=\langle g,\mu_k^{(\omega)}\rangle$ as a function on D converges to $x_k\geqq 0$ for each k, then μ^* is an extremal measure for x. It also holds that $\lim_\omega(\mu^{(\omega)},\mu^{(\omega)})=(\mu^*,\mu^*)$, $\lim_\omega\langle f,\mu^{(\omega)}\rangle=\langle f,\mu^*\rangle$ and $\lim_\omega x_k^{(\omega)}\gamma_k(\mu^{(\omega)})=x_k\gamma_k(\mu^*)$ for each k.*

25) This was suggested by Ogasawara.

PROOF. By assumption

$$x_k^{(\omega)}=\int g d\mu_k^{(\omega)} \to \int g d\mu_k^* = x_k \qquad \text{as } T \ni \mu^{(\omega)} \to \mu^*,$$

and $\mu^* \in \mathscr{E}_K(g, x)$. We take any extremal measure μ_x at x. On account of the continuity of $I(\mu_x)$ in $x_1 \geqq 0, \cdots, x_n \geqq 0$, we have

$$\begin{aligned} I(\mu_x) = \lim_\omega I(\mu^{(\omega)}) &\geqq \varliminf_\omega (\mu^{(\omega)}, \mu^{(\omega)}) - 2 \varlimsup_\omega \langle f, \mu^{(\omega)} \rangle \\ &\geqq (\mu^*, \mu^*) - 2 \langle f, \mu^* \rangle = I(\mu^*) \geqq I(\mu_x). \end{aligned}$$

Thus $I(\mu^*) = I(\mu_x)$ and it is shown that $\mu^* \in \mathscr{M}_x^*$. We see also that $x_k^{(\omega)} \gamma_k(\mu^{(\omega)}) \to x_k \gamma_k(\mu^*)$ for each k as in the proof of Theorem 2.11. Hence

$$\begin{aligned} \lim_\omega \langle f, \mu^{(\omega)} \rangle &= \lim_\omega \left\{ \sum_{k=1}^n x_k^{(\omega)} \gamma_k(\mu^{(\omega)}) - I(\mu^{(\omega)}) \right\} \\ &= \sum_{k=1}^n x_k \gamma_k(\mu^*) - I(\mu^*) = \langle f, \mu^* \rangle \end{aligned}$$

and

$$\lim_\omega (\mu^{(\omega)}, \mu^{(\omega)}) = \lim_\omega \left\{ 2 \sum_{k=1}^n x_k^{(\omega)} \gamma_k(\mu^{(\omega)}) - I(\mu^{(\omega)}) \right\} = (\mu^*, \mu^*).$$

Our lemma is established.

Let $x_k^{(0)} > 0$ for $k=1, \cdots, n$, and A be a closed ball in $x_1 > 0, \cdots, x_n > 0$ with center at $x_0 = (x_1^{(0)}, \cdots, x_n^{(0)})$. In general we denote $\sqrt{x_1^2 + \cdots + x_n^2}$ by $|x|$ for $x = (x_1, \cdots, x_n)$, not necessarily in $x_1 \geqq 0, \cdots, x_n \geqq 0$, the point $(x_1 - x_1^{(0)}, \cdots, x_n - x_n^{(0)})$ by $x - x_0$ and the half line issuing from x_0 and passing x by l_x. We have observed that $I(\mu_x)$ is the lower envelope of $\{P(\nu_\xi)\}$ and that $J_A(x) = I(\mu_x) - c_A (x_1^2 + \cdots + x_n^2) = I(\mu_x) - c_A |x|^2$ is concave on A for a large c_A. For a point $\xi = (\xi_1, \cdots, \xi_n) \in A$ different from x_0, the graph of $P(\nu_\xi) - c_A(x_1^2 + \cdots + x_n^2)$ touches the graph of $J_A(x)$. Therefore the derivative of $P(\nu_\xi) - c_A(x_1^2 + \cdots + x_n^2)$ at ξ along l_ξ is not larger than the derivative of $J_A(x)$ at x_0 along l_ξ. The former is equal to

$$\begin{aligned} &2 \sum_{k=1}^n \frac{(\mu_\xi, \mu_\xi^{(k)})}{\xi_k} \eta_k - 2 \sum_{k=1}^n \frac{\langle f, \mu_\xi^{(k)} \rangle}{\xi_k} \eta_k - 2c_A \sum_{k=1}^n \xi_k \eta_k \\ &= 2 \sum_{k=1}^n \frac{\eta_k}{\xi_k} ((\mu_\xi, \mu_\xi^{(k)}) - \langle f, \mu_\xi^{(k)} \rangle) - 2c_A \sum_{k=1}^n \xi_k \eta_k = 2 \sum_{k=1}^n \gamma_k(\mu_\xi) \eta_k - 2c_A \sum_{k=1}^n \xi_k \eta_k, \end{aligned}$$

where $\eta = (\eta_1, \cdots, \eta_n)$ is determined by $\xi - x_0 = |\xi - x_0| \eta$. We set $\gamma(\mu_x) = (\gamma_1(\mu_x), \cdots, \gamma_n(\mu_x))$ and proved that $\Gamma_x = \{\gamma(\mu_x); \mu_x \in \mathscr{M}_x^*\}$ is a compact set in E_n in Theorem 2.9. Hence for any $y_1, \cdots, y_n$ there are a measure of $\mathscr{M}_x^*$ which attains $\min\limits_{\mu \in \mathscr{M}_x^*} \sum_{k=1}^n \gamma_k(\mu) y_k$ and a measure of $\mathscr{M}_x^*$ which attains $\max\limits_{\mu \in \mathscr{M}_x^*} \sum_{k=1}^n \gamma_k(\mu) y_k$. We denote these values by $\underline{\gamma}(x, y)$ and $\overline{\gamma}(x, y)$ respectively; in the case of one dimensional problem we write $\underline{\gamma}(x)$ and $\overline{\gamma}(x)$ simply. If we denote the derivative of $I(\mu_x)$ along l_ξ by $dI(\mu_x)/dl_\xi$, we have

$$
\begin{aligned}
(2.17)\qquad & 2\underline{\gamma}(\xi, \eta)-2c_A\sum_{k=1}^{n}\xi_k\eta_k \leqq 2\overline{\gamma}(\xi, \eta)-2c_A\sum_{k=1}^{n}\xi_k\eta_k \\
& \leqq \frac{I(\mu_\xi)-I(\mu_{x_0})}{|\xi-x_0|}-c_A\frac{|\xi|^2-|x_0|^2}{|\xi-x_0|} \leqq \frac{dI(\mu_x)}{dl_\xi}\bigg|_{x=x_0}-2c_A\sum_{k=1}^{n}x_k^{(0)}\eta_k \\
& \leqq 2\underline{\gamma}(x_0, \eta)-2c_A\sum_{k=1}^{n}x_k^{(0)}\eta_k.
\end{aligned}
$$

Let ξ approach x_0 such that $(\xi_k-x_k^{(0)})/|\xi-x_0|$ tends to a certain limit y_k. Naturally $y_1^2+\cdots+y_n^2=1$. By Lemma 2.1, for any subnet $\{\mu^{(\omega)}\}$ of $\{\mu_\xi\}$ converging vaguely to a measure $\mu^*\in\mathscr{M}_{x_0}^*$, each $\gamma_k(\mu^{(\omega)})$ tends to $\gamma_k(\mu^*)$. We obtain from (2.17)

$$(2.18)\qquad \lim_{\omega}\sum_{k=1}^{n}\gamma_k(\mu^{(\omega)})\frac{\xi_k^{(\omega)}-x_k^{(0)}}{|\xi^{(\omega)}-x_0|}=\sum_{k=1}^{n}\gamma_k(\mu^*)y_k\leqq\underline{\gamma}(x_0, y),$$

where $\xi^{(\omega)}$ is determined by $\mu_{\xi^{(\omega)}}=\mu^{(\omega)}$. We have the equality in the last inequality, and in view of the arbitrariness of $\{\mu^{(\omega)}\}$, we see that

$$\lim_{\xi\to x_0}\sum_{k=1}^{n}\gamma_k(\mu_\xi)\frac{\xi_k-x_k^{(0)}}{|\xi-x_0|}=\underline{\gamma}(x_0, y),$$

whatever the values $\{\gamma_k(\mu_\xi)\}$ may be at ξ.

We shall show that $\underline{\gamma}(x_0, y)$ is continuous with respect to y. Let $y^{(p)}=(y_1^{(p)}, \cdots, y_n^{(p)})$ with $|y^{(p)}|=1$ tend to y. We can choose $x^{(p)}=(x_1^{(p)}, \cdots, x_n^{(p)})$ such that $|x^{(p)}-x_0|<1/p$, $x^{(p)}-x_0=|x^{(p)}-x_0|y^{(p)}$ and

$$|\underline{\gamma}(x^{(p)}, y^{(p)})-\underline{\gamma}(x_0, y^{(p)})|<\frac{1}{p}.$$

Since $\underline{\gamma}(x^{(p)}, y^{(p)})$ tends to $\underline{\gamma}(x_0, y)$, it follows that $\underline{\gamma}(x_0, y^{(p)})$ tends to $\underline{\gamma}(x_0, y)$. Next let us see that there is ε_r tending to zero with r such that

$$|\underline{\gamma}(\xi, y)-\underline{\gamma}(x_0, y)|<\varepsilon_{|\xi-x_0|}$$

for any $\xi\neq x_0$, where $\xi-x_0=|\xi-x_0|y$. In view of (2.17) we assume, to the contrary, that there is a sequence $\{\xi^{(p)}\}$ tending to x_0 such that $y^{(p)}=(\xi^{(p)}-x_0)/|\xi^{(p)}-x_0|$ tends to a certain limit y_0 and

$$(2.19)\qquad \lim_{p\to\infty}\{\underline{\gamma}(\xi^{(p)}, y^{(p)})-\underline{\gamma}(x_0, y^{(p)})\}=\lim_{p\to\infty}\underline{\gamma}(\xi^{(p)}, y^{(p)})-\gamma(x_0, y_0)$$

exists and is negative, where we use the fact that $\underline{\gamma}(x_0, y)$ is continuous with respect to y. Let $\mu^{(p)}\in\mathscr{M}_{\xi^{(p)}}^*$ such that $\sum_{k=1}^{n}\gamma_k(\mu^{(p)})y_k^{(p)}=\underline{\gamma}(\xi^{(p)}, y^{(p)})$ and choose a subnet $\{\mu^{(\omega)}\}$ of $\{\mu^{(p)}\}$ which converges vaguely to a measure $\mu^*\in\mathscr{M}_{x_0}^*$. By

Lemma 2.1 each $\gamma_k(\mu^{(\omega)})$ tends to $\gamma_k(\mu^*)$. Therefore

$$\lim_{p\to\infty} \underline{\gamma}(\xi^{(p)}, y^{(p)}) \geqq \underline{\gamma}(x_0, y_0).$$

This contradicts our assumption that the value of (2.19) is negative. On account of (2.17) again we have

$$I(\mu_\xi)-I(\mu_{x_0})=2\underline{\gamma}\left(\xi, \frac{\xi-x_0}{|\xi-x_0|}\right)|\xi-x_0|+\varepsilon, \ |\varepsilon|<\varepsilon_{|\xi-x_0|},$$

with ε_r tending to 0 with r. From (2.17) follows also

$$\lim_{\xi\to x_0} \frac{I(\mu_\xi)-I(\mu_{x_0})}{|\xi-x_0|}=2\lim_{\xi\to x_0} \underline{\gamma}\left(\xi, \frac{\xi-x_0}{|\xi-x_0|}\right)=2\lim_{\xi\to x_0} \bar{\gamma}\left(\xi, \frac{\xi-x_0}{|\xi-x_0|}\right)=2\underline{\gamma}(x_0, y),$$

where ξ approaches x_0 along a curve having a tangent at x_0 whose direction is determined by a unit vector y.

Changing some notations we state

THEOREM 2.14. *As a function of y, $\underline{\gamma}(x, y)$ is continuous on $|y|=1$ for any x in $x_1>0, \cdots, x_n>0$, and with any x' in $x_1>0, \cdots, x_n>0$ we have*

$$I(\mu_{x'})-I(\mu_x)=2\underline{\gamma}\left(x', \frac{x'-x}{|x'-x|}\right)|x'-x|+\varepsilon, \tag{2.20}$$

where $|\varepsilon|<\varepsilon_{|x'-x|}$ and ε_r tends to 0 with r. Let c_y be a curve terminating at x and having a tangent at x whose direction is determined by a unit vector y. As x' approaches x along c_y,

$$\lim \frac{I(\mu_{x'})-I(\mu_x)}{|x'-x|}=2\lim \underline{\gamma}\left(x', \frac{x'-x}{|x'-x|}\right)=2\lim \bar{\gamma}\left(x', \frac{x'-x}{|x'-x|}\right)=2\underline{\gamma}(x, y).$$

COROLLARY 1. *$I(\mu_x)$ is totally differentiable at x if and only if Γ_x consists of one point.*

COROLLARY 2. *$I(\mu_x)$ is continuously differentiable in $x_1>0, \cdots, x_n>0$ and $\partial I(\mu_x)/\partial x_k$ is equal to $2\gamma_k(\mu_x)$ if and only if $\{\gamma_k(\mu_x)\}$ are uniquely determined in $x_1>0, \cdots, x_n>0$.*

COROLLARY 3. *If $\{\gamma_k(\mu_x)\}$ are uniquely determined at a point x in $x_1>0, \cdots, x_n>0$, then each $\gamma_k(\mu_x)$ is continuous at x in the sense that each $\gamma_k(\mu_{x'})$ is close to $\gamma_k(\mu_x)$ for x' near x.*

We have observed before that $I(\mu_x)$ is totally differentiable a.e. in $x_1>0, \cdots, x_n>0$. This follows also from our theorem by the aid of a theorem of Rademacher [1].

Next we shall compute the derivative of $I(\mu_x)$ and see the behavior of $\gamma(\mu_x)$ on the boundary of $x_1\geqq 0, \cdots, x_n\geqq 0$. Take $x=(x_1, \cdots, x_m, 0, \cdots, 0)$ with $x_k>0$ for k, $1\leqq k\leqq m<n$, and denote by l the half line in $x_1>0, \cdots, x_n>0$, issuing from x and having the direction determined by a unit vector with components $y_1, \cdots, y_n$. We include the case $x=(0, \cdots, 0)$. For x' on l and $\mu_{x'}$ we denote by $\bar{\mu}_{x'}$ and $\bar{\bar{\mu}}_{x'}$ the restrictions of $\mu_{x'}$ to $\bigcup_{k=1}^{m} K_k$ and $\bigcup_{k=m+1}^{n} K_k$ respectively; these restrictions may be regarded as measures on K. We set $\Delta x_k = x'_k - x_k$ and $\bar{\bar{y}}=(0, \cdots, 0, y_{m+1}, \cdots, y_n)$. We shall prove that $(\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'})/|\Delta x|$ tends to 0 as x' approaches x along l. Assume that $\bar{\mu}_{x'}$ converges vaguely to $\bar{\mu}$ and $\bar{\bar{\mu}}_{x'}/|\Delta x|$ does to $\bar{\lambda}$; otherwise we take subnets and proceed in a similar manner. Take any $\mu_x \in \mathscr{M}_x^*$ and $\lambda \in \mathscr{E}_K(g, \bar{\bar{y}})$. We denote the restriction of μ_x to K_k by $\mu_x^{(k)}$ and set

$$\frac{x'}{x}\mu_x = \sum_{k=1}^{m} \frac{x'_k}{x_k}\mu_x^{(k)}.$$

It holds that

$$I(\mu_{x'}) \leqq I\left(\frac{x'}{x}\mu_x + |\Delta x|\lambda\right) \tag{2.21}$$

$$= I\left(\frac{x'}{x}\mu_x\right) + 2\left(\frac{x'}{x}\mu_x, \lambda\right)|\Delta x| + (\lambda, \lambda)|\Delta x|^2 - 2\langle f, \lambda\rangle|\Delta x|$$

$$= I(\mu_x) + 2\sum_{k=1}^{m}\gamma_k(\mu_x)\Delta x_k + 2\langle U^{\mu_x} - f, \lambda\rangle|\Delta x| + 2\left(\frac{x'}{x}\mu_x - \mu_x, \lambda\right)|\Delta x| + O(|\Delta x|^2).$$

We observe also that

$$\left(\frac{x'}{x}\mu_x - \mu_x, \lambda\right)|\Delta x| = O(|\Delta x|^2).$$

On the other hand, setting

$$\frac{x}{x'}\mu_{x'} = \sum_{k=1}^{n} \frac{x_k}{x'_k}\mu_{x'}^{(k)},$$

we see that

$$I(\mu_x) \leqq I\left(\frac{x}{x'}\mu_{x'}\right)$$

$$= I(\bar{\mu}_{x'}) - 2\sum_{k=1}^{m}\gamma_k(\bar{\mu}_{x'})\Delta x_k + \sum_{j,k=1}^{m} \frac{\Delta x_j\, \Delta x_k}{x'_j\, x'_k}(\mu_{x'}^{(j)}, \mu_{x'}^{(k)})$$

where $x'_k\gamma_k(\bar{\mu}_{x'}) = (\bar{\mu}_{x'}, \mu_{x'}^{(k)}) - \langle f, \mu_{x'}^{(k)}\rangle$, $k=1, \cdots, m$, and hence

$$I(\mu_{x'}) = I(\bar{\mu}_{x'}) + 2(\bar{\mu}_{x'}, \bar{\bar{\mu}}_{x'}) + I(\bar{\bar{\mu}}_{x'}) \tag{2.22}$$

$$\geqq I(\mu_x) + 2\sum_{k=1}^{m}\gamma_k(\bar{\mu}_{x'})\Delta x_k + 2\langle U^{\bar{\mu}_{x'}} - f, \bar{\bar{\mu}}_{x'}\rangle + (\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'}) + O(|\Delta x|^2).$$

From this and (2.21) follows

$$2\sum_{k=1}^{m}\gamma_k(\mu_x)y_k + 2\langle U^{\mu_x} - f, \lambda\rangle \tag{2.23}$$

$$\geqq 2 \sum_{k=1}^{m} \gamma_k(\bar{\mu}_{x'})y_k + 2 \langle U^{\bar{\mu}_{x'}} - f, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|} \rangle + \frac{1}{|\Delta x|} (\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'}) + O(|\Delta x|).$$

Now we set

$$\inf_{\mu \in \mathscr{M}_x^*,\ \lambda \in \mathscr{E}_{K}(g, \bar{y})} \left\{ \sum_{k=1}^{m} \gamma_k(\mu)y_k + \langle U^{\mu} - f, \lambda \rangle \right\} = m_y.$$

Given $\varepsilon > 0$, we choose μ_x and λ so that the left side of (2.23) is smaller than $2m_y + \varepsilon$ and obtain

$$(2.24) \qquad 2m_y + \varepsilon \geqq 2m_y + 2 \sum_{k=1}^{m} \left\{ \gamma_k(\bar{\mu}_{x'}) - \gamma_k(\bar{\mu}) \right\} y_k + 2 \left(\bar{\mu}_{x'} - \bar{\mu}, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|} \right) + \frac{1}{|\Delta x|} (\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'}) + O(|\Delta x|),$$

where we use the fact that $\bar{\mu} \in \mathscr{M}_x^*$ which follows by Lemma 2.1. Since

$$\underline{\lim}_{\Delta x \to 0} \gamma_k(\bar{\mu}_{x'}) = \underline{\lim}_{\Delta x \to 0} \frac{1}{x'_k} \langle U^{\bar{\mu}_{x'}} - f, \bar{\mu}_{x'}^{(k)} \rangle \geqq \frac{1}{x_k} \langle U^{\bar{\mu}} - f, \bar{\mu}^{(k)} \rangle = \gamma_k(\bar{\mu})$$

for each $k \leqq m$, it follows that

$$2 \underline{\lim}_{\Delta x \to 0} \left(\bar{\mu}_{x'} - \bar{\mu}, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|} \right) + \overline{\lim}_{\Delta x \to 0} \frac{1}{|\Delta x|} (\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'}) \leqq \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, the left side is not positive. Now we make the following assumption:[26)]

(*) Whenever the potential of a measure μ of $\mathscr{E}_{\bigcup_{k=1}^{n} K_k}$ is continuous as a function on S_μ, it is continuous on $\bigcup_{k=m+1}^{n} K_k$.

This is naturally satisfied if the continuity principle is true or the kernel is continuous outside the diagonal set. For given $\varepsilon > 0$, there is by Lusin's theorem a compact set $F \subset S_{\bar{\mu}}$ such that $\bar{\mu}(K - F) < \varepsilon$ and the restriction of $U^{\bar{\mu}}(P)$ to F is continuous. We denote by $\bar{\mu}_F$ the restriction of $\bar{\mu}$ to F. The restriction of $U^{\bar{\mu}_F}(P)$ to F is continuous and hence $U^{\bar{\mu}_F}(P)$ is continuous on $\bigcup_{k=m+1}^{n} K_k$ by our assumption (*). Hence

$$\lim_{\Delta x \to 0} \left(\bar{\mu}_F, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|} \right) = (\bar{\mu}_F, \bar{\lambda}).$$

Since

$$\left(\bar{\mu} - \bar{\mu}_F, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|} \right) < \varepsilon \sup_{\bigcup_{k=1}^{m} K_k \times \bigcup_{k=m+1}^{n} K_k} \Phi \cdot \frac{\bar{\bar{\mu}}_{x'}(K)}{|\Delta x|} < \varepsilon \text{ const.},$$

26) It is an open question whether one can prove (2.25) without condition (*). In case $m=0$, namely at the origin we have no such condition.

we have that

$$\lim_{\Delta x\to 0}\left(\bar{\mu}, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\right)=(\bar{\mu}, \lambda).$$

Consequently

$$\varliminf_{\Delta x\to 0}\left(\bar{\mu}_{x'}-\bar{\mu}, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\right)\geqq(\bar{\mu}, \lambda)-(\bar{\mu}, \lambda)=0.^{27)}$$

Therefore

$$\varlimsup_{\Delta x\to 0}\frac{1}{|\Delta x|}(\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'})\leqq 0.$$

We can conclude

(2.25)
$$\varliminf_{\Delta x\to 0}\frac{1}{|\Delta x|}(\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'})=0$$

because

$$\varliminf_{\Delta x\to 0}\frac{1}{|\Delta x|}(\bar{\bar{\mu}}_{x'}, \bar{\bar{\mu}}_{x'})\geqq\varliminf_{\Delta x\to 0}(-|\Delta x|\,|\inf_{K\times K}\Phi|)\geqq 0.$$

We infer by (2.23), (2.24) and (2. 25) that

$$\varliminf_{\Delta x\to 0}\left\{\sum_{k=1}^{m}\gamma_k(\bar{\mu}_{x'})y_k+\langle U^{\bar{\mu}_{x'}}-f, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\rangle\right\}=m_y.$$

Let us evaluate the difference

$$d(x')=\sum_{k=1}^{n}\gamma_k(\mu_{x'})y_k-\left\{\sum_{k=1}^{m}\gamma_k(\bar{\mu}_{x'})y_k+\langle U^{\bar{\mu}_{x'}}-f, \frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\rangle\right\}.$$

We recall that the kernel is bounded on $\bigcup_{k=1}^{m}K_k\times\bigcup_{k=m+1}^{n}K_k$. As $x'\to x$, $\bar{\bar{\mu}}_{x'}(K)\to 0$ and hence

$$\sum_{k=1}^{m}\left\{\gamma_k(\mu_{x'})y_k-\gamma_k(\bar{\mu}_{x'})y_k\right\}=\sum_{k=1}^{m}\frac{y_k}{x'_k}(\bar{\bar{\mu}}_{x'}, \bar{\mu}_{x'}^{(k)})=(\bar{\bar{\mu}}_{x'}, \sum_{k=1}^{m}\frac{y_k}{x'_k}\bar{\mu}_{x'}^{(k)})\to 0.$$

---

27) In general cases it can happen that $\{\mu^{(p)}\}$ supported by a compact set K_1 converges vaguely to a measure μ_0, $\{\nu^{(p)}\}$ supported by a disjoint compact set K_2 converges vaguely to ν_0 and

$$\lim_{p\to\infty}(\mu^{(p)}-\mu_0, \nu^{(p)})<0.$$

For instance, take $\{1/p\}\cup\{0\}$ for K_1 and $\{-1+1/p\}\cup\{-1\}$ for K_2, and consider $K=K_1\cup K_2$ as a subspace of the real line. We set $\Phi(1/n, -1+1/p)=\Phi(-1+1/p, 1/n)=\Phi(0, -1+1/p)=\Phi(-1+1/p, 0)=1$ $(n=p+1, \cdots)$ for each p. For other points in $K\times K$ we set $\Phi=0$. For the unit measure $\mu^{(p)}$ ($\nu^{(p)}$ resp.) at $1/p$ ($-1+1/p$ resp.) and the unit measure μ_0(ν_0 resp.) at 0 (-1 resp.) it holds that

$$\lim_{p\to\infty}(\mu^{(p)}-\mu_0, \nu^{(p)})=-1.$$

Since

$$\sum_{k=m+1}^{n}\gamma_k(\mu_{x'})y_k-\langle U^{\bar{\mu}_{x'}}-f,\frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\rangle=\langle U^{\mu_{x'}}-f,\frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\rangle-\langle U^{\bar{\mu}_{x'}}-f,\frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\rangle$$
$$=\frac{\langle\bar{\bar{\mu}}_{x'},\bar{\bar{\mu}}_{x'}\rangle}{|\Delta x|}\to 0,$$

we conclude that $d(x')\to 0$ as $x'\to x$. It follows that $\lim_{x'\to x}\sum_{k=1}^{n}\gamma_k(\mu_{x'})y_k=m_y$. We obtain from (2.21) and (2.22)

$$2\sum_{k=1}^{m}\gamma_k(\mu_x)y_k+2\langle U^{\mu_x}-f,\lambda\rangle\geqq\frac{I(\mu_{x'})-I(\mu_x)}{|\Delta x|}$$
$$\geqq 2\sum_{k=1}^{m}\gamma_k(\bar{\mu}_{x'})y_k+2\langle U^{\bar{\mu}_{x'}}-f,\frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\rangle+\frac{1}{|\Delta x|}(\bar{\bar{\mu}}_{x'},\bar{\bar{\mu}}_{x'})+O(|\Delta x|).$$

Consequently

$$\lim_{\Delta x\to 0}\frac{I(\mu_{x'})-I(\mu_x)}{|\Delta x|}=2\lim_{x'\to x}\sum_{k=1}^{n}\gamma_k(\mu_{x'})y_k \tag{2.26}$$
$$=2\lim_{\Delta x\to 0}\cdot\left\{\sum_{k=1}^{m}\gamma_k(\bar{\mu}_{x'})y_k+\langle U^{\bar{\mu}_{x'}}-f,\frac{\bar{\bar{\mu}}_{x'}}{|\Delta x|}\rangle\right\}$$
$$=2\inf_{\mu\in\mathscr{M}_x^*,\,\lambda\in\mathscr{E}_K(g,\bar{y})}\left\{\sum_{k=1}^{m}\gamma_k(\mu)y_k+\langle U^{\mu}-f,\lambda\rangle\right\}.$$

We state

THEOREM 2.15. *Let $x=(x_1,\cdots,x_m,0,\cdots,0)$, where $x_k>0$ for k, $1\leqq k\leqq m$, and denote by l the half line in $x_1>0,\cdots,x_n>0$, issuing from x. Let the direction of l be expressed by a point $y=(y_1,\cdots,y_n)$ with $|y|=1$. For a point $x'=(x_1',\cdots,x_n')$ on l we set $\Delta x_k=x_k'-x_k$ and take any $\mu_{x'}\in\mathscr{M}_{x'}^*$. Let $\bar{\mu}_{x'},\bar{\bar{\mu}}_{x'}$ be its respective restrictions to $\bigcup_{k=1}^{m}K_k$ and $\bigcup_{k=m+1}^{n}K_k$. Then we have* (2.26) *under condition* (*).

We remark that the limit along l, which lies in $x_1\geqq 0,\cdots,x_n\geqq 0$, can be computed by considering a lower dimensional problem.

QUESTION. *Do we have a relation similar to* (2.20) *at a point on the boundary of $x_1\geqq 0,\cdots,x_n\geqq 0$?*

We have called a problem to minimize $I(\mu)$ for $\mu\in\mathscr{E}_K(g,x)$ n-dimensional when K consists of $K_1,\cdots,K_n$ and $x=(x_1,\cdots,x_n)$. In order to make it clear we shall write $I_n(\mu)$ in the following paragraph. Let us consider the question how the problem to minimize $I_n(\mu)$ for $\mu\in\mathscr{E}_K(g,x)$ and the problem to minimize $I_1(\mu)$ for $\mu\in\mathscr{E}_K(g,\sum_{k=1}^{n}x_k)$ are related to each other.[28]

28) This question was raised by Ogasawara.

We take $a>0$. Since $I_n(\mu_x)$ is a continuous function of $x=(x_1, \cdots, x_n)$, it holds at some point $x_0=(x_1^{(0)}, \cdots, x_n^{(0)})$ with $\sum_{k=1}^{n} x_k^{(0)}=a$ that

$$I_1(\mu_a)=I_n(\mu_{x_0})=\min_{\sum_{k=1}^{n} x_k=a} I_n(\mu_x).$$

Since $\sum_{k=1}^{n} x_k^{(0)}=a>0$, at least one of $\{x_k^{(0)}\}$ is positive; we assume $x_n^{(0)}>0$. For a sufficiently small $t>0$, the measure, defined by $(1+t)\mu_{x_0}^{(k_0)}$ on K_{k_0}, by $(1-tx_{k_0}^{(0)}/x_n^{(0)})\mu_{x_0}^{(n)}$ on K_n and by $\mu_{x_0}^{(k)}$ on K_k for $k=1, \cdots, k_0-1, k_0+1, \cdots, n-1$, belongs to $\mathscr{E}_K(g, a)$. Hence

$$I_1(\mu_a)\leqq I_1\Big(\sum_{\substack{k=1\\ k\neq k_0}}^{n-1}\mu_{x_0}^{(k)}+(1+t)\,\mu_{x_0}^{(k_0)}+\Big(1-t\frac{x_{k_0}^{(0)}}{x_n^{(0)}}\Big)\mu_{x_0}^{(n)}\Big)$$

and

$$0\leqq\frac{1}{2}\frac{dI}{dt}\Big|_{t=0}=\langle U^{\mu_{x_0}}-f,\ \mu_{x_0}^{(k)}-\frac{x_{k_0}^{(0)}}{x_n^{(0)}}\mu_{x_0}^{(n)}\rangle=x_{k_0}^{(0)}\,\gamma_{k_0}(\mu_{x_0})-x_{k_0}^{(0)}\,\gamma_n(\mu_{x_0}).$$

This is true for any k_0 between 1 and $n-1$. Therefore, for any k, $1\leqq k\leqq n$, such that $x_k^{(0)}>0$, it holds that

$$\gamma_n(\mu_{x_0})\leqq\gamma_k(\mu_{x_0}).$$

This was derived under the condition that $x_n^{(0)}>0$. We obtain similarly

$$\gamma_j(\mu_{x_0})\leqq\gamma_k(\mu_{x_0})$$

if $x_j^{(0)}>0$ and $x_k^{(0)}>0$, and conclude that

$$\gamma_j(\mu_{x_0})=\gamma_k(\mu_{x_0}) \tag{2.27}$$

whenever $x_j^{(0)}>0$ and $x_k^{(0)}>0$. If all $x_k^{(0)}>0$, we have by Theorem 2.14 for some extremal measure μ_{x_0}

$$\begin{aligned}I_n(\mu_x)&=I_n(\mu_{x_0})+2\sum_{k=1}^{n}\gamma_k(\mu_{x_0})\Delta x_k+o(|\Delta x|)\\&=I_n(\mu_{x_0})+2\gamma_1(\mu_{x_0})\sum_{k=1}^{n}\Delta x_k+o(|\Delta x|)=I_n(\mu_{x_0})+o(|\Delta x|)\end{aligned}$$

provided that $\sum_{k=1}^{n} x_k=a$. Therefore the derivative of $I_n(\mu_x)$ at $x=x_0$ along the plane $\sum_{k=1}^{n} x_k=a$ is zero. Conversely if the derivative of $I_n(\mu_x)$ along the plane is zero at $x=x_0$ with positive coordinates, then all γ_k are equal.

If the kernel is of positive type, and if all γ_k are equal at $x_0=(x_1^{(0)}, \cdots, x_n^{(0)})$, $x_1^{(0)}>0, \cdots, x_n^{(0)}>0$, then μ_{x_0} is a 1-dimensional solution. In fact, we have

$$(\mu_{x_0}, \mu_{x_0})-\langle f, \mu_{x_0}\rangle=\sum_{k=1}^{n} x_k^{(0)}\gamma_k(\mu_{x_0})=\gamma_1(\mu_{x_0})\sum_{k=1}^{n} x_k^{(0)}=a\gamma_1(\mu_{x_0}),$$

and

$$U^{\mu_{x_0}}(P)\geqq f(P)+\gamma_k(\mu_{x_0})\, g(P) \qquad \text{p.p.p. on } K_k.$$

Hence

$$U^{\mu_{x_0}}(P)\geqq f(P)+\gamma_1(\mu_{x_0})\, g(P) \qquad \text{p.p.p. on } K.$$

By Theorem 2.3 it is concluded that μ_{x_0} minimizes $I_1(\mu)$ among $\mu\in\mathscr{E}_K(g, a)$.

With similar reasoning we can establish

THEOREM 2.16. *First consider the n-dimensional problem to minimize $I(\mu)$ for $\mu\in\mathscr{E}_K(g, x)$, where $K=\bigcup_{k=1}^{n} K_k$ and $x=(x_1, \cdots, x_n)$. Setting $\sum_{k=1}^{n_1} x_k=x'_1$, $\sum_{k=n_1+1}^{n_2} x_k=x'_2$, $\cdots$, $\sum_{k=n_{m-1}+1}^{n} x_k=x'_m$ and $\bigcup_{k=1}^{n_1} K_k=K'_1$, $\bigcup_{k=n_1+1}^{n_2} K_k=K'_2$, $\cdots$, $\bigcup_{k=n_{m-1}+1}^{n} K_k=K'_m$, consider next the m-dimensional problem to minimize $I(\mu)$ for $\mu\in\mathscr{E}_K(g, x')$ where $K=\bigcup_{j=1}^{m} K'_j$ and $x'=(x'_1, \cdots, x'_m)$. If a solution at x_0 of the n-dimensional problem gives a solution of the m-dimensional problem, then*

$$\gamma_1(\mu_{x_0})=\cdots=\gamma_{n_1}(\mu_{x_0}),\ \gamma_{n_1+1}(\mu_{x_0})=\cdots=\gamma_{n_2}(\mu_{x_0}),\ \cdots,\ \gamma_{n_{m-1}}(\mu_{x_0})=\cdots=\gamma_n(\mu_{x_0}); \tag{2.28}$$

these equalities are considered only for $\{\gamma_k\}$ which are well-defined. If all $x_0^{(k)}$ are positive, the derivative of $I(\mu_x)$ at $x=x_0$ along the $(n-m)$-dimensional plane $\sum_{k=1}^{n_1} x_k=x'_1$, $\cdots$, $\sum_{k=n_{m-1}+1}^{n} x_k=x'_m$ vanishes and this last fact guarantees (2.28) *conversely.*

If the kernel is of positive type, if (2.28) is true for x_0 and if the coordinates of x_0 are all positive, then μ_{x_0} is an m-dimensional solution.

2.5. Behavior at $x=0$ and $x=\infty$.

We consider again the family $\Pi=\{P(\nu_\xi);\ 0\leqq\xi_1<\infty, \cdots, 0\leqq\xi_n<\infty\}$ and recall that the graph of $I(\mu_x)$ is the lower envelope of Π. Let $y=(y_1, \cdots, y_n)$, $y_1>0, \cdots, y_n>0$, be a point with $|y|=1$ and l be the half line issuing from the origin and passing y. Any point ξ of l is expressed by $|\xi|y$. The graph of $I(\mu_x)$ over l is the lower envelope of

$$\Pi_y=\{P(\nu_{ry});\ 0\leqq r<\infty\}.$$

Hence the graph of $J(x)=I(\mu_x)/\rho$, $\rho=|x|$, over l is the lower envelope of

$$\Pi'_y=\left\{\frac{P(\nu_{ry})}{\rho};\ 0\leqq r<\infty\right\},$$

where

$$\frac{P(\nu_{ry})}{\rho}=\rho\sum_{j,k=1}^{n}y_j y_k(\nu_{ry}^{(j)},\ \nu_{ry}^{(k)})-2\sum_{k=1}^{n}y_k\ \langle f,\ \nu_{ry}^{(k)}\rangle$$
$$=\rho(y\nu_{ry},\ y\nu_{ry})-2\langle f,\ y\nu_{ry}\rangle,\qquad 0\leqq\rho<\infty,$$

represents a line for each r. The graph of $J(x)$ over l is therefore a concave curve.[29] Hence it has a right derivative and a left derivative at every point (this was already observed before Lemma 2.1), and they coincide with each other with a possible exception of a countable number of points. We infer also that $(y\nu_{ry},\ y\nu_{ry})$ and $\langle f,\ y\nu_{ry}\rangle$ decrease as $r\to\infty$. By (2.26)

$$2\lim_{r\to 0}\langle f,\ y\nu_{ry}\rangle=2\sup_{\nu\in\mathscr{E}_K(g,\ y)}\langle f,\ \nu\rangle=-\lim_{r\to 0}\frac{I(\mu_{ry})}{r}.$$

We can express this value in a more explicit form. We shall denote by M_k the *pseudo-maximum* of $f(P)/g(P)$ on K_k. Namely

$$M_k=\inf\left\{M;\ \frac{f(P)}{g(P)}\leqq M\quad \text{p.p.p. on } K_k\right\}.$$

If $M'_k<M_k$ and

$$B_k(M'_k)=\left\{P\in K_k;\ \frac{f(P)}{g(P)}>M'_k\right\},$$

then $\mathscr{E}_{B_k(M'_k)}\neq\{0\}$. For any $\nu\in\mathscr{E}_{\bigcup_{k=1}^{n}B_k(M'_k)}(g,\ y)$, we have

$$\langle f,\ \nu\rangle=\sum_{k=1}^{n}\int_{K_k}\frac{f}{g}\,gd\nu\geqq\sum_{k=1}^{n}M'_k\,y_k.$$

Thus

$$\sum_{k=1}^{n}M'_k\,y_k\leqq\sup_{\nu\in\mathscr{E}_K(g,\ y)}\langle f,\ \nu\rangle$$

and hence

$$\sum_{k=1}^{n}M_k\,y_k\leqq\sup_{\nu\in\mathscr{E}_K(g,\ y)}\langle f,\ \nu\rangle.$$

On the other hand, for given $\varepsilon>0$, we choose $\nu_\varepsilon\in\mathscr{E}_K(g,\ y)$ such that $\langle f,\ \nu_\varepsilon\rangle\geqq\sup_{\nu\in\mathscr{E}_K(g,\ y)}\langle f,\ \nu\rangle-\varepsilon$. Since $\mathscr{E}_{\bigcup_{k=1}^{n}B_k}=\{0\}$ for the set $B_k=\{P\in K_k;\ f/g>M_k\}$, we have $\nu_\varepsilon(\bigcup_{k=1}^{n}B_k)=0$ and

$$\langle f,\ \nu_\varepsilon\rangle=\int_{K-\bigcup_{k=1}^{n}B_k}fd\nu_\varepsilon\leqq\sum_{k=1}^{n}M_k\int_{K_k}gd\nu_\varepsilon=\sum_{k=1}^{n}M_k\,y_k.$$

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29) Ogasawara suggested the author to make use of this property of the curve.

Hence $\sum_{k=1}^{n} M_k y_k \geqq \sup_{\lambda\in\mathcal{E}_K(g,y)} \langle f, \lambda\rangle - \varepsilon$ and the equality

$$\sum_{k=1}^{n} M_k y_k = \sup_{\lambda\in\mathcal{E}_K(g,y)} \langle f, \lambda\rangle \tag{2.29}$$

follows.

We have already seen that $\lim_{r\to 0} r(y\nu_{ry}, y\nu_{ry})=0$ by (2.25). We ask whether $(y\nu_{ry}, y\nu_{ry})$ has a finite limit or not as $r\to 0$. First we assume that the V_i-value of the compact set $F_k=\{P\in K_k;\ f(P)\geqq M_k\, g(P)\}$ is finite for each k, and take $\lambda_0\in\mathcal{E}_{\bigcup_{k=1}^{n} F_k}(g, y)$ which minimizes (λ, λ) among $\lambda\in\mathcal{E}_{\bigcup_{k=1}^{n} F_k}(g, y)$. Since $I(\mu_x)\leqq I(r\lambda_0)=(\lambda_0, \lambda_0)\, r^2-2\langle f, \lambda_0\rangle r$, we have

$$(y\nu_{ry}, y\nu_{ry})r-2\langle f, y\nu_{ry}\rangle\leqq(\lambda_0, \lambda_0)\, r-2\langle f, \lambda_0\rangle.$$

By (2.29)

$$\langle f, y\nu_{ry}\rangle\leqq\sum_{k=1}^{n} M_k\, y_k=\langle f, \lambda_0\rangle,$$

whence

$$(y\nu_{ry}, y\nu_{ry})\leqq(\lambda_0, \lambda_0). \tag{2.30}$$

On the other hand, let $T=\{\nu^{(\omega)};\ \omega\in D\}$ be a subnet of the sequence $\{\nu_{y/p}\}$ converging vaguely to ν'. We observe that

$$\lim_{\omega} \langle f, y\nu^{(\omega)}\rangle\leqq\langle f, y\nu'\rangle$$

and hence

$$\sum_{k=1}^{n} M_k\, y_k\leqq\langle f, y\nu'\rangle=\int \frac{f}{g}\, gd(y\nu')\leqq\sum_{k=1}^{n} M_k\, y_k.$$

Consequently $f/g=M_k$ ν'-a.e. on K_k and it is concluded that $S_{\nu'}\subset\bigcup_{k=1}^{n} F_k$. Therefore

$$\lim_{r\to 0}\ (y\nu_{ry}, y\nu_{ry})=\lim_{\omega}\ (y\nu^{(\omega)}, y\nu^{(\omega)})\geqq(y\nu', y\nu')\geqq(\lambda_0, \lambda_0). \tag{2.31}$$

On account of (2.30) we have

$$\lim_{r\to 0}\ (y\nu_{ry}, y\nu_{ry})=(\lambda_0, \lambda_0)$$

if $V_i(F_k)<\infty$ for each k. Let us write in general $V_i^{(g,x)}(X)$ for $\inf_{\mu\in\mathcal{E}_X(g,x)} (\mu, \mu)$. It is seen from (2.31) that

$$\lim_{r\to 0}\ (y\nu_{ry},\ y\nu_{ry})=V_i^{(g,y)}(\bigcup_{k=1}^{n} F_k)=\infty$$

if $V_i(F_k)=\infty$ for some k.

We summarize the above results as

THEOREM 2.17. *For each* $y=(y_1, \cdots, y_n)$, $y_1>0, \cdots, y_n>0$, *such that* $|y|=1$, $I(\mu_{ry})/r$ *is a concave function of* r, *and* $(y\nu_{ry}, y\nu_{ry})$ *and* $\langle f, y\nu_{ry}\rangle$ *are decreasing functions of* r.

If the pseudo-maximum of $f(P)/g(P)$ *on* K_k *is denoted by* M_k, *i.e. if*

$$M_k=\inf\left\{M;\ \frac{f(P)}{g(P)}\leqq M \qquad p.p.p.\ on\ K_k\right\},$$

then

$$\sum_{k=1}^{n} M_k y_k=\lim_{r\to 0}\ \langle f, y\nu_{ry}\rangle=\sup_{\lambda\in\mathscr{E}_K(g,y)}\langle f, \lambda\rangle$$

$$=-\lim_{r\to 0}\sum_{k=1}^{n}\gamma_k(\mu_{ry})y_k=-\frac{1}{2}\lim_{r\to 0}\frac{I(\mu_{ry})}{r}=-\frac{1}{2}\left.\frac{dI(\mu_{ry})}{dr}\right|_{r=0}.$$

We have also

$$\lim_{r\to 0}\ (y\nu_{ry}, y\nu_{ry})=V_i^{(g,y)}\Big(\bigcup_{k=1}^{n}\{P\in K_k;\ f(P)\geqq M_k\, g(P)\}\Big).$$

The next question is as to the behavior of $I(\mu_x)$ and $\gamma(\mu_x)$ as $x\to\infty$. We shall prove first

LEMMA 2.2. *Let* $x_1\geqq 0, \cdots, x_n\geqq 0$. *Let* $B\subset K$ *be a* K_σ-*set and let* $\mu\in\mathscr{E}'_B(g, x)$. *If a sequence* $\{F^{(p)}\}$ *of compact sets increases to* B, *then*

$$(\mu, \mu)\geqq V_i^{(g,x)}(B)=\lim_{p\to\infty} V_i^{(g,x)}(F^{(p)}).$$

PROOF. Obviously $V_i^{(g,x)}(B)\leqq V_i^{(g,x)}(F^{(p)})$ for each p. We assume $(\mu, \mu)<\infty$ and define $\bar\mu^{(p)}$ by setting it equal to $x_k\,\mu_k^{(p)}\langle g, \mu_k^{(p)}\rangle^{-1}$ on K_k, where $\mu_k^{(p)}$ is the restriction of μ to $K_k\cap F^{(p)}$; if $\mu_k^{(p)}=0$ we set $\bar\mu^{(p)}=0$ on $K_k\cap F^{(p)}$. Then $\bar\mu^{(p)}\in\mathscr{E}_{F^{(p)}}(g, x)$ if p is sufficiently large. Since $\mu_k^{(p)}$ increases to μ on K_k, $(\bar\mu^{(p)}, \bar\mu^{(p)})$ tends to (μ, μ). Therefore

$$(\mu, \mu)=\lim_{p\to\infty}(\bar\mu^{(p)}, \bar\mu^{(p)})\geqq\lim_{p\to\infty} V_i^{(g,x)}(F^{(p)}).$$

Next let ν be any measure of $\mathscr{E}_B(g, x)$. Then $(\nu, \nu)\geqq\lim_{p\to\infty} V_i^{(g,x)}(F^{(p)})$ and hence $V_i^{(g,x)}(B)\geqq\lim_{p\to\infty} V_i^{(g,x)}(F^{(p)})$. Together with the inequality obtained at the beginning, this completes the proof.

Now we set

$$B=\{P\in K; f(P)>-\infty\}\qquad\text{and}\qquad B^{(p)}=\{P\in K; f(P)\geqq -p\}.$$

Each $B^{(p)}$ is compact and $B=\bigcup_p B^{(p)}$. Let us recall that we are still under the assumption that $V_i(B\cap K_k)<\infty$ for each k and hence $V_i^{(g,x)}(B)<\infty$. We shall prove

$$\lim_{r\to\infty}\frac{I(\mu_{ry})}{r^2}=\lim_{r\to\infty}(y\nu_{ry}, y\nu_{ry})=V_i^{(g,y)}(B). \tag{2.32}$$

Let $\nu_y^{(p)}\in\mathscr{E}_{B(p)}(g, e)$ give $(y\nu_y^{(p)}, y\nu_y^{(p)})=V_i^{(g,y)}(B^{(p)})$. By (2.16) we have

$$\overline{\lim_{r\to\infty}}\frac{I(\mu_{ry})}{r^2}\leqq(y\nu_y^{(p)}, y\nu_y^{(p)})$$

and hence

$$\overline{\lim_{r\to\infty}}\frac{I(\mu_{ry})}{r^2}\leqq V_i^{(g,y)}(B) \tag{2.33}$$

by Lemma 2.2.

On the other hand, we denote by $\nu_{ry}^{(p)}$ the restriction of ν_{ry} to $B^{(p)}$. Since $\langle f, y\nu_{ry}\rangle>-\infty$, $\nu_{ry}(K-B)=0$ and $\nu_{ry}\in\mathscr{E}'_B(g, e)$. We have by Lemma 2.2

$$(y\nu_{ry}, y\nu_{ry})\geqq\lim_{p\to\infty}V_i^{(g,y)}(B^{(p)})=V_i^{(g,y)}(B).$$

In view of the inequality

$$I(\mu_{ry})=(\mu_{ry}, \mu_{ry})-2\langle f, \mu_{ry}\rangle\geqq(y\nu_{ry}, y\nu_{ry})r^2-2r\frac{\max_K f^+}{\min_K g}\sum_{k=1}^n y_k,$$

we obtain

$$\underline{\lim_{r\to\infty}}\frac{I(\mu_{ry})}{r^2}\geqq\lim_{r\to\infty}(y\nu_{ry}, y\nu_{ry})\geqq V_i^{(g,y)}(B).$$

This relation and (2.33) give (2.32). By (2.14)

$$\lim_{r\to\infty}\frac{\overrightarrow{O\gamma(\mu_{ry})}\cdot\overrightarrow{Oy}}{r}=\frac{1}{2}\lim_{r\to\infty}\left\{\frac{I(\mu_{ry})}{r^2}+(y\nu_{ry}, y\nu_{ry})\right\}=V_i^{(g,y)}(B)$$

and

$$\lim_{r\to\infty}\frac{\langle f, y\nu_{ry}\rangle}{r}=\frac{1}{2}\lim_{r\to\infty}\left\{(y\nu_{ry}, y\nu_{ry})-\frac{I(\mu_{ry})}{r^2}\right\}=0.$$

In general, the energy of the vague limit of any subnet of $\{y\nu_{ry}\}$ is not equal to $V_i^{(g,y)}(B)$ as Example 1 given after Theorem 2.18 will show. Let ν be the limit of a vaguely convergent net $\{\nu^{(\omega)}\in\mathscr{E}_K(g, e);\ \omega\in D\}$ such that $(y\nu^{(\omega)}, y\nu^{(\omega)})$ tends to $V_i^{(g,y)}(B)$. Then $(y\nu, y\nu)=V_i^{(g,y)}(B)$ if the following condition is satisfied:

(α) $f(P)>-\infty$ ν-a. e.

To prove this, first we observe that $V_i^{(g,y)}(B) \geqq (y\nu, y\nu)$ by Proposition 4 in § 1.6. On the other hand on account of (α), ν belongs to $\mathscr{E}'_B(g, e)$ and hence by Lemma 2.2

$$(y\nu, y\nu) \geqq \lim_{p \to \infty} V_i^{(g,y)}(B_p) = V_i^{(g,y)}(B).$$

Thus we obtain the equality.

Next we shall study $\lim \langle f, y\nu_{ry} \rangle$ as $r \to \infty$. We know that $\langle f, y\nu_{ry} \rangle$ decreases as $r \to \infty$. Let $T = \{\nu^{(\omega)}; \omega \in D\}$ be a subnet of $\{\nu_{ry}\}$, converging vaguely to some measure ν_y^*. It follows that $\lim\limits_{r \to \infty} (y\nu_{ry}, y\nu_{ry}) \geqq (y\nu_y^*, y\nu_y^*)$ and

$$\lim_{r \to \infty} \langle f, y\nu_{ry} \rangle = \lim_{\omega} \langle f, y\nu^{(\omega)} \rangle \leqq \langle f, y\nu_y^* \rangle.$$

If $\langle f, y\nu_y^* \rangle = -\infty$, $\lim\limits_{r \to \infty} \langle f, y\nu_{ry} \rangle = -\infty$ too. If $\langle f, y\nu_y^* \rangle > -\infty$,

$$(y\nu_{ry}, y\nu_{ry})r - 2\langle f, y\nu_{ry} \rangle \leqq (y\nu_y^*, y\nu_y^*)r - 2\langle f, y\nu_y^* \rangle$$

by (2.16) and hence

$$0 \leqq \{(y\nu_{ry}, y\nu_{ry}) - (y\nu_y^*, y\nu_y^*)\} r \leqq 2\langle f, y\nu_{ry} \rangle - 2\langle f, y\nu_y^* \rangle.$$

Thus $\langle f, y\nu_{ry} \rangle$ decreases to $\langle f, y\nu_y^* \rangle$ and $\{(y\nu_{ry}, y\nu_{ry}) - (y\nu_y^*, y\nu_y^*)\} r$ tends to 0 as $r \to \infty$. In any case, $\lim\limits_{r \to \infty} \langle f, y\nu_{ry} \rangle = \langle f, y\nu_y^* \rangle$. We remark that each of $\langle f, y\nu_y^* \rangle$ and $(y\nu_y^*, y\nu_y^*)$ is the same for all vague limits of subnets of $\{y\nu_{ry}\}$.

In the case that $\langle f, y\nu_y^* \rangle > -\infty$, we have $V_i^{(g,y)}(B) = \lim\limits_{r \to \infty} (y\nu_{ry}, y\nu_{ry}) = (y\nu_y^*, y\nu_y^*)$ in view of (2.32). This follows also from (α) which is satisfied in virtue of $\langle f, y\nu_y^* \rangle > -\infty$. Example 2 given after Theorem 2.18 will show that (α) may be satisfied even if $\langle f, y\nu_y^* \rangle = -\infty$.

Let $\{\nu^{(\omega)}; \omega \in D\}$ be a net in $\mathscr{E}_K(g, e)$, converging vaguely to some measure ν, such that $(y\nu^{(\omega)}, y\nu^{(\omega)})$ tends to $V_i^{(g,y)}(B)$. Denoting by $\mathscr{N}_y$ the set of all such vague limits, we shall show that

$$\langle f, y\nu_y^* \rangle = \max_{\nu \in \mathscr{N}_y} \langle f, y\nu \rangle, \tag{2.34}$$

where ν_y^* is the limit of any vaguely convergent subnet of $\{\nu_{ry}\}$. We consider the above net $\{\nu^{(\omega)}\}$ and its vague limit ν. Since

$$(y\nu_{ry}, y\nu_{ry}) \geqq V_i^{(g,y)}(B) = \lim (y\nu^{(\omega)}, y\nu^{(\omega)}) \geqq (y\nu, y\nu)$$

we have

$$0 \leqq \{(y\nu_{ry}, y\nu_{ry}) - (y\nu, y\nu)\} r \leqq 2\langle f, y\nu_{ry} \rangle - 2\langle f, y\nu \rangle$$

by (2.16). Therefore

$$\langle f, y\nu \rangle \leqq \lim_{r \to \infty} \langle f, y\nu_{ry} \rangle = \langle f, y\nu_y^* \rangle$$

and (2.34) is concluded.

We state

THEOREM 2.18.

$$\lim_{r\to\infty} \frac{I(\mu_{ry})}{r^2} = \lim_{r\to\infty} (y\nu_{ry}, y\nu_{ry}) = \lim_{r\to\infty} \frac{\overrightarrow{O\gamma(\mu_{ry})}\cdot \overrightarrow{Oy}}{r} = V_i^{(g,y)}(B)$$

and

$$\lim_{r\to\infty} \frac{\langle f, y\nu_{ry}\rangle}{r} = 0$$

regardless of the choice of $\mu_{ry} \in \mathcal{M}_{ry}^*$ *at each point* ry, *where* $B = \{P \in K;\ f(P) > -\infty\}$. *Denoting by* ν_y^* *the vague limit of any vaguely convergent subnet of* $\{\nu_{ry}\}$, *we see that*

$$\lim_{r\to\infty} \langle f, y\nu_{ry}\rangle = \langle f, y\nu_y^*\rangle \geqq -\infty.$$

Let us denote by $\mathcal{N}_y$ *the class of all vague limits of vaguely converging nets in* $\mathcal{E}_K(g, e)$ *such that* $(y\nu, y\nu)$ *converges to* $V_i^{(g,y)}(B)$ *as* ν *tends to the limit along a net. Then*

$$\langle f, y\nu_y^*\rangle = \max_{\nu\in\mathcal{N}_y} \langle f, y\nu\rangle.$$

If $\langle f, y\nu_y^*\rangle > -\infty$,

$$\lim_{r\to\infty} \{(y\nu_{ry}, y\nu_{ry}) - (y\nu_y^*, y\nu_y^*)\}\, r = 0$$

and

$$I(\mu_{ry}) = (y\nu_y^*, y\nu_y^*) r^2 - 2\{\langle f, y\nu_y^*\rangle + o(1)\}\, r \qquad \textit{near } r = \infty.$$

Let $\nu \in \mathcal{N}_y$. *A sufficient condition for* $V_i^{(g,y)}(B) = (y\nu, y\nu)$ *is the following and it is satisfied if* $\langle f, y\nu\rangle > -\infty$:

(α) $f(P) > -\infty$ ν-*a.e.*

EXAMPLE 1. If (α) is not satisfied, it can happen that $\lim_{r\to\infty} (y\nu_{ry}, y\nu_{ry}) > (y\nu_y^*, y\nu_y^*)$ as the following example shows. Let K be the unit ball with center at the origin in E_3 and consider the Newtonian kernel. We set $f(P) = -(1-\overline{OP})^{-1}$ and $g(P) \equiv 1$. We consider the case $n=1$ and hence $y=1$. By Theorem 2.18 we have that (ν_x, ν_x) tends to (ν_∞, ν_∞), where ν_∞ is the uniform unit measure on ∂K. Let ν_1^* be the vague limit of some subnet of $\{\nu_x\}$. Since $\lim_{x\to\infty} (\nu_x, \nu_x) \geqq (\nu_1^*, \nu_1^*) \geqq (\nu_\infty, \nu_\infty)$, it follows that $(\nu_1^*, \nu_1^*) = (\nu_\infty, \nu_\infty)$ and hence $\nu_1^* = \nu_\infty$. Let us set

$$\Phi(P, Q)=\begin{cases} \dfrac{1}{\overline{PQ}} & \text{if } P, Q\in K-\partial K, \\ 0 & \text{if } P\cup Q\in \partial K. \end{cases}$$

For this kernel ν_x must be the same as above for every $x\geqq 0$. If we add the subscript Φ to inner products defined with respect to the new kernel, $(\nu_x, \nu_x)_\Phi=(\nu_x, \nu_x)$ but $(\nu_1^*, \nu_1^*)_\Phi=0<(\nu_1^*, \nu_1^*)=1$.

Example 2. We shall show with an example that it happens that $\langle f, \nu_y^*\rangle=-\infty$ even if $f(P)$ is finite-valued. Take $\Omega=K=[0, 1]$ on the x-axis and $\Phi(P, Q)=\log 1/\overline{PQ}$. The support of the equilibrium measure ν_∞ is identical with K. If we let $f(P)$ tend to $-\infty$ rapidly as P tends to the end points of K but set it equal to 0 at the end points, then $\langle f, \nu_\infty\rangle=-\infty$.

Next we shall apply the above theorem to study $\lim I(\mu_{ry})$ as $r\to\infty$. It is seen that $I(\mu_{ry})$ tends to ∞ or to $-\infty$ according as $V_i^{(g,y)}(B)>0$ or <0. In case $V_i^{(g,y)}(B)=0$, $I(\mu_{ry})$ tends to $-\infty$ if $\langle f, y\nu_y^*\rangle>0$ by the above theorem. If $V_i^{(g,y)}(B)=0$ and $0>\langle f, y\nu_y^*\rangle$, then $(y\nu_{ry}, y\nu_{ry})\geqq V_i^{(g,y)}(B)=0$ and

$$\frac{I(\mu_{ry})}{r}=(y\nu_{ry}, y\nu_{ry})\, r-2\langle f, y\nu_{ry}\rangle\geqq -2\langle f, y\nu_{ry}\rangle\to -2\langle f, y\nu_y^*\rangle$$

as $r\to\infty$. Therefore $I(\mu_{ry})$ tends to ∞.

We state the above results as

Theorem 2.19. *$I(\mu_{ry})$ tends to ∞ as $r\to\infty$ if $V_i^{(g,y)}(B)>0$, or if $V_i^{(g,y)}(B)=0$ and $\langle f, y\nu_y^*\rangle>0$. It tends to $-\infty$ as $r\to\infty$ if $V_i^{(g,y)}(B)<0$, or if $V_i^{(g,y)}(B)=0$ and $\langle f, y\nu_y^*\rangle>0$.*

2.6. Further study of the graph of $I(\mu_x)$.

We begin with

Theorem 2.20 *Let $y=(y_1, \cdots, y_n)$ be a variable such that $y_1\geqq 0, \cdots, y_n\geqq 0$ and $|y|=1$. In order that $I(\mu_x)$ be a parabolic quadratic surface it is necessary and sufficient that $V_i^{(g,y)}(F)$ is a quadratic form in $y_1, \cdots, y_n$ and $V_i^{(g,y)}(B)=V_i^{(g,y)}(F)$ for each y, where*

$$F=\bigcup_{k=1}^{n}\{P\in K_k;\, f(P)\geqq M_k\, g(P)\}\qquad \textit{and}\qquad B=\{P\in K;\, f(P)>-\infty\}.$$

Proof. We assume that

$$I(\mu_x)=\sum_{j,k=1}^{n} a_{jk}\, x_j\, x_k-2\sum_{k=1}^{n} b_k\, x_k \tag{2.35}$$

in $x_1>0, \cdots, x_n>0$. We set $|x|=r$, $\mu_x=x\nu_x$ and $x_k=ry_k$ for each k. Since

$$r \sum_{j,k=1}^{n} a_{jk} y_j y_k - 2 \sum_{k=1}^{n} b_k y_k$$

and

$$r(y\nu_{r'y}, y\nu_{r'y}) - 2 \langle f, y\nu_{r'y} \rangle$$

are linear in r, coincide with each other at $r=r'$ and the latter is not smaller than the former, they are identical for any $r>0$. Hence

$$(2.36) \qquad (y\nu_{r'y}, y\nu_{r'y}) = \sum_{j,k=1}^{n} a_{jk} y_j y_k.$$

By Theorems 2.17 and 2.18 we have

$$\langle f, y\nu_{ry} \rangle = \lim_{r\to 0} \langle f, y\nu_{ry} \rangle = \sum_{k=1}^{n} M_k y_k$$

and

$$(2.37) \qquad (y\nu_{ry}, y\nu_{ry}) = \lim_{r\to 0} (y\nu_{ry}, y\nu_{ry}) = V_i^{(g,y)}(F)$$
$$= \lim_{r\to\infty} (y\nu_{ry}, y\nu_{ry}) = V_i^{(g,y)}(B).$$

Thus $V_i^{(g,y)}(F) = V_i^{(g,y)}(B)$ and they are quadratic in $y_1, \cdots, y_n$.

Next we shall prove the sufficiency. We assume that

$$V_i^{(g,y)}(F) = V_i^{(g,y)}(B) = \sum_{j,k=1}^{n} a_{jk} y_j y_k$$

for each y. If $\langle f, \mu \rangle > -\infty$ for $\mu \in \mathscr{E}_K(g, y)$, then we have by Lemma 2.2

$$(\mu, \mu) \geqq V_i^{(g,y)}(B) = \sum_{j,k=1}^{n} a_{jk} y_j y_k.$$

Let $\nu_y \in \mathscr{E}_F(g, e)$ be an extremal measure which gives $(y\nu_y, y\nu_y) = V_i^{(g,y)}(F)$. It follows that

$$I(ry\nu_y) = (y\nu_y, y\nu_y) r^2 - 2 \langle f, y\nu_y \rangle r = \sum_{j,k=1}^{n} a_{jk} y_j y_k r^2 - 2 \sum_{k=1}^{n} M_k y_k r$$
$$\leqq (r\mu, r\mu) - 2r \int f d\mu = I(r\mu).$$

The inequality is true if $\langle f, \mu \rangle = -\infty$. We see that $ry\,\nu_y \in \mathscr{M}_{ry}$ and that $I(ry\nu_y) = I(\mu_{ry})$ for any y. Thus $I(\mu_x)$ is a parabolic quadratic surface.

A simple example in the case $n=1$ in which the condition is not satisfied is the following: K= a unit spherical surface in E_3, $\Phi(P, Q) = \overline{PQ}^{-1}$, $f(P) \equiv 1$. The equilibrium measure ν_∞ is the uniform measure on K and $S_{\nu_\infty} = K$. If we choose any nonconstant continuous function for $f(P)$, $I(\mu_x)$ will not be a polynomial.

If $n=1$ and $f(P)/g(P)$ is constant on K, the condition in the theorem is naturally fulfilled. However, in case $n \geqq 2$, Examples 6 and 7 in the next section will show that $I(\mu_x)$ may not be a parabolic quadratic surface even if $f(P)/g(P)$ is constant on K.

One asks probably conditions for $V_i^{(g,y)}(K)$ to be a quadratic form in $y_1, \cdots, y_n$, where $K = \bigcup_{k=1}^{n} K_k$ is a disjoint union of compact sets. We do not know, however, any condition at present.

If $I(\mu_x)$ is not entirely a polynomial, does it coincide with a parabolic quadratic surface on some $0 \leqq |x| \leqq r_0$ or on $0 < r_0 \leqq |x| < \infty$? We begin with the first problem. We assume (2.35) for x in $0 \leqq |x| \leqq r_0$. As in the proof of Theorem 2.20 it follows that $V_i^{(g,y)}(F)$ is a quadratic form in $y_1, \cdots, y_n$ and that, for any fixed y, $(y\nu_{ry}, y\nu_{ry})$ is constant on $0 < r \leqq r_0$ and $\langle f, y\nu_{ry}\rangle = \sum_{k=1}^{n} M_k y_k$. Let $\nu \in \mathscr{E}_K(g, e)$. Then

$$(ry\nu, ry\nu) - 2\langle f, ry\nu\rangle \geqq (ry\nu_{ry}, ry\nu_{ry}) - 2\langle f, ry\nu_{ry}\rangle$$
$$= V_i^{(g,y)}(F) r^2 - 2\sum_{k=1}^{n} M_k y_k r$$

for $r \leqq r_0$. It follows that

$$2\left(\sum_{k=1}^{n} M_k y_k - \langle f, y\nu\rangle\right) \geqq r_0 \{V_i^{(g,y)}(F) - (y\nu, y\nu)\} \tag{2.38}$$

for any y with $|y| = 1$. Conversely, if $V_i^{(g,y)}(F)$ is a quadratic form in $y_1, \cdots, y_n$ and if (2.38) is true for any $\nu \in \mathscr{E}_K(g, e)$ and y, then $I(\mu_x)$ is a parabolic quadratic surface in $|x| \leqq r_0$.

We restrict ourselves to the case $n=1$; we write K and M instead of K_1 and M_1. We shall prove (2.38) for some r_0 in the case that there is $\alpha > 0$ such that

$$V_i\left(\left\{P \in K;\ M - \alpha < \frac{f(P)}{g(P)} < M\right\}\right) = \infty. \tag{2.39}$$

Let $\nu \in \mathscr{E}_K(g, 1)$. We denote its restriction to F by ν_F and set $\nu' = \nu - \nu_F$. Obviously

$$\nu'\left(\left\{P \in K;\ M - \alpha < \frac{f(P)}{g(P)}\right\}\right) = 0.$$

We see that

$$M - \langle f, \nu\rangle \geqq M - M\langle g, \nu_F\rangle - (M-\alpha)\langle g, \nu'\rangle = \alpha\langle g, \nu'\rangle.$$

We shall write simply $V_i^{(g)}(X)$ for $V_i^{(g,1)}(X)$ in general and prove that $\{V_i^{(g)}(F) - (\nu, \nu)\}/\langle g, \nu'\rangle$ is bounded from above by a constant not depending on ν, provided that $\langle g, \nu'\rangle > 0$; it is obviously so if $\nu = \nu'$. Hence we assume that $\nu_F \neq 0$. Since

$$\frac{V_i^{(g)}(F)-(\nu_F+\nu',\ \nu_F+\nu')}{\langle g,\ \nu'\rangle}$$

$$=\frac{V_i^{(g)}(F)-(\nu_F,\ \nu_F)}{\langle g,\ \nu'\rangle}-2\left(\nu_F,\ \frac{\nu'}{\langle g,\ \nu'\rangle}\right)-\langle g,\ \nu'\rangle\left(\frac{\nu'}{\langle g,\ \nu'\rangle},\ \frac{\nu'}{\langle g,\ \nu'\rangle}\right)$$

$$\leqq\frac{V_i^{(g)}(F)-(\nu_F,\ \nu_F)}{\langle g,\ \nu'\rangle}+\text{const.},$$

it is sufficient to show that $\{V_i^{(g)}(F)-(\nu_F,\ \nu_F)\}/\langle g,\ \nu'\rangle$ is bounded from above. We set

$$\lambda=\frac{\nu_F}{\langle g,\ \nu_F\rangle}$$

and note that $V_i^{(g)}(F)\leqq(\lambda,\ \lambda)$. It follows that

$$\frac{V_i^{(g)}(F)-(\lambda,\ \lambda)\ \langle g,\ \nu_F\rangle^2}{1-\langle g,\ \nu_F\rangle}\leqq\frac{V_i^{(g)}(F)\ (1-\langle g,\ \nu_F\rangle^2)}{1-\langle g,\ \nu_F\rangle}$$

$$=V_i^{(g)}(F)\ (1+\langle g,\ \nu_F\rangle)\leqq\max\ \{2V_i^{(g)}(F),\ 0\}.$$

Thus

$$\frac{V_i^{(g)}(F)-(\nu,\ \nu)}{M-\langle f,\ \nu\rangle}$$

is bounded from above with respect to $\nu\in\mathscr{E}_K(g,\ 1)$ provided that $\langle g,\nu'\rangle>0$. If $\langle g,\nu'\rangle=0$, $S_\nu\subset F$ and $V_i^{(g)}(F)\leqq(\nu,\ \nu)$. Hence (2.38) is true with some positive constant r_0.

We state

THEOREM 2.21. *In order that $I(\mu_x)$ is equal to a parabolic quadratic surface on $0\leqq|x|\leqq r_0$, it is necessary and sufficient that $V_i^{(g,y)}(F)$ is a quadratic form in $y_1,\ \cdots,\ y_n$ and*

$$2\left(\sum_{k=1}^n M_k\,y_k-\langle f,\ y\nu\rangle\right)>r_0\{V_i^{(g,y)}(F)-(y\nu,\ y\nu)\}$$

for any $\nu\in\mathscr{E}_K(g,\ e)$ and any y with $|y|=1$. This is satisfied in case $n=1$ if there is $\alpha>0$ for which (2.39) *is true.*

In case $n=1$ we shall give in the next section an example (Example 3) which shows that the last condition in the above is not always necessary for (2.38), and another example (Example 4) which does not satisfy (2.38) with any r_0.

The next question is to find condition for $I(\mu_x)$ to be equal to a parabolic quadratic surface on some part $r_0\leqq|x|<\infty$. We assume that

$$(2.40)\qquad I(\mu_x)=\sum_{j,k=1}^n a_{jk}\,x_j\,x_k-2\sum_{k=1}^n b_k\,x_k+c\qquad\text{for }|x|\geqq r_0>0.$$

By Theorem 2.14 and its Corollary 1 we see that $x_k\,\gamma_k(\mu_x)$ is uniquely determined at every x in $r_0 \leqq |x| < \infty$ and that

$$2\gamma_k(\mu_x) = \frac{\partial I(\mu_x)}{\partial x_k} = 2\sum_{j=1}^{n} a_{jk}\, x_j - 2b_k \tag{2.41}$$

if $x_k > 0$. On account of (2.14) we have from (2.40) and (2.41)

$$\langle f, y\nu_{ry}\rangle = \sum_{k=1}^{n} y_k\, \gamma_k(\mu_{ry}) - \frac{I(\mu_{ry})}{r} = \sum_{k=1}^{n} b_k\, y_k - \frac{c}{r}.$$

Using Theorem 2.18, we see that

$$\sum_{k=1}^{n} b_k\, y_k = \lim_{r\to\infty} \langle f, y\nu_{ry}\rangle = \langle f, y\nu_y^*\rangle,$$

where ν_y^* is the limit of a vaguely convergent subnet of $\{\nu_{ry}\}$. Let the point $(0, \cdots, 0, 1, 0, \cdots, 0)$ on the x_k-axis be denoted by y^k. We define $\nu_{y^k}^*$ on K_k by considering a one-dimensional problem. We integrate the inequality

$$U^{\mu_x}(P) \geqq f(P) + \gamma_k(\mu_x) g(P) \qquad \text{p.p.p. on } K_k \tag{2.11}$$

and obtain

$$\int U^{\nu_{y^k}^*}\, d\mu_x = \int U^{\mu_x}\, d\nu_{y^k}^* \geqq \langle f, \nu_{y^k}^*\rangle + \gamma_k(\mu_x) = \sum_{k=1}^{n} a_{jk}\, x_j \tag{2.42}$$

$$\text{for } |x| \geqq r_0,\ x_1 > 0, \cdots, x_n > 0.$$

Let us assume that

$$U^{\nu_{y^k}^*}(P) \leqq a_{jk}\, g(P) \qquad \text{p.p.p. on } B \cap K_j \tag{2.43}$$

for each j and k. Then $\int U^{\nu_{y^k}^*}\, d\mu_x = \sum_{j=1}^{n} a_{jk}\, x_j$ and all terms are equal in (2.42). Therefore

$$U^{\mu_x}(P) = f(P) + \gamma_k(\mu_x) g(P) \qquad \nu_{y^k}^*\text{-a.e. on } K_k \tag{2.44}$$

on $|x| \geqq r_0$, $x_1 > 0, \cdots, x_n > 0$.

Conversely we assume (2.44) and that each $U^{\nu_{y^k}^*}(P)/g(P)$ is constant p.p.p. on each $B \cap K_j$; let us denote the constant by a_{jk}. It follows that $(\nu_{y^k}^*, \nu_{y^j}^*) = a_{jk} = a_{kj}$. We integrate (2.44) and obtain

$$(\mu_x, \nu_{y^k}^*) = \sum_{j=1}^{n} a_{jk}\, x_j = \langle f, \nu_{y^k}^*\rangle + \gamma_k(\mu_x).$$

On account of Theorem 2.14, the derivative at $x = (x_1, \cdots, x_n)$, $x_1 > 0, \cdots, x_n > 0$, in the direction given by $z = (z_1, \cdots, z_n)$, $|z| = 1$, is equal to

$$\frac{dI(\mu_x)}{ds_z} = 2\sum_{k=1}^{n} \gamma_k(\mu_x) z_k = 2\sum_{j,k=1}^{n} a_{jk}\, x_j\, z_k - 2\sum_{k=1}^{n} \langle f, \nu_{y^k}^*\rangle\, z_k$$

$$=\frac{d}{ds_z}\left(\sum_{j,k=1}^{n} a_{jk}\, x_j\, x_k - 2\sum_{k=1}^{n} \langle f, \nu_y^{*k}\rangle\, x_k\right).$$

Therefore

$$I(\mu_x)=\sum_{j,k=1}^{n} a_{jk}\, x_j\, x_k - 2\sum_{k=1}^{n} \langle f, \nu_y^{*k}\rangle\, x_k + \text{const}.$$

We state

THEOREM 2.22. *Let ν_y^{*k} be a measure which maximizes $\langle f, \nu\rangle$ among $\nu\in\mathcal{N}_{y^k}$. If $U^{\nu_y^{*k}}(P)\leqq a_{jk}\, g(P)$ p.p.p. on $B\frown K_j$ for each j and k and if $I(\mu_x)$ coincides with a parabolic quadratic surface on $|x|\geqq r_0>0$, then, for each k,*

$$U^{\mu_x}(P)=f(P)+\gamma_k(\mu_x)g(P) \qquad \nu_y^{*k}\text{-a.e. on } K_k$$

on $|x|\geqq r_0$, $x_1>0, \cdots, x_n>0$.

Conversely, if this is true and if each $U^{\nu_y^{*k}}(P)/g(P)$ is constant p.p.p. on each $B\frown K_j$, then $I(\mu_x)$ coincides with a parabolic quadratic surface on $|x|\geqq r_0$.

We shall give an example of continuous $f(P)$ which does not satisfy (2.44) as Example 5 in the next section.

In the same way as for Theorem 2.21, we can show

THEOREM 2.23. *In order that*

$$I(\mu_x)=\sum_{j,k=1}^{n} a_{jk}\, x_j\, x_k - 2\sum_{k=1}^{n} b_k\, x_k \qquad \text{on } r_0\leqq|x|<\infty,$$

it is necessary and sufficient that $V_i^{(g,y)}(B)$ is a quadratic form in $y_1, \cdots, y_n$ and

$$\{(y\nu, y\nu)-V_i^{(g,y)}(B)\}r_0\geqq 2(\langle f, y\nu\rangle-\langle f, y\nu_y^*\rangle)$$

for any y and $\nu\in\mathcal{E}_K(g, e)$ such that $\langle f, y\nu\rangle>-\infty$, where $\langle f, y\nu_y^*\rangle=\max_{\nu\in\mathcal{N}_y}\langle f, y\nu\rangle$.

Finally we assume that the kernel is of positive type. According to Theorem 2. 8, each $x_k\gamma_k(\mu_x)$ is single-valued. Therefore by Corollary 2 to Theorem 2.14 $I(\mu_x)$ is continuously differentiable in $x_1>0, \cdots, x_n>0$. We take x', x'', $a\geqq 0$ and $b\geqq 0$ such that $a+b=1$ and set $ax'+bx''=\bar{x}$. Then

$$a\mu_{x'}+b\mu_{x''}\in\mathcal{E}_K(g, \bar{x}) \qquad \text{and} \qquad I(\mu_{\bar{x}})\leqq I(a\mu_{x'}+b\mu_{x''}).$$

From this we obtain

$$\begin{aligned}(2.45)\qquad & I(\mu_{\bar{x}})+ab(\mu_{x'}-\mu_{x''}, \mu_{x'}-\mu_{x''})\\ &\leqq aI(\mu_{x'})+bI(\mu_{x''})+(a+b-1)\ \{a(\mu_{x'}, \mu_{x'})+b(\mu_{x''}, \mu_{x''})\}=aI(\mu_{x'})+bI(\mu_{x''}).\end{aligned}$$

Since

$$(\mu_{x'}-\mu_{x''}, \mu_{x'}-\mu_{x''})\geqq 0,$$

it follows that

$$I(\mu_{\bar{x}}) \leqq aI(\mu_{x'}) + bI(\mu_{x''}).$$

This shows that $I(\mu_x)$ is a convex function in $x_1 \geqq 0, \cdots, x_n \geqq 0$. For $a=b=1/2$, we have

$$0 \leqq \frac{1}{4} \|\mu_{x'} - \mu_{x''}\|^2 \leqq \frac{I(\mu_{x'}) + I(\mu_{x''})}{2} - I(\mu_x)$$

from (2.45). Since $I(\mu_x)$ is continuous, the right side tends to 0 as $x' - x'' \to 0$. Hence $\|\mu_{x'} - \mu_{x''}\| \to 0$. Taking

$$| \|\mu_{x'}\| - \|\mu_{x''}\| | \leqq \|\mu_{x'} - \mu_{x''}\|$$

into consideration, we see that $\|\mu_x\|$ is single-valued and continuous on $x_1 \geqq 0, \cdots, x_n \geqq 0$. Hence

$$2 \sum_{k=1}^{n} \gamma_k(\mu_x) x_k = I(\mu_x) + \|\mu_x\|^2$$

is single-valued and continuous on $x_1 \geqq 0, \cdots, x_n \geqq 0$. Since $I(\mu_x)$ is convex,

$$\frac{\partial I(\mu_{ry})}{\partial r} = \sum_{k=1}^{n} \gamma_k(\mu_{ry}) y_k$$

is an increasing function of r.

We shall investigate this case furthermore. We assume that $V_i^{(g,y)}(F)$ is finite, where $F = \bigcup_{k=1}^{n} \{P \in K_k;\ f(P) \geqq M_k g(P)\}$. Consider the class of measures of $\mathscr{E}_F(g, y)$ whose energies are equal to $V_i^{(g,y)}(F)$ and λ_y be a measure of the class such that $\langle f, \lambda_y \rangle = \max \langle f, \lambda \rangle$ for λ of the class. We have by (2.12), (2.13) and Theorem 2.17

$$\begin{aligned} \|y\nu_{ry} - \lambda_y\|^2 &= \|y\nu_{ry}\|^2 + \|\lambda_y\|^2 - 2(y\nu_{ry}, \lambda_y) \\ &\leqq \|y\nu_{ry}\|^2 + \|\lambda_y\|^2 - \frac{2}{r}\left\{ \sum_{k=1}^{n} \gamma_k(\mu_{ry}) y_k + \langle f, \lambda_y \rangle \right\} \\ &\leqq \|y\nu_{ry}\|^2 + \|\lambda_y\|^2 - 2\|y\nu_{ry}\|^2 - 2\frac{\langle f, \lambda_y \rangle - \langle f, y\nu_{ry} \rangle}{r} \\ &\leqq \|\lambda_y\|^2 - \|y\nu_{ry}\|^2 \to 0 \qquad \text{as } r \to 0. \end{aligned}$$

We defined $\mathscr{N}_y$ before as the class of measures which are the vague limits of vaguely converging nets in $\mathscr{E}_K(g, e)$ such that $(y\nu, y\nu)$ tends to $V_i^{(g,y)}(B)$ as ν tends to the limit along a net, where $B = \{P \in K;\ f(P) > -\infty\}$. Let ν_y^* be any measure maximizing $\langle f, y\nu \rangle$ among $\nu \in \mathscr{N}_y$. We integrate (2.12) and obtain

$$(\mu_x, y\nu_y^*) \geqq \langle f, y\nu_y^* \rangle + \sum_{k=1}^{n} \gamma_k(\mu_x) y_k.$$

Therefore, if $\langle f, y\nu_y^* \rangle$ is finite,

$$\|y\nu_{ry}-y\nu_y^*\|^2 \leqq \|y\nu_{ry}\|^2+\|y\nu_y^*\|^2-\frac{2}{r}\left\{\langle f, y\nu_y^*\rangle+\sum_{k=1}^{n}\gamma_k(\mu_{ry})y_k\right\}\to 0 \quad \text{as } r\to\infty$$

on account of Theorem 2.18.

We state

THEOREM 2.24. *If the kernel is of positive type, then $I(\mu_x)$ is convex and continuously differentiable, μ_x is continuous in semi-norm, and $\|\mu_x\|$ is single-valued continuous in $x_1 \geqq 0, \cdots, x_n \geqq 0$. As $r\to 0$, $\|y\nu_{ry}-\lambda_y\|\to 0$ if $V_i^{(g,y)}(F)$ is finite, and $\|y\nu_{ry}-y\nu_y^*\|\to 0$ as $r\to\infty$ if $\langle f, y\nu_y^*\rangle=\lim_{r\to\infty}\langle f, y\nu_{ry}\rangle$ is finite.*

COROLLARY. *In case $n=1$, if the kernel is of positive type, $\gamma(\mu_x)$ is an increasing single-valued function on $0\leqq x<\infty$.*

2.7. Examples.

In the first five examples, $n=1$. Namely we do not divide K into $K_1, \cdots, K_n$.

EXAMPLE 1. $\Omega=K=$ two points P_1 and P_2, $g(P)\equiv 1$, $f(P_1)=1$, $f(P_2)=2$, and $\Phi(P, Q)$ is given by

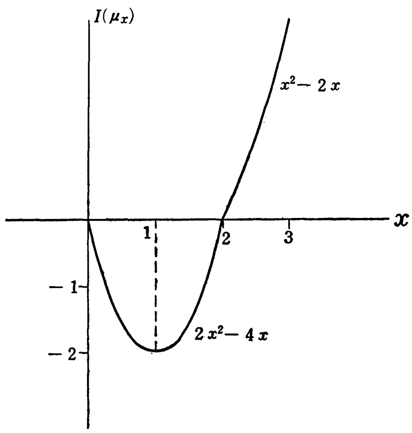
$$\begin{pmatrix} 1 & a \\ a & 2 \end{pmatrix}.$$

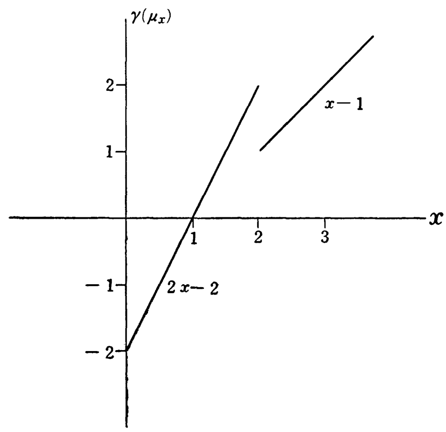
Let μ be a measure on K with total mass $x>0$, and set $x_1=\mu(\{P_1\})$ and $x_2=\mu(\{P_2\})$. We have

$$\begin{aligned} I(\mu)&=x_1^2+2x_2^2+2ax_1x_2-2x_1-4x_2 \\ &=(3-2a)\left(x_1+\frac{ax+1-2x}{3-2a}\right)^2+2x^2-4x-\frac{(ax+1-2x)^2}{3-2a}. \end{aligned}$$

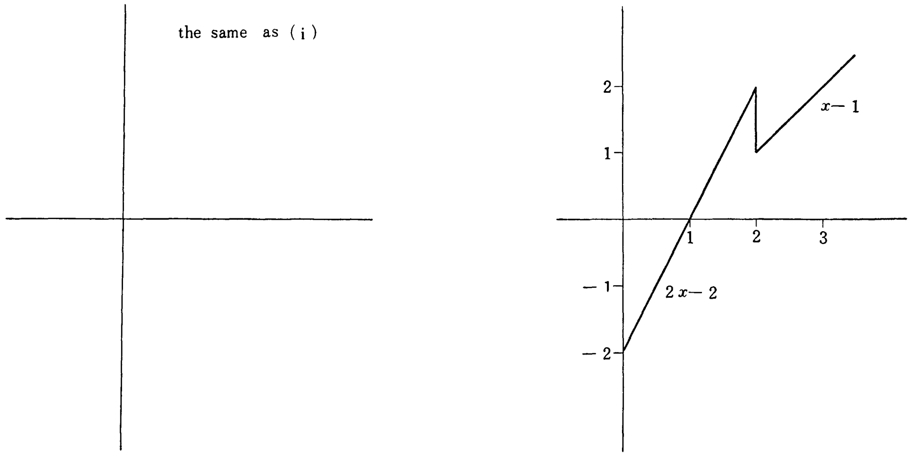
We first give the graphs of $I(\mu_x)$ and $\gamma(\mu_x)$.

(i) $a>\frac{3}{2}$

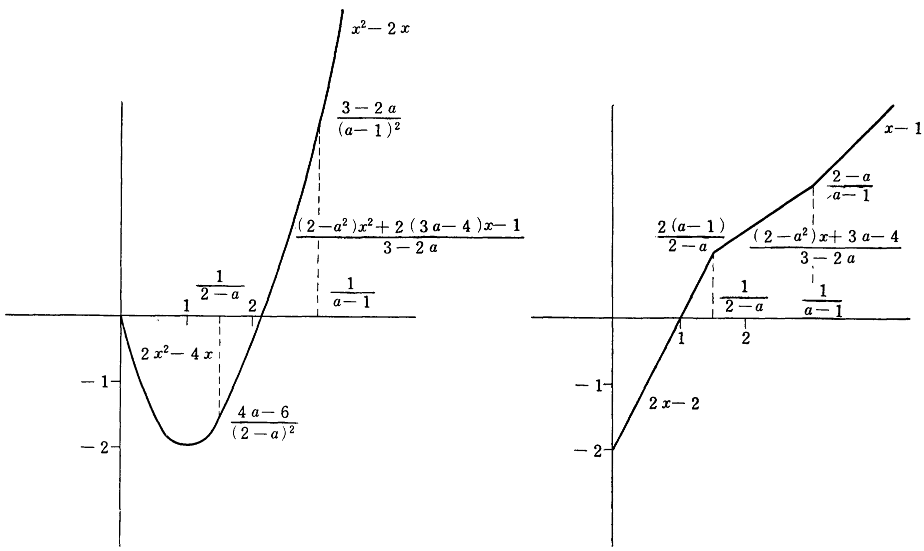




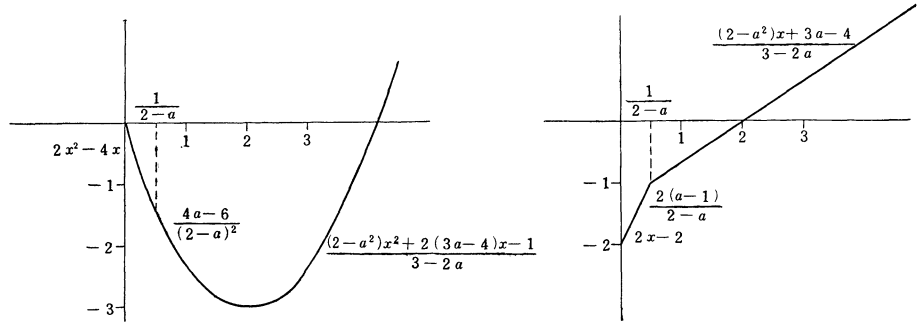
(ii) $a=\frac{3}{2}$



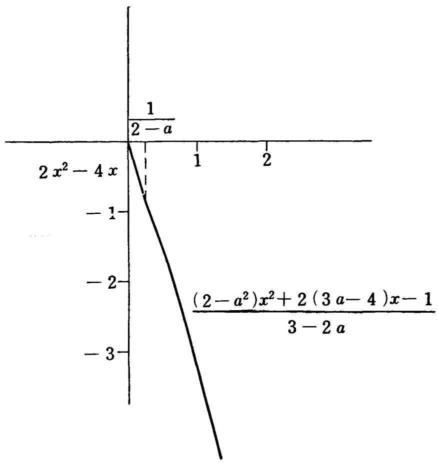
(iii) $1<a<\frac{3}{2}$

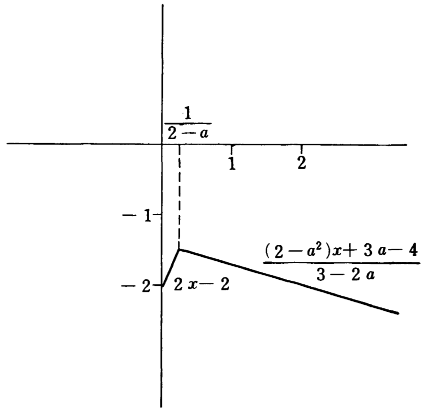


(iv) $-\sqrt{2}\leqq a\leqq 1$



(v) $a<-\sqrt{2}$





Let us examine when $\Phi(P, Q)$ is of positive type. Let μ, ν be any measures and set $\mu(\{P_1\})=x_1$, $\mu(\{P_2\})=x_2$, $\nu(\{P_1\})=x_1'$, $\nu(\{P_2\})=x_2'$. We have

$$(\mu-\nu,\ \mu-\nu)=(x_1^2+2x_2^2+2ax_1x_2)+(x_1'^2+2x_2'^2+2ax_1'x_2')$$
$$-2(x_1x_1'+2x_2x_2'+ax_1x_2'+ax_2x_1')=(x_1-x_1')^2+2(x_2-x_2')^2+2a(x_1-x_1')(x_2-x_2')$$
$$=\{(x_1-x_1')+a(x_2-x_2')\}^2+(2-a^2)(x_2-x_2')^2.$$

This is always nonnegative if and only if $a^2\leqq 2$, and, for any different μ and ν, this is positive if and only if $a^2<2$. Namely, $\Phi(P, Q)$ is of positive type for a with $|a|\leqq\sqrt{2}$, and satisfies the energy principle for a with $|a|<\sqrt{2}$.

We observe several characteristic points in the above figures.

1) In (i) $\mathscr{M}_2$ consists of two measures: a point measure at P_1 and a point measure at P_2.

2) In (ii) $\mathscr{M}_2$ consists of the segment joining the above two measures.

3) For a, $1<|a|<\sqrt{2}$, the kernel satisfies the energy principle but $d^2I(\mu_x)/dx^2$ does not everywhere exist.

4) In (v) $\gamma(\mu_x)$ is continuous but not increasing.

EXAMPLE 2. $\Omega=K=3$ points P_1, P_2 and P_3, $g(P)\equiv 1$, $f(P_1)=3/2$, $f(P_2)=2$, $f(P_3)=3$, $\Phi(P, Q)$ is given by

$$\begin{pmatrix} 1 & a_1 & a_3 \\ a_1 & 2 & a_2 \\ a_3 & a_2 & 4 \end{pmatrix}.$$

For μ with total mass x, we set $x_i=\mu(\{P_i\})$, $i=1, 2, 3$. We have

$$I(\mu)=x_1^2+2x_2^2+4x_3^2+2a_1x_1x_2+2a_2x_2x_3+2a_3x_3x_1-3x_1-4x_2-6x_3.$$

Fixing x_3, we substitute $x-x_1-x_3$ for x_2 and differentiate $I(\mu)$ twice partially with respect to x_1. It follows that

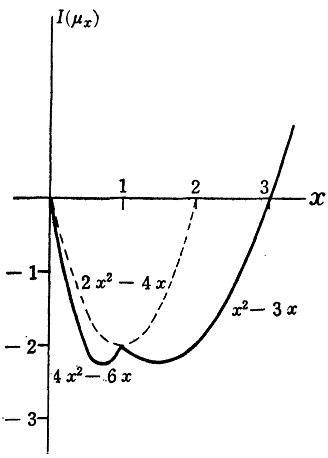
$$\frac{\partial^2 I(\mu)}{\partial x_1^2} = 6 - 4a_1.$$

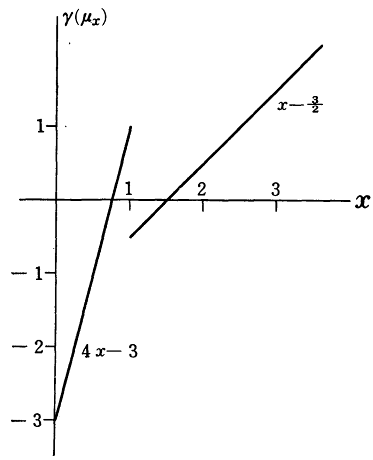
If $a_1 > 3/2$, this is negative and the minimum is taken when $x_1 = 0$ or $x_2 = 0$. By a similar reasoning we can assert that the minimum is taken when $x_1 = x$ or $x_2 = x$ or $x_3 = x$. Namely, the graph of $I(\mu_x)$ is equal to the lower envelope of three parabolas $x^2 - 3x$, $2x^2 - 4x$, $4x^2 - 6x$. It is equal to the lower envelope of $4x^2 - 6x$ and $x^2 - 3x$.

The graph of $\gamma(\mu_x)$ is equal to

$$\begin{array}{ll} 4x-3 & \text{for } 0 \leqq x < 1, \\ 1,\ 0,\ -3/2 & \text{for } x = 1, \\ x - 3/2 & \text{for } 1 > x. \end{array}$$

Thus $\gamma(\mu_{x_2})$ has 3 values at $x=1$. Both curves $4x^2 - 6x$ and $x^2 - 3x$ have the same minimum value $-9/4$.





In a similar way we can give an example in which $I(\mu_x)$ is equal to the lower envelope of any finite number of parabolas passing through the origin.

Question. *Can $\gamma(\mu_x)$ have an infinite number of points of multivalency, clustering at a finite point or tending to $+\infty$?*[30]

Example 3. We shall solve the variational problem in the following case: $\Omega = K =$ the ball $\{P; \overline{OP} \leqq 2\}$ in E_3, $\Phi(P, Q) = 1/\overline{PQ}$, $g(P) \equiv 1$, $f(P) = 1$ on the ball $K_0 = \{P; \overline{OP} \leqq 1\}$ and $= 1/\overline{OP}$ outside K_0 on K; if ν^* is the uniform unit measure on ∂K_0, $f(P) = U^{\nu^*}(P)$ on K. Our problem is to find $\nu_x \in \mathscr{E}_K(g, 1)$ which gives $I(x\nu_x) = \min I(\mu)$ for $\mu \in \mathscr{E}_K(g, x)$.

30) Ogasawara told the author that he has an affirmative answer.

Since the energy principle is satisfied, ν_x is unique and hence its distribution depends only on the radius $\overline{OP}=r$. We denote the mass $\nu_x(\{P;\ \overline{OP}\leqq r\})$ by $\nu_x(r)$. The potential

$$\int_{0<r_1\leqq\overline{OQ}\leqq r_2\leqq 2}\frac{1}{\overline{PQ}}\,d\nu_x(Q)$$

is constant on any spherical surface with center at 0 and harmonic in $\overline{OP}<r_1$. Therefore it is constant there and equals

$$\int_{r_1\leqq r\leqq r_2}\frac{d\nu_x(r)}{r}.$$

By taking the mean on the spherical surface passing P we see that

$$\int_{r_1\leqq\overline{OQ}\leqq r_2\leqq\overline{OP}}\frac{1}{\overline{PQ}}\,d\nu_x(Q)=\frac{\nu_x(\{Q;\ r_1\leqq\overline{OQ}\leqq r_2\})}{\overline{PO}}.$$

Therefore

$$\int_0^2\frac{d\nu_x(Q)}{\overline{PQ}}=\int_{\overline{OP}<r\leqq 2}\frac{d\nu_x(r)}{r}+\frac{\nu_x(\overline{OP})}{\overline{OP}}$$

$$=\frac{\nu_x(r)}{r}\Big|_{\overline{OP}}^{2}+\int_{\overline{OP}<r\leqq 2}\frac{\nu_x(r)}{r^2}\,dr+\frac{\nu_x(\overline{OP})}{\overline{OP}}=\frac{1}{2}+\int_{\overline{OP}<r\leqq 2}\frac{\nu_x(r)}{r^2}\,dr.$$

Hence

$$(\nu_x,\ \nu_x)=\frac{1}{2}+\int_0^2 d\nu_x(P)\int_{\overline{OP}<r\leqq 2}\frac{\nu_x(r)}{r^2}\,dr=\frac{1}{2}+\int_0^2\frac{\nu_x^2(r)}{r^2}\,dr$$

and

$$(\nu_x,\ \nu^*)=\frac{1}{2}+\int_1^2\frac{\nu_x(r)}{r^2}\,dr.$$

We have

$$(\nu_x,\ \nu_x)-\frac{2}{x}(\nu_x,\ \nu^*)=\int_0^1\frac{\nu_x^2(r)}{r^2}\,dr+\int_1^2\frac{\nu_x^2(r)-2\nu_x(r)/x}{r^2}\,dr+\frac{1}{2}-\frac{1}{x}$$

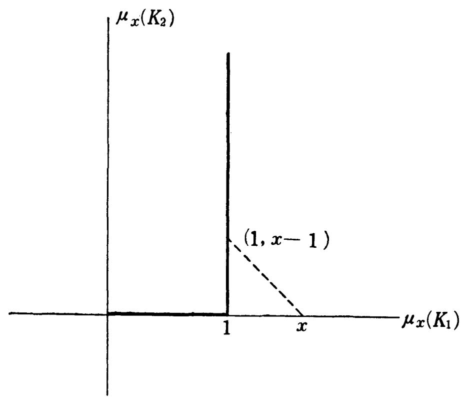
$$=\int_0^1\frac{\nu_x^2(r)}{r^2}\,dr+\int_1^2\frac{\left(\nu_x(r)-\dfrac{1}{x}\right)^2}{r^2}\,dr-\frac{1}{2x^2}-\frac{1}{x}+\frac{1}{2}.$$

The first integral attains its minimum when $\nu_x(r)=0$ on $0\leqq r<1$, and the second integral does when $\nu_x(r)=\min(1,\ 1/x)$ on $1\leqq r<2$. Therefore it is concluded that ν_x is the uniform unit measure on ∂K_0 when $x\leqq 1$ and it is the sum of the uniform measure on ∂K_0 with total mass $1/x$ and the one on ∂K with total mass $1-1/x$ when $x>1$. We find

$$I(\mu_x)=\begin{cases} x^2-2x & \text{on } 0\leqq x\leqq 1, \\ \dfrac{x^2}{2}-x-\dfrac{1}{2} & \text{on } 1<x<\infty. \end{cases}$$

This example shows that (2.39) is not always necessary for (2.38) to be true.

We write K_1 and K_2 for ∂K_0 and ∂K respectively, and take the same kernel, $f(P)$ and $g(P)$ as above. The extremal measure μ_x is then the same for $K_1 \cup K_2$. The graph given below has relation to the problem treated in Theorem 2.16.



Example 4. We shall show by an example that if $f(P)$ decreases slowly as P goes away from F, then (2.38) does not hold for any $x_0>0$. We take everything the same as in Example 3 except for $f(P)$: $f(P)=1$ on K_0 and $=1-\exp(1-\overline{OP})^{-1}$ on $K-K_0$. Let $\nu(r)$ denote the uniform unit measure on the surface $\overline{OP}=r$, $1<r<2$. We have $\langle f, \nu^*\rangle-\langle f, \nu(r)\rangle=1-\{1-\exp(1-r)^{-1}\}=\exp(1-r)^{-1}$. If we set $(r-1)^{-1}=t$, it is equal to e^{-t}. It follows that

$$\frac{\langle f, \nu^*\rangle-\langle f, \nu(r)\rangle}{(\nu^*, \nu^*)-(\nu(r), \nu(r))}=\frac{e^{-t}}{1-\dfrac{1}{r}}=\frac{rt}{e^t}\to 0$$

as $r\to 1$ and hence as $t\to\infty$. Thus (2.38) is not satisfied with any $x_0>0$.

Example 5. $\Omega=E_3$, $K=$ the unit spherical surface with center at the origin 0. We shall give a continuous function $f(P)$ on K with the property that, for any constant c, $f(P)+c$ is not equal even a.e. on K to any Newtonian potential of a measure on K; a.e. here is understood in the 2-dimensional sense.

Prior to the construction we remark two facts. First, for any bounded Borel function $h(P)$ given on K we regard the Poisson integrals as solutions of the interior and exterior Dirichlet problems. They tend to $h(P)$ as the variable approaches any point P of continuity of $h(P)$ on K. We shall denote by I_h the function in E_3 obtained by extending $h(P)$ by the Poisson integrals. If, in particular, $h(P)=$ const. c on K, then the interior solution is constantly c and the exterior solution is equal to $c/\overline{OP}$. Secondly, if $h_n(P)$ tends to $h(P)$ monotonously with possible exception of a finite number of points on K, then I_{h_n} tends to I_h monotonously.

We take an arbitrary point P_0 on K and denote by C_n the spherical surface with center P_0 and radius $1/n$. We shall denote the mean value over C_n of I_h by $M_n(h)$. If the boundary value on K is equal to a positive constant c, then

$$M_n(c)=\frac{c}{\omega_n}\int_{C_n'} d\sigma_n+\frac{c}{\omega_n}\int_{C_n''}\frac{1}{\overline{OQ}}\,d\sigma_n(Q)$$

where ω_n is the surface area of C_n, $d\sigma_n$ is a surface element and C_n', C_n'' are the parts of C_n inside and outside K respectively. We see that

$$M_n(c)\geqq\frac{c}{\omega_n}\int_{C_n}\frac{1}{1+\dfrac{1}{n}}\,d\sigma_n(Q)=\frac{cn}{n+1}.$$

In particular $M_n(1/\sqrt{n})\geqq\sqrt{n}(n+1)^{-1}$. By the second remark given above, we can find a neighborhood N_n of P_0 on K such that, if the boundary value $h(P)$ is nonnegative continuous and not smaller than $1/\sqrt{n}$ outside N_n, then $M_n(h)\geqq\sqrt{n}\,\{2(n+1)\}^{-1}$. We assume that N_n is the intersection of K with a ball around P_0 and that the radius decreases strictly as $n\to\infty$. Now we set

$$f(P)=\begin{cases}1 & \text{on } K-N_1,\\ \dfrac{1}{\sqrt{n}} & \text{on } \partial N_n,\end{cases}$$

and define it entirely on K such that it is continuous and $f(P)\geqq 1/\sqrt{n}$ outside N_n. Then

$$M_n(f)>\frac{\sqrt{n}}{2(n+1)}.$$

We shall show that this $f(P)$ is a required one. We assume that there is a constant c and a measure μ on K such that

$$f(P)+c=U^{\mu}(P)\qquad\text{a.e. on } K; \tag{2.46}$$

a.e. is understood here in the 2-dimensional sense. Let B be the part of K where the equality holds. Since $U^{\mu}(P)$ is lower semicontinuous,

$$f(P)+c=\varliminf_{Q\in B,\,Q\to P} U^{\mu}(Q)\geqq U^{\mu}(P)$$

at every point P of K. At any point P_1 of $B\cap S_{\mu}$, it holds that

$$f(P_1)+c \geqq \overline{\lim_{P\in K,\, P\to P_1}}\, U^{\mu}(P).$$

It is well known that

$$\overline{\lim_{P\in S_{\mu},\, P\to P_1}}\, U^{\mu}(P)=\overline{\lim_{P\to P_1}}\, U^{\mu}(P).$$

Therefore

$$f(P_1)+c \geqq \overline{\lim_{P\to P_1}}\, U^{\mu}(P).$$

On account of the lower semicontinuity of $U^{\mu}(P)$, it follows that

$$\lim_{P\to P_1} U^{\mu}(P)=f(P_1)+c.$$

This is evident if $P_1 \in B-S_{\mu}$. Now $U^{\mu}(P)$ and I_{f+c} have the same boundary limits a.e. on K and vanish at the point at infinity. Hence they coincide in the whole space.

We shall compare $M_n(f+c)$ with the value of $f+c$ at P_0, namely with $f(P_0)+c=c$. We see that

$$M_n(f+c)=M_n(f)+M_n(c)>\frac{\sqrt{n}}{2(n+1)}+c-\frac{c}{n+1}=c+\frac{1}{n+1}\left(\frac{\sqrt{n}}{2}-c\right).$$

If n is sufficiently large, then $M_n(f+c)>c$. This shows that the mean value of $U^{\mu}(P)$ over C_n is greater than its value at the center of C_n. This is impossible because $U^{\mu}(P)$ is superharmonic in the whole space.

Let us see how this example is related to Theorem 2.19. We consider the Newtonian kernel in E_3, take a unit spherical surface for K and $f(P)$ of Example 5, and set $g(P)\equiv 1$ on K. Then the support of the equilibrium measure on K coincides with K. We have seen that

$$U^{\mu_x}(P)<f(P)+\gamma(\mu_x)$$

on a subset of K of positive 2-dimensional measure. Consequently, for some continuous function $f(P)$, the part of $I(\mu_x)$ corresponding to any interval $x_0 \leqq |x| < \infty$ does not coincide with any polynomial.

If we allow $f(P)$ to be discontinuous it is rather easy to construct an example. We take the same Ω, K and $g(P)$ as above. We divide K into two semispheres and set $f(P)=1$ on one closed semisphere C_1 and $f(P)=0$ on the rest C_2 of K. Suppose that there are a constant c and a measure μ supported by K such that (2.46) is true. For the same reason as above we conclude that $U^{\mu}(P)$ coincides with I_{f+c}. Let P_0 be any point of the border of C_1. By the lower semicontinuity of $U^{\mu}(P)$ it holds that $U^{\mu}(P_0)\leqq c$. On the other hand we set $\check{f}=1-f$ on K and see that the mean value $M_n(f+\check{f})=2M_n(f)$ around P_0 tends to $1/2$. Therefore $M_n(f+c)=M_n(f)+M_n(c)$ tends to $c+1/2>U^{\mu}(P_0)$. This contradicts the superharmonicity of $U^{\mu}(P)$.

EXAMPLE 6. $K_1=$ two points P_1, P_2, $K_2=$ one point P_3, $K=K_1\cup K_2$; hence $n=2$. $f(P)\equiv 0$, $g(P)\equiv 1$, $\Phi(P, Q)$ is given by

$$\begin{pmatrix} a & d & e \\ d & b & f \\ e & f & c \end{pmatrix}.$$

For $y=(y_1, y_2)$, $y_1\geqq 0$, $y_2\geqq 0$ such that $y_1^2+y_2^2=1$ and for $\mu\in\mathscr{E}_K(g, y)$, we set $\mu(\{P_1\})=y_1'$ and $\mu(\{P_2\})=y_1''$ and have

$$I(\mu)=ay_1'^2+by_1''^2+2dy_1'y_1''+cy_2^2+2ey_1'y_2+2fy_1''y_2$$
$$=\begin{cases}(a+b-2d)(y_1'-Y)^2+by_1^2+cy_2^2+2fy_1y_2-(a+b-2d)Y^2 & \text{if } a+b\neq 2d,\\ -2\{(b-d)y_1+(f-e)y_2\}y_1'+by_1^2+cy_2^2+2fy_1y_2 & \text{if } a+b=2d,\end{cases}$$

where

$$Y=\frac{(b-d)y_1+(f-e)y_2}{a+b-2d}.$$

We see that $\min\limits_{0\leqq y_1'\leqq y_1} I(\mu)$ is a quadratic form in y_1 and y_2 in the following cases:

(1) $a=\min(a, b)\leqq d$ and $e\leqq f$,
(2) $b=\min(a, b)\leqq d$ and $f\leqq e$,
(3) $d\leqq\min(a, b)$ and $e=f$.

In other cases the graph of $I(\mu_x)$ is not a parabolic quadratic surface but consists of two or three pieces of different parabolic quadratic surfaces.

EXAMPLE 7. Let K_1 and K_2 be the surfaces of mutually disjoint unit balls in E_3, and d be the distance between the centers. Considering the Newtonian kernel, we shall show that $V_i^{(1,y)}(K_1\cup K_2)$ is not a quadratic form in y_1 and y_2 at least for large d, where $y=(y_1, y_2)$ and $y_1^2+y_2^2=1$.

We shall use the following classical result in the electrostatic theory (see Smythe [1], pp. 118–119, for instance): There is a sequence of point measures inside K_1 with the total mass

$$m_1=1+\frac{1}{d^2-1}+\frac{1}{(d^2-1)^2-d^2}+\cdots \tag{2.47}$$

and a sequence of negative point measures inside K_2 with the total mass

$$-m_2=-\frac{1}{d}-\frac{1}{d(d^2-2)}-\cdots \tag{2.48}$$

such that the potential of the whole measure is equal to 1 on K_1 and 0 on K_2. By sweeping-out processes we can find a positive measure μ_1 supported by K_1

and a negative measure $-\mu_2$ supported by K_2 which together give the same effect on and outside $K=K_1\cup K_2$. The total masses remain unchanged. We interchange μ_1 and $-\mu_2$ and denote the resulting measures on K_1 and K_2 by $-\nu_1$ and ν_2 respectively; the total masses of $-\nu_1$ and ν_2 are $-m_2$ and m_1 respectively. It follows that

$$\frac{\mu_1-\nu_1+\nu_2-\mu_2}{2(m_1-m_2)}$$

is a unit measure on K whose potential is constantly $(2m_1-2m_2)^{-1}$ on K. Because of the uniqueness of measures which give the same potential, it is concluded that the measure is the nonnegative equilibrium measure on K. The value of $V_i^{(1,y)}(K)$ for $y=(1/\sqrt{2}, 1/\sqrt{2})$ is equal to

$$\frac{1}{m_1-m_2}.$$

Given $y_1>0$, $y_2>0$, $y_1^2+y_2^2=1$, let us solve

$$\begin{cases} tm_1-sm_2=y_1, \\ -tm_2+sm_1=y_2 \end{cases}$$

in t and s. The solution is

$$t=\frac{m_1y_1+m_2y_2}{m_1^2-m_2^2}, \quad s=\frac{m_2y_1+m_1y_2}{m_1^2-m_2^2},$$

and the potential of $\lambda=t(\mu_1-\mu_2)-s(\nu_1-\nu_2)$ is equal to t on K_1 and to s on K_2. We shall show that, if $|y_1-y_2|$ is small, λ is the extremal nonnegative measure whose energy is equal to $V_i^{(1,y)}(K)$, where $y=(y_1, y_2)$.

There is a number a_0, $0<a_0<1$, such that the a_0-niveau surface of the harmonic function $h_0(P)$, equal to 1 on K and to 0 at the point at infinity, consists of two closed surfaces, one F_1 enclosing K_1 and the other F_2 enclosing K_2. We recall that $\gamma_1(\mu_y)$ and $\gamma_2(\mu_y)$ are continuous with respect to y. Hence there exists y_0, $0<y_0<1/\sqrt{2}$, such that, for any $y=(y_1, \sqrt{1-y_1^2})$ with $y_1\in[y_0, 1/\sqrt{2}]$,

$$\left|\gamma_i(\mu_y)-\frac{1}{\sqrt{2}(m_1-m_2)}\right|<\frac{1-a_0}{2\sqrt{2}(m_1-m_2)} \qquad (i=1, 2).$$

Suppose now that $K_1\cap S_{\mu_y}\neq\emptyset$, and consider

$$h(P)=U^{\mu_y}(P)-\frac{1}{\sqrt{2}(m_1-m_2)}h_0(P)-\frac{1-a_0}{2\sqrt{2}(m_1-m_2)}.$$

This is harmonic outside K and takes a negative value at the point at infinity. Since

$$\max_K h(P) \leqq \max_{S_{\mu_y}} U^{\mu_y}(P) - \frac{1}{\sqrt{2}(m_1-m_2)} - \frac{1-a_0}{2\sqrt{2}(m_1-m_2)}$$
$$= \max_{i=1,2} \gamma_i(\mu_x) - \frac{1}{\sqrt{2}(m_1-m_2)} - \frac{1-a_0}{2\sqrt{2}(m_1-m_2)} \leqq 0,$$

$h(P) \leqq 0$ outside K by the maximum principle for harmonic function. Hence

$$U^{\mu_y}(P) \leqq \frac{1}{\sqrt{2}(m_1-m_2)} h_0(P) + \frac{1-a_0}{2\sqrt{2}(m_1-m_2)} = \frac{1+a_0}{2\sqrt{2}(m_1-m_2)}$$

on $F_1 \cup F_2$. On $S_{\mu_y} \cap K_1$ we have

$$U^{\mu_y}(P) = \gamma_1(\mu_y) > \frac{1}{\sqrt{2}(m_1-m_2)} - \frac{1-a_0}{2\sqrt{2}(m_1-m_2)} > \frac{1+a_0}{2\sqrt{2}(m_1-m_2)}.$$

Therefore

$$U^{\mu_y}(P) < \gamma_1(\mu_y)$$

in the domain bounded by $S_{\mu_y} \cap K_1$ and F_1. This contradicts the fact that $U^{\mu_y}(P) \geqq \gamma_1(\mu_y)$ p.p.p. on K_1 which was shown in Theorem 2.7. Consequently $S_{\mu_y} \supset K_1$. Similarly we see that $S_{\mu_y} \supset K_2$ and hence S_{μ_y} coincides with K.[31] It is shown that $U^{\mu_y}(P) = \gamma_1(\mu_y)$ on K_1 and $= \gamma_2(\mu_y)$ on K_2. By means of the energy principle we can conclude that μ_y is equal to the above $\lambda = t(\mu_1 - \mu_2) - s(\nu_1 - \nu_2)$. Its energy is given by

$$ty_1 + sy_2 = \frac{m_1 y_1^2 + 2m_2 y_1 y_2 + m_1 y_2^2}{m_1^2 - m_2^2}.$$

If $V_i^{(1,y)}(K)$ were a quadratic form in y_1 and y_2, then the coefficient of y_1^2 would be equal to $V_i^{(1,1)}(K_1) = 1$. Therefore

$$m_1 = m_1^2 - m_2^2. \tag{2.49}$$

We substitute (2.47) and (2.48) in (2.49), expand it into a series in $1/d$ and find that (2.49) is not true in general. It is now proved that $V_i^{(1,y)}(K)$ is not a quadratic form in y_1 and y_2 at least for large d.

2.8. Unconditional variation.

Let K be a compact set with $\mathscr{E}_K \neq \{0\}$. We fix an upper semicontinuous function $f(P) < \infty$ which does not have the property that $f(P) = -\infty$ p.p.p. on K, and fix an arbitrary positive continuous function $g(P)$ on K. We denote by $I(\mu_x)$ the conditional minimum of $I(\mu) = (\mu, \mu) - 2\langle f, \mu \rangle$ under the condition $\langle g, \mu \rangle = x$ as before. If there exsits an extremal measure μ^* which gives the finite unconditional minimum of $I(\mu)$, μ^* must be one of the conditional

31) This reasoning is due to Leja [2].

extremal measures, which minimize $I(\mu)$ under the condition $\langle g, \mu\rangle=\langle g, \mu^*\rangle$, and $I(\mu^*)$ must be the minimum value of the continuous function $I(\mu_x)$ of x. Conversely, if $I(\mu_x)$ has a finite minimum value at x_0, then any conditional extremal measure μ_{x_0} is an unconditional extremal measure. Thus the unconditional problem is reduced to finding a finite minimum value of $I(\mu_x)$. Things are the same in case $n\geqq 2$. However, finding a minimum point of a curve of $I(\mu_x)$ in case $n=1$ is much easier than finding a minimum point of a surface in case $n\geqq 2$. For this reason we shall restrict ourselves to the case $n=1$ in this section.

By Theorem 2.19 we have

THEOREM 2.25. *There is an extremal measure which gives the finite unconditional minimum for $I(\mu)$ if $V_i(B)>0$ or if $V_i(B)=0$ and $\langle f, \nu^*\rangle<0$, where ν^* maximizes $\langle f, \nu\rangle$ among $\nu\in\mathcal{N}$; $\nu\in\mathcal{N}$ if ν is the vague limit of a net $\{\nu^{(\omega)}\}$ in $\mathcal{E}_K(g, 1)$ such that $(\nu^{(\omega)}, \nu^{(\omega)})$ tends to $V_i^{(g)}(B)$. There is no finite unconditional minimum if $V_i(B)<0$ or if $V_i(B)=0$ and $\langle f, \nu^*\rangle>0$.*

The case that $V_i(B)=0$ and $\langle f, \nu^*\rangle=0$ is delicate. In the special case that $I(\mu_x)=bx+c$ on some $x_0\leqq x<\infty$, $\langle f, \nu_x\rangle=-b/2-c/x$ on $x_0\leqq x<\infty$ and $b>0$ or $=0$ or <0 according as $\langle f, \nu^*\rangle<0$ or $=0$ or >0. As an example, we consider

$$I(\mu_x)=x_1^2+2x_2^2-2\sqrt{2}x_1 x_2-2x_1+2\alpha x_2$$

with the same notations x_1 and x_2 as in Example 1 of § 2.7. According as $\alpha<\sqrt{2}$, $=\sqrt{2}$, $>\sqrt{2}$, we have $b<0$, $b=0$, $b>0$.

THEOREM 2.26. *Assume that $(\mu, \mu)>0$ whenever $S_\mu\subset K$, $\mu\not\equiv 0$ and $\langle f, \mu\rangle \geqq 0$. Then there is no $x_0>0$ such that $I(\mu_{x_0})=\min_{0\leqq x<\infty} I(\mu_x)$ if and only if $f(P)\leqq 0$ p.p.p. on K.*

PROOF. First we suppose that there is a compact set $F\subset K$ with $\mathcal{E}_F\not\equiv\{0\}$ such that $f(P)>0$ on F. By Proposition 1 of Chapter I, there is a compact subset F_1 with $\mathcal{E}_{F_1}\not\equiv\{0\}$ of F on which $f(P)>\alpha>0$. We take $\lambda\in\mathcal{E}_{F_1}(g, 1)$. By assumption $(\lambda, \lambda)>0$ and, for any $x>0$,

$$\begin{aligned} I(x\lambda)&=x^2(\lambda, \lambda)-2x\langle f, \lambda\rangle\leqq x^2(\lambda, \lambda)-2x\alpha\,(\max_K g)^{-1} \\ &=(\lambda, \lambda)\left(x-\frac{\alpha}{(\lambda, \lambda)\max_K g}\right)^2-\frac{\alpha^2}{(\lambda, \lambda)\,(\max_K g)^2}\,. \end{aligned}$$

For $x=\alpha\,\{(\lambda, \lambda)\max_K g\}^{-1}$, $I(x\lambda)=-\alpha^2\{(\lambda, \lambda)\,(\max_K g)^2\}^{-1}<0$ and hence $\min_{0\leqq x<\infty} I(\mu_x)<0$. Since $I(\mu_x)\to\infty$ with x, there is $x_0>0$ which gives $I(\mu_{x_0})=\min_{0\leqq x<\infty} I(\mu_x)$.

Conversely, if $f(P)\leqq 0$ p.p.p. on K then $\int fd\mu\leqq 0$ for any $\mu\in\mathcal{E}_K$. For $x>0$ we have

$$I(\mu_x)=(\mu_x, \mu_x)-2\langle f, \mu_x\rangle \geqq \max\{(\mu_x, \mu_x), -\langle f, \mu_x\rangle\}>0.$$

Therefore $x_0=0$.

THEOREM 2.27. *If $x_0>0$ and $I(\mu_x)$ has a local minimum value at x_0, then $\gamma(\mu_{x_0})=0$ and $dI(\mu_x)/dx$ at x_0 is equal to zero.*

PROOF. If $I(\mu_{x_0})$ is a local minimum value, $I'_-(\mu_x)|_{x=x_0}\leqq 0$ and $I'_+(\mu_x)|_{x=x_0} \geqq 0$. However, by Theorem 2.14, $I'_+(\mu_x)=2\underline{\gamma}(x)\leqq 2\overline{\gamma}(x)=I'_-(\mu_x)$. Hence, at $x=x_0$, $I'_+(\mu_x)=I'_-(\mu_x)=0$ and $\gamma(\mu_x)=I'(\mu_x)=0$.

REMARK. There may be many points at which $\min_{0\leqq x<\infty} I(\mu_x)$ is attained. In example 2 of § 2.7 the minimum value $-9/4$ is attained at $x=3/4$ and $3/2$. By modifying the same example it is easy to give an example in which a local minimum value exists and is different from the minimum value.

So far we considered the unconditional problem on a compact set. Now we consider a general set. We shall not take any $g(P)$ in advance.

THEOREM 2.28. *Consider $A\in\mathfrak{A}$ with $\mathscr{E}_A\neq\{0\}$ such that (μ, ν) is well-defined for any $\mu\in\mathscr{E}'_A$ and $\nu\in\mathscr{E}_A$. Let $f(P)$ be an $\mathfrak{A}$-measurable function on A such that $\langle f, \nu\rangle$ is defined for any $\nu\in\mathscr{E}_A$ and $\mu'\neq 0$ be a measure of $\mathscr{E}'_A$ with the following properties:*

$I(\mu')$ is finite,

There is an $\mathfrak{A}$-measurable positive function $g(P)$ on A such that $\langle g, \mu'\rangle <\infty$, that $\langle g, \nu\rangle$ is defined and finite for any $\nu\in\mathscr{E}_A$ and that

$$I(\mu')=\min_{\lambda\in\mathscr{E}'_A(g, \langle g, \mu'\rangle)} I(\lambda),$$

There is t_0, $0<t_0<1$, such that

$$I((1+t)\mu')\geqq I(\mu') \qquad \text{if } |t|\leqq t_0.$$

Then

$$U^{\mu'}(P)\geqq f(P) \qquad p.p.p. \text{ on } A, \tag{2.50}$$

and

$$U^{\mu'}(P)=f(P) \qquad \mu'\text{-a. e.} \tag{2.51}$$

If, in addition, $f(P)$ is upper semicontinuous on A, then $U^{\mu'}(P)\leqq f(P)$ everywhere on $S_{\mu'}\cap A$ and $U^{\mu'}(P)\geqq f(P)$ p.p.p. on A except on a set which is the intersection of an F_σ-set with A.

If the kernel is of positive type and there are measures μ' and μ'' of $\mathscr{E}'_A$ which satisfy (2.50) *and* (2.51), *then $\|\mu'-\mu''\|=0$ and $I(\mu')=I(\mu'')$. If the energy principle is satisfied, there is at most one measure of $\mathscr{E}'_A$ which satisfies the two inequalities.*

PROOF. We have

$$I((1+t)\mu')=(1+t)^2(\mu', \mu')-2(1+t)\langle f, \mu'\rangle \geqq I(\mu').$$

Hence

$$0=\frac{dI((1+t)\mu')}{dt}\Big|_{t=0}=2(\mu', \mu')-2\langle f, \mu'\rangle$$

and

$$\gamma(\mu')=\{(\mu', \mu')-\langle f, \mu'\rangle\}/\langle g, \mu'\rangle=0.$$

Therefore by Theorem 2.1 $U^{\mu'}(P)\geqq f(P)$ p.p.p. on A and $U^{\mu'}(P)=f(P)$ μ'-a.e. If $f(P)$ is upper semicontinuous on A, then $U^{\mu'}(P)\leqq f(P)$ everywhere on $S_{\mu'}\frown A$ and $U^{\mu'}(P)=f(P)$ on A except a set, which is the intersection of an F_σ-set with A.

If μ' satisfies (2.51), then $(\mu', \mu')=\langle f, \mu'\rangle$ and

$$I(\mu')=(\mu', \mu')-2\langle f, \mu'\rangle=-(\mu', \mu').$$

If the kernel is of positive type and (2.50) and (2.51) are true for μ' and μ'' of $\mathscr{E}'_A$, then

$$\begin{aligned}0&\leqq\|\mu'-\mu''\|^2=\|\mu'\|^2+\|\mu''\|^2-2(\mu', \mu'')\\&\leqq\|\mu'\|^2+\|\mu''\|^2-2\langle f, \mu''\rangle=\|\mu'\|^2-\|\mu''\|^2.\end{aligned}$$

Therefore $\|\mu''\|\leqq\|\mu'\|$. Similarly $\|\mu'\|\leqq\|\mu''\|$, and hence $\|\mu'\|=\|\mu''\|$ and $\|\mu'-\mu''\|=0$. Consequently

$$I(\mu')=-(\mu', \mu')=-(\mu'', \mu'')=I(\mu'').$$

If the energy principle is satisfied, $\mu'=\mu''$ follows from $\|\mu'-\mu''\|=0$.

Corollary. *Consider the same A and $f(P)$ as in the theorem. Let $\mu'\not\equiv 0$ be a measure of $\mathscr{E}'_A$ which gives the unconditional finite minimum to $I(\mu)$ among $\mu\in\mathscr{E}'_A$ for which $\langle f, \mu\rangle$ is defined. Then $U^{\mu'}(P)\geqq f(P)$ p.p.p. on A and $U^{\mu'}(P)=f(P)$ μ'-a.e. If $f(P)$ is upper semicontinuous on A, then $U^{\mu'}(P)\leqq f(P)$ on $S_{\mu'}\frown A$.*

Remark 1. Given a compact set K with $\mathscr{E}_K\not\equiv\{0\}$, a positive continuous function $g(P)$ on K and an upper semicontinuous finite-valued function $f(P)$ on K, we consider $I(\mu_x)$ as a function of x as before. If $x_0>0$ and $I(\mu_{x_0})$ is a local minimum value, then μ_{x_0} satisfies the three conditions required on μ' in Theorem 2.28.

Remark. 2. Let us assume the same as in Theorem 2.26. Then we can give a different proof to that theorem by means of (2.50) and (2.51).

Remark. 3. If we do not assume the positivity of type of the kernel, two inequalities can be true for μ_{x_0} which does not give a local minimum of $I(\mu_x)$. For instance, if we consider

$$I(\mu)=2x_1^2+4x_2^2+2\alpha x_1 x_2-4x_1-6x_2$$

with a sufficiently large α under the same circumstances as in Example 1 of § 2.7,

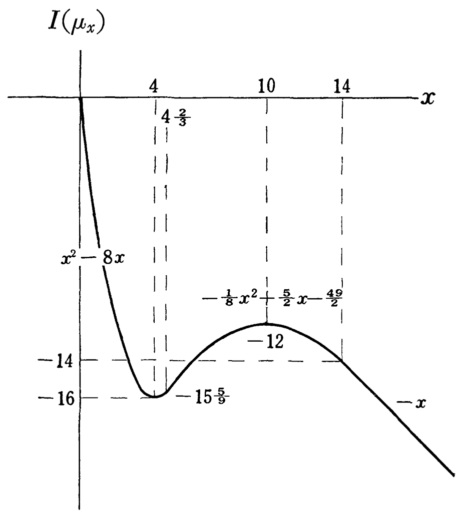
$$I(\mu_x)=\begin{cases} 4x^2-6x & \text{for } 0\leqq x\leqq 1, \\ 2x^2-4x & \text{for } 1\leqq x<\infty. \end{cases}$$

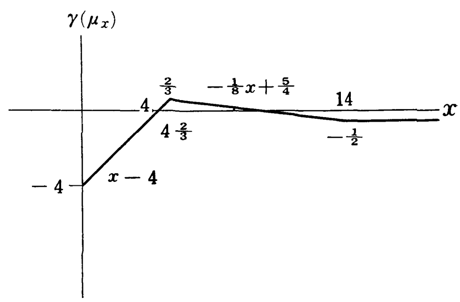
The minimum is taken at $x=3/4$, but at $x=1$ we have two extremal measures and the corresponding values of γ are equal to 1 and 0. For the latter measure the above two inequalities are true but $I(\mu_1)=-2>I(\mu_{3/4})=-9/4$; see Example 2 of § 2.7 too.

Another example is

$$I(\mu)=x_1^2+\frac{x_1 x_2}{2}-8x_1-x_2.$$

See the graph.





REMARK 4. If $f(P)\equiv 0$ in Theorem 2.7, we have

$$U^{\mu_x}(P)\leqq\gamma(\mu_x)g(P) \qquad \text{on } S_{\mu_x}$$

and

$$U^{\mu_x}(P)\geqq\gamma(\mu_x)g(P) \qquad \text{p.p.p. on } K.$$

We know that $I(\mu_x)=x^2(\nu_\infty,\nu_\infty)$ with ν_∞ giving $(\nu_\infty,\nu_\infty)=\min\limits_{\nu\in\mathscr{E}_K(g,1)}(\nu,\nu)$, and that $\gamma(\mu_x)=x(\nu_\infty,\nu_\infty)$. If $(\nu_\infty,\nu_\infty)>0$, we set $\lambda=\mu_x/\gamma(\mu_x)=\mu_x\{x(\nu_\infty,\nu_\infty)\}^{-1}$ and obtain

$$U^\lambda(P)\leqq g(P) \qquad \text{on } S_\lambda=S_{\mu_x}$$

and

$$U^\lambda(P)\geqq g(P) \qquad \text{p.p.p. on } K.$$

Thus we can find a measure λ which satisfies two inequalities (2.50) and (2.51). However, this method is less general than that in Theorem 2.28, because $g(P)$ must be positive on K in the present case.

2.9. Multiple variational problem.

For each k, $1 \leq k \leq n$, let $\Phi_k(P, Q)$ be a kernel,[32] A_k be an $\mathfrak{A}$-measurable set wth $\mathscr{E}_{A_k, \Phi_k} \neq \{0\}$ in Ω such that $(\mu, \nu)_{\Phi_k}$ and $(\nu, \mu)_{\Phi_k}$ are well-defined for any $\mu \in \mathscr{E}'_{A_k, \Phi_k}$ and any $\nu \in \mathscr{E}_{A_k, \Phi_k}$, $f_k(P)$ be an $\mathfrak{A}$-measurable function on A_k such that $\langle f_k, \nu \rangle$ is defined for any $\nu \in \mathscr{E}_{A_k, \Phi_k}$ and $g_k(P)$ be an $\mathfrak{A}$-measurable function on A_k such that $\langle g_k, \nu \rangle$ is defined and finite for any $\nu \in \mathscr{E}_{A_k, \Phi_k}$. So far we have mostly assumed that kernels are symmetric. Although we shall come back soon to this assumption, kernels considered here may not be symmetric. Further consider for each pair j and k, $j \neq k$, a function $h_{jk}(P, Q)$ on $A_j \times A_k$ with the property that $\int h_{jk}\, d(\mu \otimes \nu)$ and $\int h_{jk}\, d(\nu \otimes \mu)$ are always well-defined for product measures $\mu \otimes \nu$ and $\nu \otimes \mu$ of $\mu \in \mathscr{E}'_{A_j, \Phi_j}$ and $\nu \in \mathscr{E}_{A_k, \Phi_k}$ respectively.

This section concerns itself with the problem to minimize

$$(2.52) \qquad \sum_{k=1}^{n} (\mu^{(k)}, \mu^{(k)})_{\Phi_k} + \sum_{\substack{j,k=1 \\ j \neq k}}^{n} \iint h_{jk}(P, Q)\, d\mu^{(j)}(Q)\, d\mu^{(k)}(P) - 2 \sum_{k=1}^{n} \langle f_k, \mu^{(k)} \rangle$$

among $\mu^{(k)} \in \mathscr{E}'_{A_k, \Phi_k}(g_k, x_k, f_k)$, $k=1, \cdots, n$, for which (2.52) has a meaning. We give

THEOREM 2.29. *With the above notations, suppose that there are extremal measures $\{\mu_k^*\}$ giving a finite minimum to expression* (2.52) *among $\mu^{(k)} \in \mathscr{E}'_{A_k, \Phi_k}$ (g_k, x_k, f_k), $k=1, \cdots, n$, for which* (2.52) *has a meaning. Then we have, if $x_k \neq 0$,*

$$(2.53) \qquad \hat{U}_{\Phi_k}^{\mu_k^*}(P) \geq f_k(P) - \frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^{n} \int \left\{ h_{jk}(P, Q) + h_{kj}(Q, P) \right\} d\mu_j^*(Q) + \gamma_k g_k(P)$$

Φ_k-p.p.p. on A_k with

$$x_k \gamma_k = (\mu_k^*, \mu_k^*)_{\Phi_k} + \langle \frac{1}{2} \sum_{\substack{j=1 \\ j \neq k}}^{n} \int \left\{ h_{jk}(P, Q) + h_{kj}(Q, P) \right\} d\mu_j^*(Q) - f_k, \mu_k^* \rangle,$$

and the equality holds μ_k^*-a. e. in (2.53). *If $\int \left\{ h_{jk}(P, Q) + h_{kj}(Q, P) \right\} d\mu_j^*(Q)$, $j=1, \cdots, k-1, k+1, \cdots, n$, and $-f_k$ are lower semicontinuous on A and if g_k is continuous on A_k, then the inverse inequality is true on $S_{\mu_k^*} \cap A_k$.*

PROOF. We observe that μ_k^* minimizes

$$(\mu, \mu)_{\Phi_k} + \langle \sum_{\substack{j=1 \\ j \neq k}}^{n} \int \left\{ h_{jk}(P, Q) + h_{kj}(Q, P) \right\} d\mu_j^*(Q) - 2f_k, \mu \rangle$$

among $\mu \in \mathscr{E}'_{A_k, \Phi_k}(g_k, x_k, f_k)$. Applying Theorem 2.1 in the case $n=1$ we obtain the conclusions.

We can prove the following theorem in the same way as Theorem 2.6.

---

32) This idea of considering different kernels is due to Ninomiya [11]. When more than one kernels are considered, they will be specified as subscripts.

THEOREM 2.30. *In addition to the properties assumed at the beginning of the present section, suppose that* $\{A_k\}$ *are compact sets* $\{K_k\}$, *each* $h_{jk}(P, Q)$ *and each* $-f_k(P)$ *are* $>-\infty$ *and lower semicontinuous on* $K_j\times K_k$, $j\neq k$, *and* K_k *respectively, each* $g_k(P)$ *is positive continuous and* $x_k\geqq 0$. *Then* (2.52) *is always equal to* ∞ *or there exist* $\{\mu_k^*\}$ *which give finite minimum to* (2.52).

We remark that, in the case where each A_k is a compact set K_k and where each $h_{jk}(P, Q)$ is $>-\infty$ and lower semicontinuous on $K_j\times K_k$, our problem is equivalent to that of § 2.2. In fact, we regard $K=K_1+\cdots+K_n$ as a sum space and define a kernel on $K\times K$ by

$$\Phi(P, Q)=\begin{cases}\Phi_k(P, Q) & \text{if } P, Q\in K_k,\\ h_{jk}(P, Q) & \text{if } P\in K_j,\ Q\in K_k \text{ and } j\neq k.\end{cases}$$

We also define $f(P)$ by $f_k(P)$ on K_k and $g(P)$ by $g_k(P)$ on $K_k(k=1, \cdots, n)$. Then (2.52) is equal to $I(\mu)$, and a measure on K which is equal to $\mu^{(k)}\in\mathscr{E}_{K_k,\Phi_k}(g_k, x_k, f_k)$ on K_k, $k=1,\cdots, n$, belongs to $\mathscr{E}_{K,\Phi}(g, x, f)$ with $x=(x_1,\cdots, x_n)$. We can write (2.53) in the form

$$\hat{U}_{\Phi}^{\mu^*}(P)\geqq f(P)+\gamma_k(\mu^*)g_k(P) \qquad \text{p.p.p. on } S_{\mu^*}\cap K_k,$$

where $\mu^*=\mu_k^*$ on K_k and

$$x_k\,\gamma_k(\mu^*)=\sum_{j=1}^{n}\int_{K_k}(\hat{U}_{\Phi}^{\mu_j^*}-f)\,d\mu_k^*, \qquad k=1, \cdots, n.$$

This inequality may be identified with (2.11). It follows also from Theorem 2.7 that the equality in (2.53) holds μ_k-a. e. and that the inverse inequality is true on $S_{\mu_k^*}$ if $f_k(P)<\infty$ is upper semicontinuous and $g_k(P)$ is positive and continuous on K_k.

Conversely let $K_1, \cdots, K_n$ be mutually disjoint compact sets, and $f(P)$ and $g(P)$ be functions on $\bigcup_{k=1}^{n} K_k$. If we take the restrictions of $\Phi(P, Q)$ to $K_k\times K_k$ and to $K_j\times K_k$, $j\neq k$, for $\Phi_k(P, Q)$ and $h_{jk}(P, Q)$ respectively, and take the restrictions of $f(P)$ and $g(P)$ to K_k for $f_k(P)$ and $g_k(P)$ respectively, then the problem to minimize $I(\mu)$ for $\mu\in\mathscr{E}_K(g, x)$, $x=(x_1, \cdots, x_n)$, is transformed to a problem in the present section.

Next we are interested in minimizing

$$(\mu, \mu)_{\Phi}+(\nu, \nu)_{\Psi}-2(\mu, \nu)_{\Theta} \tag{2.54}$$

with symmetric kernels Φ, Ψ and Θ. If $\Psi=c\Phi$ and $\Theta=\Phi$, (2.54) is the expression which appeared in the definition of energy principles given near the end of § 1.2. In the rest of our chapter we shall consider only symmetric kernels without mentioning the symmetry sometimes. As a corollary of Theorem 2.29 we obtain

THEOREM 2.31. *Let* $\Phi(P, Q)$, $\Psi(P, Q)$ *and* $\Theta(P, Q)$ *be symmetric kernels, and* A_1 *and* A_2 *be sets of* $\mathfrak{A}$ *with* $\mathscr{E}_{A_1,\Phi}\neq\{0\}$ *and* $\mathscr{E}_{A_2,\Psi}\neq\{0\}$ *such that* $(\mu, \nu)_{\Phi}$ *is well-*

defined for any $\mu\in\mathscr{E}'_{A_1,\Phi}$ *and any* $\nu\in\mathscr{E}_{A_1,\Phi}$, $(\mu,\nu)_\Psi$ *is well-defined for any* $\mu\in\mathscr{E}'_{A_2,\Psi}$ *and any* $\nu\in\mathscr{E}_{A_2,\Psi}$ *and* $(\mu,\nu)_\Theta$ *is well-defined for any* $\mu\in\mathscr{E}'_{A_1,\Phi}$ *and any* $\nu\in\mathscr{E}_{A_2,\Psi}$ *as well as for any* $\mu\in\mathscr{E}_{A_1,\Phi}$ *and any* $\nu\in\mathscr{E}'_{A_2,\Psi}$. *Let* $g_1(P)$ ($g_2(P)$ *resp.*) *be an* $\mathfrak{A}$*-measurable function on* A_1 (A_2 *resp.*) *such that* $\langle g_1,\nu\rangle$ ($\langle g_2,\nu\rangle$ *resp.*) *is defined and finite for any* $\nu\in\mathscr{E}_{A_1,\Phi}$ ($\mathscr{E}_{A_2,\Psi}$ *resp.*), *and let* x_1 *and* x_2 *be non-vanishing numbers. If there exist* μ^* *and* ν^* *which give finite minimum to* (2.54) *among* $\mu\in\mathscr{E}'_{A_1,\Phi}(g_1,x_1)$ *and* $\nu\in\mathscr{E}'_{A_2,\Psi}(g_2,x_2)$, *then*

$$U_\Phi^{\mu^*}(P)\geqq U_\Theta^{\nu^*}(P)+\gamma_1 g_1(P) \qquad \Phi\text{-p.p.p. on } A_1 \tag{2.55}$$

and

$$U_\Psi^{\nu^*}(P)\geqq U_\Theta^{\mu^*}(P)+\gamma_2 g_2(P) \qquad \Psi\text{-p.p.p. on } A_2 \tag{2.56}$$

with

$$x_1\gamma_1=(\mu^*,\mu^*)_\Phi-(\mu^*,\nu^*)_\Theta \qquad \text{and} \qquad x_2\gamma_2=(\nu^*,\nu^*)_\Psi-(\mu^*,\nu^*)_\Theta$$

and the equalities hold in (2.55) *and* (2.56) μ^**-a. e. and* ν^**-a. e. respectively.*

If, in addition, the closures A_1^a *and* A_2^a *of* A_1 *and* A_2 *are compact, if* $\Theta(P,Q)$ *is continuous on* $A_1^a\times A_2^a$ *and* $A_2^a\times A_1^a$ *and if* $g_1(P)$ *and* $g_2(P)$ *are continuous on* A_1 *and* A_2 *respectively, then the inverse inequalities are true in* (2.55) *and* (2.56) *on* $S_{\mu^*}\cap A_1$ *and* $S_{\nu^*}\cap A_2$ *respectively.*

THEOREM 2.32. *Let* $\Phi(P,Q)$, $\Psi(P,Q)$ *and* $\Theta(P,Q)$ *be symmetric kernels and assume that* $\Theta(P,Q)$ *is continuous outside the diagonal set. Let* K_1 *and* K_2 *be mutually disjoint compact sets with* $\mathscr{E}_{K_1,\Phi}\neq\{0\}$ *and* $\mathscr{E}_{K_2,\Psi}\neq\{0\}$ *respectively. Let* $g_1(P)$ *and* $g_2(P)$ *be positive continuous functions on* K_1 *and* K_2 *respectively, and* x_1 *and* x_2 *be nonnegative. Then there exist* μ^* *and* ν^* *which give finite minimum to* (2.54) *among* $\mu\in\mathscr{E}_{K_1,\Phi}(g_1,x_1)$ *and* $\nu\in\mathscr{E}_{K_2,\Psi}(g_2,x_2)$.

We can state theorems corresponding to Theorems 2.29 and 2.30 in the unconditional case. However, we shall be contended with giving the following theorems which will be needed later. We use Corollary of Theorem 2.28 and obtain

THEOREM 2.33. *Let* $\Phi(P,Q)$, $\Psi(P,Q)$ *and* $\Theta(P,Q)$ *be symmetric kernels, and* A_1 *and* A_2 *be sets of* $\mathfrak{A}$ *with* $\mathscr{E}_{A_1,\Phi}\neq\{0\}$ *and* $\mathscr{E}_{A_2,\Psi}\neq\{0\}$ *such that* $(\mu,\nu)_\Phi$ *is well-defined for any* $\mu\in\mathscr{E}'_{A_1,\Phi}$ *and* $\nu\in\mathscr{E}_{A_1,\Phi}$, $(\mu,\nu)_\Psi$ *is well-defined for any* $\mu\in\mathscr{E}'_{A_2,\Psi}$ *and any* $\nu\in\mathscr{E}_{A_2,\Psi}$ *and* $(\mu,\nu)_\Theta$ *is well-defined for any* $\mu\in\mathscr{E}'_{A_1,\Phi}$ *and any* $\nu\in\mathscr{E}_{A_2,\Psi}$ *as well as for any* $\mu\in\mathscr{E}_{A_1,\Phi}$ *and any* $\nu\in\mathscr{E}'_{A_2,\Psi}$. *Let* $g(P)$ *be an* $\mathfrak{A}$*-measurable function on* A_2 *such that* $\langle g,\nu\rangle$ *is finite for any* $\nu\in\mathscr{E}_{A_2,\Psi}$. *If there exist* μ^* *and* ν^* *which give the finite minimum to* (2.54) *among* $\mu\in\mathscr{E}'_{A_1,\Phi}$ *and* $\nu\in\mathscr{E}'_{A_2,\Psi}(g,1)$, *then* $U_\Phi^{\mu^*}(P)\geqq U_\Theta^{\nu^*}(P)$ Φ*-p.p.p. on* A_1, $U_\Phi^{\mu^*}(P)=U_\Theta^{\nu^*}(P)$ μ^**-a.e.*,

$$U_\Psi^{\nu^*}(P)\geqq U_\Theta^{\mu^*}(P)+\{(\nu^*,\nu^*)_\Psi-(\nu^*,\mu^*)_\Theta\}g(P) \qquad \Psi\text{-p.p.p. on } A_2 \tag{2.57}$$

and the equality holds there ν^**-a. e.*

If, in addition, the closures A_1^a and A_2^a of A_1 and A_2 are compact, if $\Theta(P, Q)$ is continuous on $A_1^a \times A_2^a$ and $A_2^a \times A_1^a$ and if $g(P)$ is continuous on A_2, then $U_\Phi^{\mu^*}(P) \leqq U_\Theta^{\nu^*}(P)$ on $S_{\mu^*} \cap A_1$ and the inverse inequality is true in (2.57) on $S_{\nu^*} \cap A_2$.

THEOREM 2.34. *Let $\Phi(P, Q)$, $\Psi(P, Q)$ and $\Theta(P, Q)$ be symmetric kernels, K_1 and K_2 be mutually disjoint compact sets with $\mathscr{E}_{K_1, \Phi} \neq \{0\}$ and $\mathscr{E}_{K_2, \Psi} \neq \{0\}$, and assume that $\Theta(P, Q)$ is continuous outside the diagonal set and $(\mu, \mu)_\Phi > 0$ for every $\mu \neq 0$ supported by K_1. Let $g(P)$ be a positive continuous function on K_2. Then there exist μ^* and ν^* which give the finite minimum to (2.54) among $\mu \in \mathscr{E}_{K_1, \Phi}$ and $\nu \in \mathscr{E}_{K_2, \Psi}(g, 1)$.*

2.10 Applications to energy principles.

Ninomiya [1; 4; 5; 6; 8; 9] considered the variational problem to minimize the expression

$$\frac{(\mu, \mu)(\nu, \nu)}{(\mu, \nu)^2}$$

and applied the results to prove the following theorems in case $\Phi = \Psi = \Theta$. We shall use Theorems 2.33 and 2.34 instead in the proof.

THEOREM 2.35. *Let $\Phi(P, Q)$, $\Psi(P, Q)$ and $\Theta(P, Q)$ be symmetric kernels. If (2.54) is nonnegative for any $\mu \in \mathscr{E}_\Phi$ and $\nu \in \mathscr{E}_\Psi$ with compact support, then the following condition is satisfied:*

[A_1] *Whenever μ and ν have compact supports and $U_\Phi^\mu(P) \leqq U_\Theta^\nu(P)$ on S_μ, $U_\Theta^\mu(P) \leqq U_\Psi^\nu(P)$ is true at at least one point of S_ν.*

THEOREM 2.36. *Let $\Phi(P, Q)$, $\Psi(P, Q)$ and $\Theta(P, Q)$ be symmetric kernels. If (2.54) is positive for any different measures $\mu \in \mathscr{E}_\Phi$ and $\nu \in \mathscr{E}_\Psi$ with compact support, the following condition is satisfied:*

[A_2] *Whenever $\mu \in \mathscr{E}_\Phi$ and $\nu \in \mathscr{E}_\Psi$ are different and have compact supports and $U_\Phi^\mu(P) \leqq U_\Theta^\nu(P)$ is true μ-a. e., $U_\Theta^\mu(P) < U_\Psi^\nu(P)$ is true on a set with positive ν-measure.*

PROOF for both theorems. We write

$$(\mu, \mu)_\Phi + (\nu, \nu)_\Psi - 2(\mu, \nu)_\Theta = \int (U_\Phi^\mu - U_\Theta^\nu)\, d\mu - \int (U_\Theta^\mu - U_\Psi^\nu)\, d\nu. \tag{2.58}$$

If this is nonnegative and $U_\Phi^\mu(P) \leqq U_\Theta^\nu(P)$ on S_μ, then $\int (U_\Theta^\mu - U_\Psi^\nu)\, d\nu \leqq 0$ and $U_\Theta^\mu(P) \leqq U_\Psi^\nu(P)$ at at least one point of S_ν. If (2.58) is positive and $U_\Phi^\mu(P) \leqq U_\Theta^\nu(P)$ μ-a. e., then $\int (U_\Theta^\mu - U_\Psi^\nu) d\nu < 0$ and $U_\Theta^\mu(P) < U_\Psi^\nu(P)$ on a set with positive ν-measure.

Next we discuss the sufficiency of conditions.

THEOREM 2.37. *Let $\Phi(P, Q)$ $\Psi(P, Q)$ and $\Theta(P, Q)$ be symmetric kernels such that $\Phi \geqq \Theta$ and $\Psi \geqq \Theta$. Assume that $\Theta(P, Q)$ is continuous outside the diagonal set, that both Φ and Ψ are pseudo-positive and that at least one of Φ and Ψ is strictly pseudo-positive. If* [A_1] *is satisfied,* (2.54) *is nonnegative for any $\mu \in \mathscr{E}_\Phi$ and $\nu \in \mathscr{E}_\Psi$ with compact support.*

PROOF. First we consider the case that $S_\mu \cap S_\nu = \emptyset$. We may assume that Φ is strictly pseudo-positive and ν is a unit measure. It is sufficient to prove under [A_1] that (2.54) is nonnegative for $\mu^* \in \mathscr{E}_{S_\mu, \Phi}$ and $\nu^* \in \mathscr{E}_{S_\nu, \Psi}(1, 1)$ obtained in Theorem 2.34 in the case $g(P) \equiv 1$. Since $U_\Phi^{\mu^*}(P) \leqq U_\Theta^{\nu^*}(P)$ on S_{μ^*} by Theorem 2.33, $U_\Theta^{\mu^*}(P) \leqq U_\Psi^{\nu^*}(P)$ at at least one point of S_{ν^*} by [A_1]. The inverse inequality of (2.57) being true everywhere on S_{ν^*}, it follows that $(\mu^*, \nu^*)_\Theta \leqq (\nu^*, \nu^*)_\Psi$. Theorem 2.33 gives $(\mu^*, \mu^*)_\Phi = (\mu^*, \nu^*)_\Theta$ and there follows

$$
\begin{aligned}
&(\mu^*, \mu^*)_\Phi + (\nu^*, \nu^*)_\Psi - 2(\mu^*, \nu^*)_\Theta \\
&\geqq (\mu^*, \nu^*)_\Theta + (\mu^*, \nu^*)_\Theta - 2(\mu^*, \nu^*)_\Theta = 0.
\end{aligned}
$$

Next let $\mu \not\equiv 0$ be a measure which vanishes outside a relatively compact Borel set B_1, and ν be a unit measure which vanishes outside a relatively compact Borel set B_2 disjoint from B_1. Let $\{K_m^{(1)}\}$ be a sequence of compact sets such that $\mu(K_m^{(1)})$ tends to $\mu(B_1)$ as $m \to \infty$ and $\{K_m^{(2)}\}$ be a similar sequence taken in connection with ν. We denote by μ_m and ν_m the restrictions of μ and ν to $K_m^{(1)}$ and $K_m^{(2)}$ respectively. It follows that

$$(\mu, \mu)_\Phi + (\nu, \nu)_\Psi - 2(\mu, \nu)_\Theta = \lim_{m \to \infty} \{(\mu_m, \mu_m)_\Phi + (\nu_m, \nu_m)_\Psi - 2(\mu_m, \nu_m)_\Theta\} \geqq 0.$$

Finally let μ and ν be any measures with compact support. We can decompose $S_\mu \cup S_\nu$ into mutually disjoint Borel sets B_1 and B_2 such that $\mu(B) \geqq \nu(B)$ for any Borel subset $B \subset B_1$ and $\mu(B') \leqq \nu(B')$ for any Borel subset $B' \subset B_2$. We denote by μ' the restriction of $\mu - \nu$ to B_1 and by ν' the one to B_2 of $\nu - \mu$. It follows that $\mu - \mu' = \nu - \nu' = \lambda$ is a nonnegative measure. We see that

$$
\begin{aligned}
(2.59) \qquad &(\mu, \mu)_\Phi + (\nu, \nu)_\Psi - 2(\mu, \nu)_\Theta \\
&= (\mu' + \lambda, \mu' + \lambda)_\Phi + (\nu' + \lambda, \nu' + \lambda)_\Psi - 2(\mu' + \lambda, \nu' + \lambda)_\Theta \\
&= (\mu', \mu')_\Phi + (\nu', \nu')_\Psi - 2(\mu', \nu')_\Theta + 2 \langle U_\Phi^{\mu'} - U_\Theta^{\mu'} + U_\Psi^{\nu'} - U_\Theta^{\nu'}, \lambda \rangle + (\lambda, \lambda)_{\Phi + \Psi - 2\Theta} \geqq 0,
\end{aligned}
$$

because $\Phi \geqq \Theta$ and $\Psi \geqq \Theta$.

THEOREM 2.38. *Let $\Phi(P, Q)$, $\Psi(P, Q)$ and $\Theta(P, Q)$ be symmetric kernels such that $\Phi(P, Q)$ is strictly pseudo-positive and assume that* [A_2] *is satisfied and that* (2.54) *is nonnegative for any $\mu \in \mathscr{E}_\Phi$ and $\nu \in \mathscr{E}_\Psi$ with compact support. Then* (2.54) *is positive provided that $\mu \not\equiv \nu$.*

PROOF. We suppose that (2.54) vanishes for different $\mu^* \in \mathscr{E}_\Phi$ and $\nu^* \in \mathscr{E}_\Psi$ with compact support. We may assume that $\nu^* \in \mathscr{E}_\Psi(1, 1)$. Since μ^* minimizes

(2.54), $U_{\Phi}^{\mu^*}(P)=U_{\Theta}^{\nu^*}(P)$ μ^*-a.e. by Theorem 2.33. This and [A_2] give $U_{\Theta}^{\mu^*}(P)$ $<U_{\Psi}^{\nu^*}(P)$ on a set of positive ν^*-measure. We use (2.57) which follows from the fact that ν^* minimizes (2.54), and see that $(\nu^*, \nu^*)_{\Psi}>(\nu^*, \mu^*)_{\Theta}$. Therefore

$$(\mu^*, \mu^*)_{\Phi}+(\nu^*, \nu^*)_{\Psi}-2(\mu^*, \nu^*)_{\Theta}>(\mu^*, \nu^*)_{\Theta}+(\mu^*, \nu^*)_{\Theta}-2(\mu^*, \nu^*)_{\Theta}=0.$$

This is impossible and the theorem is concluded.

We shall apply the above results to c-energy principles.

THEOREM 2.39. *Consider a nonnegative kernel $\Phi(P, Q)$ which is strictly pseudo-positive and continuous outside the diagonal set, and let $c\geqq 1$; in case $c=1$, the kernel can be negative. In order that the weak c-energy principle $(\mathrm{E}')_c$ be true,* [A_1] *is necessary and sufficient where $\Psi=c\Phi$ and $\Theta=\Phi$ are taken. In order that the restricted c-energy principle $(\mathrm{E}^*)_c$ be true,* [A_2] *is necessary and sufficient where $\Psi=c\Phi$ and $\Theta=\Phi$ are taken.*

PROOF. It is sufficient to point out that (2.59) is equal to

$$(\mu', \mu')+c(\nu', \nu')-2(\mu', \nu')+(c-1)(2\nu'+\lambda, \lambda). \tag{2.60}$$

We shall give a different application of Theorem 2.33.

THEOREM 2.40. *Consider a kernel which is strictly pseudo-positive and satisfies the weak c-energy principle, and assume $(\mu^*, \mu^*)+c(\nu^*, \nu^*)-2(\mu^*, \nu^*)=0$ for $c\geqq 1$ and for $\mu^*, \nu^*\in\mathscr{E}$ having compact supports. Then*

$$U^{\nu^*}(P)\leqq U^{\mu^*}(P)\leqq cU^{\nu^*}(P) \qquad p.p.p. \text{ in } \Omega, \tag{2.61}$$

$U^{\mu^*}(P)=U^{\nu^*}(P)$ μ^*-a. e. and $U^{\mu^*}(P)=cU^{\nu^*}(P)$ ν^*-a. e. If $c>1$ and the kernel is nonnegative, then Ω is divided into mutually disjoint Borel sets B_1 and B_2 such that $\mu^*(\Omega-B_1)=\nu^*(\Omega-B_2)=0$.

PROOF. Since the kernel is strictly pseudo-positive, either $\mu^*\equiv\nu^*\equiv 0$ or $\mu^*\not\equiv 0$, $\nu^*\not\equiv 0$. We assume that $\mu^*\not\equiv 0$, $\nu^*\not\equiv 0$. Let K be any compact set containing S_{μ^*} and set

$$\bar{\mu}=\frac{\mu^*}{\mu^*(\Omega)} \qquad \text{and} \qquad \bar{\nu}=\frac{\nu^*}{\nu^*(\Omega)}.$$

Then $\bar{\mu}\in\mathscr{E}_K$ and $\bar{\nu}\in\mathscr{E}_{S_{\nu^*}}(1, 1)$. Since $\bar{\mu}$ and $\bar{\nu}$ minimize (2.54), we have $U^{\bar{\mu}}(P)$ $\geqq U^{\bar{\nu}}(P)$ p.p.p. on K by Theorem 2.33. Consequently $U^{\mu^*}(P)\geqq U^{\nu^*}(P)$ p.p.p. on K. Because of the arbitrary character of K, $U^{\mu^*}(P)\geqq U^{\nu^*}(P)$ p.p.p. in Ω. We can write

$$(\mu^*, \mu^*)+c(\nu^*, \nu^*)-2(\mu^*, \nu^*)=c\left\{c\left(\frac{\mu^*}{c}, \frac{\mu^*}{c}\right)+(\nu^*, \nu^*)-2\left(\frac{\mu^*}{c}, \nu^*\right)\right\}.$$

Hence $U^{\mu^*}(P)/c\leqq U^{\nu^*}(P)$ p.p.p. on any compact set containing S_{ν^*}. It now follows that $U^{\mu^*}(P)\leqq cU^{\nu^*}(P)$ p.p.p. in Ω. Consequently (2.61) is derived. It follows that $(\mu^*, \mu^*) \geqq (\mu^*, \nu^*)$ and $c(\nu^*, \nu^*) \geqq (\mu^*, \nu^*)$ and hence

$$0 = (\mu^*, \mu^*) + c(\nu^*, \nu^*) - 2(\mu^*, \nu^*) \geqq (\mu^*, \nu^*) + (\mu^*, \nu^*) - 2(\mu^*, \nu^*) = 0.$$

Therefore $(\mu^*, \mu^*) = (\mu^*, \nu^*)$ and $c(\nu^*, \nu^*) = (\mu^*, \nu^*)$ and it is seen that $U^{\mu^*}(P) = U^{\nu^*}(P)$ μ^*-a. e. and $U^{\mu^*}(P) = cU^{\nu^*}(P)$ ν^*-a. e. Next we assume that $c > 1$ and the kernel is nonnegative. In (2.60) we have $(2\nu' + \lambda, \lambda) = 0$, which concludes $\lambda \equiv 0$ because the kernel is strictly pseudo-positive. Hence $\mu^* \equiv \mu'$ and $\nu^* \equiv \nu'$. This completes the proof.

As an immediate application of Theorem 2.39 we state

THEOREM 2.41. *Assume that the kernel is nonnegative, symmetric, continuous outside the diagonal set and strictly pseudo-positive. If the restricted c-dilated domination principle* $(\mathrm{U}_{\mathrm{d}}^*)_c$ *is true, then the weak c-energy principle* $(\mathrm{E}')_c$ *is satisfied for* $c \geqq 1$ *(in case* $c = 1$ *the kernel can be negative).*

We shall prove another result of Ninomiya [11] in a generalized form.

THEOREM 2.42. *Let* $\Phi(P, Q)$ *be a nonnegative symmetric kernel which is continuous outside the diagonal set and strictly pseudo-positive, and* $c \geqq 1$; *if* $c = 1$, $\Phi(P, Q)$ *can be negative. Let* $\mathscr{F}$ *be a class of functions such that, for any compact set K, we can find* $f \in \mathscr{F}$, *defined at least on K, with the following property:* $f(P)$ *has a positive lower bound on K, and* $U^{\mu}(P) \leqq cf(P)$ *on K whenever* $\mu \in \mathscr{E}_K$ *and* $U^{\mu}(P) \leqq f(P)$ *on* S_{μ}. *Then* $\Phi(P, Q)$ *satisfies the weak* c^2*-energy principle.*

PROOF. It is enough to show that $(\mu^*, \mu^*) + c^2(\nu^*, \nu^*) - 2(\mu^*, \nu^*) \geqq 0$ for $\mu^* \in \mathscr{E}_{K_1}$ and $\nu^* \in \mathscr{E}_{K_2}(1, 1)$ obtained in Theorem 2.34, where K_1 and K_2 are mutually disjoint compact sets whose V_i-values are finite. We have by Theorem 2.33

$$c^2 U^{\nu^*}(P) \leqq U^{\mu^*}(P) + c^2(\nu^*, \nu^*) - (\mu^*, \nu^*) \qquad \text{on } S_{\nu^*}. \tag{2.62}$$

This shows that $U^{\nu^*}(P)$ is bounded on S_{ν^*}. By our assumption there is $f \in \mathscr{F}$ such that $f(P)$ is defined on $S_{\mu^*} \cup S_{\nu^*}$ and have properties described above. We determine α by

$$\sup_{P \in S_{\nu^*}} \{U^{\nu^*}(P) - \alpha f(P)\} = 0. \tag{2.63}$$

Then α is finite positive and it follows by our assumption that $U^{\nu^*}(P) \leqq c\alpha f(P)$ on S_{μ^*}. According to Theorem 2.33 it holds that $U^{\mu^*}(P) \leqq U^{\nu^*}(P)$ on S_{μ^*} and hence by our assumption again it follows that $U^{\mu^*}(P) \leqq c^2 \alpha f(P)$ on S_{ν^*}. By substituting this inequality into (2.62) we obtain

$$c^2 U^{\nu^*}(P) \leqq c^2 \alpha f(P) + c^2(\nu^*, \nu^*) - (\mu^*, \nu^*) \qquad \text{on } S_{\nu^*}.$$

This and (2.63) give

$$0 \leqq c^2(\nu^*, \nu^*) - (\mu^*, \nu^*).$$

Since $(\mu^*, \mu^*)=(\mu^*, \nu^*)$ is seen on account of Theorem 2.33, we have

$$(\mu^*, \mu^*)+c^2(\nu^*, \nu^*)-2(\mu^*, \nu^*)\geqq(\mu^*, \nu^*)+(\mu^*, \nu^*)-2(\mu^*, \nu^*)=0.$$

COROLLARY. *Let $\Phi(P, Q)$ be nonnegative, symmetric, continuous outside the diagonal set and strictly pseudo-positive. Let $\mathscr{F}$ be a non-empty class of positive functions in Ω such that each function of $\mathscr{F}$ has a positive lower bound on every compact set. If the $\mathscr{F}$-relative c-dilated maximum principle $(\mathrm{U}_{\mathscr{F}})_c$, $c\geqq 1$, is true, then the weak c^2-energy principle $(\mathrm{E}')_{c^2}$ is satisfied; in case $c=1$ the kernel can be negative.*

REMARK 1. There is a kernel which satisfies $(\mathrm{U})_1=(\mathrm{F})$ but is not pseudo-positive. Therefore we need at least the pseudo-positivity in the above assumption. An example is given by

$$\begin{pmatrix} -1 & -a \\ -a & -1 \end{pmatrix} \qquad \text{with } a>1.$$

Another example is

$$\int_0^1\int_0^1 \log\frac{1}{|x-y|}\,dxdy+\int_6^7\int_6^7 \log\frac{1}{|x-y|}\,dxdy+2\int_0^1\int_6^7 \log\frac{1}{|x-y|}\,dxdy<0.$$

REMARK 2. From Corollary it follows that $(\mathrm{U}_\mathrm{d}^*)_c$ implies $(\mathrm{E}')_{c^2}$. However, Theorem 2.41 gives the better result $(\mathrm{U}_\mathrm{d}^*)_c\rightarrow(\mathrm{E}')_c$.

In the preceding paper Ohtsuka [7] was proved $(\mathrm{E})\not\rightarrow(\mathrm{C})$. We now state

$$\begin{array}{ccccccccc}
(\mathrm{C}) & \substack{\not\leftarrow\\ \not\rightarrow} & (\mathrm{E})_c & \substack{\rightarrow\\ \not\leftarrow} & (\mathrm{E}^*)_c & \substack{\rightarrow\\ \not\leftarrow} & (\mathrm{E}')_c & \substack{\not\rightarrow\\ (\leftarrow)} & (\mathrm{U})_{\sqrt{c}}, \\
 & & & & & & \not\downarrow\,(\uparrow) & & \\
 & & & & & & (\mathrm{U}_\mathrm{d}^*)_c & &
\end{array}$$

where $(\leftarrow)$ means that this is true if the kernel is nonnegative, symmetric, continuous outside the diagonal set and strictly pseudo-positive. It is easy to see that $(\mathrm{E}')_c$ means $(\mathrm{E}')_{c'}$ for any $c'>c$ provided that the kernel is pseudo-positive; if the kernel is strictly pseudo-positive, then $(\mathrm{E}')_c\rightarrow(\mathrm{E}^*)_{c'}$ for any $c'>c$.

In order to complement the above schema we give

Example to show $(\mathrm{U})_{\sqrt{c}}\not\rightarrow(\mathrm{E}^*)_c$ and $(\mathrm{U}_\mathrm{d})_c\not\rightarrow(\mathrm{E}^*)_c$ $(c\geqq 1)$. Take two points P_1 and P_2 for Ω, and define $\Phi(P, Q)$ by

$$\begin{pmatrix} 1 & \sqrt{c} \\ \sqrt{c} & 1 \end{pmatrix}.$$

Evidently $(\mathrm{U})_{\sqrt{c}}$ is satisfied. We can check easily that $(\mathrm{U}_\mathrm{d})_c$ is satisfied but neither $(\mathrm{U}_\mathrm{d})_{c'}$ nor $(\mathrm{E}')_{c'}$ with any $c'<c$. It follows that $(\mathrm{E}')_c$ is true. Let us take the unit point measure at P_1 for μ and the point measure with total mass

$1/\sqrt{c}$ at P_2 for ν. Then $(\mu, \mu)+c(\nu, \nu)-2(\mu, \nu)=0$ and hence $(\mathrm{E}^*)_c$ is not true.

In Theorem 2.41 (the above Corollary resp.) we saw that the weak $c(c^2$ resp.)-energy principle holds. Our example shows that we can not replace $c(c^2$ resp.) by smaller number in general.

2.11. Maximum and domination principles.

We were already concerned with $(\mathrm{U})_c$ and $(\mathrm{U}_{\mathrm{d}}^*)_c$ in the preceding section. In the present section we still assume that kernels are symmetric, and continue to generalize other Ninomiya's results in [11]. He proved Theorem 2.43 in case $\Phi=\theta$ and Theorems 2.44 and 2.45 in case $c=1$.

Let us consider the following principles:

(i) (Φ, Ψ, θ)-*domination principle*: If $U_\Phi^\mu(P)\leqq U_\Theta^\nu(P)$ on S_μ for $\mu\in\mathscr{E}_\Phi$ and ν both with compact support, then $U_\Phi^\mu(P)\leqq U_\Psi^\nu(P)$ in $\Omega-S_\mu$.

(ii) (Φ, Ψ, θ)-*light domination principle*: If $U_\Phi^\mu(P)\leqq\theta(P, Q)$ on S_μ for $\mu\in\mathscr{E}_\Phi$ and a point Q, then $U_\Phi^\mu(P)\leqq\Psi(P, Q)$ in $\Omega-S_\mu$.

(iii) (Φ, Ψ, θ)-*sweeping-out principle*: For any compact set K with $V_{i,\Phi}(K)<\infty$ and ν with compact support, there is a measure μ supported by K such that $U_\Phi^\mu(P)\geqq U_\Theta^\nu(P)$ Φ-p.p.p. on K and $U_\Phi^\mu(P)\leqq U_\Psi^\nu(P)$ in $\Omega-K$.

(iv) (Φ, Ψ, θ)-*light sweeping-out principle*: For any compact set K with $V_{i,\Phi}(K)<\infty$ and any Q, there is a measure μ supported by K such that $U_\Phi^\mu(P)\geqq\theta(P, Q)$ Φ-p.p.p. on K and $U_\Phi^\mu(P)\leqq\Psi(P, Q)$ in $\Omega-K$.

It is evident that (i)$\rightarrow$(ii) and (iii)$\rightarrow$(iv). In what follows in this section we shall assume that $\Phi(P, Q)$ is strictly pseudo-positive and continuous outside the diagonal set. We shall prove

(iv)$\rightarrow$(i). Assume that $\mu\in\mathscr{E}_\Phi$ and ν have compact supports and $U_\Phi^\mu(P)\leqq U_\Theta^\nu(P)$ on S_μ. For $P_0\notin S_\mu$ there is μ_0 supported by S_μ such that $U_\Phi^{\mu_0}(P)\geqq\Phi(P, P_0)$ Φ-p.p.p. on S_μ and $U_\Phi^{\mu_0}(P)\leqq\Phi(P, P_0)$ on S_{μ_0} by Corollary of Theorem 2.28. For a point Q, we choose a sequence $\{\theta_m\}$ of continuous symmetric kernels on $(S_{\mu_0}\cup\{Q\})\times(S_{\mu_0}\cup\{Q\})$ increasing to θ. There is a measure μ_m supported by S_{μ_0} such that $U_\Phi^{\mu_m}(P)\geqq\theta_m(P, Q)$ Φ-p.p.p. on S_{μ_0} and $U_\Phi^{\mu_m}(P)\leqq\theta_m(P, Q)\leqq\theta(P, Q)$ on S_{μ_m}. By (ii) we have $U_\Phi^{\mu_m}(P)\leqq\Psi(P, Q)$ in $\Omega-S_{\mu_m}$, and

$$U_\Theta^{\mu_0}(P)=\lim_{m\to\infty}U_{\Theta_m}^{\mu_0}(Q)\leqq(\mu_m, \mu_0)_\Phi=(\mu_0, \mu_m)_\Phi\leqq U_\Phi^{\mu_m}(P_0)\leqq\Psi(P_0, Q).$$

Hence

$$U_\Phi^\mu(P_0)\leqq(\mu_0, \mu)_\Phi=(\mu, \mu_0)_\Phi\leqq(\nu, \mu_0)_\Theta=(\mu_0, \nu)_\Theta\leqq U_\Psi^\nu(P_0).$$

Likewise we can prove (vi)$\rightarrow$(i).

We shall establish the following lemma in order to derive (ii)$\rightarrow$(iv).

Lemma 2.3. *Furthermore assume that $\Phi(P, Q)$ is continuous in the extended sense, $\Psi(P, Q)$ is locally bounded outside the diagonal set and $\theta(P, Q)$ is positive on the diagonal set. If* (ii) *is true, $\Phi(P, Q)$ satisfies the continuity princi-*

ple.

PROOF. Let μ be a measure with compact support on which $U_{\Phi}^{\mu}(P)$ is bounded. We set

$$V_{\Phi}(\mu)=\sup_{P\in S_{\mu}} U_{\Phi}^{\mu}(P).$$

Since $\Phi(P, Q)$ is bounded from below on $K\times K$ for any compact set K in Ω, there is a finite number v such that $V_{\Phi}(\nu)\leqq v$ for any restriction ν of μ to a subset in Ω. Let P_0 be a point of S_{μ} which is not isolated in Ω and take an open neighborhood N_{P_0} of P_0 such that $a=\inf \Theta>0$ on $N_{P_0}\times N_{P_0}$. Let $N'\subset N_{P_0}$ be a compact neighborhood of P_0 such that $N_{P_0}-N'\neq\emptyset$ and denote the restriction of μ to N' by μ'. We have, for any point $Q\in N_{P_0}-N'$,

$$U_{\Phi}^{\mu'}(P)\leqq \frac{v}{a}\Theta(P, Q) \qquad \text{on } S_{\mu'}.$$

By (ii) it follows that $U_{\Phi}^{\mu'}(P)\leqq a^{-1}v\Psi(P, Q)$ in $\Omega-S_{\mu'}$. We infer that $U_{\Phi}^{\mu'}(P)$ is bounded on N'. Since $U_{\Phi}^{\mu-\mu'}(P)$ is bounded in a neighborhood of P_0, $U_{\Phi}^{\mu}(P)$ is likewise bounded in a neighborhood of P_0. Consequently $U_{\Phi}^{\mu}(P)$ is bounded on any compact set in Ω. It is concluded that $\Phi(P, Q)$ satisfies the continuity principle on account of (IV) of § 1.3.

(ii)$\rightarrow$(iv) under the assumptions in Lemma 2.3 and the assumption that $\Phi(P, Q)$ is nonnegative in $\Omega\times\Omega$ and positive on the diagonal set, that $\Theta(P, P)=\infty$ implies always $\Phi(P, P)=\infty$ and that $\Theta(P, Q)$ is finite outside the diagonal set. Let K be a compact set with $V_{i,\Phi}(K)<\infty$. We choose a sequence $\{\Theta_m\}$ of continuous symmetric kernels on $(K\cup\{Q\})\times(K\cup\{Q\})$ increasing to Θ. There is a measure μ_m supported by K such that $U_{\Phi}^{\mu_m}(P)\geqq\Theta_m(P, Q)$ Φ-p.p.p. on K and $U_{\Phi}^{\mu_m}(P)\leqq\Theta_m(P, Q)\leqq\Theta(P, Q)$ on S_{μ_m}. By (ii) we have $U_{\Phi}^{\mu_m}(P)\leqq\Psi(P, Q)$ in $\Omega-S_{\mu_m}$. Suppose that $\mu_m(K)$ is not uniformly bounded. Then there is a point $P_0\in K$ such that, for any neighborhood V of P_0, $\mu_m(V)$ is not bounded. Let V_0 be a neighborhood such that $\Phi(P, P')\geqq a>0$ on $V_0\times V_0$. If Q is isolated in Ω and $\Theta(Q, Q)=\infty$, then $\Phi(Q, Q)=\infty$ by our assumption and $\Phi(Q, Q)\mu_m(\{Q\})\leqq U_{\Phi}^{\mu_m}(Q)$. This shows that $Q\notin S_{\mu_m}$ for every m, because if $Q\in S_{\mu_m}$ then $U_{\Phi}^{\mu_m}(Q)\leqq\Theta_m(Q, Q)<\infty$ but $\Phi(Q, Q)\mu_m(\{Q\})=\infty$. Consequently $P_0\neq Q$. If $P_0\neq Q$ in general, we have

$$a\mu_m(V_0)\leqq U_{\Phi}^{\mu_m}(P_0)\leqq \max\,\{\Psi(P_0, Q), \Theta(P_0, Q)\}.$$

This is a contradiction and hence $\mu_m(K)$ is uniformy bounded. If $P_0=Q$ is isolated in Ω and $\Theta(Q, Q)<\infty$, $a\mu_m(V_0)\leqq\Theta(Q, Q)$. If $P_0=Q$ is not isolated in Ω, we take any $P\neq P_0$ in V_0 and have $a\mu_m(V_0)\leqq\max\,\{\Psi(P, Q), \Theta(P, Q)\}$. In any case $\mu_m(V_0)$ is uniformly bounded and a contradiction arises. We can choose a subnet $T=\{\mu^{(\omega)}; \omega\in D\}$ of $\{\mu_m\}$ converging vaguely to a measure μ_0. By Theorems 1.15 and 1.16

$$U_{\Phi}^{\mu_0}(P)=\underline{\lim}_{\omega}\ U_{\Phi}^{\mu^{(\omega)}}(P)\geqq \lim_{m\to\infty}\ \Theta_m(P,Q)=\Theta(P,Q) \qquad \Phi\text{-p.p.p. on } K$$

and

$$U_{\Phi}^{\mu_0}(P)\leqq \underline{\lim}_{\omega}\ U_{\Phi}^{\mu^{(\omega)}}(P)\leqq \Psi(P,Q) \qquad \text{in } \Omega-K.$$

Remark 1. If there is no isolated point Q with $\Phi(Q, Q)=\infty$ or if $\Phi(P, Q)$ is positive and Ω contains at least two points, then we need not assume that $\Theta(P, P)=\infty$ implies $\Phi(P, P)=\infty$. For a proof under the second condition we refer to the following proof of (i)$\rightarrow$(iii).

Remark 2. Let Ω consist of two points, Φ be given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

and $\Psi=\Theta$ be given by

$$\begin{pmatrix} 1 & 0 \\ 0 & \infty \end{pmatrix}.$$

Then (ii) is true but not (iv). Thus we can not drop the condition that $\Theta(P, P)=\infty$ implies $\Phi(P, P)=\infty$ in general.

(i)$\rightarrow$(iii) under the assumption in Lemma 2.3 and the assumption that Ω contains at least two points, that $\Phi(P, Q)$ is positive in $\Omega\times\Omega$ and that $\Theta(P, Q)$ is locally bounded outside the diagonal set. Let K be a compact set with $V_{i,\Phi}(K)<\infty$ and ν be a measure with compact support. We choose a sequence $\{\Theta_m\}$ of continuous symmetric kernels on $(K\cup S_\nu)\times(K\cup S_\nu)$ increasing to Θ. There exists a measure μ_m supported by K such that $U_{\Phi}^{\mu_m}(P)\geqq U_{\check{\Theta}_m}^{\nu}(P)$ Φ-p.p.p. on K and $U_{\Phi}^{\mu_m}(P)\leqq U_{\check{\Theta}_m}^{\nu}(P)\leqq U_{\check{\Theta}}^{\nu}(P)$ on S_{μ_m}. By (i) we have $U_{\Phi}^{\mu_m}(P)\leqq U_{\check{\Psi}}^{\nu}(P)$ in $\Omega-S_{\mu_m}$. If $\mu_m(K)$ is uniformly bounded, we can choose a subnet of $\{\mu_m\}$ which converges vaguely to a measure μ_0 and obtain

$$U_{\Phi}^{\mu_0}(P)\geqq U_{\check{\Theta}}^{\nu}(P) \qquad \Phi\text{-p.p.p. on } K$$

and

$$U_{\Phi}^{\mu_0}(P)\leqq U_{\check{\Psi}}^{\nu}(P) \qquad \text{in } \Omega-K.$$

In case $\Omega\neq K$ we take any point $P_0\notin K$ and a compact neighborhood N_{P_0} of P_0 disjoint from K. We denote the restrictions of ν to $\Omega-N_{P_0}$ and N_{P_0} by ν_1 and ν_2 respectively. For ν_1 we can find $\mu^{(1)}$ which satisfies the two inequalities, because the total mass of the measure μ_m corresponding to ν_1 and Θ_m is bounded on account of the inequality

$$\inf_{Q\in K}\ \Phi(P_0, Q)\mu_m(K)\leqq U_{\check{\Psi}}^{\nu_1}(P_0)<\infty$$

and the above discussion applies. By a similar reasoning we find $\mu^{(2)}$ satisfying the two inequalities with ν_2. The measure $\mu^{(1)}+\mu^{(2)}$ satisfies the required condition.

In the case $\Omega=K$, we take two different points P_0 and Q_0. We choose compact neighborhoods of P_0 and Q_0 respectively so that they are disjoint from each other. We denote their respective interiors by N_1 and N_2. By the above discussion we know that there is a measure μ_1 supported by $K-N_1$, which satisfies $U_{\Phi}^{\mu_1}(P)\geqq U_{\Theta}^{\nu}(P)$ Φ-p.p.p. on $K-N_1$, and a similar measure μ_2 supported by $K-N_2$. Then $U_{\Phi}^{\mu_1+\mu_2}(P)\geqq U_{\Theta}^{\nu}(P)$ Φ-p.p.p. on $(K-N_1)\cup(K-N_2)=K$ and it is concluded that $\mu_1+\mu_2$ is a desired measure.

We state these results as the following theorem:

THEOREM 2.43. *Let $\Phi(P, Q)$ be strictly pseudo-positive and continuous outside the diagonal set. Then the (Φ, Ψ, Θ)-light sweeping-out principle implies the (Φ, Ψ, Θ)-domination principle and the latter is equivalent to the (Φ, Ψ, Θ)-light domination principle. If, in addition, $\Phi(P, Q)$ is nonnegative in $\Omega\times\Omega$, positive on the diagonal set and continuous in the extended sense, if $\Psi(P, Q)$ is locally bounded outside the diagonal set, if $\Theta(P, Q)$ is positive on the diagonal set and finite outside the diagonal set and if $\Theta(P, P)=\infty$ implies always $\Phi(P, P)=\infty$, then the above three principles are equivalent.*

If, furthermore, Ω contains at least two points, if $\Phi(P, Q)$ is positive in $\Omega\times\Omega$ and if $\Theta(P, Q)$ is locally bounded outside the diagonal set, then the (Φ, Ψ, Θ)-light sweeping-out principle is equivalent to the (Φ, Ψ, Θ)-sweeping-out principle.

REMARK. In (iii) and (iv) $U_{\Phi}^{\mu}(P)$ is not dominated by any potential on K. It is preferable to have additional condition to bound $U^{\mu}(P)$ from above on K. Let us say that the modified (iv) is true if, in addition to the properties of $U^{\mu}(P)$ in (iv), $U_{\Phi}^{\mu}(P)\leqq\Theta(P, Q)$ on S_{μ} and $U_{\Phi}^{\mu}(P)\leqq\Psi(P, Q)$ in $\Omega-S_{\mu}$. If we assume, in addition to the above assumptions which guarantee (i)$\rightleftarrows$(ii)$\rightleftarrows$(iv), that $\Theta(P, Q)$ is continuous in the extended sense and finite outside the diagonal set we can prove by means of Lemma 1.10 that the modified (iv) is equivalent to them. In fact, with the notations used in the proof of (ii)$\to$(iv), we have

$$0\geqq \varlimsup_{m\to\infty}\ \sup_{G\cap S_{\mu_m}}\{U_{\Phi}^{\mu_m}(P)-\Theta(P, Q)\}\geqq \sup_{G\cap S_{\mu}}\{U_{\Phi}^{\mu}(P)-\Theta(P, Q)\};$$

where $G=\Omega$ in case $\Theta(Q, Q)<\infty$ and $G=\Omega-\{Q\}$ in case $\Theta(Q, Q)=\infty$. It follows that $U^{\mu}(P)\leqq\Theta(P, Q)$ everywhere on S_{μ}. Another inequality holds by (ii). However, it is still open whether or not one can add to (iii) the inequalities $U_{\Phi}^{\mu}(P)\leqq U_{\Theta}^{\nu}(P)$ on S_{μ} and $U_{\Phi}^{\mu}(P)\leqq U_{\Psi}^{\nu}(P)$ in $\Omega-S_{\mu}$ while preserving the equivalence to other principles.

We can apply Theorem 2.43 to principles $(\mathrm{U})_c$, $(\mathrm{U}_{\mathrm{d}}^{*})_c$, $(\mathrm{E}_{\mathrm{q}})_c$ and $(\mathrm{S})_c$. However some assumptions can be weakened and sometimes independent proofs will be given. First we obtain by taking $\Psi=c\Phi$ and $\Theta=\Phi$

THEOREM 2.44. *Let $\Phi(P, Q)$ be strictly pseudo-positive and continuous out-*

side the diagonal set. In order that it satisfy $(\mathrm{U_d})_c$, *the following condition is necessary and sufficient*:

[Q_2] *Let* μ *be any measure with compact support and* P_0 *be any point outside* S_μ. *If* $U^\mu(P)\leqq\Phi(P, P_0)$ *on* S_μ, *then* $U^\mu(P)\leqq c\Phi(P, P_0)$ *everywhere in* Ω.

THEOREM 2.45. *Let* $\Phi(P, Q)$ *be strictly pseudo-positive and continuous outside the diagonal set. In order that it satisfy* $(\mathrm{U})_c$ *the following condition is necessary*:

[Q_1] *Let* P_0 *be any point and* λ *be any measure with compact support* S_λ *not containing* P_0. *If*

$$U^\mu(P)\leqq\Phi(P, P_0) \qquad \text{on } S_\lambda,$$

then $\lambda(\Omega)\leqq c$.
If, in addition, Φ *is nonnegative,* [Q_1] *is sufficient for* $(\mathrm{U})_c$.

PROOF. We suppose that $(\mathrm{U})_c$ is satisfied. Let P_0 be any point and λ be a measure with compact support $S_\lambda\not\ni P_0$ such that

$$U^\lambda(P)\leqq\Phi(P_0, P) \qquad \text{on } S_\lambda.$$

Let λ_0 be a unit measure of $\mathscr{E}_{S_\lambda}$ for which $U^{\lambda_0}(P)\geqq V_i(S_\lambda)$ p.p.p. on S_λ and $U^{\lambda_0}(P)\leqq V_i(S_\lambda)$ everywhere on S_{λ_0}. Since the kernel is strictly pseudo-positive, $V_i(S_\lambda)=(\lambda_0, \lambda_0)>0$. By $(\mathrm{U})_c$

$$U^{\lambda_0}(P)\leqq cV_i(S_\lambda) \qquad \text{in } \Omega.$$

Therefore

$$\lambda(\Omega)V_i(S_\lambda)\leqq\int U^{\lambda_0}(P)d\lambda(P)=\int U^\lambda(P)d\lambda_0(P)\leqq\int\Phi(P_0, P)d\lambda_0(P)$$
$$=U^{\lambda_0}(P_0)\leqq cV_i(S_\lambda),$$

whence $\lambda(\Omega)\leqq c$.

Conversely, we assume [Q_1] and that $\Phi\geqq 0$. We take ν with compact support and any point $P_0\notin S_\nu$. Since the kernel is strictly pseudo-positive, there exists an extremal measure $\mu^*\in\mathscr{E}_{S_\nu}$ satisfying $U^{\mu^*}(P)\geqq\Phi(P_0, P)$ p.p.p. on S_ν and $U^{\mu^*}(P)\leqq\Phi(P_0, P)$ everywhere on S_{μ^*}. From the latter inequaltiy it follows that $\mu^*(\Omega)\leqq c$ by [Q_1]. We have

$$U^\nu(P_0)\leqq\int U^{\mu^*}d\nu=\int U^\nu d\mu^*\leqq\sup_{S_\nu} U^\nu\cdot\mu^*(\Omega)\leqq c\sup_{S_\nu} U^\nu.$$

Thus $(\mathrm{U})_c$ is proved.

We shall see relations between $(\mathrm{U})_c$ and $(\mathrm{U_d})_c$. We showed already in § 1.3 that $(\mathrm{F})\nrightarrow(\mathrm{U_d^*})_c$ for any $c\geqq 1$ (see Example for $(\mathrm{F})\nrightarrow(\mathrm{D}^*)$) and that $(\mathrm{D})\nrightarrow(\mathrm{U})_c$ for any $c\geqq 1$ (see (1.13)). We shall now consider a convolution kernel in E_3. We denote the distance from the origin to $x=(x_1, \cdots, x_n)$ in E_n by $|x|$.

Let $\varphi(x)=\varphi(x_1, \cdots, x_n)$ be a nonnegative function in E_n with the following properties:

It is symmetric: $\varphi(x)=\varphi(-x)$,

It is continuous outside the origin and $\underline{\lim}_{x\to 0}\ \varphi(x)\geqq\varphi(0)>0$,

$\int_{|x|>a}\varphi(x)dx<\infty$ for some $a>0$,

$\iint\varphi(x-y)d\mu(x)d\mu(y)>0$ for any $\mu\neq 0$ with compact support.

We set, for $P=x$ and $Q=y$,

$$\Phi(P, Q)=\varphi(x-y)$$

and take it as a kernel in E_3. Then, by Theorem 2.45, [Q_1] is necessary and sufficient for $(\mathrm{U})_c$ to be true. We can prove, in the same way as for Theorem 8 of Ninomiya [8],

THEOREM 2.46. *If the above $\Phi(P, Q)$ satisfies $(\mathrm{U_d})_{c_1}$ and if*

$$\underline{\lim}_{r\to\infty}\ \frac{\int_{|x|\leqq r}\varphi(x)dx}{\int_{|x|\leqq r-\rho}\varphi(x)dx}\leqq c_2$$

for every $\rho>0$, then $(\mathrm{U})_{c_1 c_2}$ is satisfied.

COROLLARY 1. *If the above $\Phi(P, Q)$ satisfies $(\mathrm{U_d})_{c_1}$ and if $0<\int_{E_n}\varphi(x)dx<\infty$, then $(\mathrm{U})_{c_1}$ is satisfied.*

COROLLARY 2. *If the above $\Phi(P, Q)$ satisfies $(\mathrm{U_d})_{c_1}$ and $\varphi(x)$ is a decreasing function of $|x|$, then $(\mathrm{U})_{c_1}$ is satisfied.*

Next we shall be concerned with α-kernels in E_n. We consider Kelvin transformation.[33] We fix a point P_0 and transform $P(\neq P_0)$ to P', lying on the half line which issues from P_0 and passes through P, such that $\overline{P_0P}\cdot\overline{P_0P'}=1$. Given a measure μ with $S_\mu\not\ni P_0$ and a μ-measurable set A, we set

$$\mu'(A')=\int_A\frac{1}{\overline{P_0Q}^\alpha}\,d\mu(Q)$$

for the transform A' of A. For such measure μ' it holds that

$$U^{\mu'}(P')=\int\frac{1}{\overline{P'Q'}^\alpha}\,d\mu'(Q')=\overline{PP_0}^\alpha\int\frac{1}{\overline{PQ}^\alpha}\,d\mu(Q),$$

because

$$\overline{P'Q'}=\frac{\overline{PQ}}{\overline{P_0P}\cdot\overline{P_0Q}}\,.$$

33) M. Riesz [1] used Kelvin's transformation to study α-potentials.

Now suppose that $\overline{PQ}^{-\alpha}$ satisfies $(\mathrm{U})_c$. Then for any P

$$c \geqq \frac{U^{\mu'}(P')}{\sup\limits_{T'\in S_{\mu'}} U^{\mu'}(T')} = \frac{\overline{PP}_0^{\alpha}\, U^{\mu}(P)}{\sup\limits_{T\in S_\mu} \overline{TP}_0^{\alpha}\cdot U^{\mu}(T)}. \tag{2.64}$$

Consequently if $U^{\mu}(P)\leqq \overline{PP}_0^{-\alpha}$ on S_μ, then $U^{\mu}(P)\leqq c\overline{PP}_0^{-\alpha}$ in E_n. This shows that $(\mathrm{U_d})_c$ is satisfied in virtue of Theorem 2.44. By the identity in (2.64), we can prove similarly that if $\overline{PQ}^{-\alpha}$ satisfies $(\mathrm{U_d})_c$ then it does $(\mathrm{U})_c$.

We state

THEOREM 2.47. *For α-kernel in E_n, if $(\mathrm{U})_c$ is satisfied then $(\mathrm{U_d})_c$ is satisfied, and vice versa.*

We know that there is an α-kernel which does not satisfy (FV) (Kunugui's example stated in § 1.5). It is seen by the identity in (2.64) that kernel does not satisfy (DV).

Leaving discussions on α-kernels, let us be finally concerned with $(\mathrm{E_q})_c$ and $(\mathrm{S}^*)_c$.

THEOREM 2.48. *Always $(\mathrm{U})_c$ implies $(\mathrm{E_q})_c$. If the kernel is strictly pseudo-positive and continuous outside the diagonal set, then $(\mathrm{E_q})_c$ implies $(\mathrm{U})_c$ for $c\geqq 1$.*

PROOF. Let K be a compact set with $V_i(K)<\infty$, and μ_1 be a unit extremal measure which minimizes (μ, μ) among $\mu\in\mathscr{E}_K(1, 1)$. We know that $U^{\mu_1}(P)\geqq(\mu_1, \mu_1)$ p.p.p. on K and $U^{\mu_1}(P)\leqq(\mu_1, \mu_1)$ on S_{μ_1}. By $(\mathrm{U})_c$ it follows that $U^{\mu_1}(P)\leqq c(\mu_1, \mu_1)$ in Ω. Thus $(\mathrm{E_q})_c$ is satisfied. The latter half of the theorem follows from Theorem 2.43 if we take $\Psi=c$ and $\theta=1$.

THEOREM 2.49. *Always $(\mathrm{S}^*)_c$ implies $(\mathrm{U_d^*})_c$, and if the kernel is nonnegative, $(\mathrm{S}^*)_c$ implies $(\mathrm{U_d})_c$.*

If, furthermore, the kernel is positive on the diagonal set, continuous in the extended sense and finite outside the diagonal set, then $(\mathrm{U_d^*})_c$ implies $(\mathrm{S}^*)_c$.

PROOF. We assume $(\mathrm{S}^*)_c$ and that $U^{\mu}(P)\leqq U^{\nu}(P)$ on S_μ for $\mu\in\mathscr{E}$ and ν with compact support. There is for any $P_0\notin S_\mu$, a measure μ_{P_0} supported by S_μ such that $U^{\mu_{P_0}}(P)\geqq\Phi(P, P_0)$ p.p.p. on S_μ and $U^{\mu_{P_0}}(P)\leqq c\Phi(P, P_0)$ in Ω. We have

$$U^{\mu}(P_0)\leqq(\mu_{P_0}, \mu)=(\mu, \mu_{P_0})\leqq(\nu, \mu_{P_0})=(\mu_{P_0}, \nu)\leqq cU^{\nu}(P_0).$$

This shows that $(\mathrm{U_d^*})_c$ is true. We can similarly conclude $(\mathrm{U_d})_c$ if the kernel is nonnegative.

To prove the latter half, take any compact set K with $V_i(K)<\infty$ and any measure ν with compact support. Let ν_K denote the restriction of ν to K. As in the proof of (i)$\rightarrow$(iii) we find a measure μ supported by K with the property that $U^{\mu}(P)\geqq U^{\nu-\nu_K}(P)$ p.p.p. on K and $U^{\mu}(P)\leqq cU^{\nu-\nu_K}(P)$ in Ω, although it is not certain that the total mass of μ is finite. To prove this we first observe that the continuity principle is true in virtue of Lemma 2.3. By (IV) of § 1.3 (B_K) is satisfied; namely the boundedness principle is valid on every compact set. Let μ_1 be a unit measure on K which gives $(\mu_1, \mu_1)=V_i(K)>0$. It holds that $U^{\mu_1}(P)\geqq(\mu_1, \mu_1)$ p.p.p. on K and $U^{\mu_1}(P)\leqq(\mu_1\ \mu_1)$ on S_{μ_1}. By (B_K) it is bounded on S_ν, say $U^{\mu_1}(P)<M$ on S_ν. It follows that

$$\begin{aligned}(\mu_1, \mu_1)\mu(K)\leqq(\mu_1, \mu)=(\mu, \mu_1)\leqq c(\nu-\nu_K, \mu_1)\\ =c(\mu_1, \nu-\nu_K)\leqq cM\nu(\Omega).\end{aligned}$$

This shows that $\mu(K)$ is finite. Since $U^{\mu+\nu_K}(P)\geqq U^\nu(P)$ p.p.p. on K and $U^{\mu+\nu_K}(P)\leqq cU^\nu(P)$ in Ω, $(S^*)_c$ is true.

2.12. Notes and questions.

It is well known that Gauss variation is useful in the sweeping-out process. It was used also to prove the potential representation of a superharmonic function by Frostman [2]. The first paper which discussed the variation itself from a general point of view seems to be Kametani [1; 3]. Recently Polish mathematicians Gorski, Leja and Siciak use general Gauss variation in their works on transfinite diameters probably without knowing the results of Kametani. In all these papers space is euclidean and function f is defined and continuous on a compact set. Leja [1] was the motivation for the author to investigate the n-dimensional problem (see § 2.2 of our paper).

After our manuscript was completed, Choquet [8] was published. It deals with problems which are partially common to our § 2.10.

Open questions.

2.1. The support S_{μ_x} of an extremal measure μ_x on K, defined in § 2.2, does not coincide with K generally. We ask when $K=S_{\mu_x}$. Every compact set of finite V_i-value contains such compact set; S_{μ_x} itself is such a set. This question depends on given f, g and x generally.

2.2. Can we improve the coefficient 2 in the inequality $V_i(X)-m\leqq2(\check{V}_i(X)-m)$ in Corollary 2 of Theorem 2.7? What is the best possible value?

2.3. Question stated at footnote 26).

2.4. Question stated after Theorem 2.15.

2.5. Let $K=\bigcup_{k=1}^{n} K_k$ be a disjoint union of compact sets. When is $V_i^{(g,y)}(K)$ quadratic in $y_1, \cdots, y_n$?

2.6. Are the conditions $\Phi\geqq\theta$ and $\Psi\geqq\theta$ really necessary in Theorem 2.37?

2.7. Let us add to (iii) in § 2.11 the condition that $U_\Phi^\mu(P)\leqq U_\Theta^\nu(P)$ on S_μ and $U_\Phi^\mu\leqq U_\Psi^\nu(P)$ in $\Omega-S_\mu$. Is this modified (iii) equivalent to (i) ($\rightleftarrows$(ii)$\rightleftarrows$(iii)$\rightleftarrows$(iv))?

Chapter III. Inner and outer problems.

3.1. Inner variational problem.

Let $X_1, \cdots, X_n$ be non-empty sets in Ω. We shall say that they are *Φ-separate* if they are mutually disjoint and $\Phi(P, Q)$ is bounded on each $X_j \times X_k$, $j \neq k$. Assume that they are Φ-separate and that $V_i(X_k) < \infty$ for each k. Let $f(P) < \infty$ be an upper semicontinuous function on $X = \bigcup_{k=1}^{n} X_k$ and $g(P)$ be a positive continuous function on X. Let $x_1 \geqq 0, \cdots, x_n \geqq 0$ and $x = (x_1, \cdots, x_n)$. We denoted in the preceding chapter by μ_x a conditional extremal measure on K consisting of mutually disjoint compact sets $\{K_k\}$. In this chapter we shall denote it by $\mu_K(g, x, f)$ or by $\mu_K(x)$ or simply by μ_K. As before we set for any $\mu \in \mathscr{E}_K(g, x)$

$$x_k \gamma_k(\mu) = \int_{K_k} \hat{U}^\mu \, d\mu - \int_{K_k} f d\mu \tag{3.1}$$

if $x_k > 0$. If $x_k = 0$, we do not define $\gamma_k(\mu)$ itself but set $x_k \gamma_k(\mu) = 0$. The *inner variational problem* is to consider

$$\inf_{\mu \in \mathscr{E}_X(g, x)} I(\mu) = I_X^i(g, x, f) = I_X^i(x) = I_X^i,$$

where $\mathscr{E}_X(g, x)$ is equal to $\{\mu \in \mathscr{E};\, S_\mu$ is decomposed into compact sets $K_1, \cdots, K_n$ such that $K_k \subset X_k$ and $\int_{K_k} g d\mu = x_k$ for each $k\}$. We shall call the restriction of μ to K_k *the restriction of μ to X_k*, and write $\int_{X_k} g d\mu$ for $\int_{K_k} g d\mu$. If there is a subset $Y_k \subset X_k$ with $V_i(Y_k) < \infty$ for each k such that $f(P)$ is finite on Y_k, we see $I_X^i < \infty$ without difficulty. In other cases, including the case that $\mathscr{E}_X(g, x) = \emptyset$, we set $I_X^i = \infty$. For $\mu \in \mathscr{E}_X(g, x)$ we define $x_k \gamma_k(\mu)$ by (3.1), where $\bigcup_{k=1}^{n} K_k = S_\mu$.

Let us recall that $\mathfrak{A}$ denotes the class of sets which are measurable for all measures. Since $\mu(A) = \sup \mu(K)$ for compact $K \subset A$ if $A \in \mathfrak{A}$, I_A^i is equal to $\inf I(\mu)$ for $\mu \in \mathscr{E}'_A(g, x, f) = \{\mu \in \mathscr{E};\, \mu(\Omega - A) = 0, \int_{A_k} g d\mu = x_k$ for each k and $\langle f, \mu \rangle$ is defined$\}$.

Let $\{\mu^{(m)}\}$ be a sequence of measures in $\mathscr{E}_X(g, x)$ such that $I(\mu^{(m)})$ tends to I_X^i. We are interested in finding a limiting measure of $\{\mu^{(m)}\}$ and its property. We shall discuss this problem in two manners. One is under the assumption of the continuity principle and the other is, essentially speaking, under the assumption of the completeness of some subclass of $\mathscr{E}$.

We assume in this section that the kernel $\hat{\Phi}$ satisfies the continuity principle. If some of $\{x_k\}$ vanish the problem reduces to a lower dimensional case and hence hereafter we assume that $x_1 > 0, \cdots, x_n > 0$ except in § 3.8 and

§ 3.9. We shall state the conditions precisely. Let X consist of relatively compact Φ-separate sets $\{X_k\}$ such that $V_i(X_k)<\infty$ for each k, $f(P)<\infty$ be an upper semicontinuous function on X such that it is finite on some set $Y_k\subset X_k$ with $V_i(Y_k)<\infty$ for each k and $g(P)$ be a continuous function with a positive lower bound on X. Assume that $I_X^i>-\infty$ and let $I(\mu^{(m)})$ tend to I_X^i. We choose a subsequence $\{\mu^{(m_p)}\}$ of $\{\mu^{(m)}\}$ such that each $\lim_{p\to\infty}\gamma_k(\mu^{(m_p)})$ exists. Since $\int_{X_k} gd\mu^{(m)}=x_k$, $\mu^{(m)}(\Omega)\leqq(\inf_X g)^{-1}\sum_{k=1}^n x_k$ and $\{\mu^{(m)}\}$ is bounded in $\mathscr{M}$. By Proposition 3 in § 1.6 we can find a subnet $T=\{\mu^{(\omega)};\ \omega\in D\}$ of $\{\mu^{(m_p)}\}$ such that the restriction $\mu_k^{(\omega)}$ of $\mu^{(\omega)}$ to X_k converges vaguely to some measure for each k; in this chapter the subscript k of a measure will always mean the restriction of the measure to the k-th component of a set. This measure $\mu_k^{(\omega)}$ is supported by the closure X_k^a of X_k in Ω and will be denoted by $\mu_{X_k}^i$. We set $\mu_X^i=\sum_{k=1}^n\mu_{X_k}^i$ and write γ for the point in E_3 with coordinates $\{\gamma_k=\lim_\omega\gamma_k(\mu^{(\omega)})\}$. This measure μ_X^i will also be denoted by $\mu_X^i(g,x,f)$ or by $\mu_X^i(x)$.

An alternative condition to ensure that $\{\mu^{(m)}\}$ is bounded in $\mathscr{M}$ is $V_i(X)>0$. In fact, on account of the continuity principle, there is a measure $\nu\in\mathscr{E}_X(g,x)$ which gives a continuous potential $\hat{U}^\nu(P)$ and finite $I(\nu)$. Since $(\mu^{(m)}+\nu)/2\in\mathscr{E}_X(g,x)$,

$$I_X^i\leqq I\left(\frac{\mu^{(m)}+\nu}{2}\right)=\frac{1}{2}I(\mu^{(m)})+\frac{1}{2}I(\nu)-\frac{1}{4}(\mu^{(m)}-\nu,\mu^{(m)}-\nu)$$

and hence $(\mu^{(m)}-\nu,\mu^{(m)}-\nu)$ is bounded from above. It holds that

$$(\mu^{(m)}-\nu,\mu^{(m)}-\nu)\geqq V_i(X)\{\mu^{(m)}(\Omega)\}^2-2\sup\hat{U}^\nu\cdot\mu^{(m)}(\Omega)+(\nu,\nu)$$

and it is seen that $\mu^{(m)}(\Omega)$ is bounded.

We write

$$\begin{aligned}I(\mu^{(\omega)})&=\sum_{k=1}^n\{(\mu_k^{(\omega)},\mu_k^{(\omega)})-2\langle f,\mu_k^{(\omega)}\rangle\}+\sum_{\substack{j,k=1\\ j\neq k}}^n(\mu_j^{(\omega)},\mu_k^{(\omega)})\\&=\sum_{k=1}^n I(\mu_k^{(\omega)})+\sum_{\substack{j,k=1\\ j\neq k}}^n(\mu_j^{(\omega)},\mu_k^{(\omega)}).\end{aligned}$$

We note that the last sum of mutual energies is a bounded quantity. This shows that each $I(\mu_k^{(\omega)})$ is bounded from below, because if $I(\mu_{k_0}^{(\omega)})$ were not so we should see that $I(\sum_{\substack{k=1\\ k\neq k_0}}^n\mu_k^{(\omega_0)}+\mu_{k_0}^{(\omega)})$ is not bounded from below for any fixed ω_0. This contradicts our assumption $I_X^i>-\infty$. Consequently each $I(\mu_k^{(\omega)})$ is bounded. Since

$$(\mu_k^{(\omega)},\mu_k^{(\omega)})\geqq\min_{Q,P\in X^a}\Phi(P,Q)\,(\mu_k^{(\omega)}(\Omega))^2,$$

$(\mu_k^{(\omega)}, \mu_k^{(\omega)})$ is bounded from below. It follows that

$$(3.2) \qquad 2x_k \gamma_k(\mu^{(\omega)}) = I(\mu_k^{(\omega)}) + 2\sum_{\substack{j=1 \\ j\neq k}}^{n} (\mu_j^{(\omega)}, \mu_k^{(\omega)}) + (\mu_k^{(\omega)}, \mu_k^{(\omega)})$$

is bounded from below. Consequently $\lim_{\omega} \sum_{k=1}^{n} x_k \gamma_k(\mu^{(\omega)})$ exists and equals $\sum_{k=1}^{n} x_k \gamma_k$.

Now by our assumption there is a compact subset K_k of X_k for each k on which $f(P)$ is bounded and whose V_i-value is finite. We shall use $\nu \in \mathscr{E}_{\bigcup_{k=1}^{n} K_k}(g, x)$ which gives a continuous potential $\hat{U}^{\nu}(P)$. We have

$$I_X^i \leqq I\left(\frac{1}{2}\mu^{(\omega)} + \frac{1}{2}\nu\right)$$
$$= \frac{3}{4} I(\mu^{(\omega)}) - \frac{1}{2}\sum_{k=1}^{n} x_k \gamma_k(\mu^{(\omega)}) + \frac{1}{4}(\nu, \nu) + \frac{1}{2}\int \hat{U}^{\nu} d\mu^{(\omega)} - \langle f, \nu\rangle.$$

It follows that

$$I_X^i \leqq \frac{3}{4} I_X^i - \frac{1}{2} \lim_{\omega} \sum_{k=1}^{n} x_k \gamma_k(\mu^{(\omega)}) + \frac{1}{4}(\nu, \nu) + \frac{1}{2}\int \hat{U}^{\nu} d\mu_X^i - \langle f, \nu\rangle.$$

Therefore

$$\lim_{\omega} \sum_{k=1}^{n} x_k \gamma_k(\mu^{(\omega)}) = \sum_{k=1}^{n} x_k \gamma_k < \infty.$$

We have already seen that each $\gamma_k(\mu^{(\omega)})$ is bounded from below. Hence each $\gamma_k(\mu^{(\omega)})$ must be bounded. It follows from (3.2) that each $\langle f, \mu_k^{(\omega)}\rangle$ and $(\mu_k^{(\omega)}, \mu_k^{(\omega)})$ are bounded, say, $|\langle f, \mu_k^{(\omega)}\rangle| < M$ and $|(\mu_k^{(\omega)}, \mu_k^{(\omega)})| < M$.

We set

$$H_k = \{P \in X_k;\ \hat{U}^{\mu_X^i}(P) - f(P) < \gamma_k g(P)\}.$$

Let us suppose that $V_i(H_{k_0}) < \infty$. We can find a compact set $K_0 \subset H_{k_0}$ with $V_i(K_0) < \infty$ and a positive number η by Proposition 1 in § 1.1 such that

$$(3.3) \qquad \hat{U}^{\mu_X^i}(P) - f(P) < \gamma_k g(P) - \eta \qquad \text{on } K_0.$$

We observe that $f(P)$ is bounded from below on K_0. Let $\nu \in \mathscr{E}_{K_0}(g, x_{k_0})$ be a measure which gives a continuous potential $\hat{U}^{\nu}(P)$ in Ω. For any t, $0 \leqq t \leqq 1$, it holds that

$$I_X^i \leqq I(\mu^{(\omega)} - t\mu_{k_0}^{(\omega)} + t\nu)$$
$$= I(\mu^{(\omega)}) - 2tx_{k_0}\gamma_{k_0}(\mu^{(\omega)}) + t^2(\mu_{k_0}^{(\omega)}, \mu_{k_0}^{(\omega)}) + 2t\int \left\{\hat{U}^{\mu^{(\omega)}} - t\hat{U}^{\mu_{k_0}^{(\omega)}}\right\} d\nu + t^2(\nu, \nu) - 2t\langle f, \nu\rangle$$

and follows that

$$I_X^i \leqq I_X^i - 2tx_{k_0}\gamma_{k_0} + 2t\int\left\{\hat{U}^{\mu_X^i} - t\hat{U}^{\mu_{X_{k_0}}^i}\right\} d\nu + t^2(\nu, \nu) - 2t\langle f, \nu\rangle + t^2 M.$$

Cancelling I_X^i, dividing the rest by t and letting $t \to 0$, we obtain

$$\int \hat{U}^{\mu_X^i} d\nu - \langle f, \nu\rangle \geqq x_{k_0}\gamma_{k_0}.$$

On the other hand (3.3) gives us the following contradicting inequality:

$$\int \hat{U}^{\mu_X^i} d\nu - \langle f, \nu\rangle < x_{k_0}\gamma_{k_0} - \eta\nu(\Omega).$$

It is now proved that

$$\hat{U}^{\mu_X^i}(P) - f(P) \geqq \gamma_k\, g(P) \qquad \text{p.p.p. on } X_k$$

for each k.

In case $f(P)$ is defined and continuous on X^a, the equality

$$I_X^i + \langle f, \mu_X^i\rangle = \sum_{k=1}^n x_k \gamma_k$$

follows from

$$I(\mu^{(\omega)}) + \langle f, \mu^{(\omega)}\rangle = \sum_{k=1}^n x_k \gamma_k(\mu^{(\omega)}).$$

We state these results as

THEOREM 3.1. *Let X be a relatively compact set consisting of Φ-separate sets $X_1, \cdots, X_n$ such that $V_i(X_k) < \infty$ for each k, $f(P) < \infty$ be an upper semicontinuous function on X such that $f(P) > -\infty$ on some set $Y_k \subset X_k$ with $V_i(Y_k) < \infty$ for each k, and $g(P)$ be a positive continuous function on X. Assume that $\hat{\Phi}$ satisfies the continuity principle, and that one or both of the following conditions is satisfied:*

(a$_1$) *$g(P)$ has a positive lower bound on X,*

(a$_2$)$_i$ *$V_i(X) > 0$.*

Assume also for $x = (x_1, \cdots, x_n)$, $x_1 > 0, \cdots, x_n > 0$, that

$$I_X^i = \inf_{\mu \in \mathscr{E}_X(g, x)} I(\mu) > -\infty,$$

and let $\{\mu^{(m)}\}$ be a sequence of measures in $\mathscr{E}_X(g, x)$ which gives

$$\lim_{m\to\infty} I(\mu^{(m)}) = I_X^i.$$

Then there is a subnet $\{\mu^{(\omega)}; \omega \in D\}$ of $\{\mu^{(m)}\}$ such that the restriction $\mu_k^{(\omega)}$ of $\mu^{(\omega)}$ to X_k converges vaguely to a measure $\mu_{X_k}^i$, $\lim_\omega \gamma_k(\mu^{(\omega)})$ exists and is finite for each k. Setting $\sum_{k=1}^n \mu_{X_k}^i = \mu_X^i$ and $\lim_\omega \gamma_k(\mu^{(\omega)}) = \gamma_k$, we have

$$(3.4) \qquad \hat{U}^{\mu^i_X}(P)-f(P)\geqq \gamma_k\, g(P) \qquad p.p.p.\ on\ X_k.$$

If $f(P)$ is defined and continuous on X^a,

$$I^i_X+\langle f,\ \mu^i_X\rangle=\sum_{k=1}^{n} x_k\,\gamma_k.$$

In the above definition of μ^i_X we started from $\{\mu^{(m)}\}$ in $\mathscr{E}_X(g, x)$ for which $\lim\limits_{m\to\infty} I(\mu^{(m)})=I^i_X$. It follows that

$$\lim_{m\to\infty} I(\mu^{(m)})\geqq \lim_{m\to\infty} I(\mu_{S_{\mu(m)}})\geqq I^i_X$$

and hence

$$\lim_{m\to\infty} I(\mu_{S_{\mu(m)}})=I^i_X.$$

Thus there is always an increasing sequence $\{K^{(m)}\}$ of compact subsets of X such that $I(\mu_{K(m)})$ tends to I^i_X. If we restrict ourselves to vague limits of subnets of $\{\mu_{K(m)}\}$, we have

THEOREM 3.2. *Let $X_1, \cdots, X_n$ be any mutually disjoint sets with finite V_i-value, $f(P)<\infty$ be an upper semicontinuous function on some set $Z\supset X=\sum\limits_{k=1}^{n} X_k$ such that $f(P)>-\infty$ on some set $Y_k\subset X_k$ with $V_i(Y_k)<\infty$ for each k and $g(P)$ be a positive continuous function on Z. If there is a net $\{K^{(\omega)}\}$ of compact subsets of X such that $\mu_{K^{(\omega)}_k}$ converges vaguely to μ_k and $\gamma_k(\mu_{K(\omega)})$ tends to a finite number γ_k for each k, then*

$$(3.5) \qquad \hat{U}^{\mu}(P)-f(P)\leqq \gamma_k\, g(P) \qquad on\ S_{\mu_k}\cap Z,$$

where $\mu=\sum\limits_{k=1}^{n} \mu_k$.

PROOF. We recall that

$$\hat{U}^{\mu_{K^{(\omega)}}}(P)-f(P)\leqq \gamma_k(\mu_{K(\omega)})g(P) \qquad \text{on}\ \ S_{\mu_{K(\omega)}}\cap X_k=S_{\mu_{K^{(\omega)}_k}}$$

if $I^i_{K^{(\omega)}}$ is finite, according to Theorem 2.7. By Lemma 1.10 it follows that

$$0\geqq \varlimsup_{\omega}\ \sup_{S_{\mu_{K^{(\omega)}_k}}} \{\hat{U}^{\mu_{K(\omega)}}-f-\gamma_k(\mu_{K^{(\omega)}_k})g\}\geqq \sup_{S_{\mu_k}\cap Z} \{\hat{U}^{\mu}-f-\gamma_k\, g\}.$$

Thus

$$\hat{U}^{\mu}(P)-f(P)\leqq \gamma_k\, g(P) \qquad \text{on}\ \ S_{\mu_k}\cap Z.$$

In the special case $f\equiv 0$, we have

COROLLARY. *Let us consider the special case where $f(P)\equiv 0$, $x=1$ and $n=1$.*

Under the same conditions on X, Φ *and* $g(P)$ *as in Theorem* 3.1, *let* $\{K^{(m)}\}$ *be a sequence of compact subsets of* X *which gives*

$$\lim_{m\to\infty} (\mu_{K^{(m)}}, \mu_{K^{(m)}}) = V_i^{(g)}(X).^{34)}$$

Then, for the limit μ_X^i *of any vaguely converging subnet of* $\{\mu_{K^{(m)}}\}$, *it holds that*

$$\hat{U}^{\mu_X^i}(P) \geqq V_i^{(g)}(X)g(P) \qquad p.p.p. \text{ on } X \tag{3.6}$$

and that

$$\hat{U}^{\mu_X^i}(P) \leqq V_i^{(g)}(X)g(P) \qquad \text{on } S_{\mu_X^i} \cap Z \tag{3.7}$$

if $g(P)$ *is defined and positive continuous on some set* $Z \supset X$.

For $g(P)$ defined on the closure X^a, we shall call a measure satisfying (3.6) and (3.7) a *weak inner g-equilibrium measure* and, in case $g(P)\equiv 1$, a *weak inner equilibrium measure.* Its potential will have the corresponding nominations.

We shall give criteria for a relatively closed subset of an open set to be an F_σ-set.

LEMMA 3.1. *Let* G *be an open set in* Ω *and* B *be a relatively closed subset of* G *with* $V_i(B)=\infty$. *If one or both of the following conditions is satisfied, then* B *is an* F_σ*-set.*

(b_1) G *is an* F_σ*-set,*

(b_2) *The kernel* $\Phi(P, Q)$ *has the following two properties: For every point* $P\in G$ *and every neighborhood* N_P *of* P, *the kernel is bounded from above on* $\{P\} \times (G-N_P)$, *and, at each point* $P\in \partial G$ *with* $\Phi(P, P)<\infty$, *there is a neighborhood* N_P *of* P *in* Ω *such that* $\Phi(Q, Q)<\infty$ *for every* $Q\in N_P\cap G$.

PROOF. If $G=\bigcup_m F^{(m)}$, $B=\bigcup_m (B\cap F^{(m)})$ is an F_σ-set. Next assuming (b_2), we set

$$J^{(p)} = \{P\in G;\ \Phi(P, Q)\leqq p \text{ for all } Q\in\partial G\}.$$

This is a relatively closed subset of G and $\bigcup_p J^{(p)}=G$. Suppose that $B\cap J^{(p)}$ is not a closed set. Let $P_0\in\partial G$ be a point of accumulation of $B\cap J^{(p)}$. If $\Phi(P_0, P_0)=\infty$, $\Phi(P, Q)\to\infty$ as P, $Q\to P_0$ by the lower semicontinuity of Φ. This is impossible, because if so there would exist $P\in J^{(p)}$ for which $\Phi(P, P_0)>p$. Therefore $\Phi(P_0, P_0)<\infty$. By our assumption there is a point $P'\in B\cap J^{(p)}$ for which $\Phi(P', P')<\infty$. However, such a point P' has a finite V_i-value, contradicting the fact that $V_i(B)=\infty$. Thus each $B\cap J^{(p)}$ is a closed set and $B=\bigcup_p (B\cap J^{(p)})$ is an F_σ-set.

34) We recall that $V_i^{(g)}(X)=\inf(\mu, \mu)$ for $\mu\in \mathscr{E}_X(g, 1)$.

REMARK. (b_2) is satisfied if the kernel is continuous in the extended sense and bounded from above outside the diagonal set.

In case a component of X is an open set, we prove

THEOREM 3.3. *Assume that a measure* μ *and constants* $\{\gamma_k\}$ *satisfy* (3.4) *and that* $X_{k_0}=G_{k_0}$ *is an open set. Assume one or both of* (b_1) *and* (b_2) *of Lemma* 3.1 *for* G_{k_0}. *Then*

$$\hat{U}^{\mu}(P)-f(P)\geqq\gamma_{k_0}g(P) \qquad q.p.\ in\ G_{k_0}$$

and the exceptional set H *in* G_{k_0} *is a* K_σ*-set.*[35]

PROOF. We set

$$H^{(m)}=\left\{P\in G_{k_0};\ \hat{U}^{\mu}(P)-f(P)\leqq\gamma_{k_0}g(P)-\frac{1}{m}\right\}.$$

Certainly $H=\bigcup_m H^{(m)}$. Since the function on the left is lower semicontinuous, $H^{(m)}$ is a relatively closed subset of G_{k_0} with $V_i(H^{(m)})=\infty$. By Lemma 3.1 it is a K_σ-set; we write $H^{(m)}=\bigcup_p K^{(m,p)}$. Certainly $V_i(K^{(m,p)})=\infty$ for each m and p. We see that $V_i(K^{(m,p)})=V_e(K^{(m,p)})$ in virtue of Theorem 1.4 and that $V_e(H)=\infty$ on account of Proposition 2 in § 1.1. Thus the theorem is concluded.

We write γ_k for $\lim_{m\to\infty}\gamma_k(\mu^{(m)})$ so far as $\mu^{(m)}\in\mathscr{E}_X(g,x)$ and $I(\mu^{(m)})$ tends to I_X^i provided the limit, finite or infinite, exists for each k; it is not required that $\mu^{(m)}$ converges vaguely. Finally in this section we shall study the set of points $\gamma=(\gamma_1,\cdots,\gamma_n)$ in E_n. The set will be denoted by $\Gamma_X^i(x)$ or simply by Γ_X^i. We shall prove

THEOREM 3.4. *The set* Γ_X^i *is compact in* E_n *under the assumptions in Theorem* 3.1.

PROOF. Let $\{\gamma^{(m)}\}$ be a sequence of points in Γ_X^i such that each sequence $\{\gamma_k^{(m)}\}$ of components converges to a finite number or diverges to $+\infty$ or to $-\infty$. For each m we choose a measure $\mu^{(m)}\in\mathscr{E}_X(g,x)$ such that $\overline{\gamma^{(m)}\gamma(\mu^{(m)})}<1/m$ and $I(\mu^{(m)})<I_X^i+1/m$. Hence $I(\mu^{(m)})$ tends to I_X^i and $\{\mu^{(m)}\}$ is a sequence in which we were interested in Theorem 3.1. For each k, $\gamma_k(\mu^{(m)})$ tends to a finite number or diverges to $+\infty$ or to $-\infty$. We showed in the proof of Theorem 3.1 that the last two cases do not happen. The point with coordinates $\{\lim_{m\to\infty}\gamma_k(\mu^{(m)})\}$ belongs to Γ_X^i and $\{\gamma^{(m)}\}$ converges to it. Thus Γ_X^i must be compact in E_n.

3.2. Inner problem for kernels of positive type.

Throughout this section we consider a kernel of positive type, mostly

35) *A* K_σ-set is a countable union of compact sets.

without mentioning it. We prove first

THEOREM 3.5. *Let X consist of Φ-separate sets $X_1, \cdots, X_n$ in Ω such that $V_i(X_k)<\infty$ for each k. Let $f(P)<\infty$ be an upper semicontinuous function on X such that $f(P)>-\infty$ on some subset $Y_k \subset X_k$ with $V_i(Y_k)<\infty$ for each k, and $g(P)$ be a positive continuous function on X. Assume that $I_X^i>-\infty$ and that, for a sequence $\{\mu^{(m)}\}$ of measures in $\mathscr{E}_X(g, x)$,*

$$\lim_{m\to\infty} I(\mu^{(m)})=I_X^i \tag{3.8}$$

and $\mu^{(m)}$ converges strongly to some measure $\mu_X^i \in \mathscr{E}$. Unless the kernel is non-negative on eack $X_j \times X_k$, $j \neq k$, we assume also one or both of the following conditions:

(a$_1$) *$g(P)$ has a positive lower bound on X,*

(a$_2$)$_i$ *$V_i(X)>0$.*

Then, any sequence of measures in $\mathscr{E}_X(g, x)$ on which I tends to I_X^i converges strongly to μ_X^i and $\gamma_k(\mu^{(m)})$ tends to a finite value for each k. If it is denoted by γ_k, then

$$I_X^i+(\mu_X^i, \mu_X^i)=2\sum_{k=1}^n x_k \gamma_k \tag{3.9}$$

and each γ_k does not depend on the choice of $\{\mu^{(m)}\}$. It holds also that

$$U^{\mu_X^i}(P)-f(P) \geqq \gamma_k g(P) \qquad p.p.p. \text{ on } X_k. \tag{3.10}$$

If a measure $\mu \in \mathscr{E}$ and finite constants $\{c_k\}$ satisfy (3.9) and (3.10) replacing μ_X^i and $\{\gamma_k\}$, then $c_k=\gamma_k$ for each k and $\mu^{(m)}$ converges strongly to μ. If (μ, μ_X^i) is defined, $\|\mu-\mu_X^i\|=0$ and if, furthermore, the energy principle is satisfied, $\mu=\mu_X^i$.

Before the proof we give

LEMMA 3.2. *Let X, $f(P)$, $g(P)$, I_X^i and $\{\mu^{(m)}\}$ be as above, and assume $\lim_{m\to\infty}\gamma_k(\mu^{(m)})=\gamma_k$ for each k and (3.9). Assume also that*

$$U^{\mu}(P)-f(P) \geqq c_k g(P) \qquad p.p.p. \text{ on } X_k \tag{3.11}$$

for a measure $\mu \in \mathscr{E}$ and finite constants $\{c_k\}$. Then

$$J(\mu)=2\sum_{k=1}^n x_k c_k-(\mu, \mu) \leqq I_X^i.$$

If, in addition, (μ, μ_X^i) is defined, we have

$$\|\mu_X^i-\mu\|^2 \leqq I_X^i - J(\mu).$$

Proof. We integrate (3.11) with respect to $\mu^{(m)}$ and obtain

$$(\mu, \mu^{(m)})-\langle f, \mu^{(m)}\rangle \geqq \sum_{k=1}^{n} x_k c_k.$$

It follows that

$$0\leqq \|\mu^{(m)}-\mu\|^2 \leqq (\mu^{(m)}, \mu^{(m)})+(\mu, \mu)-2\sum_{k=1}^{n} x_k c_k-2\langle f, \mu^{(m)}\rangle$$
$$=I(\mu^{(m)})-J(\mu).$$

Therefore $J(\mu)\leqq \lim_{m\to\infty} I(\mu^{(m)})=I_X^i$. If (μ, μ_X^i) is defined, we have

$$\|\mu_X^i-\mu\|^2=\lim_{m\to\infty}\|\mu^{(m)}-\mu\|^2\leqq \lim_{m\to\infty} I(\mu^{(m)})-J(\mu)=I_X^i-J(\mu).$$

Proof of Theorem 3.5. Let $\{\nu^{(m)}\}$ be a similar sequence in $\mathscr{E}_X(g, x)$ for which $I(\nu^{(m)})$ tends to I_X^i as $m\to\infty$. Since

$$\frac{1}{2}(\mu^{(m)}+\nu^{(m)})\in \mathscr{E}_X(g, x),$$

$$I_X^i\leqq I\left(\frac{\mu^{(m)}+\nu^{(m)}}{2}\right)=\frac{1}{2}I(\mu^{(m)})+\frac{1}{2}I(\nu^{(m)})-\frac{1}{4}\|\mu^{(m)}-\nu^{(m)}\|^2.$$

It follows that

$$\lim_{m\to\infty}\|\mu^{(m)}-\nu^{(m)}\|^2\leqq \lim_{m\to\infty}\{2I(\mu^{(m)})+2I(\nu^{(m)})-4I_X^i\}=0.$$

Consequently $\nu^{(m)}$ converges strongly to μ_X^i.

Now we choose a subsequence $\{m_p\}$ such that $\lim_{p\to\infty}\gamma_k(\mu^{(m_p)})$ exists for each k. We shall denote this limit by γ_k. Suppose that there is a set $A\subset X_{k_0}$ with $V_i(A)<\infty$ on which

$$U^{\mu_X^i}(P)-f(P)<\gamma_{k_0} g(P).$$

We take $\gamma\in\mathscr{E}_A(g, x_{k_0})$ and have

$$(\mu_X^i, \nu)-\langle f, \nu\rangle < x_{k_0}\gamma_{k_0}. \tag{3.12}$$

We proceed under the assumption that there is a finite constant M such that $\|\mu_k^{(m)}\|<M$ for every m and k. For any t, $0\leqq t\leqq 1$, it holds that

$$I_X^i\leqq I(\mu^{(m)}-t\mu_{k_0}^{(m)}+t\nu)\leqq I(\mu^{(m)})-2tx_{k_0}\gamma_{k_0}(\mu^{(m)})+2t(\mu^{(m)}, \nu)$$
$$-2t\langle f, \nu\rangle+t^2(\|\mu_{k_0}^{(m)}\|+\|\nu\|)^2.$$

Since $\mu^{(m)}$ converges weakly to μ_X^i, we have

$$I_X^i\leqq I_X^i-2tx_{k_0}\gamma_{k_0}+2t(\mu_X^i, \nu)-2t\langle f, \nu\rangle+t^2(M+\|\nu\|)^2.$$

Cancelling I_X^i, dividing the rest by t and letting $t\to 0$, we obtain

$$0\leqq(\mu_X^i, \nu)-\langle f, \nu\rangle-x_{k_0}\gamma_{k_0}.$$

This contradicts (3.12), and (3.10) is concluded. To see that γ_k is finite, we take $\lambda \in \mathscr{E}_{Y_k}(g, 1)$ such that $f(P)$ is bounded from below on S_λ. We have by (3.10) that $(\mu_X^i, \lambda) - \langle f, \lambda \rangle \geqq \gamma_k$. This shows that each $\gamma_k < \infty$. The finiteness is seen from

$$2 \sum_{k=1}^{n} x_k \gamma_k = 2 \lim_{p \to \infty} \sum_{k=1}^{n} x_k \gamma_k(\mu^{(m_p)}) = \lim_{m \to \infty} \{I(\mu^{(m)}) + (\mu^{(m)}, \mu^{(m)})\} = I_X^i + (\mu_X^i, \mu_X^i).$$

This gives (3.9) too.

Now we shall show the existence of above M. If the kernel is nonnegative on each $X_j \times X_k$, $j \neq k$, $\|\mu_k^{(m)}\| \leqq \|\mu^{(m)}\|$ and this is bounded. In case the kernel is not always nonnegative on each $X_j \times X_k$, $j \neq k$, we assume (a_1). Then the total mass of $\mu^{(m)}$ is bounded. It is so too if we assume $(a_2)_i$ because $(\mu^{(m)}(\Omega))^2 \leqq \|\mu^{(m)}\|^2 V_i^{-1}(X)$. From the identity

$$\sum_{k=1}^{n} \|\mu_k^{(m)}\|^2 = \|\mu^{(m)}\|^2 - \sum_{\substack{j,k=1 \\ j \neq k}}^{n} (\mu_j^{(m)}, \mu_k^{(m)})$$

we see that each $\|\mu_k^{(m)}\|$ is bounded.

Suppose that μ and $\{c_k\}$ satisfy (3.9) and (3.10) replacing μ_X^i and $\{\gamma_k\}$. It holds that $J(\mu) = I_X^i$. Since $|\, \|\mu^{(m)}\| - \|\mu\|\,|^2 \leqq \|\mu^{(m)} - \mu\|^2 \leqq I(\mu^{(m)}) - I_X^i$ as is seen in the proof of Lemma 3.2, we have $\|\mu\| = \|\mu_X^i\|$ and hence $\sum_{k=1}^{n} x_k c_k = \sum_{k=1}^{n} x_k \gamma_k$. We observe also that $\mu^{(m)}$ converges strongly to μ. We integrate the inequality $U^\mu(P) - f(P) \geqq c_k g(P)$ with respect to $\mu_k^{(m_p)}$ and obtain

$$x_k c_k \leqq (\mu, \mu_k^{(m_p)}) - \langle f, \mu_k^{(m_p)} \rangle = (\mu, \mu_k^{(m_p)}) - (\mu^{(m_p)}, \mu_k^{(m_p)}) + x_k \gamma_k(\mu^{(m_p)}).$$

Since $\|\mu_k^{(m_p)}\| < M$,

$$\begin{aligned} \overline{\lim_{p \to \infty}}\, |(\mu, \mu_k^{(m_p)}) - (\mu^{(m_p)}, \mu_k^{(m_p)})| &\leqq \overline{\lim_{p \to \infty}}\, \|\mu - \mu^{(m_p)}\|\, \|\mu_k^{(m_p)}\| \\ &\leqq \lim_{p \to \infty} M \|\mu - \mu^{(m_p)}\| = 0. \end{aligned}$$

Hence $x_k(c_k - \gamma_k) \leqq 0$. It is concluded that $c_k = \gamma_k$ because $\sum_{k=1}^{n} x_k c_k = \sum_{k=1}^{n} x_k \gamma_k$. In particular, it follows that $\lim_{m \to \infty} \gamma_k(\mu^{(m)})$ exists regardless of the choice of $\{\mu^{(m)}\}$. Then Γ_X^i, defined in § 3.1, consists of a single point. If (μ, μ_X^i) is defined, $\|\mu - \mu_X^i\| = 0$ by Lemma 3.2, and if, furthermore, the energy principle is true then, $\mu = \mu_X^i$.

As is seen in the proof we may replace the condition that Φ is bounded on each $X_j \times X_k$, $j \neq k$, by the weaker condition that Φ is bounded from below on each $X_j \times X_k$, $j \neq k$. The same remark applies to some of subsequent theorems.

We shall denote γ_k by $\gamma_{X_k}^i$ for each k; this notation will be used only if γ_k is uniquely determined.

For later use we give

LEMMA 3.3. *Besides the conditions in Lemma* 3.2 *assume one or both of* (a_1) *and* $(a_2)_i$ *stated in Theorem* 3.5, *unless the kernel is nonnegative in each* $X_j \times X_k$, $j \neq k$. *Then we have*

$$(c_k - \gamma^i_{X_k})\, x_k \leqq a \|\mu - \mu^i_X\|$$

with

$$a = \begin{cases} \sqrt{\|\mu^i_X\|^2 + n(n-1)\, c\left(\dfrac{x}{\inf\limits_X g}\right)^2} & \text{if } (a_1) \text{ is true,} \\ \sqrt{1 + \dfrac{n(n-1)c}{V_i(X)}}\, \|\mu^i_X\| & \text{if } (a_2)_i \text{ is true,} \\ \|\mu^i_X\| & \text{if } \Phi \geqq 0 \text{ on each } X_j \times X_k,\ j \neq k, \end{cases}$$

where

$$c = \max_{\substack{1 \leqq j, k \leqq n \\ j \neq k}} \ \sup_{X_j \times X_k} (-\Phi(P, Q)).$$

PROOF. Like in the proof of the above theorem we have

$$(c_k - \gamma^i_{X_k}) x_k \leqq \varliminf_{m \to \infty} \|\mu_k^{(m)}\| \cdot \|\mu - \mu^i_X\|.$$

If we examine the reasoning in the above proof to show $\|\mu_k^{(m)}\| < M < \infty$, it is easily seen that $\varliminf_{m \to \infty} \|\mu_k^{(m)}\| \leqq a$.

In the special case $f \equiv 0$, we have

COROLLARY. *Let X be any set with $V_i(X) < \infty$ in Ω, and $g(P)$ be a positive continuous function on X. Assume that, for a sequence $\{K^{(m)}\}$ of compact subsets of X, $V_i^{(g)}(K^{(m)})$ tends to $V_i^{(g)}(X)$ and $\mu_{K^{(m)}}$ converges strongly to a measure μ^i_X. Then $V_i^{(g)}(X) = \|\mu^i_X\|^2$ and*

$$U^{\mu^i_X}(P) \geqq V_i^{(g)}(X) g(P) \qquad p.p.p. \text{ on } X.$$

If $g(P)$ is defined and positive continuous on $Z \supset X$, and if μ^i_X is the vague limit of some subnet of $\{\mu_{K^{(m)}}\}$, then

$$U^{\mu^i_X}(P) \leqq V_i^{(g)}(X) g(P) \qquad \text{on } S_{\mu^i_X} \cap Z.$$

If μ satisfies $\|\mu\|^2 = V_i^{(g)}(X)$ and $U^\mu(P) \geqq V_i^{(g)}(X) g(P)$ p.p.p. on X and (μ, μ^i_X) is defined, then $\|\mu - \mu^i_X\| = 0$. If the energy principle is satisfied, $\mu = \mu^i_X$.

In the proof we need Theorem 3.2. We shall call μ, which satisfis $\|\mu\|^2 = V_i^{(g)}(X)$ and $U^\mu(P) \geqq V_i^{(g)}(X) g(P)$ p.p.p. on X, an *inner* g-*equilibrium measure* for X. In case $g(P) \equiv 1$, we simply call it an *inner equilibrium measure.* Its potential will have the corresponding nominations.

We prove next

THEOREM 3.6. *If I_X^i is finite and every strong Cauchy net in $\mathscr{E}_X(g,x)$ is strongly convergent, then any sequence $\{\mu^{(m)}\}$ of measures in $\mathscr{E}_X(g, x)$ for which $\lim_{m\to\infty} I(\mu^{(m)})=I_X^i$, converges strongly to some measure.*

PROOF. Since $(\mu^{(m)}+\mu^{(p)})/2\in\mathscr{E}_X(g, x)$, we have

$$I_X^i \leqq I\left(\frac{\mu^{(m)}+\mu^{(p)}}{2}\right)=\frac{1}{2}\, I(\mu^{(m)})+\frac{1}{2}\, I(\mu^{(p)})-\frac{1}{4}\, \|\mu^{(m)}-\mu^{(p)}\|^2$$

and observe that $\{\mu^{(m)}\}$ is a Cauchy sequence. It converges strongly to some measure by assumption.

The next question is as to the strong convergence of a Cauchy net. We defined in Chapter I the following notion of Fuglede [1]: A kernel is called consistent if it is of positive type and any strong Cauchy net converging vaguely to a measure converges strongly to the same measure. He called a kernel of positive type K-consistent if any strong Cauchy net, whose elements are supported by a fixed compact set and which converges vaguely to a measure, converges strongly to the same measure.

In terms of these notions we shall give several sufficient conditions for any strong Cauchy net in $\mathscr{E}_X(g, x)$ to be strongly convergent.

(i) *The kernel is consistent and $g(P)$ has a positive lower bound on every relatively compact subset of X.*

(ii) *The kernel is consistent and $V_i(X)>0$.*

(iii) *The kernel is nonnegative consistent and $V_i(K\frown X)>0$ for every compact set K in Ω.*

(vi) *X is relatively compact, the kernel is K-consistent and $g(P)$ has a positive lower bound on X.*

(v) *X is relatively compact, the kernel is K-consistent and $V_i(X)>0$.*

To prove that (i) is sufficient, let K be any compact set in Ω and $\{\mu_\omega\}$ be a strong Cauchy net in $\mathscr{E}_X(g, x)$. Since

$$\sum_{k=1}^{n} x_k \geqq \int_K g d\mu_\omega \geqq \inf_K\, g\cdot\mu_\omega(K),$$

$\mu_\omega(K)$ is bounded from above by a constant which may depend on K. By Proposition 3 of § 1.6 there is a subnet of $\{\mu_\omega\}$ which converges vaguely to some measure μ_X^i. By our assumption that the kernel is consistent, the subnet converges strongly to μ_X^i and $\{\mu_\omega\}$ does too.

In case (ii) is satisfied, let $\{\mu_\omega\}$ be a strong Cauchy net. As is well known (μ_ω, μ_ω) is bounded for $\omega\geqq\omega_0$, where ω_0 is some element. We have

$$(\mu_\omega, \mu_\omega)\geqq V_i(\cancel{X})\mu_\omega^2(\Omega)$$

and hence $\mu_\omega(\Omega)$, $\omega\geqq\omega_0$, is bounded. The rest is the same as in the first case. Likewise we can see that each of (iii), (iv) and (v) is a sufficient condition.

In general $\langle g, \mu^i_{X_k}\rangle \leqq \varliminf_{m\to\infty} \langle g, \mu_{K_k^{(m)}}\rangle = x_k$ as $\mu_{K_k^{(m)}}$ converges vaguely to $\mu^i_{X_k}$. In a special case we can show the equality. In fact we prove

THEOREM 3.7. *Let $X, X_1, \cdots, X_n$ be as in Theorem 3.5, $f(P)$ be a continuous function with compact support defined on the closure X^a of X, and $g(P)$ be a positive continuous function on X^a. Assume that the kernel is consistent and nonnegative on each $X_j\times X_k$, $j\neq k$, and that $0<I^i_{X_k}(g, x_k, f)+V_i^{(g,x_k)}(X_k)<\infty$ for each k. Then $I(\mu^i_X)=I^i_X(g, x, f)$ and $\langle g, \mu^i_{X_k}\rangle = x_k$ for each k. If, in addition, the kernel is nonnegative in $\Omega\times\Omega$ or X is relatively compact in Ω, then $x_k\gamma^i_{X_k} = (\mu^i_X, \mu^i_{X_k}) - \langle f, \mu^i_{X_k}\rangle$ for each k.*

PROOF. Let $\{K^{(m)}\}$ be an increasing sequence of compact subsets of X such that $I(\mu_{K^{(m)}})$ tends to $I^i_X = I^i_X(g, x, f)$, and set $K_k^{(m)} = K^{(m)}\cap X_k$. We see by Lemma 3.2 that $\{\mu_{K(m)}\}$ is a Cauchy sequence. It converges strongly to μ^i_X by (i). Furthermore we observe that there is a subnet $\{\mu_k^{(\omega)}\}$ of $\{\mu_{K_k^{(m)}}\}$ converging vaguely to a measure $\mu^i_{X_k}$ for each k and $\mu^i_X = \sum_{k=1}^n \mu^i_{X_k}$. Since $f(P)$ is continuous on X^a and has a compact support, $\langle f, \mu_k^{(\omega)}\rangle$ tends to $\langle f, \mu^i_{X_k}\rangle$ for each k. Using the relation $\lim_{m\to\infty}\|\mu_{K(m)}\| = \|\mu^i_X\|$ we observe that

$$I^i_X = \lim_{m\to\infty} I(\mu_{K(m)}) = \lim_{\omega}\ \{(\mu^{(\omega)}, \mu^{(\omega)}) - 2\langle f, \mu^{(\omega)}\rangle\} = I(\mu^i_X).$$

Hence by (3.9)

$$\begin{aligned} 2\sum_{k=1}^n x_k\gamma_k &= I^i_X + (\mu^i_X, \mu^i_X) = I(\mu^i_X) + (\mu^i_X, \mu^i_X) \\ &= 2(\mu^i_X, \mu^i_X) - 2\langle f, \mu^i_X\rangle. \end{aligned}$$

On account of Theorem 3.2, (3.5) holds everywhere on $S_{\mu^i_{X_k}}$. We integrate it with respect to $\mu^i_{X_k}$ and obtain

$$(\mu^i_X, \mu^i_X) - \langle f, \mu^i_X\rangle \leqq \sum_{k=1}^n \gamma_k\langle g, \mu^i_{X_k}\rangle.$$

Therefore we derive

$$\sum_{k=1}^n x_k\gamma_k \leqq \sum_{k=1}^n \langle g, \mu^i_{X_k}\rangle\gamma_k.$$

Since

$$2x_k\gamma_k(\mu_{K^{(m)}}) = 2(\mu_{K^{(m)}}, \mu_{K_k^{(m)}}) - 2\langle f, \mu_{K_k^{(m)}}\rangle$$

$$= I(\mu_{K_k^{(m)}}) + (\mu_{K_k^{(m)}}, \mu_{K_k^{(m)}}) + 2\sum_{\substack{j=1\\ j\neq k}}^n (\mu_{K_j^{(m)}}, \mu_{K_k^{(m)}}) \geqq I^i_X(g, x_k, f) + V_i^{(g,x_k)}(X_k) > 0$$

by our assumption, $\gamma_k = \lim_{m\to\infty} \gamma_k(\mu_{K^{(m)}}) > 0$ for each k. Taking into consideration

$$\langle g, \mu^i_{X_k}\rangle \leqq \varliminf_{m\to\infty} \langle g, \mu_{K_k^{(m)}}\rangle = x_k,$$

we now conclude that $\langle g, \mu^i_{X_k}\rangle = x_k$ for each k.

If the kernel is nonnegative in $\Omega\times\Omega$ or X is relatively compact in Ω, then we have

$$x_k\, \gamma^i_{X_k} \geqq (\mu^i_X, \mu^i_{X_k}) - \langle f, \mu^i_{X_k}\rangle$$

by Proposition 4 of § 1.6. Since we obtain an equality by summing up both sides for $k=1, \cdots, n$, each must be an equality. Thus our proof is completed.

Fuglede [1] proved a special case in Lemma 4.2.1. In his case $n=1$, $x=1$ and $f\equiv 0$; then $I^i_X(g, x, f) + V^{(g)}_i(X) = 2V^{(g)}_i(X)$.

We remark that, although $S_{\mu^i_X} \subset X^a$ and $\langle g, \mu^i_{X_k}\rangle = x_k$ for each k, we can not always write $\mu^i_X \in \mathscr{E}_{X^a}(g, x)$ because $S_{\mu^i_X}$ may not be compact; see the definition of $\mathscr{E}_{X^a}(g, x)$ in § 2.1.

For later use we shall prove the following well known

LEMMA 3.4. *Take $X\subset\Omega$ and $\nu, \lambda\in\mathscr{E}$ for which (ν, λ) is defined. If*

$$U^\nu(P) \geqq U^\lambda(P) + t$$

for $t\geqq 0$ on X, then

$$V_i(X) \geqq t^2\|\nu-\lambda\|^{-2}.$$

PROOF. We may assume that $V_i(X)<\infty$. We take a compact set $K\subset X$ with $V_i(K)<\infty$, and denote by μ_K a unit extremal measure satisfying $\|\mu_K\|^2 = V_i(K)$. Then

$$t \leqq (\nu, \mu_K) - (\lambda, \mu_K) \leqq \|\nu-\lambda\|\,\|\mu_K\|.$$

Hence

$$V_i(X) = \inf_{K\subset X} \|\mu_K\|^2 \geqq t^2\|\nu-\lambda\|^{-2}.$$

LEMMA 3.5.[36] *Let G_0 be an open set in Ω on whose product the kernel is bounded from below and A be a set of the form $\bigcup_m (F^{(m)}\cap G^{(m)})$ where each $F^{(m)}$ is a closed set and $G^{(m)}$ is an open subset of G_0. Assume that there is an inner equilibrium measure μ^i_A for A in case $V_i(A)<\infty$; in case $V_i(A)=\infty$ we have no such requirement. Assume also one or both of the following conditions:*

$(\mathrm{b}_1)^*$ *Every open subset of G_0 is a K_σ-set,*

$(\mathrm{b}_2)^*$ *G_0 is a countable union of relatively compact open sets, the set*

$$\Delta_\infty = \{P \in \Omega;\ \Phi(P, P) = \infty\}$$

is closed and, for every point $P \in G_0$ and every neighborhood N_P of P, the kernel is bounded from above on $\{P\} \times (G_0 - N_P)$.
Then

$$V_i(A) = V_e(A).$$

Proof. First we consider the case where $V_i(A) < \infty$. Since the kernel is of positive type, $V_i(A) \geqq V_e(A) \geqq 0$ and hence we may also assume that $V_i(A) > 0$. By our assumption there is a measure μ_A^i such that $\|\mu_A^i\|^2 = V_i(A)$ and

$$U^{\mu_A^i}(P) \geqq V_i(A) \qquad \text{p.p.p. on } A.$$

The exceptional set H is equal to

$$\bigcup_{m > (V_i(A))^{-1}} \left[\left\{ P \in F^{(m)};\ U^{\mu_A^i}(P) \leqq V_i(A) - \frac{1}{m} \right\} \cap G^{(m)} \right].$$

The set inside [ ] is a relatively closed subset of $G^{(m)}$ whose V_i-value is infinite. By Lemma 3.1 it is an F_σ-set; hence H is so too. It follows that H is a K_σ-set on account of $(\mathrm{b}_1)^*$ or $(\mathrm{b}_2)^*$. We know by Theorem 1.14 that, for every compact set K, $V_i(K) = V_e(K)$. By Proposition 2 of Chapter I it follows that $V_e(H) = \infty$. The set $A - H$ is contained in the open set

$$G^{(m)} = \left\{ P \in G_0;\ U^{\mu_A^i}(P) > V_i(A) - \frac{1}{m} \right\}.$$

By the above lemma we have

$$V_e(A - H) \geqq V_i(G^{(m)}) \geqq \left(V_i(A) - \frac{1}{m} \right)^2 \|\mu_A^i\|^{-2} = \left(V_i(A) - \frac{1}{m} \right)^2 (V_i(A))^{-1}.$$

It follows that $V_e(A - H) \geqq V_i(A)$. Since $V_e(H) = \infty$, we have

$$V_e(A) = V_e(A - H)$$

by Proposition 2 of Chapter I. Consequently $V_i(A) = V_e(A)$.

Next we consider the case where $V_i(A) = \infty$. Evidently $V_i(F^{(m)} \cap G^{(m)}) = \infty$ for each m. By Lemma 3.1 we observe as in the first case that each $F^{(m)} \cap G^{(m)}$ is and hence $A = \bigcup_m (F^{(m)} \cap G^{(m)})$ is a K_σ-set. Therefore $V_e(A) = \infty$ by Proposition 2.

Condition $(\mathrm{b}_1)^*$ or condition $(\mathrm{b}_2)^*$ is imposed in Lemma 3.5 in order to reduce open or closed sets to K_σ-sets. If we consider a consistent kernel we can replace $(\mathrm{b}_1)^*$ and $(\mathrm{b}_2)^*$ by (b_1) and (b_2). It will be proved as Lemma 3.8 at the end of the next section.

3.3. Outer variational problem.

Let X consist of mutually disjoint non-empty sets $X_1, \cdots, X_n$. We shall say that open sets $G_1, \cdots, G_n$ *separate* $X_1, \cdots, X_n$ if $\{G_k\}$ are mutually disjoint and $G_k \supset X_k$ for each k. We shall say then that $X_1, \cdots, X_n$ are *separate* by open sets. If, in addition, $\Phi(P, Q)$ is bounded on each $G_j \times G_k$, $j \neq k$, $X_1, \cdots, X_n$ are separable by Φ-separate open sets. In this section we are concerned with the *outer variational problem* to discuss

$$\sup_{\{G_k\}} I_G^i(g, x, f) = I_X^e(g, x, f) = I_X^e(x) = I_X^e,$$

where $X = \bigcup_{k=1}^{n} X_k$, $G = \bigcup_{k=1}^{n} G_k$ and $\{G_k\}$ separate $\{X_k\}$.

In the special case that X is a compact set we can prove

THEOREM 3.8. *Let K consist of mutually disjoint compact sets $K_1, \cdots, K_n$ in Ω and G_0 be an open set containing K. Let $f(P) < \infty$ be defined and upper semicontinuous in G_0 and $g(P)$ be defined and positive continuous in G_0. Then*

$$I_K^i(g, x, f) = I_K^e(g, x, f).$$

PROOF. The function

$$\Psi(P, Q) = \Phi(P, Q) - \frac{f(P)g(Q) + f(Q)g(P)}{\sum_{k=1}^{n} x_k}$$

is lower semicontinuous and does not take $-\infty$ in $G_0 \times G_0$. Therefore it may be taken for a kernel in G_0. It holds that

$$\iint \Psi(P, Q) d\mu(Q) d\mu(P) = (\mu, \mu) - 2\langle f, \mu \rangle = I(\mu)$$

for $\mu \in \mathscr{E}_{G_0}(g, x)$. Therefore our theorem follows from Theorem 1.14 applied to the kernel $\Phi(P, Q)$.

Like in § 3.1 and § 3.2 we shall investigate extremal measures in two cases.

(1) THEOREM 3.9. *Let X consist of relatively compact sets $X_1, \cdots, X_n$ which are separable by Φ-separate open sets and $G_0 \supset X$ be an open set in Ω. Let $f(P) < \infty$ be defined and upper semicontinuous in G_0 and $g(P)$ be defined and positive continuous in G_0. Assume that $\Phi(P, Q)$ satisfies the continuity principle, that I_X^e is finite and that one or both of the following conditions is satisfied:*

(a_1) *$g(P)$ has a positive lower bound on G_0,*[37]

(a_2)$'_e$ *$V_e(X) > 0$, and $f(P)(1 + g(P))^{-1}$ is bounded from above on G_0.*[38]

Then, for any sequence $\{G^{(m)}\}$, $X \subset G^{(m)} \subset G_0$, of relatively compact sets such that $G^{(m)}$ can be divided into Φ-separate open sets $G_1^{(m)}, \cdots, G_n^{(m)}$ separating $X_1, \cdots, X_n$ and $I_{G^{(m)}}^i$ tends to I_X^e, there is a subnet $\{\mu^{(\omega)}; \omega \in D\}$ of $\{\mu_{G^{(m)}}^i\}$ such that $\mu_k^{(\omega)}$ con-

37) This condition is not completely the same as (a_1) of Theorem 3.5 but we use the same letter.

38) The question as to whether we can replace (a_2)$'_e$ by (a_2)$_e$ $V_e(X) > 0$ is open.

verges vaguely to some measure $\mu^e_{X_k}$ *for each* k *and each* $\lim_\omega \gamma_k^{(\omega)}$ *exists, where* $\gamma^{(\omega)}=(\gamma_1^{(\omega)}, \cdots, \gamma_n^{(\omega)})$ *is suitably chosen in* $\Gamma^i_{G^{(m)}}$, m *being determined by the equality* $\mu^{(\omega)}=\mu^i_{G^{(m)}}$. *This limit is not equal to* $-\infty$, *and if we set* $\sum_{k=1}^n \mu^e_{X_k}=\mu^e_X$ *and denote the above limit by* γ_k, *then we have*

$$\hat{U}^{\mu^e_X}(P)-f(P)\geqq\gamma_k\, g(P) \qquad p.p.p.\ on\ X_k.$$

If, furthermore, $f(P)$ *is defined and continuous on* G^a_0, *then*

$$I^e_X+\langle f,\ \mu^e_X\rangle=\sum_{k=1}^n x_k\,\gamma_k. \tag{3.13}$$

Proof. By Theorem 3.1 it holds that

$$U^{\mu^i_{G^{(m)}}}(P)-f(P)\geqq\gamma_k^{(m)}g(P) \qquad \text{p.p.p. on } G_k^{(m)}$$

with some $\gamma^{(m)}=(\gamma_1^{(m)}, \cdots, \gamma_n^{(m)})\in\Gamma^i_{G^{(m)}}$. It is easy to see that $\mu^i_{G^{(m)}}(\Omega)$ is bounded if (a_1) is assumed. Let us assume $(a_2)'_e$. There is a finite number M' such that

$$f(P)<M'\,(1+g(P)) \qquad \text{on } G_0.$$

We conclude that $\mu^i_{G^{(m)}}(\Omega)$ is bounded in this case too because

$$I(\mu)\geqq V_i(G^{(m)})\mu^2(\Omega)-2M'\ \{\mu(\Omega)+\sum_{k=1}^n x_k\}$$

for any $\mu\in\mathscr{E}_{G^{(m)}}(g, x)$. Consequently there is a subnet $T=\{\mu^{(\omega)};\ \omega\in D\}$ of $\{\mu^i_{G^{(m)}}\}$ such that $\mu_k^{(\omega)}$ converges vaguely to some measure $\mu^e_{X_k}$ and $\gamma_k^{(\omega)}$ tends to a limit which will be denoted by γ_k, where $\gamma_k^{(\omega)}$ is equal to $\gamma_k^{(m)}$ with m determined by $\mu^{(\omega)}=\mu^i_{G^{(m)}}$. According to Theorem 1.16 it holds that

$$\varliminf_{\omega}\ \hat{U}^{\mu^{(\omega)}}(P)=\hat{U}^{\mu^e_X}(P) \qquad \text{p.p.p. in } \Omega.$$

Therefore

$$\hat{U}^{\mu^e_X}(P)-f(P)\geqq\gamma_k\, g(P) \qquad \text{p.p.p. on } X_k.$$

It will be shown near the end of § 3.4 that each $\gamma_k>-\infty$. Equality (3.13) follows from the similar equality in Theorem 3.1.

If $g(P)$ is defined and positive continuous on G^a_0 and each $\mu^i_{G^{(m)}}$ is the vague limit of some subnet of a sequence $\{\mu_{K^{(m,p)}}\}$, $K^{(m,p)}\subset G^{(m)}$, then we have inequality (3.5) for μ^e_X.

The measure μ^e_X will be also denoted by $\mu^e_X(g, x, f)$ or by $\mu^e_X(x)$. It is not sure that the support of μ^e_X is contained in X^a. See Problem 3.4 in § 3.11 in this connection.

If we use Theorems 3.3 and 1.17 we can prove

THEOREM 3.10. *In addition to the conditions required in the first part of Theorem* 3.9, *we assume that* $\Phi(P, Q)$ *is continuous outside the diagonal set and that* $\Phi(P, P)=\infty$ *at each point* P *of* $\mathbf{O}_\infty$ *which is defined with respect to* $\hat{\Phi}$. *Assume also one of the following conditions*:

$(\mathrm{b}_1)^*$ *Every open subset of* G_0 *is a* K_σ*-set,*

$(\mathrm{b}_2)^*$ *The set* $\Delta_\infty=\{P;\ \Phi(P, P)=\infty\}$ *is closed.*

Then each γ_k *is finite and for each* k

$$\hat{U}^{\mu^e_X}(P)-f(P)\geqq\gamma_k\, g(P) \qquad q.p.\ on\ X_k. \tag{3.14}$$

The finiteness of each γ_k will be proved at the end of § 3.4.

In case a positive continuous function $g(P)$ is defined on an open set $G_0\supset X$, we write $V^{(g)}_e(X)$ for inf $V^{(g)}_i(G)$, where G is an open set such that $X\subset G\subset G_0$. We have

COROLLARY. *We consider the special case* $n=1, x=1$ *and* $f\equiv 0$ *in the theorem. Then*

$$\hat{U}^{\mu^e_X}(P)\geqq V^{(g)}_e(X)g(P) \qquad q.p.\ on\ X. \tag{3.15}$$

If $g(P)$ *is defined and continuous on* G^a_0, *then*

$$\hat{U}^{\mu^e_X}(P)\leqq V^{(g)}_e(X)g(P) \qquad on\ S_{\mu^e_X}. \tag{3.16}$$

For $g(P)$ defined on G^a_0 we shall call a measure satisfying (3.15) and (3.16) a *weak outer g-equilibrium measure* and, in case $g(P)\equiv 1$, a *weak outer equilibrium measure.* Its potential will have the corresponding nominations.

We define $\Gamma^e_X(x)=\Gamma^e_X$ as the set of $\gamma=(\gamma_1, \cdots, \gamma_n)$ appeared in Theorem 3.10 and can prove as before

THEOREM 3.11. *The set* Γ^e_X *is compact in* E_n *under the assumption of Theorem* 3.10.

(2) In the following Theorems 3.12, 3.13 and 3.15, we consider X consisting of sets $X_1, \cdots, X_n$ which are separable by Φ-separate open sets $G_1, \cdots, G_n$ in Ω, an upper semicontinuous function $f(P)<\infty$ in an open set $G_0\supset X$ and a positive continuous function $g(P)$ in G_0. We assume also that the kernel is of positive type and that $I^e_X=I^e_X(g, x, f)$ is finite.

We shall denote by $\mathscr{E}^i_{G_0}(g, x)$ the closure of $\mathscr{E}_{G_0}(g, x)$ with respect to the strong topology and prove

THEOREM 3.12. *Assume that every strong Cauchy net in* $\mathscr{E}^i_{G_0}(g, x)$ *is strongly convergent and that* $(\mu^i_G, \mu^i_{G'})$ *is defined for any open subsets* G *and* G' *of* G_0 *both including* X. *Unless the kernel is nonnegative on each* $G_j\times G_k$, $j\neq k$, *assume also one or both of the following conditions*:

(a_1) $g(P)$ *has a positive lower bound on* G_0,

$(\mathrm{a}_2)_e$ $V_e(X)>0$.

Then, for any sequence $\{G^{(m)}\}$, $X\subset G^{(m)}\subset G_0$, *of open sets such that each* $G^{(m)}$ *is decomposed into open sets separating* $X_1, \cdots, X_n$ *and* $I^i_{G^{(m)}}$ *tends to* $I^e_{X^{(m)}}$, $\mu^i_{G^{(m)}}$ *converges strongly to some measure* $\mu^e_X=\mu^e_X(g, x, f)$ *and* $\gamma^i_{G^{(m)}_k}$ *tends to a finite constant* γ_k *for each* k. *This measure* μ^e_X *is a strong limit for any sequence of open sets of the above character, and each* γ_k *does not depend on the choice of* $\{G^{(m)}\}$. *We have also*

$$I^e_X+(\mu^e_X, \mu^e_X)=2\sum_{k=1}^{n} x_k\gamma_k \tag{3.17}$$

and

$$U^{\mu^e_X}(P)-f(P)\geqq\gamma_k g(P) \qquad \textit{p.p.p. on } X_k. \tag{3.18}$$

We may require furthermore that the support of μ^e_X *is contained in* X^a.

PROOF. By Therem 3.5 we have

$$U^{\mu^i_{G^{(m)}}}(P)-f(P)\geqq\gamma^i_{G^{(m)}_k} g(P) \qquad \text{p.p.p. in } G^{(m)}_k. \tag{3.19}$$

For $G=G^{(m)}\frown G^{(p)}$ it holds that

$$0\leqq\|\mu^i_G-\mu^i_{G^{(m)}}\|^2\leqq I^i_G-I^i_{G^{(m)}}\leqq I^e_X-I^i_{G^{(m)}}$$

in virtue of Lemma 3.2. Similarly

$$\|\mu^i_G-\mu^i_{G^{(p)}}\|^2\leqq I^e_X-I^i_{G^{(p)}}$$

and hence

$$\|\mu^i_{G^{(m)}}-\mu^i_{G^{(p)}}\|\leqq\sqrt{I^e_X-I^i_{G^{(m)}}}+\sqrt{I^e_X-I^i_{G^{(p)}}}.$$

Thus $\{\mu^i_{G^{(m)}}\}$ form a strong Cauchy sequence and this converges strongly to some measure μ^e_X by our assumption. The fact that μ^e_X is a strong limit for any sequence like $\{G^{(m)}\}$ follows if we mix two such sequences.

We set $G^{(1)}\frown\cdots\frown G^{(m)}=D^{(m)}$. Since

$$U^{\mu^i_{G^{(m)}}}(P)-f(P)\geqq\gamma_k(\mu^i_{G^{(m)}})g(P) \qquad \text{p.p.p. on } G^{(m)}_k,$$

by Lemma 3.3 it follows that

$$\{\gamma_k(\mu^i_{G^{(m)}})-\gamma_k(\mu^i_{D^{(m+p)}})\}\,x_k\leqq a\,\|\mu^i_{G^{(m)}}-\mu^i_{D^{(m+p)}}\|,$$

where a is a nonnegative constant not depending on the choice of $\{G^{(m)}\}$. It follows that

$$\varliminf_{m\to\infty}\ \gamma_k(\mu^i_{G^{(m)}})\leqq\varlimsup_{m\to\infty}\ \gamma_k(\mu^i_{G^{(m)}})\leqq\varliminf_{m\to\infty}\ \gamma_k(\mu^i_{D^{(m)}})\ (>-\infty)$$

for each k. We infer by (3.9) that

$$I_X^e+(\mu_X^e, \mu_X^e)=\lim_{m\to\infty}\{I^i_{G^{(m)}}+(\mu^i_{G^{(m)}}, \mu^i_{G^{(m)}})\}\leqq 2\sum_{j\neq k} x_j \overline{\lim_{m\to\infty}}\ \gamma_j(\mu^i_{G^{(m)}})+2x_k \varliminf_{m\to\infty}\gamma_k(\mu^i_{G^{(m)}})$$
$$\leqq 2\sum_{j=1}^n x_j \overline{\lim_{m\to\infty}}\ \gamma_j(\mu^i_{G^{(m)}})\leqq 2\sum_{j=1}^n x_j \varliminf_{m\to\infty}\gamma_j(\mu^i_{D^{(m)}})\leqq \lim_{m\to\infty}\{I^i_{D^{(m)}}+(\mu^i_{D^{(m)}}, \mu^i_{D^{(m)}})\}$$
$$=I_X^e+(\mu_X^e, \mu_X^e).$$

It results that $\lim_{m\to\infty}\gamma_k(\mu^i_{G^{(m)}})$ exists and is finite for each k and that, if γ_k denotes this limit, (3.17) holds. The independence of γ_k of the choice of $\{G^{(m)}\}$ is concluded by mixing two such sequences.

Let $H^{(m)}\subset G^{(m)}$ be the set where (3.19) does not hold. By Proposition 1 of Chapter I, $V_i(\bigcup_m H^{(m)})=\infty$ and, for $P\in X_k-\bigcup_m H^{(m)}$, it follows naturally that

$$\varliminf_{m\to\infty} U^{\mu^i_{G^{(m)}}}(P)-f(P)\geqq\gamma_k\, g(P).$$

In view of Theorem 1.18 we have

$$U^{\mu_X^e}(P)-f(P)\geqq\gamma_k\, g(P) \qquad \text{p.p.p. on } X_k.$$

Finally we shall show that we can choose μ_X^e so that $S_{\mu_X^e}\subset X^a$. We denote by D the directed set consisting of all open sets G such that $X\subset G\subset G_0$ and G is decomposed into open sets separating $X_1, \cdots, X_n$; the direction in D is defined by the inclusion, namely $G\leqq G'$ if and only if $G\supset G'$. We see that $\{\mu^i_G;\ G\in D\}$ form a Cauchy net in $\mathscr{E}^i_{G_0}(g, x)$. It converges strongly to a measure μ by our assumption. Since $\bigcap_{G\in D} G^a=X^a$, $S_\mu\subset X^a$. There is a sequence $\{\tilde{G}^{(m)}\}$ of open sets in D such that $\mu^i_{\tilde{G}^{(m)}}$ converges strongly to μ and $I^i_{\tilde{G}^{(m)}}$ tends to I^e_X. As observed already μ is a strong limit of the original sequence $\{\mu^i_{G^{(m)}}\}$, and hence we may take μ for μ_X^e.

In case γ_k is uniquely determined like in this theorem, it will be denoted by $\gamma^e_{X_k}$.

If $g(P)$ is defined and positive continuous on G_0^a, if each $\mu^i_{G^{(m)}}$ is the vague limit of some subnet of a sequence $\{\mu_{K^{(m,p)}}\}$, $K^{(m,p)}\subset G$ and if $\{\mu^i_{G^{(m)}}\}$ contains a subnet converging vaguely to μ_X^e, then we have inequality (3.5) for μ_X^e.

By Lemma 3.2 we obtain immediately

THEOREM 3.13. *If I_X^i and I_X^e are finite under the assumptions of Theorem 3.12 and if (μ_X^i, μ_X^e) is defined, we have*

$$\|\mu_X^i-\mu_X^e\|^2\leqq I_X^i-I_X^e.$$

In the special case that X is a compact set K we have seen that $I_K^i=I_K^e$ in Theorem 3.8. We shall prove

THEOREM 3.14. *Let K consist of mutually disjoint compact sets $K_1, \cdots, K_n$ and $G_0\subset\Omega$ be an open set containing K. Let $f(P)<\infty$ be defined and upper semicontinuous in G_0 such that $f(P)>-\infty$ on some set $Y_k\subset K_k$ with $V_i(Y_k)<\infty$*

for each k, and $g(P)$ be defined and positive continuous in G_0. Consider a kernel of positive type, and assume that every strong Cauchy net in $\mathcal{E}_{G_0}(g, x)$ is strongly convergent. Then

$$\|\mu_K-\mu_K^e\|=0 \qquad and \qquad \gamma_k(\mu_K)=\gamma_{K_k}^e \qquad\qquad for\ each\ k.$$

PROOF. By Theorem 3.13 the first equality follows immediately. The second equality follows from the equality $I_X^i=I_K^e$, (3.18) and the last paragraph of Therem 3.5.

Corresponding to Theorem 3.7 we state

THEOREM 3.15. *Consider a consistent kernel. Let G_0 be an open set, $f(P)$ be a continuous function with compact support on G_0^a, and $g(P)$ be a positive continuous function on G_0^a. Assume that $(\mu_G^i, \mu_{G'}^i)$ is defined for any open subsets G and G' of G_0 both including a fixed set X, that G_0 consists of mutually disjoint open sets $G_1, \cdots, G_n$ such that the kernel is nonnegative and finite on each $G_j\times G_k$, $j\neq k$, and that $0<I_{X_k}^e(g, x_k, f)+V_e^{(g,x_k)}(X_k)<\infty$ for sach k where $X_k=X\cap G_k$. Under these conditions $I(\mu_X^e)=I_X^e(g, x, f)$ and $\langle g, \mu_{X_k}^e\rangle=x_k$ for each k. If, in addition, the kernel is nonnegative in $\Omega\times\Omega$ or X is relatively compact in Ω, then $\gamma_{X_k}^e=(\mu_X^e, \mu_{X_k}^e)-\langle f, \mu_{X_k}^e\rangle$ for each k.*

COROLLARY. *If, in paricular, each X_k is a closed set F_k,*

$$I_F^i(g, x, f)=I_F^e(g, x, f), \tag{3.20}$$

where $F=\bigcup_{k=1}^{n} F_k$.

PROOF. As proved in Therem 3.12 we may assume that $S_{\mu_{F_k}^e}\subset F_k$. Since $\langle g, \mu_{F_k}^e\rangle=x_k$, $I_F^i(g, x, f)\leqq I(\mu_F^e)=I_F^e(g, x, f)$. The inverse inequality $I_F^i(g, x, f)\geqq I_F^e(g, x, f)$ being evident, we conclude the equality.

Fuglede [1] proved $V_i(F_0)=V_e(F_0)$ for F_0 with $V_e(F_0)>0$ under the additional assumption that the space is normal; see his Lemma 4.2.2. He showed also that we can not replace $V_e(F_0)>0$ by $V_e(F_0)=0$. See his Example 10 in § 8.3.

In order to show that the inequality (3.9) is true q.p. on X_k we shall prove two lemmas.

Lemma 3.6. *Let G_0 be an open set in Ω such that the kernel $\Phi(P, Q)$ is bounded from below on $G_0\times G_0$, and $X\subset G_0$ be any set. Suppose that the kernel is of positive type and that, for any sequence $\{F^{(m)}\}$ of closed sets and any sequence of $\{G^{(m)}\}$ of open subsets of G_0, each having a positive V_i-value, we have*

$$V_i(\bigcup_m (F^{(m)}\cap G^{(m)}))=V_e(\bigcup_m (F^{(m)}\cap G^{(m)}).$$

Let μ, ν be measures of $\mathcal{E}$ such that (μ, ν) is defined. If $t>0$ and $U^\mu(P)\geqq U^\nu(P)+t$ q.p. on X, then

$$V_e(X) \geqq t^2 \|\mu - \nu\|^{-2}.$$

PROOF.[39] In view of Proposition 2 of Chapter I we may assume that $U^\mu(P) \geqq U^\nu(P) + t$ everywhere on X. For any number s, $0 < s < t$, X is contained in the set

$$H = \{P \in G_0;\ U^\mu(P) > U^\nu(P) + s\}.$$

We set, for every rational number r,

$$F_r = \{P;\ U^\nu(P) \leqq r\}$$

and

$$G_r = \{P \in G_0;\ U^\mu(P) > r + s\},$$

and have $H = \bigcup_r (F_r \cap G_r)$. By Lemma 3.4 we have $V_i(G_r) \geqq (r+s)^2 \|\mu\|^{-2} > 0$. Consequently $V_i(H) = V_e(H)$ by our assumption. Hence from Lemma 3.4 follows $V_e(H) \geqq s^2 \|\mu - \nu\|^{-2}$. Since s may be taken arbitrarily close to t, the required inequality follows.

Lemma 3.7. *Consider the same G_0 and $\Phi(P, Q)$ as in Lemma 3.6. Then, for any sequence $\{\mu_n\}$ converging strongly to μ_0, we have*

$$\varliminf_{n \to \infty} U^{\mu_n}(P) \leqq U^{\mu_0}(P) \qquad q.p. \text{ in } G_0.$$

Proof. We set

$$H = \{P \in G_0;\ U^{\mu_0}(P) < \varliminf_{n \to \infty} U^{\mu_n}(P)\},$$

$$H_{p,q} = \left\{P \in G_0;\ U^{\mu_0}(P) \leqq \inf_{p<n} U^{\mu_n}(P) - \frac{1}{q}\right\}$$

and

$$B_{n,q} = \left\{P \in G_0;\ U^{\mu_0}(P) \leqq U^{\mu_n}(P) - \frac{1}{q}\right\}.$$

Then

$$H = \bigcup_{p,q} H_{p,q} \qquad \text{and} \qquad H_{p,q} = \bigcap_{p<n} B_{n,q}.$$

By Lemma 3.6

$$V_e(H_{p,q}) \geqq V_e(B_{n,q}) \geqq \frac{1}{q^2} \|\mu_n - \mu_0\|^{-2} \qquad \text{for each } n > p.$$

The right side tends to ∞ as $n \to \infty$ and $V_e(H_{p,q}) = \infty$. Consequently $V_e(H) = \infty$

39) cf. Fuglede [1], Lemma 3.2.3.

by Proposition 2 in § 1.1.

If we use Lemma 3.7 instead of Theorem 1.18, we can improve Theorem 3.12 under some additional condition.

THEOREM 3.16. *Let G_0 be an open set in Ω on whose product the kernel is bounded from below, and X be any subset of G_0. Let $f(P)<\infty$ be defined and upper semicontinuous in G_0, and $g(P)$ be defined and positive continuous in G_0. Consider a kernel of positive type, and assume that I_X^e is finite, that every strong Cauchy net in $\mathscr{E}^i_{G_0}(g, x)$ is strongly convergent, that $(\mu_G^i, \mu_{G'}^i)$ is defined for any open subsets G and G' of G_0 both including X, and that one or both of* $(b_1)^*$ *and* $(b_2)^*$ *of Lemma* 3.5 *is true. Unless the kernel is nonnegative in each $G_j \times G_k$, $j \neq k$, assume also one or both of* (a_1) *and* $(a_2)_e$ *of Theorem* 3.12. *Then with $\{\gamma_k\}$, defined in Theorem* 3.12, *it holds that*

$$U^{\mu_X^e}(P)-f(P) \geqq \gamma_k\, g(P) \qquad q.p. \text{ on } X_k.$$

COROLLARY. *We consider the special case that $n=1$, $x=1$ and $f(P)\equiv 0$ in the theorem. Then $\|\mu_X^e\|^2 = V_e^{(g)}(X)$ and*

$$U^{\mu_X^e}(P) \geqq V_e^{(g)}(X) g(P) \qquad q.p. \text{ on } X.$$

We shall call a measure satisfying these relations an *outer g-equilibrium measure* and, in case $g(P)\equiv 1$, an *outer equilibrium measure.* Its potential will have the corresponding nominations.

REMARK. If we consider a consistent kernel, the following conditions may replace $(b_1)^*$ and $(b_2)^*$ required in Theorem 3.16:

(b_1) Every open subset of G_0 is an F_σ-set,

(b_2) Δ_∞ is closed and, for every point $P\in G_0$ and for every neighborhood N_P of P, the kernel is bounded from above on $\{P\}\times(G_0-N_P)$.

To justify the assertion it will be sufficient, in view of Lemma 3.6, to prove

LEMMA 3.8. *Consider a consistent kernel. Let G_0 be an open set in Ω on whose product the kernel is bounded from below, $\{G^{(m)}\}$ be a sequence of open subsets of G_0, each having a positive V_i-value and $\{F^{(m)}\}$ be a sequence of closed sets. Assume one or both of* (b_1) *and* (b_2). *Then*

$$V_i\Big(\bigcup_m (F^{(m)}\cap G^{(m)})\Big)=V_e\Big(\bigcup_m (F^{(m)}\cap G^{(m)})\Big).$$

PROOF. Set $A=\bigcup_m (F^{(m)}\cap G^{(m)})$ and consider the case $0<V_i(A)<\infty$ first. There exists an inner equilibrium measure μ_A^i for A. With the same notations as in the proof of Lemma 3.5 it holds that

$$H=\bigcup_{m>(V_i(A))^{-1}}\Big[\Big\{P\in F^{(m)};\ U^{\mu_A^i}(P)\leqq V_i(A)-\frac{1}{m}\Big\}\cap G^{(m)}\Big].$$

The set inside [ ] is an F_σ-set by Lemma 3.1 and hence H is so too. Since

$V_i(G^{(m)})>0$, H can be written as $\bigcup_p E^{(p)}$ with closed sets $\{E^{(p)}\}$, each having a positive V_e-value. Now by (3.20) we see that $V_i(E^{(p)})=V_e(E^{(p)})$ for each p. By Proposition 2 of Chapter I it follows that $V_e(H)=\infty$. The rest of the proof is similar to that of Lemma 3.5.

3.4. Sets with $I_X^{i}(g, x, f)=\infty$ or with $I_X^e(g, x, f)=\infty$.

We shall give conditions for $I_X^i(g, x, f)=\infty$ and for $I_X^e(g, x, f)=\infty$. Even if the kernel is not of positive type, we may use the notation $\|\mu\|$ for $\sqrt{(\mu, \mu)}$ provided that $(\mu, \mu)\geqq 0$. We are still under the condition that $x_1>0, \cdots, x_n>0$.

In § 3.1 we set $I_X^i=\infty$ unless each X_k contains Y_k with $V_i(Y_k)<\infty$ on which $f(P)$ is finite. This is justified by

THEOREM 3.17. *Let X consist of Φ-separate sets $X_1, \cdots, X_n$ in Ω, $f(P)<\infty$ be an upper semicontinuous function defined on X and $g(P)$ be a positive continuous function on X. Then $I(\mu)=(\mu, \mu)-2\langle f, \mu\rangle=\infty$ for any μ such that S_μ is decomposed into compact sets $K_1, \cdots, K_n$, each $K_k\subset X_k$, and that $\int_{K_k} g d\mu=x_k$ for each k, if and only if $f(P)=-\infty$ p.p.p. on some X_k.*

The proof is easy.

By Proposition 1 in Chapter I we have

COROLLARY. *Let $\{A^{(m)}\}$ be a sequence of sets of $\mathfrak{A}$ such that each $A^{(m)}$ is decomposed into $\mathfrak{A}$-measurable sets $A_1^{(m)}, \cdots, A_n^{(m)}$ with the property that each $A_k^{(m)}$ increases with m and $\{\bigcup_m A_k^{(m)}\}$ are Φ-separate. Set $A=\bigcup_m A^{(m)}$. Let X be a set such that $A_k^{(1)}\cap X\neq\emptyset$ for each k, $f(P)<\infty$ be an upper semicontinuous function on $A\cap X$ and $g(P)$ be a positive continuous function on $A\cap X$. If $I_{A^{(m)}\cap X}^i(g, x, f)=\infty$ for each m, then $I_{A\cap X}^i(g, x, f)=\infty$.*

The proof of the corresponding result in the case where $I_X^e(g, x, f)$ is in question is not as simple as the preceding one. We shall prove several lemmas first.

LEMMA 3.9. *Let B be a Borel set in Ω. Then, for any μ such that $\mu(\Omega-B)=0$ and (μ, μ) is defined, we have*

$$V_i(B)\mu^2(B)\leqq(\mu, \mu).$$

PROOF. First we take μ with compact $S_\mu\subset B$. The inequality evidently holds if $\mu\equiv 0$. Otherwise $\mu/\mu(B)$ is a unit measure and

$$\frac{(\mu, \mu)}{\mu^2(B)}\geqq V_i(B).$$

In case $S_\mu\not\subset B$ but $\mu(\Omega-B)=0$ and (μ, μ) is defined, we approximate μ by the restrictions of μ to compact subsets of B and obtain the inequality.

LEMMA 3.10. *Let G_0 be an open set in Ω with $V_i(G_0)>0$ such that the kernel is bounded from below on $G_0 \times G_0$, and $B \subset G_0$ be a Borel set. Then, for any μ with $\mu(\Omega - G_0)=0$ and for the restriction μ_B of μ to B, we have*

$$\|\mu_B\| \leqq \sqrt{1+\frac{m^-}{V_i(G_0)}}\ \|\mu\|,$$

where $m^- = \max\,(0,\ -\inf_{G_0 \times G_0} \Phi)$.

PROOF. We have $\mu^2(\Omega) \leqq \|\mu\|^2\ V_i^{-1}(G_0)$ by Lemma 3.9. Therefore

$$\begin{aligned}\|\mu_B\|^2 &= \|\mu\|^2 - (\mu - \mu_B,\ \mu + \mu_B) \leqq \|\mu\|^2 + m^-\ \mu^2(\Omega) \\ &\leqq \|\mu\|^2 + m^- \frac{\|\mu\|^2}{V_i(G_0)} = \left(1+\frac{m^-}{V_i(G_0)}\right)\|\mu\|^2.\end{aligned}$$

If we assume that the kernel is of positive type, we shall obtain an inequality which is sharper in a sense than the above one. In fact we can prove the following:

Consider a kernel of positive type. Let G_0 be an open set in Ω with $V_i(G_0)>0$ such that the kernel is bounded from below on $G_0 \times G_0$, and $B \subset G$ be a Borel set such that

$$V_i(B) \geqq \frac{(m^-)^2}{c^2\, V_i(G_0)} \tag{3.21}$$

with a positive constant c. Then, for any μ with $\mu(\Omega - G_0)=0$ and for the restriction μ_B of μ to B, we have

$$\|\mu_B\| \leqq (1+2c)\|\mu\|.$$

PROOF. Since the kernel is of positive type, $\|\mu - \mu_B\| \geqq 0$. Therefore, by Lemma 3.9,

$$\begin{aligned}\|\mu\|^2 &\geqq \|\mu_B\|^2 + 2\,(\mu_B,\ \mu - \mu_B) \geqq \|\mu_B\|^2 - 2m^- \mu_B(\Omega)\,\mu(\Omega) \\ &\geqq \|\mu_B\|^2 - \frac{2m^- \|\mu_B\|\,\|\mu\|}{\sqrt{V_i(B)\,V_i(G_0)}}.\end{aligned}$$

By (3.12) it follows that

$$\|\mu\|^2 \geqq \|\mu_B\|^2 - 2c\,\|\mu_B\|\,\|\mu\| = (\|\mu_B\| - c\,\|\mu\|)^2 - c^2\|\mu\|^2.$$

Hence

$$\|\mu_B\| \leqq (\sqrt{1+c^2}+c)\,\|\mu\| \leqq (1+2c)\|\mu\|.$$

LEMMA 3.11. *Let G_0 be an open set in Ω on whose product the kernel is bounded from below. Let X be a set in G_0 with $V_e(X)=\infty$, and $h(P)$ be a nonnegative finite-valued upper semicontinuous function defined in G_0. Then, given*

ε, $0<\varepsilon<1$, *we can find an open set* G_ε *such that* $X\subset G_\varepsilon\subset G_0$ *and*

$$\langle h, \mu\rangle \leqq \varepsilon\|\mu\|$$

for every μ *with* $S_\mu\subset G_\varepsilon$.

PROOF. In this proof we shall use subscript p to indicate a term of a sequence. We choose an open set G_0' such that $X\subset G_0'\subset G_0$ and $V_i(G_0')>m^-$. We set

$$D_p=\{P\in G_0';\ h(P)<p\} \qquad \text{and} \qquad X_p=X\cap D_p.$$

Certainly $X=\bigcup_p X_p$. Since $V_e(X_p)=\infty$ for each p, we can find an open set $G_p\subset D_p$ such that $X_p\subset G_p$ and

$$\sqrt{V_i(G_p)}>\frac{p2^{p+1}}{\varepsilon}.$$

Let μ be any measure with S_μ contained in $\bigcup_p G_p$. If μ_p denotes the restriction of μ to G_p,

$$\langle h, \mu\rangle \leqq \sum_p \int_{G_p} hd\mu \leqq \sum_p p\mu(G_p) \leqq \sum_p \frac{p\|\mu_p\|}{\sqrt{V_i(G_p)}} \leqq \sum_p \frac{\varepsilon\|\mu_p\|}{2^{p+1}},$$

where we make use of Lemma 3.9. Lemma 3.10 is applied to obtain

$$\|\mu_p\|\leqq\sqrt{2}\,\|\mu\|.$$

This inequality yields

$$\langle h, \mu\rangle \leqq \varepsilon \sum_p \frac{\|\mu_p\|}{2^{p+1}} \leqq \varepsilon\|\mu\|.$$

Hence we may take $\bigcup_p G_p$ for G_ε.

We begin with the case $n=1$.

THEOREM 3.18. *Consider the case* $n=1$. *Let* G_0 *be an open set in* Ω *on whose product the kernel is bounded from below,* X *be any subset of* G_0, $f(P)<\infty$ *be an upper semicontinuous function on* G_0 *and* $g(P)$ *be a positive continuous function* G_0. *Assume also at least one of the following conditions*: (a_1) $g(P)$ *has a positive lower bound on* G_0, $(a_2)_e$ $V_e(X)>0$; *if the kernel is nonnegative in* $G_0\times G_0$, *we do not need any of these assumptions. If* $f(P)=-\infty$ *q.p. on* X, *then* $I_X^e(g, x, f)=\infty$. *Conversely if* $I_X^e(g, x, f)=\infty$ *and the* V_i*-value and the* V_e*-value coincide for the intersection of every open subset of* G_0 *with every closed set,*[40] *then* $f(P)=-\infty$ *q.p. on* X; *in the converse we need neither* (a_1) *nor* $(a_2)_e$.

PROOF. First we assume that $V_e(X)=\infty$. We may suppose $V_i(G_0)>1$.

40) This is so if the kernel is of positive type and $(b_1)^*$ or $(b_2)^*$ of Lemma 3.5 is true.

We set

$$D_p=\{P\in G_0;\ \frac{f(P)}{g(P)}<p \qquad \text{and} \qquad g(P)<p\},^{41)}$$

and $X_p=X\cap D_p$. Each D_p is open and $\bigcup_p D_p=G_0$. Since $V_e(X_p)=\infty$ for each p, we can find for given ε, $0<\varepsilon<1$, an open set G_p such that

$$X_p\subset G_p\subset D_p \qquad \text{and} \qquad V_i(G_p)\geqq\left(\frac{2^p p^2}{\varepsilon}\right)^2 \max\ \{1,\ (m^-)^2\},$$

where $m^-=\max\ (0,\ -\inf_{G_0\times G_0} \Phi)$ as before.

We shall show

$$\sum_p \|\mu_p\|^2\leqq 3\,\|\mu\|^2$$

for any $\mu\in\mathscr{E}_{\bigcup_p G_p}(g,\ x)$, where μ_p denotes the restriction of μ to $D'_p=G_p-\bigcup_{k=1}^{p-1} G_k$. We set

$$\mu'_p=\mu-\sum_{k=1}^{p}\mu_k.$$

We observe that

$$(\mu'_p,\ \mu'_p)=(\mu_{p+1},\ \mu_{p+1})+(\mu'_{p+1},\ \mu'_{p+1})+2(\mu_{p+1},\ \mu'_{p+1})$$

and

$$\begin{aligned}(\mu_{p+1},\ \mu'_{p+1})&\geqq -m^-\mu_{p+1}(\Omega)\,\mu'_{p+1}(\Omega)\geqq -m^-\frac{\|\mu_{p+1}\|\,\|\mu'_{p+1}\|}{\sqrt{V_i(G_{p+1})\,V_i(G_0)}}\\ &\geqq\frac{-\varepsilon\|\mu_{p+1}\|\ \|\mu'_{p+1}\|}{2^{p+1}(p+1)^2}\geqq\frac{-\varepsilon}{2^{p+2}(p+1)^2}\,(\|\mu_{p+1}\|^2+\|\mu'_{p+1}\|^2).\end{aligned}$$

Hence

$$\|\mu'_p\|^2\geqq\left(1-\frac{\varepsilon}{2^{p+1}(p+1)^2}\right)(\|\mu_{p+1}\|^2+\|\mu'_{p+1}\|^2).$$

It follows that

$$\|\mu_{p+1}\|^2+\|\mu'_{p+1}\|^2\leqq\left\{1-\frac{\varepsilon}{2^{p+1}(p+1)^2}\right\}^{-1}\|\mu'_p\|^2\leqq\left\{1+\frac{\varepsilon}{2^{p}(p+1)^2}\right\}\|\mu'_p\|^2.$$

Let us evaluate $\|\mu'_p\|$. Since $\|\mu_{p+1}\|\geqq 0$,

41) Since the case $n=1$ is concerned in this theorem, we may use lower subscripts to denote sets without indicating any components.

$$\|\mu'_{p+1}\|^2 \leqq \left\{1+\frac{\varepsilon}{2^p(p+1)^2}\right\}\|\mu'_p\|^2 \leqq \left(1+\frac{1}{2^p}\right)\|\mu'_p\|^2 \leqq \left(1+\frac{1}{2^p}\right)\left(1+\frac{1}{2^{p+1}}\right)\|\mu'_{p-1}\|^2$$
$$\leqq \prod_{k=1}^{p}\left(1+\frac{1}{2^k}\right)\|\mu\|^2 < 4\|\mu\|^2.$$

Therefore

$$\|\mu_{p+1}\|^2+\|\mu'_{p+1}\|^2 \leqq \|\mu'_p\|^2+\frac{4\varepsilon}{2^p(p+1)^2}\|\mu\|^2 \leqq \|\mu'_p\|^2+\frac{1}{2^p}\|\mu\|^2$$

for $p \geqq 1$. We have also

$$\|\mu_1\|^2+\|\mu'_1\|^2 \leqq \|\mu\|^2.$$

Adding these inequalities for $p=1, \cdots, q$, we obtain

$$\sum_{p=1}^{q+1}\|\mu_p\|^2 \leqq \|\mu\|^2\left\{2+\sum_{p=1}^{q}\frac{1}{2^p}\right\} < 3\|\mu\|^2.$$

It follows that

$$\sum_p \|\mu_p\|^2 \leqq 3\|\mu\|^2.$$

Secondly we shall show that $x < \varepsilon\sqrt{\sum_p \|\mu_p\|^2}$. Since $\mu_p(\Omega-D'_p)=0$, $\mu_p(D'_p) \leqq \frac{\varepsilon}{2^p p^2}\|\mu_p\|$. Hence

$$\int g d\mu_p \leqq p\mu_p(D'_p) \leqq \frac{\varepsilon}{2^p p}\|\mu_p\|$$

and

$$\sum_p p\int g d\mu_p \leqq \varepsilon\sum_p \frac{\|\mu_p\|}{2^p} \leqq \varepsilon\sqrt{\sum_p \|\mu_p\|^2}\sqrt{\sum_p \frac{1}{2^{2p}}} < \varepsilon\sqrt{\sum_p \|\mu_p\|^2}.$$

Naturally

$$x=\sum_p \int g d\mu_p \leqq \varepsilon\sqrt{\sum_p \|\mu_p\|^2}.$$

Making use of $\sum_p \|\mu_p\|^2 \leqq 3\|\mu\|^2$, we have

$$I(\mu)=\|\mu\|^2-2\int f d\mu \geqq \frac{1}{3}\sum_p \|\mu_p\|^2-2\sum_p \int \frac{f}{g} g d\mu_p$$
$$\geqq \frac{1}{3}\sum_p \|\mu_p\|^2-2\sum_p p\int g d\mu_p \geqq \frac{1}{3}\sum_p \|\mu_p\|^2-2\varepsilon\sqrt{\sum_p \|\mu_p\|^2}$$
$$=\frac{1}{3}\left(\sqrt{\sum_p \|\mu_p\|^2}-3\varepsilon\right)^2-3\varepsilon^2 > \frac{1}{3}\left(\frac{x}{\varepsilon}-3\varepsilon\right)^2-3\varepsilon^2.$$

By the arbitrariness of $\mu \in \mathscr{E}_{\underset{p}{\cup} G_p}(g, x)$, it follows that

$$I^e_{\underset{p}{\cup} X_p}(g, x, f) \geqq I^i_{\underset{p}{\cup} G_p}(g, x, f) \geqq \frac{x^2}{3\varepsilon^2} - 2x.$$

By letting $\varepsilon \to 0$, we obtain

$$I^e_X(g, x, f) = I^e_{\underset{p}{\cup} X_p}(g, x, f) = \infty.$$

Next we assume that $V_e(X) < \infty$ but $f(P) = -\infty$ q.p. on X. If the kernel is not always nonnegative we assume furthermore at least one of (a_1) and $(a_2)_e$. We set

$$B_p = \{P \in G_0;\, f(P) < -pg(P)\}.$$

This is an open set and $V_e(X - B_p) = \infty$. Making use of Lemma 3.11, we take an open set B'_p such that

$$X - B_p \subset B'_p \subset G_0 \qquad \text{and} \qquad \langle f^+ + pg, \mu \rangle \leqq \|\mu\|$$

for every μ with $\mu(\Omega - B'_p) = 0$. If $V_e(X) > 0$, we may assume $V_i(G_0) > 0$ and, for any λ with $S_\lambda \subset G_0$ and its restriction λ_p to B'_p, we have

$$\|\lambda_p\| \leqq \sqrt{1 + \frac{m^-}{V_i(G_0)}}\, \|\lambda\|$$

by Lemma 3.10. If $g(P)$ has a positive lower bound on G_0, the total mass of any measure $\lambda \in \mathscr{E}_{G_0}(g, x)$ is bounded: $\lambda(\Omega) \leqq a < \infty$. Hence

$$\begin{aligned} \|\lambda_p\|^2 &= \|\lambda\|^2 - (\lambda + \lambda_p, \lambda - \lambda_p) \\ &\leqq \|\lambda\|^2 + m^- \lambda^2(\Omega) \leqq \|\lambda\|^2 + m^- a^2. \end{aligned}$$

If the kernel is nonnegative in $G_0 \times G_0$, obviously $\|\lambda_p\| \leqq \|\lambda\|$. We take any $\nu \in \mathscr{E}_{B_p \cup B'_p}(g, x)$. Then we have, with the restriction ν_p of ν to B'_p,

$$\begin{aligned} I(\nu) &= \|\nu\|^2 - 2\langle f, \nu \rangle = \|\nu\|^2 - 2\langle f, \nu_p \rangle - 2\langle f, \nu - \nu_p \rangle \\ &\geqq \|\nu\|^2 + 2p\langle g, \nu - \nu_p \rangle - 2\langle f, \nu_p \rangle \geqq \|\nu\|^2 + 2px - 2\langle f^+ + pg, \nu_p \rangle \\ &\geqq \|\nu\|^2 + 2px - 2\|\nu_p\| \geqq \|\nu\|^2 + 2px - 2\sqrt{1 + \frac{m^-}{V_i(G_0)}}\, \|\nu\| \\ &= \left(\|\nu\| - \sqrt{1 + \frac{m^-}{V_i(G_0)}} \right)^2 + 2px - 1 - \frac{m^-}{V_i(G_0)} \geqq 2px - 1 - \frac{m^-}{V_i(G_0)} \end{aligned}$$

or

$$\begin{aligned} I(\nu) &\geqq \|\nu\|^2 + 2px - 2\sqrt{\|\nu\|^2 + m^- a^2} = (\sqrt{\|\nu\|^2 + m^- a^2} - 1)^2 + 2px - m^- a^2 - 1 \\ &\geqq 2px - m^- a^2 - 1. \end{aligned}$$

Consequently

$$I^i_{B_p\cup B'_p}(g, x, f)\geqq 2px-1-\frac{m^-}{V_i(G_0)} \qquad \text{or} \qquad 2px-m^-a^2-1$$

and hence

$$I^e_X(g, x, f)=\infty.$$

To discuss the converse we assume that there is $Y\subset X$ on which $f(P)>-\infty$ and which has $V_e(Y)<\infty$. We may assume then that $Y=X$. The set

$$E_p=\left\{P\in G_0;\ g(P)>\frac{1}{p}\right\}$$

is an open set and $\bigcup_p E_p=G_0$. If $V_e(X\cap E_p)=\infty$ for each p, then $V_e(X)=\infty$, contradicting our assumption. Therefore $V_e(X\cap E_{p_0})<\infty$ for some p_0. It will be sufficient to show $I^e_{X\cap E_{p_0}}(g, x, f)<\infty$. Hence we assume from the beginning that $g(P)>\alpha>0$ on G_0.

The set

$$H_q=\{P\in G_0;\ f(P)\geqq -q\}$$

is closed relatively to G_0 and $G_0=\bigcup_q H_q\cup\{P\in G_0;\ f(P)=-\infty\}$. We see similarly that $V_e(X\cap H_{q_0})<\infty$ for some q_0. We take a decreasing sequence $\{G_p\}$ of open sets such that

$$X\cap H_{p_0}\subset G_p\subset G_0 \quad \text{and} \quad \lim_{p\to\infty} I^i_{G_p}(g, x, f)=I^e_{X\cap H_{q_0}}(g, x, f).$$

Let $\mu\in\mathscr{E}_{G_p\cap H_{q_0}}(g, x)$ satisfy $(\mu, \mu)\leqq V^{(g,x)}_i(G_p\cap H_{q_0})+1$. Then $V^{(g,x)}_i(G_p\cap H_{q_0})\leqq x^2\alpha^{-2}V_i(G_p\cap H_{q_0})$ and

$$\begin{aligned} I^i_{G_p}(g, x, f)&\leqq I^i_{G_p\cap H_{q_0}}(g, x, f)\leqq I(\mu)=(\mu, \mu)-2\langle f, \mu\rangle \\ &\leqq V^{(g,x)}_i(G_p\cap H_{q_0})+1+2q_0\,\mu(\Omega)\leqq \frac{x^2}{\alpha^2}V_i(G_p\cap H_{q_0})+1+2q_0\frac{x}{\alpha} \\ &=\frac{x^2}{\alpha^2}V_e(G_p\cap H_{q_0})+1+2q_0\frac{x}{\alpha}\leqq\frac{x^2}{\alpha^2}V_e(X\cap H_{q_0})+1+2q_0\frac{x}{\alpha}, \end{aligned}$$

because $V_i(G_p\cap H_{q_0})=V_e(G_p\cap H_{q_0})$ by our assumption. Consequently

$$I^e_X(g, x, f)\leqq I^e_{X\cap H_{q_0}}(g, x, f)=\lim_{p\to\infty} I^i_{G_p}(g, x, f)\leqq\frac{x^2}{\alpha^2}V_e(X\cap H_{q_0})+1+2q_0\frac{x}{\alpha}<\infty.$$

Our theorem is completly proved.

Next we consider the case $n\geqq 2$. Let $X_1, \cdots, X_n$ be subsets of an open set G_0 and assume that they are separable by Φ-separate open sets. The condition that each $I^e_{X_k}>-\infty$ and $I^e_{X_1}=\infty$ are not sufficient to have $I^e_X=\infty$. As an example we take the point $P_1=(0, 0, 2)$ in E_3 for X_1, the ball $\overline{OP}<1$ for $X_2=G_2$, and the ball $\overline{PP}_1<1/2$ for G_1. We set $\Phi(P, Q)=1/\overline{PQ}$ on $G_1\times G_1$, $\Phi(P, Q)=-1$ on $G_1\times X_2$ and on $X_2\times G_1$, $\Phi(P, Q)=0$ on $X_2\times X_2$, $f(P)\equiv 0$, $g(P)=1$ on G_1 and $=1$

$-\overline{OP}$ on X_2. Then $I^e_{X_1}=\infty$ and $I^e_{X_2}=0$. We shall show that $I^e_{X_1\cup X_2}=-\infty$. Let G be any open subset of G_1 containing P_1 and take any $\mu\in\mathscr{E}_{G\cup X_2}(g, x)$. Denoting the restrictions of μ to G and X_2 by μ_1 and μ_2 respectively, we have

$$I(\mu)=I(\mu_1)+I(\mu_2)+2\,(\mu_1, \mu_2)=I(\mu_1)-2\mu_1(\Omega)\mu_2(\Omega).$$

It holds that $x_1=\int g d\mu_1=\mu_1(\Omega)$ and $x_2=\int g d\mu_2=\int (1-\overline{OQ})\, d\mu_2(Q)$. If we take μ_2 arbitrarily close to $\overline{OP}=1$, then $\mu_2(\Omega)$ becomes arbitrarily large. This shows that $I^i_{G\cup X_2}=-\infty$.

In order to avoid this situation we assume at least one of (a_1) and $(a_2)'_e$ in Theorem 3.9; we may replace $V_e(X)>0$ in $(a_2)'_e$ by $V_e(X_k)>0$ $(k=1, \cdots, n)$. Take Φ-separate open subsets $G^{(0)}_1, \cdots, G^{(0)}_n$ of G_0 separating $X_1, \cdots, X_n$. We may assume that $G_0=\bigcup_{k=1}^{n} G^{(0)}_k$. In case $I^e_{X_k}(g, x_k, f)<\infty$, we consider, for each k, the subclass $\mathscr{E}_k$ of $\mathscr{E}_{G^{(0)}_k}(g, x_k)$ such that $I(\mu_k)<(I^e_{X_k}(g, x_k, f))^+ +1$ for any $\mu_k\in\mathscr{E}_k$, where $(I^e_{X_k})^+=\max\,(I^e_{X_k}, 0)$. We see by (a_1) or $(a_2)'_e$ that each $\mu_k(\Omega)$ is bounded on $\mathscr{E}_k$. Let G be any open subset of G_0 containing X such that $\mathscr{E}_k\cap\mathscr{E}_G\neq\emptyset$ for each k. We set $\mu=\sum_{k=1}^{n}\mu_k$ for $\mu_k\in\mathscr{E}_k\cap\mathscr{E}_G$, $k=1, \cdots, n$, and observe that

$$I^i_G(g, x, f)\leqq I(\mu)=\sum_{k=1}^{n} I(\mu_k)+\sum_{\substack{j,k=1\\ j\neq k}}^{n}\iint \Phi\, d\mu_j\, d\mu_k\leqq\sum_{k=1}^{n}(I^e_{X_k}(g, x_k, f))^+ +O(1).$$

Therefore $I^e_X(g, x, f)$ is not equal to ∞. We see also that $I^e_X(g, x, f)=-\infty$ if any one of $I^e_{X_k}(g, x_k, f)$ is $-\infty$.

Conversely assume that $I^e_X(g, x, f)$ is finite and that one or both of (a_1) and $(a_2)'_e$ in Theorem 3.9 is true. We may assume $V_i(G_0)>0$ and that G_0 consists of Φ-separable open sets $G^{(0)}_1, \cdots, G^{(0)}_n$ separating $X_1, \cdots, X_n$. We denote by $\mathscr{E}_0$ the subclass of $\mathscr{E}_{G_0}(g, x)$ such that $I(\mu)\leqq I^e_X(g, x, f)+1$ on $\mathscr{E}_0$. We have seen that each $I^e_{X_k}(g, x_k, f)>-\infty$. There is G'_0, $X\subset G'_0\subset G_0$, such that $I^i_{G_k}(g, x_k, f)$ is bounded from below for each k and every G, $X\subset G\subset G'_0$, where $G_k=G\cap G^{(0)}_k$. By our assumption $\mu(\Omega)$ is bounded on $\mathscr{E}_0$ and hence

$$\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\mu_j, \mu_k)=O(1),$$

where μ_k is the restriction of μ to $G^{(0)}_k$. For an open set G such that $X\subset G\subset G'_0$, we take any $\mu\in\mathscr{E}_G(g, x)\cap\mathscr{E}_0$. We have

$$\begin{aligned}\sum_{k=1}^{n} I^i_{G_k}(g, x_k, f)&\leqq\sum_{k=1}^{n} I(\mu_k)=I(\mu)-\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\mu_j, \mu_k)\\ &\leqq I^e_X(g, x, f)+O(1).\end{aligned}$$

Since each $I^i_{G_k}(g, x_k, f)$ is bounded from below, each $I^i_{G_k}(g, x_k, f)$ is bounded. Hence each $I^e_{X_k}(g, x_k, f)$ is finite. It is also seen that if $I^e_X(g, x, f)=-\infty$, at least one of $I^e_{X_k}(g, x_k, f)$ is $-\infty$.

We consider a kernel of positive type. Assume, for each k, that every strong Cauchy net in $\mathscr{E}^i_{G^{(0)}_k}(g, x_k)$ is strongly convergent and that $(\mu^i_G, \mu^i_{G'})$ is defined for any open subsets G and G' of $G^{(0)}_k$ both including X_k such that $I^i_G(g, x_k, f)$ and $I^i_{G'}(g, x_k, f)$ are finite. Then we may assume $V_e(X_k)>0$ $(k=1, \cdots, n)$ instead of (a_1) or $(\mathrm{a}_2)'_e$. If the kernel is nonnegative in each $G_j\times G_k$, $j\neq k$, we do not need such condition. In fact, we assume that each $I^e_{X_k}(g, x_k, f)$ is finite and take Φ-separate open subsets $G^{(1)}_1, \cdots, G^{(1)}_n$ of G_0 separating $X_1, \cdots, X_n$ such that $\|\mu^i_{G_k}\|\leqq\|\mu^e_{X_k}\|+1/2$ holds for any open subset G_k of $G^{(1)}_k$ containing X_k $(k=1, \cdots, n)$; see Theorem 3.12. For each G_k, we find $\nu_{G_k}\in\mathscr{E}_{G_k}(g, x)$ such that

$$I(\nu_{G_k})\leqq I^e_{X_k}(g, x, f)+1 \qquad \text{and} \qquad \|\nu_{G_k}\|\leqq\|\mu^i_{G_k}\|+\frac{1}{2}.$$

We write G^* for $\bigcup_{k=1}^{n} G_k$ and have

$$I^i_{G^*}(g, x, f)\leqq I(\sum_{k=1}^{n}\nu_{G_k})=\sum_{k=1}^{n}I(\nu_{G_k})+\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\nu_{G_j}, \nu_{G_k})$$
$$\leqq\sum_{k=1}^{n}I(\nu_{G_k})+\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\|\mu^e_{X_j}\|+1)(\|\mu^e_{X_k}\|+1)\leqq\sum_{k=1}^{n}I^e_{X_k}(g, x, f)+O(1).$$

Consequently $I^e_X(g, x, f)<\infty$. Next assume that each $I^e_{X_k}(g, x_k, f)<\infty$ but some of them are $-\infty$. We consider $I^e_{X_k}(g, x_k, -f^-)$ for each k where $f^-=\max(0, -f)$. It is naturally nonnegative, and finite on account of Theorem 3.18 under the assumption that the kernel is bounded from below on $G_0\times G_0$ and that $V_i(G\cap F)=V_e(G\cap F)$ for every open subset $G\subset G_0$ and every closed set F. Consequently

$$\infty>I^e_X(g, x, -f^-)\geqq I^e_X(g, x, f).$$

Conversely assume that $I^e_X(g, x, f)$ is finite. We suppose that G_0 consists of Φ-separate open sets $G^{(0)}_1, \cdots, G^{(0)}_n$. We consider still a kernel of positive type, and suppose $(\mathrm{a}_2)_e$ $V_e(X)>0$, that every strong Cauchy net in $\mathscr{E}^i_{G_0}(g, x)$ is strongly convergent and that $(\mu^i_G, \mu^i_{G'})$ is defined for any open subsets G and G' of G_0 both including X. If the kernel is nonnegative in each $G_j\times G_k$, $j\neq k$, we need not condition $(\mathrm{a}_2)_e$. We choose an open subset G' of G_0 containing X with the property that $|I^e_X-I^i_G|\leqq 1$ and $\|\mu^e_X-\mu^i_G\|\leqq 1$ for any open set G such that $X\subset G\subset G'$. We shall denote by $\mathscr{E}'$ the class of measures μ in $\bigcup_{X\subset G\subset G'}\mathscr{E}_G(g, x)$ such that $|I^e_X-I(\mu)|\leqq 2$ and $\|\mu^e_X-\mu\|\leqq 2$. If $(\mathrm{a}_2)_e$ is true, we may assume $V_i(G')>0$. It follows that $\mu(\Omega)$ is bounded on $\mathscr{E}'$. Hence $\|\mu_k\|$ is bounded because

$$\sum_{k=1}^{n}\|\mu_k\|^2=\|\mu\|^2-\sum_{\substack{j,k=1\\ j\neq k}}^{n}(\mu_j, \mu_k).$$

It is evidently so if the kernel is nonnegative in each $G_j\times G_k$, $j\neq k$. Let M

be an upper bound for $\|\mu_k\|$, $k=1,\cdots,n$. Suppose that there is $\{\mu^{(m)}\}$ in $\mathscr{E}'$ such that $I(\mu_{k_0}^{(m)})$ tends to $-\infty$ as $m\to\infty$. We have

$$I_X^e-1\leqq I_{G'}^i\leqq I(\mu_{k_0}^{(m)}+\sum_{k\neq k_0}\mu_k^{(1)})\leqq I(\mu_{k_0}^{(m)})+\sum_{k\neq k_0}I(\mu_k^{(1)})+n(n-1)M^2.$$

This is impossible and it is shown that $I(\mu_k)$ is bounded from below on $\mathscr{E}'$. From the inequality

$$\sum_{k=1}^n I(\mu_k)=I(\mu)-\sum_{\substack{j,k=1\\ j\neq k}}^n(\mu_j,\mu_k)\leqq I_X^e+2+n(n-1)M^2$$

valid for any $\mu\in\mathscr{E}'$, it follows that $I(\mu_k)$ is bounded on $\mathscr{E}'$. Since $\mathscr{E}_G(g,x)\neq\emptyset$ for any G, $X\subset G\subset G'$, $I^i_{G\cap G_k^{(0)}}(g,x_k,f)$ $(\leqq I(\mu_k))$ is bounded from above by a constant not depending on G, $X\subset G\subset G'$. Consequently $I^e_{X_k}(g,x_k,f)<\infty$ for each k.

So we state

Theorem 3.19. *Let G_0 be an open set in Ω, $X_1,\cdots,X_n$ be subsets of G_0 which are separable by Φ-separate open sets, $f(P)<\infty$ be an upper semicontinuous function on G_0 and $g(P)$ be a positive continuous function on G_0. Set $X=\bigcup_{k=1}^n X_k$ and assume at least one of conditions* (a_1) *and* $(\mathrm{a}_2)'_e$ *of Theorem* 3.9; *we may replace $V_e(X)>0$ in* $(\mathrm{a}_2)'_e$ *by $V_e(X_k)>0$ $(k=1,\cdots,n)$. If $I^e_{X_k}(g,x_k,f)<\infty$ for each k, then $I^e_X(g,x,f)<\infty$, and if any one of $I^e_{X_k}(g,x_k,f)$ is $-\infty$ then $I^e_X(g,x,f)=-\infty$. Conversely $I^i_X(g,x,f)<\infty$ implies $I^e_{X_k}(g,x_k,f)<\infty$ for each k and $I^e_X(g,x,f)=-\infty$ implies $I^e_{X_k}(g_k,x_k,f)=-\infty$ for some k under the assumption of one or both of* (a_1) *and* $(\mathrm{a}_2)'_e$ *in Theorem* 3.9.

Next consider a kernel of positive type. If, for each k, every strong Cauchy net in $\mathscr{E}^i_{G_k^{(0)}}(g,x_k)$ is strongly convergent and $(\mu^i_G,\mu^i_{G'})$ is defined for any open subsets G and G' of G_k such that $I^i_G(g,x_k,f)$ and $I^i_{G'}(g,x_k,f)$ are finite, and if $V_e(X_k)>0$ $(k=1,\cdots,n)$, then $-\infty<I^e_{X_k}(g,x_k,f)<\infty$ $(k=1,\cdots,n)$ imply $I^e_X(g,x,f)<\infty$; some $I^e_{X_k}(g,x_k,f)$ may be $-\infty$ if the kernel is bounded from below on $G_0\times G_0$ and if $V_i(G\frown F)=V_e(G\frown F)$ for every open $G\subset G_0$ and every closed F. Conversely $-\infty<I^e_X(g,x,f)<\infty$ implies $I^e_{X_k}(g,x_k,f)<\infty$ for each k if every strong Cauchy net in $\mathscr{E}^i_{G_0}(g,x)$ is strongly convergent and $(\mu^i_G,\mu^i_{G'})$ is defined for any open subsets G and G' of G_0 both including X and if $V_e(X)>0$.

In view of Proposion 2 in Chapter I we have

Corollary. *Let G_0 be an open set in Ω such that the kernel is bounded from below on $G_0\times G_0$, $\{X^{(m)}\}$ be an increasing sequence of subsets of G_0 such that each $X^{(m)}$ is decomposed into $X_1^{(m)},\cdots,X_n^{(m)}$ which are separable by Φ-separate open sets, $f(P)<\infty$ be an upper semicontinuous function on G_0 and $g(P)$ be a positive continuous function on G_0. Set $\bigcup_m X^{(m)}=X$ and $\bigcup_m X_k^{(m)}=X_k$, and assume one of conditions* (a_1) *and* $(\mathrm{a}_2)'_e$ *of Theorem* 3.9. *Assume also that the*

V_i-value and the V_e-value coincide for the intersection of every open subset of G_0 with every closed set. Then $I_X^e(g, x, f)=\infty$ if $I_{X^{(m)}}^e(g, x, f)=\infty$ for each m.

QUESTION. *Let G_0, $\{X^{(m)}\}$ be as above. Consider a kernel of positive type, and assume that every strong Cauchy net in $\mathscr{E}_{G_0}(g, x)$ is strongly convergent and that $(\mu_G^i, \mu_{G'}^i)$ is defined for every open subsets G and G' of G_0 both including X such that I_G^i and $I_{G'}^i$ are finite. Assume* $(a_2)_e$ *$V_e(X)>0$ and that $V_i(G\cap F)=V_e(G\cap F)$ for every open $G\subset G_0$ and every closed F. Is it true that $I_X^e(g, x, f)=\infty$ if each $I_{X^{(m)}}^e(g, x, f)=\infty$?*

If this is true we can replace $(a_2)_e'$ by $(a_2)_e$ in Theorems 3.20 and 3.22. Let us see why this question remains open. If $I_{X^{(m)}}^e(g, x, f)=\infty$ for each m, there is k_0 such that $I_{X_{k_0}^{(m)}}^e(g, x_{k_0}, f)=\infty$ for each m. Hence by Theorem 3.18 $I_{X_{k_0}}^e(g, x_{k_0}, f)=\infty$. But we can not assure that $I_{X_k}^e(g, x_k, f)$ is finite for other k. Under these circumstances it is not certain that $I_X^e(g, x, f)=\infty$ as Example 1 will tell.

Coming back to Theorem 3.19 we observe that we did not assert there that $I_X^e(g, x, f)<\infty$ implies $I_{X_k}^e(g, x_k, f)<\infty$ in the case of kernel of positive type. In fact this is not always true as Example 1 will show. We shall give several examples to supplement the theorem.

EXAMPLE 1. In E_3 take $P_1=(0, 0, 2)$ for X_1, the ball $\overline{OP}<1$ for $X_2=G_2$ and the ball $\overline{P_1P}<1/2$ for G_1. We consider the Newtonian kernel and set $f(P)=0$ in G_1 and $=(1-\overline{OP})^{-1}$ in G_2, $g(P)=1$ in G_1 and $=1-\overline{OP}$ in G_2. Then $I_{X_1}^e(g, 1, f)=\infty$, $I_{X_2}^e(g, 1, f)=\infty$ and $I_{X_1\cup X_2}^e(g, (1,1), f)=-\infty$. Consequently $I_X^e(g, x, f)=-\infty$ does not always mean that each $I_{X_k}^e(g, x_k, f)<\infty$.

EXAMPLE 2. Take X_1, $X_2=G_2$ as above and regard $X_1\cup G_2$ as space Ω; X_1 is then taken for G_1. We define Φ, f and g as in Example 1, the space being restricted to $X_1\cup G_2$. We see easily that $I_{X_1}^e(g, 1, f)=\infty$, $I_{X_2}^e(g, 1, f)=-\infty$ and $I_{X_1\cup X_2}^e(g, (1,1), f)=\infty$. Consequently the fact that some $I_{X_k}^e(g, x_k, f)=-\infty$ does not always mean that $I_X^e(g, x, f)<\infty$.

EXAMPLE 3. Let Ω consist of two sequences of points on the x-axis: $X_1=\{1/2, 1/3, \dots\}$ and $X_2=\{1-1/3, 1-1/4, \dots\}$. We regard Ω as a subspace of the x-axis and set $\Phi(P, Q)\equiv 1$ in $\Omega\times\Omega$. This is naturally of positive type and every strong Cauchy net in $\mathscr{E}$ is convergent. Also $V_i(\Omega)>0$. We set $2f(P)=k+1$ and $g(P)=1/k$ both at $P=1/k$ and $P=1-1/k$. Let us see that, for $K_1^{(m)}=\{1/2, \dots, 1/m\}$, $\min_{\mu\in\mathscr{E}_{K_1^{(m)}}(g,1)} I(\mu)$ is attained by the point measure at $1/m$ with mass m. We set $\mu(\{1/k\})=k\xi_k$ for $\mu\in\mathscr{E}_{K_1^{(m)}}(g, 1)$ and have

$$(\mu, \mu)-2\langle f, \mu\rangle=(\sum_{k=2}^{n} k\xi_k)^2-\sum_{k=2}^{n} k(k+1)\xi_k.$$

This takes its extremal values at some of $(1, 0, \dots, 0)$, $(0, 1, 0, \dots, 0)$, $\dots$, $(0, \dots, 0, 1)$ in the $(\xi_2, \dots, \xi_m)$ space under the condition $\sum_{k=2}^{m}\xi_k=1$, $\xi_k\geqq 0$. The value at the k-th point is $k^2-k(k+1)=-k$. Hence the minimum value is $-m$. Consequently $I^e_{X_1}(g, 1, f)=-\infty$. Likewise $I^e_{X_2}(g, 1, f)=-\infty$. To show that $I^e_\Omega(g, (1,1), f)$ is finite, take any $\mu_1\in\mathscr{E}_{X_1}(g, 1)$ and $\mu_2\in\mathscr{E}_{X_2}(g, 1)$ and set $\mu_1(\{1/k\})=k\xi_k$ and $\mu_2(\{1-1/j\})=j\eta_j$. It holds with some m and p that

$$I(\mu_1+\mu_2)=(\sum_{k=2}^m k\xi_k)^2-\sum_{k=2}^m k(k+1)\xi_k+(\sum_{j=3}^p j\eta_j)^2-\sum_{j=3}^p j(j+1)\eta_j+2(\sum_{k=2}^m k\xi_k, \sum_{j=3}^p j\eta_j)$$
$$\geqq(\sum_{k=2}^m k\xi_k)^2-\sum_{k=2}^m \{k(k+1)-3k\}\xi_k+(\sum_{j=3}^p j\eta_j)^2-\sum_{j=3}^p \{j(j+1)-2j\}\eta_j,$$

because $\mu_1(\Omega)=\sum_{k=2}^m k\xi_k\geqq 2$ and $\mu_2(\Omega)=\sum_{j=3}^p j\eta_j\geqq 3$. We see that the last side takes its minimum when one of ξ_k and one of η_j are equal to 1 and other ξ_k's and η_j's vanish. The minimum value is equal to $4+3=7$. Actually $I(\mu_1+\mu_2)=7$ when $\xi_2=1$, $\xi_3=\cdots=0$, and $\eta_3=1$, $\eta_4=\cdots=0$. Thus $I^e_\Omega(g, (1,1), f)=7$. Hence it is shown that $-\infty<I^e_X(g, x, f)<\infty$ does not always mean that at least one of $I^e_{X_k}(g, x_k, f)$ is finite.

EXAMPLE 4. We take the same $\Omega=X_1\cup X_2$ and $g(P)$ as above. We set $\Phi(P, Q)=1$ on $X_1\times X_1$ and $X_2\times X_2$ and $=-1/2$ on $X_1\times X_2$ and $X_2\times X_1$, and set $2f(P)=k$ at $P=1/k$ and $P=1-1/k$. The kernel is of positive type, every strong Cauchy net in $\mathscr{E}$ is convergent and $V_i(\Omega)>0$. We see as above that $I^e_{X_1}(g, 1, f)=I^e_{X_2}(g, 1, f)=0$. Next take $\mu_1\in\mathscr{E}_{X_1}(g, 1)$ and $\mu_2\in\mathscr{E}_{X_2}(g, 1)$ arbitrarily and set $\mu_1(\{1/k\})=k\xi_k$ and $\mu_2(\{1-1/j\})=j\eta_j$. It holds with some m and p that

$$I(\mu_1+\mu_2)=(\sum_{k=2}^m k\xi_k)^2-\sum_{k=2}^m k^2\xi_k+(\sum_{j=3}^p j\eta_j)^2-\sum_{j=3}^p j^2\eta_j-(\sum_{k=2}^p k\xi_k)(\sum_{j=3}^p j\eta_j)$$

and follows that $I^e_\Omega(g, (1,1), f)=-\infty$. Thus $I^e_X(g, x, f)=-\infty$ does not always mean that one of $I^e_{X_k}(g, 1, f)$ is $-\infty$.

Now we shall prove that each $\gamma_k>-\infty$ in Theorem 3.9. We shall use the notations used to show Theorem 3.19. Let G be an open set such that $X\subset G\subset G_0'$. We have, for any $\mu\in\mathscr{E}_G(g, x)\cap\mathscr{E}_0$, $I^i_{G_k}(g, x_k, f)\leqq I(\mu_k)$ for each k and

$$\sum_{k=1}^n I(\mu_k)\leqq I^e_X(g, x, f)+O(1)$$

as in the proof of Theorem 3.19. Since we may suppose that each $I^i_{G_k}(g, x_k, f)$ is bounded from below, each $I(\mu_k)$ is bounded. Accordingly

$$2x_k\gamma_k(\mu)=I(\mu_k)+(\mu_k, \mu_k)+2\sum_{\substack{j=1\\ j\neq k}}^n(\mu_j, \mu_k)$$

is bounded from below for each k. Since we can approximate each γ_k by $\gamma_k(\mu)$, $\mu\in\bigcup_{X\subset G\subset G_0'}\mathscr{E}_G(g, x)\cap\mathscr{E}_0$, for each k, it is concluded that each $\gamma_k>-\infty$.

Next we shall prove that γ_k in Theorem 3.10 is finite for each k. We have already seen that it is $>-\infty$. From our assumption that $I_X^e(g, x, f)$ is finite and from Theorem 3.19 it follows that $f(P)>-\infty$ on some set $Y_k \subset X_k$ with $V_e(Y_k)<\infty$. In view of Theorem 1.10, $U^{\mu_X^e}(P)$ is finite q.p. in Ω. Hence there is a set $Z_k \subset Y_k$ with $V_e(Z_k)<\infty$ such that

$$U^{\mu_X^e}(P)-f(P)<\infty \qquad \text{on } Z_k.$$

If $\gamma_k=\infty$, it follows from (3.14) that

$$U^{\mu_X^e}(P)-f(P)=\infty \qquad \text{q.p. on } X_k.$$

Namely, the V_e-value of the subset of X, on which the left side is $<\infty$, is infinite. This is a contradiction and it is concluded that γ_k is finite.

3.5. Change of sets.

In this section we are interested in the behavior, as X varies, of $I_X^i(g, x, f)$, $I_X^e(g, x, f)$, $\mu_X^i(g, x, f)$ and $\mu_X^e(g, x, f)$. We shall assume in this and the next sections the positivity of the kernel without mentioning it explicitly; only exception is in Theorem 3.21. Simpler notations I_X^i, I_X^e, μ_X^i and μ_X^e will be used.

LEMMA 3.12. *Let $\Phi(P, Q)$ be a kernel in $\Omega\times\Omega$. Let G_0 be an open set in Ω on whose product* inf $\Phi=m>-\infty$, *X consist of $X_1, \cdots, X_n$ which are separable by Φ-separate open subsets $G_1, \cdots, G_n$ of G_0, $f(P)<\infty$ be an upper semicontinuous function defined in G_0 and $g(P)$ be a positive continuous function in G_0. Assume that I_X^e is finite, that every strong Cauchy net in $\mathscr{E}^i_{G_0}(g, x)$ is strongly convergent and that $(\mu_G^i, \mu_{G'}^i)$ is defined for any open subsets G and G' of G_0 both including X. If the kernel is not always nonnegative in each $G_j\times G_k$, $j\neq k$, assume also at least one of the following conditions*: (a_1) *$g(P)$ has a positive lower bound on G_0,* $(\mathrm{a}_2)_{\mathrm{e}}$ *$V_e(X)>0$. Let μ_X^e and $\gamma^e_{X_1}, \cdots, \gamma^e_{X_n}$ be the measure and the constants obtained in Theorem 3.12. If, for a measure $\mu\in\mathscr{E}$ and for constants $\{c_k\}$, it holds that*

$$U^\mu(P)-f(P)\geqq c_k\, g(P) \qquad \text{q.p. on } X_k \ (k=1, \cdots, n),$$

then

$$J(\mu)=2\sum_{k=1}^n x_k\, c_k-(\mu, \mu)\leqq I_X^e.$$

If, furthermore, (μ, μ_X^e) is defined,

$$\|\mu-\mu_X^e\|^2\leqq I_X^e-J(\mu) \tag{3.22}$$

and

$$(c_k-\gamma^e_{X_k})\, x_k\leqq a\|\mu-\mu_X^e\|, \tag{3.23}$$

where

$$a=\begin{cases} \sqrt{\|\mu_X^e\|^2+\dfrac{m^- x^2}{(\inf_{G_0} g)^2}} & \textit{if } (\mathrm{a}_1) \textit{ is assumed,} \\ \|\mu_X^e\| \sqrt{1+\dfrac{m^-}{V_e(X)}} & \textit{if } (\mathrm{a}_2)_e \textit{ is assumed,} \\ \|\mu_X^e\| & \textit{if } \Phi \geqq 0 \textit{ on each } G_j\times G_k,\ j\neq k. \end{cases}$$

PROOF. Let $0<\varepsilon<V_e(X)$ and set

$$D_\varepsilon^{(k)}=\{P\in G_k;\ U^\mu(P)-f(P)>(c_k-\varepsilon)g(P)\}.$$

This is an open set and $V_e(X_k-D_\varepsilon^{(k)})=\infty$. We take $f^++|c_k|g$ for h in Lemma 3.11 and choose an open set $G_\varepsilon^{(k)}\supset X_k-D_\varepsilon^{(k)}$ having the property required there. First we assume (a_1). Then, the total mass of measure of $\mathscr{E}_{\bigcup_{k=1}^{n} G_k}(g, x)$ is bounded by $a_0=(\inf_{G_0} g)^{-1}x$. We require that $G_\varepsilon^{(k)}$ satisfies $G_\varepsilon^{(k)}\subset G_k$ and

$$V_i(G_\varepsilon^{(k)})>\max\left\{\left(\frac{a_0\, m^-}{\varepsilon}\right)^2,\ \left(\frac{m^-\mu(\Omega)}{\varepsilon}\right)^2\right\}.$$

Naturally

$$G_k\supset G_\varepsilon^{(k)}\cup D_\varepsilon^{(k)}\supset X_k.$$

We take an open set G' such that

$$X=\bigcup_{k=1}^{n} X_k\subset G'\subset\bigcup_{k=1}^{n}(G_\varepsilon^{(k)}\cup D_\varepsilon^{(k)}),\qquad |I_X^e-I_{G'}^i|<\varepsilon,\qquad \|\mu_X^e-\mu_{G'}^i\|<\varepsilon$$

and

$$|\gamma_k-\gamma_{G'_k}^i|<\varepsilon \qquad \text{for each } k,$$

and take a compact set $K\subset G'$ such that

$$|I_{G'}^i-I_K|<\varepsilon,\qquad \|\mu_{G'}^i-\mu_K\|<\varepsilon$$

and

$$|\gamma_{G'_k}^i-\gamma_k(\mu_K)|<\varepsilon \qquad \text{for each } k.$$

We denote the restrictions of μ_K to G_k and $G_\varepsilon^{(k)}$ respectively by $\mu_K^{(k)}$ and $\mu_1^{(k)}$ and set $\mu_2^{(k)}=\mu_K^{(k)}-\mu_1^{(k)}$. By the definition of $D_\varepsilon^{(k)}$ we have

$$(c_k-\varepsilon)\langle g,\mu_2^{(k)}\rangle\leqq(\mu,\mu_2^{(k)})-\langle f,\mu_2^{(k)}\rangle.$$

If $(\mu,\mu_1^{(k)})<0$, then by Lemma 3.9 it follows that

$$(\mu,\ \mu_1^{(k)}) \geqq -m^-\ \mu(\Omega)\ \mu_1^{(k)}(\Omega) \geqq \frac{-m^-\ \mu(\Omega)\|\mu_1^{(k)}\|}{\sqrt{V_i(G_\varepsilon^{(k)})}} \geqq -\varepsilon\|\mu_1^{(k)}\|.$$

This is evidently true if $(\mu,\ \mu_1^{(k)}) \geqq 0$. Hence in any case

$$\begin{aligned}(\mu,\ \mu_K^{(k)}) - \langle f,\ \mu_K^{(k)}\rangle - x_k\,c_k &\geqq (\mu,\ \mu_1^{(k)}) - \langle f,\ \mu_1^{(k)}\rangle - c_k\langle g,\ \mu_1^{(k)}\rangle - \varepsilon\langle g,\ \mu_2^{(k)}\rangle \\ &\geqq -\langle f^+ + |c_k|\,g,\ \mu_1^{(k)}\rangle - \varepsilon x_k - \varepsilon\|\mu_1^{(k)}\|.\end{aligned}$$

By Lemma 3.11

$$\langle f^+ + |c_k|\,g,\ \mu_1^{(k)}\rangle \leqq \varepsilon\|\mu_1^{(k)}\|$$

and hence

$$(3.24) \qquad (\mu,\ \mu_K^{(k)}) - \langle f,\ \mu_K^{(k)}\rangle - x_k\,c_k \geqq -2\varepsilon\|\mu_1^{(k)}\| - \varepsilon x_k.$$

We have

$$\begin{aligned}\|\mu_K\|^2 = \|\mu_1^{(k)}\|^2 + (\mu_K + \mu_1^{(k)},\ \mu_K - \mu_1^{(k)}) &\geqq \|\mu_1^{(k)}\|^2 - m^-\ \mu_K^2(\Omega) \\ &\geqq \|\mu_1^{(k)}\|^2 - m^-\ a_0^2\end{aligned}$$

and

$$\|\mu_1^{(k)}\| \leqq \sqrt{\|\mu_K\|^2 + m^-\ a_0^2} \leqq \sqrt{(\|\mu_X^e\| + 2\varepsilon)^2 + m^-\ a_0^2}.$$

We shall denote the last quantity by α. From (3.24) and from

$$-\langle f,\ \mu_K^{(k)}\rangle = x_k\,\gamma_k(\mu_K) - (\mu_K,\ \mu_K^{(k)}),$$

we obtain

$$\{\gamma_k(\mu_K) - c_k\}\,x_k - (\mu_K - \mu,\ \mu_K^{(k)}) \geqq -2\varepsilon\alpha - \varepsilon x_k$$

and

$$(\mu,\ \mu_K) - 2\langle f,\ \mu_K\rangle - \sum_{k=1}^n x_k\,c_k \geqq -2n\varepsilon\alpha - \varepsilon\sum_{k=1}^n x_k.$$

We have

$$\begin{aligned}0 \leqq \|\mu - \mu_K\|^2 &\leqq (\mu,\ \mu) + (\mu_K,\ \mu_K) - 2\langle f,\ \mu_K\rangle - 2\sum_{k=1}^n x_k\,c_k + 2n\varepsilon\alpha + \varepsilon\sum_{k=1}^n x_k \\ &= I_K - J(\mu) + 2n\varepsilon\alpha + \varepsilon\sum_{k=1}^n x_k < I_X^e - J(\mu) + 2\varepsilon + 2n\varepsilon\alpha + \varepsilon\sum_{k=1}^n x_k\end{aligned}$$

and

$$\{c_k - \gamma_k(\mu_K)\}\,x_k \leqq (\mu - \mu_K,\ \mu_K^{(k)}) + 2\varepsilon\alpha + \varepsilon x_k.$$

It follows that $J(\mu) \leqq I_X^e$. If $(\mu,\ \mu_X^e)$ is defined,

$$\|\mu - \mu_X^e\| \leqq \|\mu - \mu_K\| + \|\mu_K - \mu_G^i\| + \|\mu_G^i - \mu_X^e\|$$

$$< \sqrt{I_X^e - J(\mu) + 2\varepsilon + 2n\varepsilon\alpha + \varepsilon \sum_{k=1}^{n} x_k} + 2\varepsilon$$

and

$$\{c_k - \gamma_k(\mu_K)\}\, x_k \leqq \|\mu - \mu_K\|\ \|\mu_K^{(k)}\| + 2\varepsilon\alpha + \varepsilon x_k.$$

Since $\|\mu_K^{(k)}\|^2 \leqq \|\mu_K\|^2 + m^- a_0^2 \leqq \alpha^2$,

$$(c_k - \gamma_{X_k}^e - 2\varepsilon) x_k \leqq (\|\mu - \mu_X^e\| + 2\varepsilon)\, \alpha + 2\varepsilon\alpha + \varepsilon x_k.$$

By letting $\varepsilon \to 0$ we conclude (3.22) and (3.23).

Next we assume $(a_2)_e$. We require that $V_i(\bigcup_{k=1}^{n} G_k) > V_e(X) - \varepsilon$ and that $G_\varepsilon^{(k)}$ satisfies $G_\varepsilon^{(k)} \subset G_k$ and

$$V_i(G_\varepsilon^{(k)}) \geqq \left(\frac{m^- \mu(\Omega)}{\varepsilon}\right)^2.$$

We use the same μ_K and $\mu_1^{(k)}$ as above and see that both $\|\mu_1^{(k)}\|$ and $\|\mu_K^{(k)}\|$ are dominated by

$$\sqrt{1 + \frac{m^-}{V_i(\bigcup_{k=1}^{n} G_k)}}\, \|\mu_K\|.$$

Denoting $\sqrt{1 + m^-(V_e(X) - \varepsilon)^{-1}}\, (\|\mu_X^e\| + 2\varepsilon)$ by α this time, we have

$$(c_k - \gamma_{X_k}^e - 2\varepsilon)\, x_k \leqq (\|\mu - \mu_X^e\| + 2\varepsilon)\, \alpha + 2\varepsilon\alpha + \varepsilon x_k.$$

By letting $\varepsilon \to 0$ we have (3.23) with $a = \|\mu_X^e\|\ \sqrt{1 + m^-(V_e(X))^{-1}}$. We obtain (3.22) in the same way as above.

If the kernel is nonnegative in each $G_j \times G_k$, $j \neq k$, then $\|\mu_1^{(k)}\| \leqq \|\mu_K^{(k)}\| \leqq \|\mu_K\|$ and (3.22) and (3.23) are concluded similarly.

Corollary. *If*

$$I_X^e = 2 \sum_{k=1}^{n} x_k\, c_k - (\mu, \mu),$$

then $c_k = \gamma_{X_k}^e$ *for each* k *and* $\mu_{G^{(m)}}^i$ *converges strongly to* μ, *where* $X \subset G^{(m)} \subset G_0$ *and* $I_{G^{(m)}}^i$ *tends to* I_X^e. *If, furthermore,* (μ, μ_X^e) *is defined, then* $\|\mu - \mu_X^e\| = 0$.

Proof. We can conclude the strong convergence of $\mu_{G^{(m)}}^i$ to μ in view of the inequality

$$\|\mu - \mu_K\|^2 < \varepsilon\,(2 + 2n\alpha + \sum_{k=1}^{n} x_k)$$

in the above proof. The equality $c_k = \gamma_{X_k}^e$ follows from the identity $\sum_{k=1}^{n} x_k\, c_k$

$=\sum_{k=1}^{n} x_k \gamma^e_{X_k} = I^e_X + \|\mu^e_X\|^2$ if $c_k \leqq \gamma^e_{X_k}$. This is seen by

$$(c_k - \gamma^e_{X_k} - 2\varepsilon) x_k \leqq \|\mu - \mu_K\| \, \|\mu^{(k)}_K\| + \varepsilon(2\alpha + x_k) \leqq \sqrt{2\varepsilon + 2n\varepsilon\alpha + \varepsilon \sum_{k=1}^{n} x_k \, \alpha} + \varepsilon(2\alpha + x_k).$$

If (μ, μ^e_X) is defined, $\|\mu - \mu^e_X\| = 0$ because μ and μ^e_X are strong limits of the same sequence $\{\mu^i_{G^{(m)}}\}$.

Next we prove

THEOREM 3.20. *Let $\{A^{(m)}\}$ be a sequence of sets of $\mathfrak{A}$ such that each $A^{(m)}$ is decomposed into $\mathfrak{A}$-measurable sets $A^{(m)}_1, \cdots, A^{(m)}_n$ with the property that each $A^{(m)}_k$ increases to a set A_k as $m \to \infty$ and $A_1, \cdots, A_n$ are Φ-separate. Let X be an arbitrary set, $f(P) < \infty$ be an upper semicontinuous function on $A \cap X$ and $g(P)$ be a positive continuous function on $A \cap X$, where $A = \bigcup_m A^{(m)}$. Assume that every strong Cauchy net in $\mathscr{E}_{A \cap X}(g, x)$ is strongly convergent and that $(\mu^i_{A^{(m)} \cap X}, \mu^i_{A^{(p)} \cap X})$ is defined for any m and p provided both $I^i_{A^{(m)} \cap X}$ and $I^i_{A^{(p)} \cap X}$ are finite. If $\lim_{m \to \infty} I^i_{A^{(m)} \cap X} < \infty$, then $I^i_{A^{(m)} \cap X}$ tends to $I^i_{A \cap X}$. If $I^i_{A \cap X}$ is finite and if $(\mu^i_{A^{(m)} \cap X}, \mu^i_{A \cap X})$ is defined for each m, $\mu^i_{A^{(m)} \cap X}$ converges strongly to $\mu^i_{A \cap X}$ and each $\gamma^i_{A^{(m)}_k \cap X}$ tends to $\gamma^i_{A_k \cap X}$.*

Next let G_0 be an open set in Ω such that the kernel is bounded from below on $G_0 \times G_0$, assume that the above $f(P)$ and $g(P)$ are defined in G_0, and let $\{X^{(m)}\}$ be a sequence of subsets of G_0 increasing to X which consists of $X_1, \cdots, X_n$ separable by Φ-separate open subsets $G_1, \cdots, G_n$ of G_0. Assume that every strong Cauchy net in $\mathscr{E}^i_{G_0}(g, x)$ is strongly convergent, that $(\mu^i_G, \mu^i_{G'})$ is defined for any open subsets G and G' of G_0 such that I^i_G and $I^i_{G'}$ are finite and that $(\mu^e_{X^{(m)}}, \mu^e_{X^{(p)}})$ is defined for any m and p provided $I^e_{X^{(m)}}$ and $I^e_{X^{(p)}}$ are finite. Assume one or both of

$(\mathrm{b}_1)^*$ *Every open subset of G_0 is a K_σ-set,*

$(\mathrm{b}_2)^*$ *G_0 is a countable union of relatively compact open sets, $\Delta_\infty = \{P; \Phi(P, P) = \infty\}$ is closed and, for every point $P \in G_0$ and every neighborhood N_P of P, the kernel is bounded from above on $\{P\} \times (G_0 - N_P)$,*

and, unless the kernel is nonnegative on each $G_j \times G_k$, $j \neq k$, assume also one or both of

(a_1) *$g(P)$ has a positive lower bound on G_0,*

$(\mathrm{a}_2)'_e$ *$V_e(X) > 0$ and $f(1+g)^{-1}$ is bounded from above on G_0.*

Then $I^e_{X^{(m)}}$ tends to I^e_X. If I^e_X is finite and if $(\mu^e_{X^{(m)}}, \mu^e_X)$ is defined for each m, $\mu^e_{X^{(m)}}$ converges strongly to μ^e_X and each $\gamma^e_{X^{(m)}_k}$ tends to $\gamma^e_{X_k}$.

PROOF. First we assume that $\lim_{m \to \infty} I^i_{A^{(m)} \cap X}$ is finite. We may assume that all $I^i_{A^{(m)} \cap X}$ are finite. For $m < p$ we have $I^i_{A^{(m)} \cap X} \geqq I^i_{A^{(p)} \cap X}$ and

$$\|\mu^i_{A^{(m)} \cap X} - \mu^i_{A^{(p)} \cap X}\|^2 \leqq I^i_{A^{(m)} \cap X} - I^i_{A^{(p)} \cap X}$$

by Lemma 3.2. As $m, p \to \infty$, $I^i_{A^{(m)} \cap X} - I^i_{A^{(p)} \cap X}$ tends to 0 because of the assumption that $\lim_{m \to \infty} I^i_{A^{(m)} \cap X}$ is finite. Consequently $\{\mu^i_{A^{(m)} \cap X}\}$ form a Cauchy sequence and the existence of its strong limit μ_0 is concluded. We choose a subsequence $\{m_q\}$ such that $\lim_{q\to\infty} \gamma^i_{A_k^{(m_q)}\cap X}$ exists for each k. Denoting this by γ_k, we have by (3.10) and Theorem 1.16

$$U^{\mu_0}(P)-f(P)\geqq\gamma_k\, g(P) \qquad \text{p.p.p. on } A_k\cap X.$$

From this we see as in the proof of Theorem 3.5 that each γ_k is finite. It follows that

$$2\sum_{k=1}^n x_k\, \gamma^i_{A_k^{(m)}\cap X}=I^i_{A^{(m)}\cap X}+(\mu^i_{A^{(m)}\cap X},\ \mu^i_{A^{(m)}\cap X})\to 2\sum_{k=1}^n x_k\, \gamma_k=\lim_{m\to\infty} I^i_{A^{(m)}\cap X}+(\mu_0,\ \mu_0).$$

By Lemm 3.2 we have

$$I^i_{A\cap X}\geqq 2\sum_{k=1}^n x_k\, \gamma_k-(\mu_0,\ \mu_0)=\lim_{m\to\infty} I^i_{A^{(m)}\cap X}.$$

On the other hand, $I^i_{A^{(m)}\cap X}\geqq I^i_{A\cap X}$ and hence $I^i_{A\cap X}=\lim_{m\to\infty} I^i_{A^{(m)}\cap X}$. On account of Theorem 3.5 it follows that $\gamma_k=\gamma^i_{A_k\cap X}$ for each k and that, if $\mu^{(m)}$ converges strongly to $\mu^i_{A\cap X}$ and $S_{\mu^{(m)}}\subset A\cap X$, then $\mu^{(m)}$ converges strongly to μ_0. Since $\mu^i_{A^{(m)}\cap X}$ converges strongly to μ_0, it is derived that $\mu^i_{A^{(m)}\cap X}$ converges strongly to $\mu^i_{A\cap X}$. We conclude also that $\gamma^i_{A_k^{(m)}\cap X}$ tends to $\gamma^i_{A_k\cap X}$ for each k.

If $\lim_{m\to\infty} I^i_{A^{(m)}\cap X}=-\infty$, $I^i_{A^{(m)}\cap X}\geqq I^i_{A\cap X}=-\infty$ and $\lim_{m\to\infty} I^i_{A^{(m)}\cap X}=I^i_{A\cap X}$.

The latter half of the theorem is inferred if we use Lemma 3.7, Theorem 3.16, Lemma 3.12 and its corollary, and Corollary of Theorem 3.19.

Remark 1. If $I^i_{A^{(m)}\cap X}$ tends to ∞ then $I^i_{A\cap X}=\infty$ in a special case where Corollary of Theorem 3.17 applies.

Remark 2. If the kernel is consistent, we may replace respectively $(\mathrm{b}_1)^*$ and $(\mathrm{b}_2)^*$ by

(b_1) Every open subset of G_0 is an F_σ-set,

(b_2) Δ_∞ is closed and, for every point $P\in G_0$ and for every neighborhood N_P of P, the kernel is bounded from above on $\{P\}\times(G_0-N_P)$.

See the remark given to Theorem 3.16.

We might propose next to prove the corresponding results in case $X^{(m)}$ decreases to X. However, it is not true in general. For instance, we consider the decreasing sequence $\{G^{(m)}\}: G^{(m)}=\{1<\overline{OP}<1+1/m\}$ in the euclidean 3-space. The Newtonian capacity of $G^{(m)}$ is equal to $1/V_i(G^{(m)})=1+1/m$. But $\bigcap_m G^{(m)}$ is an empty set and its capacity is 0. Thus $\lim_{m\to\infty}(1+1/m)=1\neq 0$ and a counter-example is given.

We can prove only

Theorem 3.21. *Consider a kernel which may not be of positive type. Let* $K_1, \cdots, K_n$ *be mutually disjoint compact sets and, for each* k, $\{K_k^{(m)}\}$ *be a sequence* *of compact sets decreasing to K_k. Set $K=\bigcup_{k=1}^{n} K_k$ and $K^{(m)}=\bigcup_{k=1}^{n} K_k^{(m)}$. Let $f(P)<\infty$ be an upper semicontinuous function on $K^{(1)}$ and $g(P)$ be a positive continuous function on $K^{(1)}$. Then $I_{K^{(m)}}$ tends to I_K. Assume next that the kernel is of positive type and that every strong Cauchy net in $\mathscr{E}_{K^{(1)}}(g, x)$ is strongly convergent. Then if I_K is finite, $\mu_{K^{(m)}}$ converges strongly to μ_K and $\gamma_k(\mu_{K^{(m)}})$ tends to $\gamma_k(\mu_K)$ for each k.*

PROOF. Since $\mu_{K^{(m)}}(\Omega)\leqq x/\min_{K^{(1)}} g$, there is a subnet $T=\{\mu^{(\omega)};\ \omega\in D\}$ of $\{\mu_{K^{(m)}}\}$ which converges vaguely to some measure $\mu_0\in\mathscr{E}_K(g, x)$. Hence $I_K \leqq I(\mu_0)$. It is easily seen that $\lim_{m\to\infty} I_{K^{(m)}}\geqq I(\mu_0)$. On the other hand we have $I_{K^{(m)}}\leqq I_K$ and obtain

$$\lim_{m\to\infty} I_{K^{(m)}}=I(\mu_0)=I_K.$$

Consequently we may write μ_K for μ_0. If the kernel is of positive type and every strong Cauchy net in $\mathscr{E}_{K^{(m)}}(g, x)$ is strongly convergent, then

$$\|\mu_{K^{(m)}}-\mu_{K^{(p)}}\|^2\leqq I_{K^{(m)}}-I_{K^{(p)}}$$

for $m<p$ by Lemma 3.2. If $\lim_{m\to\infty} I_{K^{(m)}}$ is finite, $\{\mu_{K^{(m)}}\}$ form a Cauchy sequence and hence it converges strongly to some measure μ_0'. By Lemma 3.2 again we have

$$\|\mu_K-\mu_0'\|^2\leqq I_K-\lim_{m\to\infty} I_{K^{(m)}}=0.$$

Therefore $\mu_{K^{(m)}}$ converges strongly to μ_K. We shall denote the restrictions of $\mu^{(\omega)}$ and μ_K to $K_k^{(\omega)}$ and to K_k by $\mu_k^{(\omega)}$ and μ_{K_k} respectively; $\mu^{(\omega)}$ is equal to some $\mu_{K^{(m)}}$ and $K^{(\omega)}$ means $K_k^{(m)}$ for this m. We have

$$\varliminf_{\omega}\ x_k\gamma_k(\mu^{(\omega)})\geqq\varliminf_{\omega}\ (\mu^{(\omega)}, \mu_k^{(\omega)})-\varlimsup_{\omega}\ \langle f, \mu_k^{(\omega)}\rangle\geqq(\mu_K, \mu_{K_k})-\langle f, \mu_{K_k}\rangle=x_k\gamma_k(\mu_K).$$

Since

$$2\sum_{k=1}^{n} x_k\,\gamma_k(\mu_K)=I(\mu_K)+(\mu_K, \mu_K)=2\ \lim_{\omega}\ \sum_{k=1}^{n} x_k\,\gamma_k(\mu^{(\omega)}),$$

the equality $\lim_{\omega}\gamma_k(\mu^{(\omega)})=\gamma_k(\mu_K)$ follows for each k. The equality $\lim_{m\to\infty}\gamma_k(\mu_{K^{(m)}})=\gamma_k(\mu_K)$ is concluded by the arbitrariness in choosing a subnet and the uniqueness of $\gamma_k(\mu_K)$.

Finally if $\lim_{m\to\infty} I_{K^{(m)}}=\infty$, $I_{K^{(m)}}\leqq I_K=\infty$.

Under stronger conditions we can prove a similar theorem for closed sets.

THEOREM 3.22. *Let $F^{(m)}$ consist of Φ-separate closed sets $F_1^{(m)}, \cdots, F_n^{(m)}$ such that $F_k^{(m)}$ decreases to F_k as $m\to\infty$ for each k, $f(P)$ be a continuous function with* *compact support on* $F^{(1)}$, *and* $g(P)$ *be a positive continuous function on* $F^{(1)}$. *Assume that the kernel is consistent and nonnegative on each* $F_j^{(1)}\times F_k^{(1)}$, $j\neq k$, *that* $(\mu_{F^{(m)}}^i, \mu_{F^{(p)}}^i)$ *is defined for any* m *and* p *provided* $I_{F^{(m)}}^i$ *and* $I_{F^{(p)}}^i$ *are finite, and that* $0<I_{F_k^{(1)}}^i(g, x_k, f)+V_i^{(g,x_k)}(F_k^{(1)})$ *for each* k. *Then* $I_{F^{(m)}}^i$ *tends to* I_F^i *as* $m\to\infty$, *where* $F=\bigcup_{k=1}^n F_k$. *If* $I_F^i<\infty$ *and if* $(\mu_F^i, \mu_{F^{(m)}}^i)$ *is defined for each* m, $\mu_{F^{(m)}}^i$ *converges strongly to* μ_F^i *and* $\gamma_{F_k^{(m)}}^i$ *tends to* $\gamma_{F_k}^i$ *for each* k.

Proof. If $\lim_{m\to\infty} I_{F^{(m)}}^i$ is finite, $\{\mu_{F^{(m)}}^i\}$ form a Cauchy sequence by Lemma 3.2. A subnet $\{\mu^{(\omega)}\}$ of $\{\mu_{F^{(m)}}^i\}$ converges vaguely to a measure μ_0 supported by F and $\mu_{F^{(m)}}^i$ converges strongly to μ_0 by (i) in p. 296. Since $f(P)$ is continuous on its compact support, $\langle f, \mu^{(\omega)}\rangle$ tends to $\langle f, \mu_0\rangle$. We recall that $I_{F^{(m)}}^i=I(\mu_{F^{(m)}}^i)$ and $\gamma_{F_k^{(m)}}^i=\gamma_k(\mu_{F^{(m)}}^i)$ for each k on account of Theorem 3.7. We observe

$$\lim_{m\to\infty} I_{F^{(m)}}^i=\lim_{m\to\infty}(\mu_{F^{(m)}}^i, \mu_{F^{(m)}}^i)\ -2\lim_{\omega}\langle f, \mu^{(\omega)}\rangle=(\mu_0, \mu_0)-\langle f, \mu_0\rangle=I(\mu_0)$$

and conclude

$$x_k \lim_{\omega} \gamma_k(\mu^{(\omega)})=(\mu_0, \mu_k^{(0)})-\langle f, \mu_k^{(0)}\rangle$$

as in the proof of Theorem 3.2 for each k, where $\mu_k^{(0)}$ is the restriction of μ_0 to F_k. We can infer $U^{\mu_0}(P)-f(P)\leqq\lim_{\omega}\gamma_k(\mu^{(\omega)})g(P)$ on $S_{\mu_k^{(0)}}$ for each k by Theorem 3.2 and Lemma 1.10, and derive $\langle g, \mu_k^{(0)}\rangle=x_k$ like in Theorem 3.7. We can find a sequence $\{\nu^{(p)}\}$ of restrictions of μ_0 to compact sets such that $\langle g, \nu_k^{(p)}\rangle$ tends to x_k for each k and $I(\nu^{(p)})$ tends to $I(\mu_0)$, where $\nu_k^{(p)}$ is the restriction of $\nu^{(p)}$ to F_k. It follows easily that $I_F^i\leqq I(\mu_0)$. Since $I_{F^{(m)}}^i\leqq I_F^i$, we infer $I_F^i=I(\mu_0)=\lim_{m\to\infty} I_{F^{(m)}}^i$. We observe also that μ_0 may be taken for μ_F^i and obtain by Theorem 3.5

$$x_k\,\gamma_{F_k}^i=(\mu_F^i, \mu_{F_k}^i)-\langle f, \mu_{F_k}^i\rangle=(\mu_0, \mu_k^{(0)})-\langle f, \mu_k^{(0)}\rangle=x_k \lim_{\omega}\gamma_k(\mu^{(\omega)}).$$

By the arbitrariness in choosing $\{\mu^{(\omega)}\}$ we conclude $\gamma_{F_k}^i=\lim_{m\to\infty}\gamma_k(\mu_{F^{(m)}}^i)=\lim_{m\to\infty}\gamma_{F_k^{(m)}}^i$.

3.6. Coincidence of $I_X^i(g, x, f)$ and $I_X^e(g, x, f)$.

We shall apply Choquet's method [1; 7] concerning *capacitability*. Let $\mathfrak{H}$ be a class of sets in Ω which is closed under any formations of finite union and countable intersection. Let S be the set of all finite sequences of integers $\geqq 1$: $S=\{s=(n_1, \cdots, n_p)\}$, and $\sum$ be the set of all infinite sequences of integers $\geqq 1$: $\sum=\{\sigma=(n_1, n_2, \cdots)\}$. We write $s<\sigma$ (or $s<s'$) when s is some first section of σ (or of s'). A *determining system* on $\mathfrak{H}$ is defined by an application δ of S into $\mathfrak{H}$: $s\to H_s$ such that $s<s'$ implies $H_s\supset H_{s'}$. We extend δ to $\sum$ by setting $H_\sigma=\bigcap_{s<\sigma} H_s$ for $\sigma\in\sum$. An $\mathfrak{H}$-*Souslinian set* is equal to $H(\delta)=\bigcup_{\sigma} H_\sigma$. One can show that the class of all $\mathfrak{H}$-Souslinian sets is closed under any formations of countable union and countable intersection. Any element of the smallest class of sets with this closed character, containing $\mathfrak{H}$, is called an $\mathfrak{H}$-*Borelian set.* Hence every $\mathfrak{H}$-Borelian set is $\mathfrak{H}$-Souslinian. For further properties of $\mathfrak{H}$-Souslinian sets we refer to Choquet [6] and Sion [1; 2; 3].

Let φ be a real-valued increasing set function defined on the class of all subsets of Ω. Choquet [7] called it an *abstract capacity* on $(\Omega, \mathfrak{H})$ if it satisfies 1) $\varphi(\bigcap_n H_n)=\lim_{n\to\infty} \varphi(H_n)$ whenever $H_n \in \mathfrak{H}$ decreases, and 2) $\varphi(\bigcup_n X_n)=\lim_{n\to\infty} \varphi(X_n)$ whenever X_n increases. A set X is $(\varphi, \mathfrak{H})$-*capacitable* by definition if $\varphi(X) =\sup \varphi(H)$ where $H \in \mathfrak{H}$ and $H \subset X$. He proved that all $\mathfrak{H}$-Souslinian sets are $(\varphi, \mathfrak{H})$-capacitable.

We take $I_X^e(g, x, f)$ for $\varphi(X)$ and the class $\mathfrak{K}$ of all compact sets K for $\mathfrak{H}$. Let us write simply I_X^i, μ_X^i, γ_X^i, etc. for $I_X^i(g, x, f)$, $\mu_X^i(g, x, f)$, $\gamma_X^i(g, x, f)$, etc. respectively. Since $I_K^i=I_K^e$ by Theorem 3.8,

$$\sup_{H\subset X} \varphi(H)=\sup_{K\subset X} I_K^i=I_X^i$$

and the $(\varphi, \mathfrak{K})$-capacitability is equivalent to $I_X^i=I_X^e$. The above requirements 1) and 2) are satisfied on account of Theorems 3.21 and 3.20 respectively. Consequently we can apply Choquet's result. Making use also of Theorem 3.13 and Corollary to Lemma 3.12 we obtain

THEOREM 3.23. *Consider a kernel of positive type. Let G_0 be an open set in Ω such that the kernel is bounded from below on $G_0 \times G_0$, $f(P)<\infty$ be an upper semicontinuous function in G_0 and $g(P)$ be a positive continuous function in G_0. Assume that every strong Cauchy net in $\mathscr{E}_{G_0}^i(g, x)$ is strongly convergent, and that $(\mu_X^e, \mu_{X'}^e)$ is defined for any subsets X and X' of G_0 such that I_X^e and $I_{X'}^e$ are finite. As to conditions* (a_1), $(a_2)_e$, $(b_1)^*$, $(b_2)^*$ *assume the same as in Theorem* 3.20. *Then, for any $\mathfrak{K}$-Souslinian set A in G_0, $I_A^i=I_A^e$. If this value is finite and (μ_A^i, μ_A^e) is defined, $\|\mu_A^i-\mu_A^e\|=0$ and $\gamma_{A_k}^i=\gamma_{A_k}^e$ for each k.*

If we take Remark 2 given to Theorem 3.20 into consideration, we have

THEOREM 3.24. *Consider a consistent kernel and let G_0, $f(P)$ and $g(P)$ be as in the preceding theorem. Assume that $(\mu_X^e, \mu_{X'}^e)$ is defined for any subsets X and X' of G_0 such that I_X^e and $I_{X'}^e$ are finite, and assume one or both of*

(b_1) *Every open set is an F_σ-set,*

(b_2) *Δ_∞ is closed and, for every point P and for every neighborhood N_P of P, the kernel is bounded from above on $\{P\}\times(\Omega-N_P)$.*

Then we obtain the same conclusions as in the preceding theorem.

Another possibility of $\mathfrak{H}$ is the class $\mathfrak{F}_0$ of all closed sets $F=\bigcup_{k=1}^n F_k$ with positive $I_{F_1}^e, \cdots, I_{F_n}^e$. It seems that one can show $I_A^i=I_A^e$ for every $\mathfrak{F}_0$-Souslinian subset A of G_0 by the aid of Corollary of Theorem 3.15, Theorem 3.22 and Remark 2 given to Theorem 3.20. However, we have no handy condition under which $F_1, F_2 \in \mathfrak{F}_0$ implies $F_1 \cup F_2 \in \mathfrak{F}_0$. It is assured if we limit ourselves to the case where the kernel is nonnegative and of positive type, $n=1$ and $f\leqq 0$; it will be seen in Theorem 3.27. We can now state

THEOREM 3.25. *Consider a nonnegative consistent kernel in $\Omega\times\Omega$ and let G_0 be an open set in Ω. Let $f(P)\leqq 0$ be a continuous function with compact support in G_0^a and $g(P)$ be a positive continuous function in G_0^a. Assume one or both of* (b_1) *and* (b_2). *Then, for every $\mathfrak{F}_0$-Souslinian subset A of G_0, $I_A^i=I_A^e$. If this value is finite, $\|\mu_A^i-\mu_A^e\|=0$ and $\gamma_{A_k}^i=\gamma_{A_k}^e$ for each k.*

In Theorem 4.5 of Fuglede [1] it is stated that every σ-finite Borel set is capacitable. A *σ-finite set* means a set covered by a countable number of sets each of which is of finite outer capacity. In his paper, our (b_1) and the normalcy of the space are assumed; under (b_1) the $\mathfrak{F}$-Borel class, the $\mathfrak{F}$-Borelian class, the $\mathfrak{G}$-Borel class and the $\mathfrak{G}$-Borelian class all coincide with each other, where $\mathfrak{F}$ ($\mathfrak{G}$ resp.) is the class of all closed (open resp.) sets and the $\mathfrak{F}$-Borel ($\mathfrak{G}$-Borel) class is the smallest class which contains $\mathfrak{F}$ ($\mathfrak{G}$ resp.) and is closed under any formations of difference and countable union. Fuglede [1] gave an example (Example 10, § 8.3) which shows the necessity of σ-finiteness in his theorem.

The following result is due to Fuglede:[42] A set A is $\mathfrak{F}_0$-Souslinian if and only if A is $\mathfrak{F}$-Souslinian and covered by $\bigcup_n F_n$, $F_n\in\mathfrak{F}_0$. This shows that our theorem is an extension of his Theorem 4.5.

Finally in this section we prove

THEOREM 3.26. *There are a K_σ-set K_σ and a G_δ-set G_δ such that $K_\sigma\subset X\subset G_\delta$ and*

$$I_{K_\sigma}^i=I_X^i \qquad \textit{and} \qquad I_{G_\delta}^j=I_X^e.$$

PROOF. We shall prove only the first equality; the second can be proved in a similar fashion. There is a sequence $\{K^{(m)}\}$ of compact sets such that $K^{(m)}\subset X$ and

$$\lim_{m\to\infty} I_{K^{(m)}}^i=I_X^i.$$

If we set $\bigcup_m K^{(m)}=K_\sigma$, then $K_\sigma\subset X$, $I_{K^{(m)}}^i\geqq I_{K_\sigma}^i\geqq I_X^i$ and the equality $I_{K_\sigma}^i=I_X^i$ follows.

However, even for the Newtonian capacity, above $G_\delta-K_\sigma$ may not be of capacity zero. For instance, if X is a ball, if K_σ is the inside of the ball and if G_δ is the closed ball, then the capacity of $G_\delta-K_\sigma$ is equal to the capacity of the ball. The question then arises for any X with $I_X^i=I_X^e$ whether we can find K_σ and G_δ such that $K_\sigma\subset X\subset G_\delta$ and $I_{G_\delta-K_\sigma}^e=\infty$. The author does not know the answer even for the Newtonian capacity.

3.7. Inequalities for $I_X^i(g, x, f)$ and $I_X^e(g, x, f)$.

Let $f(P)<\infty$ be upper semicontinuous on $X\subset\Omega$, $g(P)$ be positive con-

42) This with a proof was informed to the author in a letter dated May 15, 1961. According to a later letter, Fuglede [2] will contain the proof.

tinuous on X and x be a positive number. If X is an open set,

$$\Psi(P, Q)=x\frac{\Phi(P, Q)}{g(P)g(Q)}-\frac{f(P)}{g(P)}-\frac{f(Q)}{g(Q)}$$

can be taken for a kernel in X. In the general case it does not belong to the range of kernels which we are concerned with.

We still assume that $\Phi(P, Q)$ is symmetric and shall prove inequalities similar to (1.1) and (1.2).

THEOREM 3.27. *Let* $\{A^{(p)}\}$ *be* $\mathfrak{A}$*-measurable sets in* Ω, X *be any set in* Ω, $f(P)<\infty$ *be an upper semicontinuous function defined on* $Y=\bigcup_p A^{(p)}\cap X$, $g(P)$ *be a positive continuous function defined on* Y *and* $x>0$. *If* $\Psi(P, Q)\geqq m$ *on* $Y\times Y$, *then*

$$\frac{1}{I_Y^i(g, x, f)-xm}\leqq\sum_p\frac{1}{I_{A^{(p)}\cap X}^i(g, x, f)-xm}; \tag{3.25}$$

here we do not talk about components of Y *and* $A^{(p)}\cap X$, *namely* 1*-dimensional problems are considered.*

PROOF. We shall write simply I_Y^i and so on. We may assume that $\mathscr{E}_{A^{(p)}\cap X}(g, x)\neq\{0\}$ for each p. Let $\mu\in\mathscr{E}_Y(g, x)$. For each p we choose a compact set $K^{(p)}\subset A^{(p)}\cap X$ such that $\int_{A^{(p)}\cap X-K^{(p)}}gd\mu<\varepsilon/2^p$ and denote the restriction of μ to $K^{(p)}$ by $\mu^{(p)}$. It follows that

$$\begin{aligned} x\sup_{P\in S_\mu}\frac{U^\mu(P)-f(P)}{g(P)}-\langle f, \mu\rangle-xm&=\sup_{P\in S_\mu}\int g(Q)\{\Psi(P, Q)-m\}\,d\mu(Q)\\ &\geqq\sup_{P\in S_{\mu^{(p)}}}\int_{K^{(p)}}g(Q)\{\Psi(P, Q)-m\}\,d\mu(Q)\\ &=\sup_{P\in S_{\mu^{(p)}}}\frac{xU^{\mu^{(p)}}(P)-\langle g, \mu^{(p)}\rangle f(P)}{g(P)}-\langle f, \mu^{(p)}\rangle-m\langle g, \mu^{(p)}\rangle. \end{aligned}$$

If $\mu^{(p)}\not\equiv 0$, $x\mu^{(p)}\langle g, \mu^{(p)}\rangle^{-1}\in\mathscr{E}_{A^{(p)}\cap X}(g, x)$ and

$$I_{A^{(p)}\cap X}^i\leqq x\sup_{S_{\mu^{(p)}}}\frac{xU^{\mu^{(p)}}-\langle g, \mu^{(p)}\rangle f}{\langle g, \mu^{(p)}\rangle g}-x\frac{\langle f, \mu^{(p)}\rangle}{\langle g, \mu^{(p)}\rangle}$$

by Corollary 1 of Theorem 2.7. Hence

$$x\sup_{S_{\mu^{(p)}}}\frac{xU^{\mu^{(p)}}-\langle g, \mu^{(p)}\rangle f}{g}-x\langle f, \mu^{(p)}\rangle\geqq\langle g, \mu^{(p)}\rangle I_{A^{(p)}\cap X}^i.$$

This is true even if $\mu^{(p)}\equiv 0$. Therefore

$$x=\langle g, \mu\rangle\leqq\sum_p\langle g, \mu^{(p)}\rangle+\varepsilon$$

$$\leqq x\left(x \sup_{S_\mu} \frac{U^\mu - f}{g} - \langle f, \mu\rangle - xm\right) \sum_p \frac{1}{I^i_{A^{(p)}\cap X} - xm} + \varepsilon.$$

On account of Corollary 1 of Theorem 2.7 again, we can choose μ so that the right side is arbitrarily close to

$$x\,(I^i_Y - xm) \sum_p \frac{1}{I^i_{A^{(p)}\cap X} - xm} + \varepsilon.$$

Therefore

$$\frac{1}{I^i_Y - xm} \leqq \sum_p \frac{1}{I^i_{A^{(p)}\cap X} - xm} + \frac{\varepsilon}{x(I^i_Y - xm)},$$

whence (3.25) is derived.

By making use of (3.25), or by regarding $\Psi(P, Q)$ as a kernel in $G_0 \times G_0$ and applying (1.2) we can establish easily

THEOREM 3.28. *Let G_0 be an open set in Ω, $\{X^{(p)}\}$ be a sequence of sets in G_0, $f(P) < \infty$ be an upper semicontinuous function defined in G_0, $g(P)$ be a positive continuous function in G_0 and $x > 0$. If $\Psi(P, Q) \geqq m$ on $G_0 \times G_0$, then*

$$\frac{1}{I^e_{\bigcup_p X^{(p)}}(g, x, f) - xm} \leqq \sum_p \frac{1}{I^e_{X^{(p)}}(g, x, f) - xm}.$$

3.8. Change of conditions.

We studied how I_K and $\gamma(\mu_K)$ change as $f(P)$ or $g(P)$ or both change in Chapter II. In this section we shall see how I^i_X and I^e_X change; we shall consider symmetric kernels.

First we prove

THEOREM 3.29. *Let $\Phi(P, Q)$ be a symmetric kernel, X be a set in Ω on whose product Φ is bounded from below, $X_1, \cdots, X_n$ be a decomposition of X into Φ-separate sets such that $V_i(X_k) < \infty$ for each k, $f(P)$ be a finite-valued upper semicontinuous function on X and $g(P)$ be a positive continuous function on X. Let $\{f_p(P)\}$ be a sequence of upper semicontinuous functions on X which tends uniformly to $f(P)$ and $\{g_p(P)\}$ be a sequence of positive continuous functions on X which tends uniformly to $g(P)$. Assume one or both of*

(a$_2$)$'_i$ *$V_i(X) > 0$, and $f(1+g)^{-1}$ is bounded from above,*

(a$_3$) *$1/g$ and $f(1+g)^{-1}$ are bounded from above.*[43]

If $I^i_X(g, x, f)$ is finite for $x = (x_1, \cdots, x_n)$, $x_1 \geqq 0, \cdots, x_n \geqq 0$, then $I^i_X(g_p, x, f_p)$ tends to $I^i_X(g, x, f)$ as $p \to \infty$.

PROOF. We may assume that $x_1 > 0, \cdots, x_n > 0$. Let $K \subset X$ be a compact set for which $\mu_K(g, x, f)$ exists. According to Theorem 2.11

43) It amounts to assume that $1/g$ and f/g are bounded from above.

$$\lim_{p\to\infty} I_K^i(g_p, x, f_p)=I_K^i(g, x, f).$$

Since

$$I_X^i(g_p, x, f_p)\leqq I_K^i(g_p, x, f_p)$$

for each p, we have

$$\overline{\lim_{p\to\infty}}\ I_X^i(g_p, x, f_p)\leqq I_X^i(g, x, f). \tag{3.26}$$

Consequently we may suppose that $\{I_X^i(g_p, x, f_p)\}$ are bounded from above, say

$$I_X^i(g_p, x, f_p)\leqq M.$$

By $(a_2)_i'$ and the uniform convergence of $\{f_p\}$ and $\{g_p\}$ there is a finite number M' such that $f_p(P)\leqq M'(1+g_p(P))$ for large p. We have

$$\mu^2(\Omega)\, V_i(X)-2M'(\mu(\Omega)+x)\leqq M+1$$

for any $\mu\in\mathscr{E}_X(g_p, x)$ giving $I_{f_p}(\mu)=(\mu, \mu)-2\langle f_p, \mu\rangle\leqq M+1$. Therefore $\mu(\Omega)$ is bounded and hence $I_X^i(g_p, x, f_p)\geqq -2M'(\mu(\Omega)+x)$ is bounded from below uniformly with respect to p. It also follows that (μ, μ) is bounded from above because

$$(\mu, \mu)\leqq M+1+2M'(\mu(\Omega)+x).$$

It is seen also that (μ_j, μ_k) is bounded for any j and k, where μ_k means the restriction of μ to X_k. The same facts are true under the assumption of (a_3).

We choose $\mu^{(p)}\in\mathscr{E}_X(g_p, x)$ such that

$$I_{f_p}(\mu^{(p)})\geqq I_X^i(g_p, x, f_p)+\frac{1}{p}$$

and set

$$\bar{\mu}_k^{(p)}=\frac{x_k}{\langle g, \mu_k^{(p)}\rangle}\,\mu_k^{(p)}.$$

The measure $\bar{\mu}^{(p)}=\sum_{k=1}^n \bar{\mu}_k^{(p)}$ belongs to $\mathscr{E}_X(g, x)$ and hence

$$I_X^i(g, x, f)\leqq I_f(\bar{\mu}^{(p)})=(\bar{\mu}^{(p)}, \bar{\mu}^{(p)})-2\langle f, \bar{\mu}^{(p)}\rangle.$$

We shall compute the difference of $I_f(\bar{\mu}^{(p)})$ and $I_{f_p}(\mu^{(p)})$. We have

$$I_f(\bar{\mu}^{(p)})-I_{f_p}(\mu^{(p)})=\sum_{j,k}\left(\frac{x_j\, x_k}{\langle g, \mu_j^{(p)}\rangle\langle g, \mu_k^{(p)}\rangle}-1\right)(\mu_j^{(p)}, \mu_k^{(p)})$$
$$-2\sum_k\left\langle\frac{x_k}{\langle g, \mu_k^{(p)}\rangle}\, f-f_p,\ \mu_k^{(p)}\right\rangle.$$

Since $\{(\mu_j^{(p)}, \mu_k^{(p)})\}$ are bounded and $\langle g, \mu_k^{(p)}\rangle$ tends to $\lim_{p\to\infty} \langle g_p, \mu_k^{(p)}\rangle = x_k$, the first sum of the right side tends to zero. The second sum can be written in the form

$$(3.27)\qquad \sum_k \frac{x_k}{\langle g, \mu_k^{(p)}\rangle}\langle f-f_p, \mu_k^{(p)}\rangle + \sum_k\left(\frac{x_k}{\langle g, \mu_k^{(p)}\rangle}-1\right)\langle f_p, \mu_k^{(p)}\rangle.$$

It is evident that $\{\langle f_p, \mu_k^{(p)}\rangle\}$ are bounded from above. They are bounded from below too because

$$2\sum_k \langle f_p, \mu_k^{(p)}\rangle = 2\langle f_p, \mu^{(p)}\rangle = (\mu^{(p)}, \mu^{(p)}) - I_{f_p}(\mu^{(p)})$$

are bounded. Therefore each term in (3.27) tends to 0 as $p\to\infty$. It is now verified that

$$I_X^i(g, x, f) \leqq \varliminf_{p\to\infty} I_f(\bar{\mu}^{(p)}) = \varliminf_{p\to\infty} I_{f_p}(\mu^{(p)}) = \varliminf_{p\to\infty} I_X^i(g_p, x, f_p).$$

Combined with (3.26), this proves the theorem.

If we assume the continuity principle, we can prove

THEOREM 3.30. *If X is relatively compact in Ω and the continuity principle is satisfied, we can replace* (a_3) *in Theorem* 3.29 *by*

(a_1) *$g(P)$ has a positive lower bound on X.*

PROOF. As in Theorem 3. 29 we take $\mu^{(p)}\in\mathscr{E}_X(g_p, x, f_p)$ such that $I_{f_p}(\mu^{(p)}) \leqq \max(-p, I_X^i(g_p, x, f_p)+1/p)$. By condition ($a_1$) $\{\mu^{(p)}(\Omega)\}$ are bounded, because $g_p(P)$ has a common positive lower bound for large p. It will be sufficient to show that $\{(\mu_j^{(p)}, \mu_k^{(p)})\}$ and $\{\langle f_p, \mu_k^{(p)}\rangle\}$ are bounded.

For each k there is a measure $\nu_k\in\mathscr{E}_{X_k}(g, 1)$ which gives a continuous potential in Ω and for which $\langle f, \nu_k\rangle$ is finite. Since $\langle g, \mu_k^{(p)}\rangle$ tends to x_k as $p\to\infty$, we may suppose that $\langle g, \mu_k^{(p)}\rangle < 2x_k$. With suitable numbers $\{t_k^{(p)}\}$, $\mu^{(p)}/2+\sum_{k=1}^n t_k^{(p)}\nu_k$ belongs to $\mathscr{E}_X(g, x)$ and

$$-\infty < I_X^i(g, x, f) \leqq I_f\left(\frac{\mu^{(p)}}{2}+\sum_{k=1}^n t_k^{(p)}\nu_k\right) = \frac{1}{2} I_f(\mu^{(p)}) - \frac{1}{4}(\mu^{(p)}, \mu^{(p)})$$
$$+\sum_{k=1}^n t_k^{(p)}(\mu^{(p)}, \nu_k) + \sum_{j,k=1}^n t_j^{(p)} t_k^{(p)}(\nu_j, \nu_k) - 2\sum_{k=1}^n t_k^{(p)}\langle f, \nu_k\rangle.$$

We choose a vaguely convergent subnet $\{\mu^{(\omega)};\ \omega\in D\}$ of $\{\mu^{(p)}\}$ such that $\lim_\omega I_f(\mu^{(\omega)}) = \varliminf_{p\to\infty} I_f(\mu^{(p)})$. Since $t_k^{(p)}$ converges to a limit for each k, the last three terms tend to finite limits as $\mu^{(p)}$ varies along the subnet. We take into consideration the fact that $(\mu^{(p)}, \mu^{(p)})$ is bounded from below and see that $I_f(\mu^{(p)})$ is bounded from below. We have assumed that f_p converges uniformly to f. Therefore

$$-\infty<\varliminf_{p\to\infty} I_f(\mu^{(p)})=\varliminf_{p\to\infty} I_{f_p}(\mu^{(p)})=\varliminf_{p\to\infty} I_X^i(g_p, x, f_p).$$

On account of (3.26) it follows that $I_f(\mu^{(p)})$ is bounded.

We shall use the similar reasoning in order to show that $\varliminf_{p\to\infty} I_f(\mu_k^{(p)})>-\infty$ for each k. We have

$$-\infty<I_X^i(g, x, f)\leqq I_f\left(\frac{\mu_1^{(p)}}{2}+t_1^{(p)}\nu_1+\sum_{k=2}^n x_k\nu_k\right)$$

$$=\frac{1}{2}I_f(\mu_1^{(p)})-\frac{1}{4}(\mu_1^{(p)}, \mu_1^{(p)})+(\mu_1^{(p)}, t_1^{(p)}\nu_1+\sum_{k=2}^n x_k\nu_k)$$

$$+(t_1^{(p)}\nu_1+\sum_{k=2}^n x_k\nu_k, t_1^{(p)}\nu_1+\sum_{k=2}^n x_k\nu_k)-2\langle f, t_1^{(p)}\nu_1+\sum_{k=2}^n x_k\nu_k\rangle.$$

From this relation we can infer that $\varliminf_{p\to\infty} I_f(\mu_1^{(p)})>-\infty$. Similarly we see that $\varliminf_{p\to\infty} I_f(\mu_k^{(p)})>-\infty$ for each k, $2\leqq k\leqq n$. It follows that $\{I_f(\mu_k^{(p)})\}$ are bounded because $\{I_f(\mu^{(p)})\}$ are bounded and

$$I_f(\mu^{(p)})=\sum_{k=1}^n I_f(\mu_k^{(p)})+\sum_{\substack{j,k=1\\ j\neq k}}^n(\mu_j^{(p)}, \mu_k^{(p)}).$$

Observing that

$$I_{f_p}(\mu_k^{(p)})=I_f(\mu_k^{(p)})-\langle f-f_p, \mu_k^{(p)}\rangle,$$

we conclude that $\{I_{f_p}(\mu_k^{(p)})\}$ are bounded.

We now set $\nu_k^{(p)}=x_k\nu_k\langle g_p, \nu_k\rangle^{-1}$ and $\nu^{(p)}=\sum_{k=1}^n \nu_k^{(p)}$. This belongs to $\mathscr{E}_X(g_p, x)$ and it holds that

$$I_X^i(g_p, x, f_p)\leqq I_{f_p}\left(\frac{\mu^{(p)}+\nu^{(p)}}{2}\right)$$

$$=\frac{1}{2}I_{f_p}(\mu^{(p)})-\frac{1}{4}(\mu^{(p)}, \mu^{(p)})+\frac{1}{2}(\mu^{(p)}, \nu^{(p)})+\frac{1}{4}(\nu^{(p)}, \nu^{(p)})-\langle f, \nu^{(p)}\rangle$$

$$\leqq\frac{1}{2}I_X^i(g_p, x, f_p)-\frac{1}{4}(\mu^{(p)}, \mu^{(p)})+\frac{1}{2}\sum_{k=1}^n\frac{x_k}{\langle g_p, \nu_k\rangle}(\mu^{(p)}, \nu_k)$$

$$+\frac{1}{4}\sum_{j,k=1}^n\frac{x_j x_k}{\langle g_p, \nu_j\rangle\langle g_p, \nu_k\rangle}(\nu_j, \nu_k)-\sum_{k=1}^n\frac{x_k}{\langle g_p, \nu_k\rangle}\langle f, \nu_k\rangle+\frac{1}{2p}.$$

We choose a vaguely convergent subnet $\{\mu^{(\omega')}; \omega'\in D'\}$ of $\{\mu^{(p)}\}$ such that $\lim_{\omega'}(\mu^{(\omega')}, \mu^{(\omega')})=\varlimsup_{p\to\infty}(\mu^{(p)}, \mu^{(p)})$. We see from the above inequality that $\varlimsup_{p\to\infty}(\mu^{(p)}, \mu^{(p)})$ is a finite number. Since $\{(\mu_j^{(p)}, \mu_k^{(p)})\}$ are bounded from below, these are bounded. We have already seen that $\{I_{f_p}(\mu_k^{(p)})\}$ are bounded. Hence $\{\langle f_p, \mu_k^{(p)}\rangle\}$ are bounded too.

Next we shall be concerned with the outer problem.

THEOREM 3.31. *Let $\Phi(P, Q)$ be a symmetric kernel, G_0 be an open set on whose product Φ is bounded from below, $X_1, \cdots, X_n$ be sets separable by Φ-separate open subsets $G_1^{(0)}, \cdots, G_n^{(0)}$ of G_0, $f(P)$ be a finite-valued upper semicontinuous function on G_0 and $g(P)$ be a positive continuous function on G_0. Let $\{f_p(P)\}$ be a sequence of upper semicontinuous functions on G_0 which tends uniformly to $f(P)$ and $\{g_p(P)\}$ be a sequence of positive continuous functions on G_0 which tends uniformly to $g(P)$. Assume one or both of*

(a_2)$'_e$ *$V_e(X)>0$, and $f(1+g)^{-1}$ is bounded from above,*

(a_3) *$1/g$ and $f(1+g)^{-1}$ are bounded from above.*

If $I_X^e(g, x, f)$ is finite for $x=(x_1, \cdots, x_n)$, $x_1 \geqq 0, \cdots, x_n \geqq 0$, then $I_X^e(g_p, x, f_p)$ tends to $I_X^e(g, x, f)$ as $p \to \infty$.

PROOF. We may assume that $x_1>0, \cdots, x_n>0$ and that $V_i(G_0)>0$. By Theorem 3.29 we know that, for any G such that $X \subset G \subset G_0$, $\lim_{p \to \infty} I_G^i(g_p, x, f_p) = I_G^i(g, x, f)$. Therefore

$$I_X^e(g, x, f) = \sup_{G \supset X} I_G^i(g, x, f) \leqq \varliminf_{p \to \infty} I_X^e(g_p, x, f_p). \tag{3.28}$$

For G, $X \subset G \subset G_0$, we denote by $\mathscr{E}^{(G)}$ the class of measures μ of $\mathscr{E}_G(g, x)$ such that $I_f(\mu) \leqq I_G^i(g, x, f)+1$. It follows that $\mu(\Omega)$, every (μ_j, μ_k) and $\langle f, \mu_k \rangle$ are bounded on $\bigcup_G \mathscr{E}^{(G)}$ for the same reason as in Theorem 3.29, where μ_k means the restriction of μ to G_k. Take any $\mu \in \mathscr{E}^{(G)}$ and set

$$\mu^{(p)} = \sum_{k=1}^n \frac{x_k \mu_k}{\langle g_p, \mu_k \rangle}.$$

This belongs to $\mathscr{E}_G(g_p, x)$ and it holds that

$$I_G^i(g_p, x, f_p) \leqq I_{f_p}(\mu^{(p)}) = \sum_{j,k=1}^n \frac{x_j x_k}{\langle g_p, \mu_j \rangle \langle g_p, \mu_k \rangle} (\mu_j, \mu_k)$$

$$-2 \sum_{k=1}^n \frac{x_k}{\langle g_p, \mu_k \rangle} \langle f, \mu_k \rangle + 2 \sum_{k=1}^n \frac{x_k}{\langle g_p, \mu_k \rangle} \langle f - f_p, \mu_k \rangle.$$

We denote the difference of the last side and $I_f(\mu)$ by $a_p(\mu)$. We see that there is a number ε_p tending to 0 as $p \to \infty$ and satisfying $|a_p(\mu)| \leqq \varepsilon_p$ for any $\mu \in \bigcup_G \mathscr{E}^{(G)}$. Consequently

$$I_G^i(g_p, x, f_p) \leqq I_G^i(g, x, f) + \varepsilon_p \leqq I_X^e(g, x, f) + \varepsilon_p$$

and hence

$$I_X^e(g_p, x, f_p) \leqq I(g, x, f) + \varepsilon_p.$$

This gives

$$\varlimsup_{p\to\infty} I_X^e(g_p, x, f_p) \leqq I_X^e(g, x, f).$$

On account of (3.28) our theorem is now proved.

REMARK. In Theorems 3.29, 3.30 and 3.31 we may allow $-\infty$ to $f(P)$ if all $f_p(P)$ are identical.

3.9. Graphs of $I_X^i(g, x, f)$ and $I_X^e(g, x, f)$.

The continuity of $I_X^i(x)=I_X^i(g, x, f)$ and $I_X^e(x)=I_X^e(g, x, f)$ in $x_1>0, \cdots, x_n>0$ follows from Theorems 3.29, 3.30 and 3.31. We shall show that they are continuous on $x_1 \geqq 0, \cdots, x_n \geqq 0$ under a less general condition. We shall assume that $\Phi(P, Q)$ is symmetric.

THEOREM 3.32. *Let $\Phi(P, Q)$, X, $X_1, \cdots, X_n$ and $g(P)$ be the same as in Theorem 3.29. Let $f(P)<\infty$ be an upper semicontinuous function on X such that $f(P)>-\infty$ on some set $Y_k \subset X_k$ with $V_i(Y_k)<\infty$ for each k. Assume*
(a$_3$) *$1/g(P)$ and $f(1+g)^{-1}$ are bounded from above.*
Then $I_X^i(x)$ is continuous as a function of x in $x_1 \geqq 0, \cdots, x_n \geqq 0$.

PROOF. As the lower envelope of a family of continuous functions $\{I_K^i(x); K \subset X\}$, $I_X^i(x)$ is upper semicontinuous. Let K_0 be any compact subset of X such that $I_{K_0}^i(x)$ is continuous. By (a$_3$) we see that $\mu(\Omega)$, each (μ_j, μ_k) and each $\langle f, \mu_k\rangle$ are bounded for $\mu=\mu_K(x)$ where $K_0 \subset K \subset X$ and $|x| \leqq r$, μ_k being the restriction of μ to K_k; bounds may depend on K_0 and r. Consequently $I_X^i(x)$ is finite at each x. Let $\{x^{(p)}\}$ be a sequence of points in $|x| \leqq r$, tending to $x_0 =(x_1^{(0)}, \cdots, x_n^{(0)})$. We assume that $x_1^{(0)}>0, \cdots, x_m^{(0)}>0$, $x_{m+1}^{(0)}=\cdots=x_n^{(0)}=0$. We take $K_0 \subset K^{(1)} \subset K^{(2)} \subset \cdots$ such that

$$I(\mu_{K^{(p)}}(x^{(p)})) \leqq I_X^i(x^{(p)})+\frac{1}{p}.$$

We denote $\mu_{K^{(p)}}(x^{(p)})$ simply by $\mu^{(p)}$ and define $\nu^{(p)} \in \mathscr{E}_X(g, x_0)$ by setting it equal to $x_k^{(0)}(x_k^{(p)})^{-1}\mu_k^{(p)}$ on $K_k^{(p)}$, $k=1, \cdots, m$, and to zero elsewhere, where $\mu_k^{(p)}$ is the restriction of $\mu^{(p)}$ to $X_k \cap K^{(p)}$. We have

$$I_X^i(x_0) \leqq I(\nu^{(p)})=(\nu^{(p)}, \nu^{(p)})-2\langle f, \nu^{(p)}\rangle.$$

It holds that

$$I(\nu^{(p)})-I(\mu^{(p)})=\sum_{j,k=1}^{m}\left(\frac{x_j^{(0)}}{x_j^{(p)}}\frac{x_k^{(0)}}{x_k^{(p)}}-1\right)(\mu_j^{(p)}, \mu_k^{(p)})-2\sum_{k=1}^{m}\left(\frac{x_k^{(0)}}{x_k^{(p)}}-1\right)\langle f, \mu_k^{(p)}\rangle$$
$$-\sum_{k=m+1}^{n}\{2\sum_{j=1}^{m}(\mu_j^{(p)}, \mu_k^{(p)})+\sum_{j=m+1}^{n}(\mu_j^{(p)}, \mu_k^{(p)})-2\langle f, \mu_k^{(p)}\rangle\}.$$

The first and second sums tend to zero as $p\to\infty$ because $(\mu_j^{(p)}, \mu_k^{(p)})$ and $\langle f, \mu_k^{(p)}\rangle$ are bounded. By our assumption that $g(P)$ has a positive lower bound, $\mu_k^{(p)}$

tends to zero as $p \to \infty$ for each k, $m+1 \leqq k \leqq n$. Since there is $M < \infty$ such that $f \leqq M(1+g)$ on X, it follows that $\overline{\lim}_{p\to\infty} \langle f, \mu_k^{(p)} \rangle \leqq M \overline{\lim}_{p\to\infty} (\mu_k^{(p)}(\Omega) + x_k^{(p)}) = 0$ for k, $m+1 \leqq k \leqq n$. We have also

$$\underline{\lim}_{p\to\infty} (\mu_j^{(p)}, \mu_k^{(p)}) \geqq \underline{\lim}_{p\to\infty} \inf_{X\times X} \Phi \cdot \mu_j^{(p)}(\Omega)\, \mu_k^{(p)}(\Omega) = 0 \qquad \text{if } m+1 \leqq k \leqq n.$$

Hence $\overline{\lim}_{p\to\infty} (I(\nu^{(p)}) - I(\mu^{(p)})) \leqq 0$ and

$$I_X^i(x_0) \leqq \underline{\lim}_{p\to\infty} I(\nu^{(p)}) \leqq \underline{\lim}_{p\to\infty} I(\mu^{(p)}) \leqq \underline{\lim}_{p\to\infty} I_X^i(x^{(p)}).$$

The continuity is now concluded.

Remark. The continuity in $x_1 \geqq 0, \dots, x_n \geqq 0$ is not guaranteed in general by $(a_2)_1'$. We shall give an example in the one-dimensional case. Consider the Newtonian kernel in $\Omega - E_3$, and take the unit open ball $\overline{OP} < 1$ for X, $f(P) \equiv 1$ and $g(P) = \min(\overline{OP}^{-1} - 1, 1)$. Given $x < 1$, we denote by λ_x the unit uniform measure on the sphere $\overline{OP} = (1+x)^{-1}$. Since $\lambda_x \in \mathscr{E}_X(g, x)$, we have

$$I_X^i(x) \leqq I(\lambda_x) = (\lambda_x, \lambda_x) - 2\langle f, \lambda_x \rangle = (1+x) - 2 = x - 1.$$

Therefore $\overline{\lim}_{x\to 0} I_X^i(x) \leqq -1$. It is easy to modify this example to higher dimensional case.

Theorem 3.33. *Let $\Phi(P, Q)$, $X_1, \dots, X_n$, G_0, $G_1^{(0)}, \dots, G_n^{(0)}$ and $g(P)$ be the same as in Theorem 3.31. Let $f(P) < \infty$ be an upper semicontinuous function in G_0 and assume that $I_{X_k}^e(1)$ is finite for each k. Assume also*

(a_3) $1/g$ and $f(1+g)^{-1}$ are bounded from above.

Then $I_X^e(x)$ is continuous as a function of x in $x_1 \geqq 0, \dots, x_n \geqq 0$.

Proof. By the preceding theorem $I_G^i(x)$ is continuous in $x_1 \geqq 0, \dots, x_n \geqq 0$ for each G, $X \subset G \subset G_0$. As the upper envelope of $I_G^i(x)$, $X \subset G \subset G_0$, $I_X^e(x)$ is $> -\infty$ and lower semicontinuous. We see that $\mu(\Omega)$ is bounded and hence $\langle f, \mu \rangle$ is bounded from above by (a_3) on $\bigcup_{|x| \leqq r} \mathscr{E}_{G_0}(g, x)$ for any fixed $r > 0$; (μ, μ) is bounded from below too. By assumption $I_{G_k}^i(1)$ is bounded with respect to open set G_k, $X_k \subset G_k \subset G_k^{(0)}$, for each k. For an arbitrary $\mu_k \in \mathscr{E}_{G_k}(g, 1)$ with $I(\mu_k) < I_{G_k}^i(1) + 1$, $(\mu_k, \mu_k) = I(\mu_k) + 2\langle f, \mu_k \rangle$ is bounded from above and hence is bounded. We note that the measure $\mu' = \sum_{k=1}^n x_k \mu_k$ belongs to $\mathscr{E}_{\bigcup_{k=1}^n G_k}(g, x)$. On account of the fact that $\{G_k^{(0)}\}$ are Φ-separate, (μ', μ') and hence $I(\mu')$ have bounds which may depend on r but not on the choice of $\{G_k\}$ and $\{\mu_k\}$. Consequently $I_X^e(x)$ is bounded on $|x| \leqq r$ for any $r > 0$.

We set

$$\mathscr{E}_x^{(G)} = \{\mu \in \mathscr{E}_G(g, x);\ I(\mu) \leqq I_G^i(x) + 1\}$$

for G such that $X\subset G\subset G_0$. We see that (μ_j, μ_k) and $\langle f, \mu_k\rangle$ are bounded for $\mu\in \bigcup_{G,|x|\leqq r} \mathscr{E}_x^{(G)}$ with any $r>0$, where μ_k is the restriction of μ to G_k. Let $x^{(p)}$ tend to $x_0=(x_1^{(0)}, \cdots, x_n^{(0)})$. Assume that $x_1^{(0)}>0, \cdots, x_m^{(0)}>0, x_{m+1}^{(0)}=\cdots=x_n^{(0)}=0$. For any $\mu\in\mathscr{E}_{x_0}^{(G)}$ we define $\mu^{(p)}$ by $x_j^{(p)}(x_j^{(0)})^{-1}\mu_j$ on G_j, $1\leqq j\leqq m$, and by $x_k^{(p)}\nu_k$ on G_k, $m+1\leqq k\leqq n$, where $G_k=G\frown G_k^{(0)}$ and ν_k is any measure of $\mathscr{E}_{G_k}(g, 1)$ such that $I(\nu_k)\leqq I_{G_k}^i(1)+1$. It follows that $\nu_k(\Omega)$, (ν_k, ν_k) and $\langle f, \nu_k\rangle$ are bounded. We have

$$I_G^i(x^{(p)})\leqq I(\mu^{(p)})=\sum_{j,k=m+1}^{n}\frac{x_j^{(p)}x_k^{(p)}}{x_j^{(0)}x_k^{(0)}}(\mu_j, \mu_k)-2\sum_{k=1}^{m}\frac{x_k^{(p)}}{x_k^{(0)}}\langle f, \mu_k\rangle$$

$$+2\sum_{j=1}^{m}\sum_{k=m+1}^{n}\frac{x_j^{(p)}x_k^{(p)}}{x_j^{(0)}}(\mu_j, \nu_k)+\sum_{j,k=m+1}^{n}x_j^{(p)}x_k^{(p)}(\nu_j, \nu_k)-2\sum_{k=m+1}^{n}x_k^{(p)}\langle f, \nu_k\rangle.$$

Since $G_1^{(0)}, \cdots, G_n^{(0)}$ are Φ-separate and $\mu(\Omega)$ and $\nu_k(\Omega)$ are bounded,

$$\lim_{p\to\infty}\sum_{j=1}^{m}\sum_{k=m+1}^{n}\frac{x_j^{(p)}x_k^{(p)}}{x_j^{(0)}}(\mu_j, \nu_k)=0.$$

We see that $|I(\sum_{k=m+1}^{n}x_k^{(p)}\nu_k)|$ is bounded by a number a_p which is independent of $G_{m+1}, \cdots, G_n$ and tends to 0 as $p\to\infty$. We observe also that

$$I\Big(\sum_{k=1}^{m}\frac{x_k^{(p)}}{x_k^{(0)}}\mu_k\Big)-I(\mu)$$

is bounded by a similar number b_p. Therefore

$$I_G^i(x^{(p)})\leqq I(\mu)+c_p$$

with c_p tending to 0. It follows that

$$I_G^i(x^{(p)})\leqq I_G^i(x_0)+c_p\leqq I_X^e(x_0)+c_p,$$

and that

$$I_X^e(x^{(p)})\leqq I_X^e(x_0)+c_p.$$

Consequently

$$\overline{\lim_{p\to\infty}}\ I_X^e(x^{(p)})\leqq I_X^e(x_0)$$

which gives us

$$\lim_{p\to\infty}I_X^e(x^{(p)})=I_X^e(x_0)$$

on account of the lower semicontinuity of $I_X^e(x_0)$. Thus the proof is completed.

Under the same assumptions as in Theorem 3.32, let μ be any measure of $\mathscr{E}_X(g, x)$, where $x=(x_1, \cdots, x_n)$ and $x_1>0, \cdots, x_n>0$. The reasoning in the proof of Theorem 2.14 does not apply here and so we have to take a different way. With $x'=(x_1', \cdots, x_n')$ in $x_1>0, \cdots, x_n>0$ and $\Delta x_k=x_k'-x_k$, we have

$$I_X^i(x')\leqq I\Big(\sum_{k=1}^n \frac{x_k'}{x_k}\mu_k\Big)$$
$$=I(\mu)+2\sum_{k=1}^n \gamma_k(\mu)\,\Delta x_k+\sum_{j,k=1}^n \frac{\Delta x_j\,\Delta x_k}{x_j\,x_k}(\mu_j, \mu_k).$$

For any $\eta=(\eta_1, \cdots, \eta_n)$ we set

$$\underline{\gamma}_X^i(x, \eta)=\inf_{\gamma=(\gamma_1,\ldots,\gamma_n)\in\Gamma_X^i(x)}\sum_{k=1}^n \gamma_k\,\eta_k,$$

and define $\bar{\gamma}_X^i(x, \eta)$ in a similar fashion. We choose μ so that $I(\mu)$ is arbitrarily close to $I_X^i(x)$ and $\sum_{k=1}^n \gamma_k(\mu)\,y_k$ is close to $\underline{\gamma}_X^i(x, y)$, where y_k is defined by $\Delta x_k=|\Delta x|\,y_k=\sqrt{\Delta x_1^2+\cdots+\Delta x_n^2}\,y_k$. Since (μ_j, μ_k) is uniformly bounded, say $|(\mu_j, \mu_k)|<a$, we have

$$I_X^i(x')\leqq I_X^i(x)+2\underline{\gamma}_X^i(x, y)\,|\Delta x|+a\sum_{j,k=1}^n \frac{|\Delta x|^2}{x_j\,x_k} \tag{3.29}$$

and, by interchanging x and x',

$$I_X^i(x')\geqq I_X^i(x)+2\bar{\gamma}_X^i(x', y)\,|\Delta x|-a\sum_{j,k=1}^n \frac{|\Delta x|^2}{x_j'\,x_k'}\,. \tag{3.30}$$

Consequently

$$\bar{\gamma}_X^i(x', y)\leqq\underline{\gamma}_X^i(x, y)+O\,(|\Delta x|).$$

Therefore

$$\overline{\lim_{x'\to x}}\;\bar{\gamma}_X^i(x', y)\leqq\underline{\gamma}_X^i(x, y). \tag{3.31}$$

To prove the inverse inequality we take a sequence $\{x^{(p)}\}$ of points in $x_1>0, \cdots, x_n>0$ tending to x such that $\underline{\gamma}_X^i(x^{(p)}, y)\in\Gamma_X^i(x^{(p)})$ tends to a finite or infinite number. We choose a sequence $\{\mu^{(p)}\}$ of measures respectively in $\mathscr{E}_X(g, x^{(p)})$ such that

$$\lim_{p\to\infty}\{\sum_{k=1}^n \gamma_k(\mu^{(p)})\,y_k-\underline{\gamma}_X^i(x^{(p)}, y)\}=0$$

and $I(\mu^{(p)})$ tends to $I_X^i(x)$; this is possible because $I_X^i(x)$ is a continuous function of x by Theorem 3.31. It is easy to see that

$$\lim_{p\to\infty}\sum_{k=1}^n \gamma_k(\sum_{j=1}^n x_j(x_j^{(p)})^{-1}\,\mu_k^{(p)})\,y_k=\lim_{p\to\infty}\sum_{k=1}^n \gamma_k(\mu^{(p)})\,y_k.$$

Therefore $\lim_{p\to\infty}\sum_{k=1}^n \gamma_k(\mu^{(p)})\,y_k$ belongs to $\Gamma_X^i(x, y)$. Consequently

$$\varliminf_{x'\to x} \underline{\gamma}_X^i(x', y) \geqq \underline{\gamma}_X^i(x, y).$$

Combining this with (3.31) we obtain

$$\lim_{x'\to x} \underline{\gamma}_X^i(x', y) = \lim_{x'\to x} \bar{\gamma}_X^i(x', y) = \underline{\gamma}_X^i(x, y). \tag{3.32}$$

From (3.29) and (3.30) we have

$$\lim_{x'\to x} \frac{I_X^i(x') - I_X^i(x)}{|\Delta x|} = 2\underline{\gamma}_X^i(x, y). \tag{3.33}$$

This result is less general than that expected from Theorem 2.14. We do not know whether the result corresponding to Theorem 2.14 is true or not. Above reasoning does not apply in the case of the outer problem and the question is open in this respect too. We omit discussions corresponding to some other theorems in Chapter II.

3.10. Unconditional inner and outer problems.

We shall study the unconditional problem in the case $n=1$. The inner problem is to discuss

$$I_X^i = \inf_{\mu\in\mathscr{E}_X} I(\mu).$$

THEOREM 3.34. *Let X be a relatively compact set with $V_i(X) < \infty$ in Ω, and $f(P) < \infty$ be an upper semicontinuous function on X. Assume that the kernel $\hat{\Phi}$ satisfies the continuity principle and that $I_X^i > -\infty$. Let $\{\mu_\omega\}$ be a vaguely convergent net of measures in $\mathscr{E}_X$ for which $I(\mu_\omega)$ tends to I_X^i, and μ_X^i be the vague limit.*[44] *Then*

$$\hat{U}^{\mu_X^i}(P) \geqq f(P) \qquad \textit{p.p.p. on } X. \tag{3.34}$$

If, in addition, $f(P)$ is defined and continuous on X^a, then

$$I_X^i = -\langle f, \mu_X^i \rangle. \tag{3.35}$$

If $f(P) < \infty$ is defined and upper semicontinuous on some set $Z \supset X$ and if μ_X^i is the vague limit of a vaguely convergent subnet of a sequence of measures $\{\mu_{K_n}\}$,[45] *$K_n \subset X$, such that $I(\mu_{K_n})$ tends to I_X^i, then*

$$\hat{U}^{\mu_X^i}(P) \leqq f(P) \qquad \textit{on } S_{\mu_X^i} \cap Z. \tag{3.36}$$

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44) So far μ_X^i has been used to write $\mu_X^i(g, x, f)$. In this section it represents an unconditional extremal measure. Similar remarks are given to I_X^i, I_X^e and μ_X^e.

45) We may use the notation $\{K_n\}$ to represent a sequence because K is not divided into subsets.

In case X is an open set and one or both of conditions (b_1) *and* (b_2) *stated in Lemma* 3.1 *is true, then the exceptional set in* (3.34) *is a K_σ-set and hence* (3.34) *holds. q. p. in X.*

PROOF. If $f(P)=-\infty$ p.p.p. on X, (3.34) is evidently true. Excluding this case we set

$$H=\{P\in X;\ \hat{U}^{\mu^i_X}(P)<f(P)\}$$

and assume that there is a compact set $K\subset H$ with $V_i(K)<\infty$ and a constant $\eta>0$ such that

$$\hat{U}^{\mu^i_X}(P)<f(P)-\eta \qquad \text{on } K. \tag{3.37}$$

On account of the continuity principle there exists a unit measure $\nu\in\mathscr{E}_K$ such that $\hat{U}^{\nu}(P)$ is continuous in Ω and $\langle f, \nu\rangle$ is finite. For any $t\geqq 0$ and ω we have

$$I^i_X\leqq I(\mu_\omega+t\nu)=I(\mu_\omega)+2t\int\hat{U}^{\nu}d\mu_\omega+t^2(\nu, \nu)-2t\langle f, \nu\rangle.$$

It follows that

$$I^i_X\leqq I^i_X+2t\int\hat{U}^{\nu}d\mu^i_X+t^2(\nu, \nu)-2t\langle f, \nu\rangle.$$

Cancelling I^i_X, dividing the rest by t and letting $t\to 0$, we obtain

$$0\leqq 2\int\hat{U}^{\nu}d\mu^i_X-2\langle f, \nu\rangle$$

or

$$\langle f, \nu\rangle\leqq\int\hat{U}^{\nu}\,d\mu^i_X=\int\hat{U}^{\mu^i_X}d\nu.$$

On the other hand follows from (3.37)

$$\int\hat{U}^{\mu^i_X}d\nu<\langle f, \nu\rangle-\eta.$$

These two relations are not compatible and (3.34) is established.

We can derive (3.35) from the relation

$$I^i_X\leqq I((1+t)\mu_\omega) \qquad |t|\leqq 1.$$

The rest of our theorem is proved like in the conditional case.

REMARK 1. $I^i_X\leqq I(0)=0$.

REMARK 2. If $V_i(X)>0$ and $I(\mu_n)$ tends to $I^i_X>-\infty$, then $\{\mu_n\}$ contains a vaguely convergent subnet. In fact,

$$I_X^i \leqq I\left(\frac{\mu_n}{2}\right) = \frac{1}{2} I(\mu_n) - \frac{1}{4}(\mu_n, \mu_n)$$

and hence

$$V_i(X)\mu_n^2(\Omega) \leqq -4I_X^i + 2I(\mu_n),$$

which shows that $\mu_n(\Omega)$ is bounded.

We can prove in the same way as in the conditional case

THEOREM 3.35. *Let X be a set in Ω with $V_i(X)<\infty$ and $f(P)<\infty$ be an upper semicontinuous function on X. Consider a kernel of positive type, and assume that $I_X^i > -\infty$ and that every strong Cauchy net in $\mathscr{E}_X$ is strongly convergent. Then, for any sequence $\{\mu_n\}$ of measures in $\mathscr{E}_X$ for which $I(\mu_n)$ tends to I_X^i, μ_n converges strongly to some measure μ_X^i, and we have*

$$I_X^i = -(\mu_X^i, \mu_X^i) \tag{3.38}$$

and

$$U^{\mu_X^i}(P) \geqq f(P) \qquad p.p.p. \text{ on } X.^{46)} \tag{3.39}$$

Conversely if a measure $\mu \in \mathscr{E}$ satisfies (3.38) *and* (3.39) *replacing μ_X^i in them, then μ_n converges strongly to μ. If (μ, μ_X^i) is defined, then $\|\mu - \mu_X^i\| = 0$.*

THEOREM 3.36. *Consider a consistent kernel. Let X be a set such that $0 < V_i(X) < \infty$, and $f(P)$ be a continuous function with compact support defined on X^a. Then $I(\mu_X^i) = I_X^i$.*

The unconditional outer problem is concerning

$$\sup_G I_G^i = I_X^e,$$

where G is an open set containing X. Naturally $I_X^e \leqq 0$. We shall state several results corresponding to the conditional case, without proof except for Theorem 3.37.

THEOREM 3.37. *Let K be a compact set with $V_i(K)>0$ in Ω, and $f(P)<\infty$ be defined and upper semicontinuous in an open set $G_0 \supset K$. Then*

$$I_K^i = I_K^e.$$

PROOF. By Theorem 1.14 we know that $V_i(K) = V_e(K)$. Hence there is a relatively compact open set G_0' such that $K \subset G_0' \subset G_0$ and $V_i(G_0') > 0$. We set

46) If $f(P)<\infty$ is defined and upper semicontinuous on $Z \supset X$, if $I(\mu_{K_n})$, $K_n \subset X$, tends to I_X and if $\{\mu_{K_n}\}$ contains a net vaguely convergent to μ_X^i, then

$$U^{\mu_X^i}(P) \leqq f(P) \qquad \text{on } S_{\mu_X^i} \cap X.$$

$$G_0''=\{P\in G_0';\ f(P)<\max_K f+1\}.$$

This is an open set containing K. Let F_0 be a closed subset of G_0'' which contains K in its inside. We direct by inclusion the class of all closed subsets of F_0, each containing K in its inside, and denote the resulting directed set by D. Since

$$I_K^i\geqq V_i(G_0')\,\mu_F^2(\Omega)-2\,(\max_K f+1)\,\mu_F(\Omega) \qquad \text{for } F\in D,$$

$\mu_F(\Omega)$ is bounded for $F\in D$. We extract a vaguely convergent subnet $\{\mu_\omega;\ \omega\in D'\}$ of $\{\mu_F;\ F\in D\}$ and see for the vague limit μ_0 that

$$I(\mu_0)=(\mu_0,\mu_0)-2\langle f,\mu_0\rangle\leqq\varliminf_\omega\,(\mu_\omega,\mu_\omega)-2\,\varlimsup_\omega\,\langle f,\mu_\omega\rangle\leqq\varliminf_\omega\,I(\mu_\omega)\leqq I_K^e.$$

Since $S_{\mu_0}\subset\bigcap_{F\in D}F=K$, $I(\mu_0)\geqq I_K^i$ and hence $I_K^i\leqq I_K^e$. The inverse inequality being evident, we obtain the equality.

Remark. Without the condition $V_i(K)>0$ the conclusion is not always true. For instance, let $\Omega=E_3$, $\Phi(P,Q)\equiv 0$, $K=\{\overline{OP}\leqq 1\}$ and $f(P)=0$ on K and $=\overline{OP}-1$ outside K. Then $I_K^i=0$ but $I_K^e=-\infty$.

Theorem 3.38. *Let X be a relatively compact set with $V_e(X)>0$ in Ω, $G_0\supset X$ be an open set in Ω and $f(P)<\infty$ be an upper semicontinuous function in G_0. Assume that $\hat{\Phi}$ satisfies the continuity principle and that $I_X^e>-\infty$. Let $\{G_n\}$, $X\subset G_n\subset G_0$, be a sequence of open sets such that $I_{G_n}^i$ tends to I_X^e. Then, for the vague limit μ_X^e of any vaguely convergent subnet of $\{\mu_{G_n}\}$,*[47]

$$\hat{U}^{\mu_X^e}(P)\geqq f(P) \qquad \textit{p.p.p. on } X.$$

If $f(P)<\infty$ is defined and upper semicontinuous on G_0^a, then

$$\hat{U}^{\mu_X^e}(P)\leqq f(P) \qquad \textit{on } S_{\mu_X^e}.$$

If, in addition, $f(P)$ is continuous on G_0^a, then

$$I_X^e=-\langle f,\mu_X^e\rangle.$$

Theorem 3.39. *In addition to the conditions required in the first part of the preceding theorem, suppose that $\Phi(P,Q)$ is continuous outside the diagonal set and that $\Phi(P,P)=\infty$ at each point P of $\mathbf{O}_\infty$ which is defined with respect to $\hat{\Phi}$. Assume also one or both of* (b₁)* *and* (b₂)* *stated in Theorem* 3.10. *Then*

$$\hat{U}^{\mu_X^e}(P)\geqq f(P) \qquad \textit{q.p. on } X.$$

We shall denote the closure of $\mathscr{E}_{G_0}$ with respect to the strong topology

47) The existence is ensured by Remark 2 to Theoem 3.34.

by $\mathscr{E}^i_{G_0}$.

THEOREM 3.40. *Let X be any set in Ω and $G_0 \supset X$ be an open set in Ω. Let $f(P) < \infty$ be defined and upper semicontinuous in G_0. Assume that the kernel is of positive type, that $I^e_X > -\infty$, that every strong Cauchy net in $\mathscr{E}^i_{G_0}$ is strongly convergent and that $(\mu^i_G, \mu^i_{G'})$ is defined for any open subsets G and G' of G_0 containing X. Then, for any sequence $\{G_n\}$, $X \subset G_n \subset G_0$, of open sets such that $I^i_{G_n}$ tends to I^e_X, $\mu^i_{G_n}$ converges strongly to some measure μ^e_X. It is a strong limit for any sequence of open sets of the above character. It holds that*

$$I^e_X = -(\mu^e_X, \mu^e_X)$$

and that

$$U^{\mu^e_X}(P) \geqq f(P) \qquad p.p.p. \text{ on } X.^{48)}$$

We can choose μ^e_X so that its support is contained in X^a.

THEOREM 3.41. *Under the same assumptions as above, if (μ^e_X, μ^i_X) is defined,*

$$\|\mu^i_X - \mu^e_X\|^2 \leqq I^i_X - I^e_X.$$

THEOREM 3.42. *Let K be a compact set in Ω such that $0 < V_i(K) < \infty$ and $G_0 \supset K$ be an open set in Ω. Let $f(P) < \infty$ be defined and upper semicontinuous in G_0. Consider a kernel of positive type and assume that every strong Cauchy net in $\mathscr{E}_{G_0}$ is strongly convergent. Then*

$$\|\mu_K - \mu^e_K\| = 0.$$

THEOREM 3.43. *Consider a consistent kernel. Let G_0 be an open set such that $0 < V_i(G_0) < \infty$ and $f(P)$ be a continuous function with compact support defined on G^a_0. Assume that $(\mu^i_G, \mu^i_{G'})$ is defined for any open subsets G and G' of G_0 both including a fixed set X. Then $I(\mu^e_X) = I^e_X$.*

THEOREM 3.44. *Let G_0 be an open set in Ω on whose product the kernel is bounded from below, and X be any subset of G_0. Let $f(P) < \infty$ be defined and upper semicontinuous in G_0. Assume that the kernel is of positive type, that $I^e_X > -\infty$, that every strong Cauchy net in $\mathscr{E}^i_{G_0}$ is strongly convergent, that $(\mu^i_G, \mu^i_{G'})$ is defined for every open subsets G and G' of G_0 containing X and that one or both of* $(b_1)^*$ *and* $(b_2)^*$ *stated in Lemma 3.5 is true. Then*

$$U^{\mu^e_X}(P) \geqq f(P) \qquad q.p. \text{ on } X.$$

THEOREM 3.45. *Consider a kernel of positive type. Let $\{A_n\}$ be a sequence*

48) If $f(P) < \infty$ is defined and upper semicontinuous on G^a_0 and if $\{\mu^i_{G_n}\}$ contains a subnet vaguely convergent to μ^e_X, then

$$U^{\mu^e_X}(P) \leqq f(P) \qquad \text{on } S_{\mu^e_X}.$$

of sets of $\mathfrak{A}$ increasing to A, X be an arbitrary set, $f(P)<\infty$ be an upper semi-continuous function on $A\cap X$. Assume that every strong Cauchy net in $\mathcal{E}_{A\cap X}$ is strongly convergent and that $(\mu^i_{A_n\cap X}, \mu^i_{A_m\cap X})$ is defined for any n and m provided both $I^i_{A_n\cap X}$ and $I^i_{A_m\cap X}$ are finite. Then $I^i_{A_n\cap X}$ tends to $I^i_{A\cap X}$, and $\mu^i_{A_n\cap X}$ converges strongly to $\mu^i_{A\cap X}$ if $I^i_{A\cap X}$ is finite.

Next let G_0 be an open set in Ω such that the kernel is bounded from below on $G_0\times G_0$, assume that the above $f(P)$ is defined in G_0 and let $\{X_n\}$ be a sequence of subsets of G_0 increasing to X. Assume that every Cauchy net in $\mathcal{E}^i_{G_0}$ is strongly convergent, that $(\mu^i_G, \mu^i_{G'})$ is defined for any open subsets G and G' of G_0 containing X provided I^i_G and $I^i_{G'}$ are finite, and that one or both of $(\mathrm{b}_1)^*$ *and* $(\mathrm{b}_2)^*$ *stated in Lemma 3.5 is true. Then $I^e_{X_n}$ tends to I^e_X, and $\mu^e_{X_n}$ converges strongly to μ^e_X if I^e_X is finite.*

THEOREM 3.46. *Consider a kernel of positive type, let $\{K_n\}$ be a sequence of compact sets decreasing to K with $V_i(K)>0$ and let $f(P)<\infty$ be an upper semicontinuous function on K_1. Assume that every strong Cauchy net in $\mathcal{E}_{K_1}$ is strongly convergent. Then I_K is finite, I_{K_n} tends to I_K, and μ_{K_n} converges strongly to μ_K.*[49]

THEOREM 3.47. *Consider a consistent kernel. Let $\{F_n\}$ be a sequence of closed subsets decreasing to F such that $0<V_i(F_1)$, and assume that $(\mu^i_{F_n}, \mu^i_{F_m})$ is defined for any n and m. Let $f(P)$ be a continuous function with compact support defined on F_1. Then $I^i_{F_n}$ tends to I^i_F and, if $(\mu^i_F, \mu^i_{F_n})$ is defined for every n, $\mu^i_{F_n}$ converges strongly to μ^i_F.*

THEOREM 3.48. *Consider a kernel of positive type. Let G_0 be an open set in Ω such that the kernel is bounded from below on $G_0\times G_0$, and $f(P)<\infty$ be an upper semicontinuous function in G_0. Assume that every strong Cauchy net in $\mathcal{E}^i_{G_0}$ is strongly convergent and that $(\mu^e_X, \mu^e_{X'})$ is defined for any subsets X and X' of G_0 provided I^e_X and $I^e_{X'}$ are finite. Assume also one or both of* $(\mathrm{b}_1)^*$ *and* $(\mathrm{b}_2)^*$ *of Theorem* 3.20 $(=(\mathrm{b}_1)^*$ *and* $(\mathrm{b}_2)^*$ *of Lemma* 3.5); *if the kernel is consistent,* $(\mathrm{b}_1)^*$ *and* $(\mathrm{b}_2)^*$ *may be replaced respectively by* (b_1) *and* (b_2) *stated after Theorem* 3.20. *Then, for any $\mathfrak{K}$-Souslinian set A in G_0, $I^i_A=I^e_A$. If this value is finite, $\|\mu^i_A-\mu^e_A\|=0$.*

3.11. Notes and questions.

We assumed no additional condition when we discussed the Gauss variational problem on compact sets in Chapter II. In Chapter III, however, we have some limiting process and so we need to assume something more. For kernels of positive type we assumed the continuity principle too in the first manuscript; we recall that if a kernel of positive type satisfies the continuity

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49) We first observe that $V_i(K_n)>0$ for large n by Theorem 3.21 and then that $\mu_{K_n}(\Omega)$ is bounded for large n.

principle, then it is K-consistent and $\mathscr{E}_K$ for any compact set $K \subset \Omega$ is strongly complete according to Corollary of Theorem 1.7. We changed it to the present form under the influence of Fuglede [1]. One reason why we have not started from a consistent kernel may be seen in the fact that the kernel $\Phi \equiv 1$ is not consistent in any non-compact space.

In the outer problem one might think that Φ may be replaced in the product of an open set, where f and g are defined, by

$$\Psi(P, Q) = \Phi(P, Q) - \frac{f(P)\, g(Q) + f(Q)\, g(P)}{\sum_{n=1}^{n} x_k}$$

because of the identity

$$I(\mu) = \iint \Psi(P, Q)\, d\mu(P)\, d\mu(Q).$$

However, often we state conditions on Φ, f and g separately (although one condition requires that $f(1+g)^{-1}$ is bounded from above) and can not phrase these conditions in terms of Ψ and g only.

Open questions.

3.1. Is the inequality in Theorem 3.2 true for μ_X^i and $\mu_{X_k}^i$ obtained in Theorem 3.1 ? And similar questions in other cases.

3.2. Can we replace $(\mathrm{a}_2)_e'$ by $(\mathrm{a}_2)_e$ $V_e(X) > 0$ in Theorem 3.10 ?

3.3. Does it happen that some $\gamma_k = \infty$ in the same theorem ?

3.4. Can we require in Theorem 3.9 moreover that μ_X^e is supported by X^a ?

3.5. Is it sufficient to have $f(P)$ and $g(P)$ defined only in G_0 in Theorem 3.15 ?

3.6. Can we prove the converse part of Theorem 3.18 without the assumption concerning the coincidence of V_i-value and V_e-value ?

3.7. Question stated after Corollary of Theorem 3.19.

3.8. In case the kernel satisfies the continuity principle, is the identity $I_A^i(g, x, f) = I_A^e(g, x, f)$ true for every $\mathfrak{K}$-Souslinian set A ? See the discussion on capacitability in Kishi [3; 4; 7] in this connection.

3.9. Is the condition $f(P) \leqq 0$ necessary in Theorem 3.25 ?

3.10. Can we find, for any X, a K_σ-set K_σ and a G_δ-set G_δ such that $K_\sigma \subset X \subset G_\delta$ and $I_{G_\delta - K_\sigma}^e(g, x, f) = \infty$? See the end of § 3.6.

3.11. Is the result corresponding to Theorem 2. 14 true in the inner problem ?

3.12. Do we have the equalities corresponding to (3.32) and (3.33) in the outer problem ?

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(Principles are omitted; they are found in § 1. 2 and § 2.11)

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