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Invariant Manifolds under a Certain Transformation

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1. Introduction.

In this note, we are concerned with a manifold invariant under the real transformation Σ of the form

(1) $$z' = Cz + h(z)$$

where

z and z' are n-dimensional real vectors;

C is a real $n\times n$-matrix such that some of its characteristic roots are less than unity in absolute value;

$h(z)$ is a real continuous function of z such that $h(0)=0$ and, for any $\varepsilon > 0$, there exists a positive number $\delta = \delta(\varepsilon)$ such that

$$|h(z_1)-h(z_2)| \leqq \varepsilon |z_1 - z_2|^{1)}$$

whenever $|z_1|$, $|z_2| \leqq \delta$.

The present problem is connected closely with the conditional stability of the solutions of the differential equations, because the iteration of the transformation (1) converges on the invariant manifold in question.

The case where $h(z)$ is analytic was discussed already by M. Urabe [1]. In the present note, the same results will be proved for the more general case where $h(z)$ is a function described above.

2. The functional equation for an invariant manifold.

If we transform the variable z to $\bar{z}$ by a real linear transformation $z = P\bar{z}$, the initial transformation (1) is written as

(2) $$\bar{z}' = P^{-1}CP\bar{z} + P^{-1}h(P\bar{z}).$$

Clearly we can choose P so that $P^{-1}CP$ may have the form

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1) $|\cdots|$ means the norm such that $|v| = \sum_i |v^i|$ for a vector $v = v^i$ and $|A| = \max_j \sum_i |a_{ij}|$ for a matrix $A = (a_{ij})$.

$$P^{-1}CP=\begin{pmatrix} A & O \\ O & B \end{pmatrix},$$

where A is a real matrix whose characteristic roots are all less than unity in absolute value and B is a real matrix whose characteristic roots are all not less than unity in absolute value. For such P, decomposing the initial vectors into two component vectors, we can write the transformation (2) as follows:

$$\text{(3)} \qquad \Pi:\begin{cases} x' = Ax+f(x, y), \\ y' = By+g(x, y). \end{cases}$$

Here, due to the assumptions on $h(z)$, it is evident that

1° $f(0, 0)=0$, $g(0, 0)=0$ and $f(x, y)$, $g(x, y)$ are continuous in the region: $|x|$, $|y|\leqq\delta_0$ where δ_0 is a certain positive number;

2° for any $\varepsilon>0$, there exists a positive

$\delta=\delta(\varepsilon)\leqq\delta_0$ such that

$$|f(x_1, y_1)-f(x_2, y_2)|\leqq\varepsilon(|x_1-x_2|+|y_1-y_2|),$$

$$|g(x_1, y_1)-g(x_2, y_2)|\leqq\varepsilon(|x_1-x_2|+|y_1-y_2|),$$

whenever $|x_1|$, $|x_2|$, $|y_1|$, $|y_2|\leqq\delta$.

First, we shall seek a manifold of the from $y=Y(x)$ invariant under the above transformation Π.

The invariancy of the manifold $y=Y(x)$ means that $y'=Y(x')$ whenever $y=Y(x)$, namely that

$$\text{(4)} \qquad BY(x)+g\{x, Y(x)\} = Y[Ax+f\{x, Y(x)\}].$$

Now, from the character of the present problem, we are concerned only with the invariant manifold passing through the origin. So, at present, we may assume

$$\text{(5)} \qquad Y(0)=0.$$

Then the problem to seek a desired invariant manifold is reduced to the problem to solve the the functional equation (4) under the initial condition (5).

Since B is non-singular from its definition, let us write the equation (4) in the form

$$\text{(6)} \qquad Y(x)=-B^{-1}g\{x, Y(x)\}+B^{-1}Y[Ax+f\{x, Y(x)\}]$$

and seek its solution by the iterative method, namely by making the functions $Y_k(x)$ $(k=1, 2,\cdots,)$ successively as follows:

$$(7)\qquad \begin{cases} Y_0(x) \equiv 0, \\ Y_{k+1}(x) = -B^{-1}g\{x,\ Y_k(x)\} + B^{-1}Y_k[Ax + f\{x,\ Y_k(x)\}] \\ \hfill (k=1,\ 2, \cdots). \end{cases}$$

To prove the convergence of this iteration process, we transform the matrices B^{-1} and A to the matrices B_0^{-1} and A_0 respectively which are of the canonical forms such that

$$(8)\qquad B_0^{-1} = \begin{pmatrix} \mu_1 & 0 & 0\cdots \\ \beta_1 & \mu_2 & 0\cdots \\ 0 & \beta_2 & \mu_3 \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & & \ddots \end{pmatrix} \quad \text{and } A_0 = \begin{pmatrix} \lambda_1 & 0 & 0\cdots \\ \alpha_1 & \lambda_2 & 0\cdots \\ 0 & \alpha_2 & \lambda_3 \\ 0 & 0 & \ddots & \ddots \\ \vdots & \vdots & & \ddots \end{pmatrix}$$

Here,

$\mu_1,\ \mu_2 \cdots$ are the characteristic roots of B^{-1};

$\lambda_1,\ \lambda_2, \cdots$ are the characteristic roots of A;

$\beta_1,\ \beta_2, \cdots$ and $\alpha_1,\ \alpha_2, \cdots$ are 0 or γ, where γ is an arbitrary positive number.

Put

$$(9)\qquad B_0^{-1} = S^{-1}B^{-1}S, \quad A_0 = T^{-1}AT,$$

and

$$(10)\qquad \begin{cases} x = T\hat{x}; \\ S^{-1}Y_k(T\hat{x}) = \hat{Y}_k(\hat{x}) \qquad (k=0,\ 1,\ 2, \cdots); \\ S^{-1}g(T\hat{x},\ S\hat{y}) = \hat{g}(\hat{x},\ \hat{y}), \quad T^{-1}f(T\hat{x},\ S\hat{y}) = \hat{f}(\hat{x},\ \hat{y}). \end{cases}$$

Then the initial iterative process (7) is transformed to that of the same form as follows:

$$(11)\qquad \begin{cases} \hat{Y}_0(\hat{x}) \equiv 0, \\ \hat{Y}_{k+1}(\hat{x}) = -B_0^{-1}\hat{g}\{\hat{x},\ \hat{Y}_k(\hat{x})\} + B_0^{-1}\hat{Y}_k[A_0\hat{x} + f\{\hat{x},\ \hat{Y}_k(\hat{x})\}] \\ \hfill (k=0,\ 1,\ 2, \cdots). \end{cases}$$

Hence, by (10), to prove the convergence of the iterative process (7), we have only to prove it for the iterative process (11).

Before proving the convergence, we remark that the functions $\hat{f}(\hat{x},\ \hat{y})$ and $\hat{g}(\hat{x},\ \hat{y})$ have the same character as $f(x,\ y)$ and $g(x,\ y)$, namely that, for $\hat{x}$ and $\hat{y}$ such that $T\hat{x}$ and $S\hat{x}$ are real,

1° $\hat{f}(0,\ 0) = 0$, $\hat{g}(0,\ 0) = 0$ and $\hat{f}(\hat{x},\ \hat{y})$, $\hat{g}(\hat{x},\ \hat{y})$ are continuous in $\hat{x}$ and $\hat{y}$

provided

$$|\hat{x}|,\ |\hat{y}| \leqq \hat{\delta}_0 = \frac{\delta_0}{\max(|S|,\ |T|)};$$

2° for any $\varepsilon > 0$, there exists a positive $\hat{\delta} = \hat{\delta}(\varepsilon)^{1)} \leqq \hat{\delta}_0$

such that

$$|\hat{f}(\hat{x}_1, \hat{y}_1) - \hat{f}(\hat{x}_2, \hat{y}_2)| \leqq \varepsilon(|\hat{x}_1 - \hat{x}_2| + |\hat{y}_1 - \hat{y}_2|),$$

$$|\hat{g}(\hat{x}_1, \hat{y}_1) - \hat{g}(\hat{x}_2, \hat{y}_2)| \leqq \varepsilon(|\hat{x}_1 - \hat{x}_2| + |\hat{y}_1 - \hat{y}_2|),$$

whenever $|\hat{x}_1|,\ |\hat{x}_2|,\ |\hat{y}_1|,\ |\hat{y}_2| \leqq \hat{\delta}$.

In the following paragraphs, the possibility and convergence will be proved for the iterative process (11).

3. Possibility

We prove the possibility of the iteration process (11) by induction.

It is evident that $\hat{Y}(\hat{x})$ is defined in $|\hat{x}| \leqq \hat{\delta}$.

We take a positive number M such that, for a sufficiently small $\varepsilon < \varepsilon_0$,

$$M > L + L(|A_0| + 2\varepsilon)M + \varepsilon^2 LM^2 \tag{12}$$

where $L = |B^{-1}|$.

The possibility of such a choice of M is proved as follows. The inequality (12) is equivalent to the inequality

$$\varepsilon^2 LM^2 - \{(1 - L|A_0|) - 2\varepsilon L\}M + L < 0. \tag{13}$$

Now, by the assumption, $|A_0| < 1$ if γ is chosen sufficiently small. In addition, by the assumption, the characteristic roots of B_0^{-1} are not greater than unity in absolute value. Therefore, in case $|A_0| \neq 0$, if we choose γ sufficiently small, we may suppose that $|B_0^{-1}| < 1/|A_0|$, namely $1 - L|A_0| > 0$. In case $|A_0| = 0$, it is evident $1 - L|A| > 0$. Therefore, we may suppose the inequality $1 - L|A_0| > 0$ always holds. Then the quadratic inequality (13) is solved as follows:

$$M_2(\varepsilon) < M < M_1(\varepsilon),$$

where

1) For $\hat{\delta}(\varepsilon)$, the following value can be taken:

$$\hat{\delta}(\varepsilon) = \delta\left(\frac{\varepsilon}{\max(|S^{-1}|,\ |T^{-1}|) \cdot \max(|S|,\ |T|)}\right) \Big/ \max(|S|,\ |T|).$$

$$M_1(\varepsilon)=\frac{(1-L|A_0|)-2\varepsilon L+\sqrt{\{(1-L|A_0|)-2\varepsilon L\}^2-4\varepsilon^2L^2}}{2\varepsilon^2 L}$$

$$=\frac{1}{\varepsilon^2}\frac{1-L|A_0|}{L}-\frac{2}{\varepsilon}-\cdots,$$

$$M_2(\varepsilon)=\frac{(1-L|A_0|)-2\varepsilon L-\sqrt{\{(1-L|A_0|)-2\varepsilon L\}^2-4\varepsilon^2L^2}}{2\varepsilon^2 L}$$

$$=\frac{L}{1-L|A_0|}+\frac{2\varepsilon L^2}{(1-L|A_0|)^2}+\cdots$$

As is seen from the above formulas, $M_1(\varepsilon)\to+\infty$ and $M_2(\varepsilon)\to L/(1-L|A_0|)$ as $\varepsilon\to 0$. Consequently, for a sufficiently small positive number ε_0, there actually exists a constant M such that $M_2(\varepsilon)<M<M_1(\varepsilon)$ for $\varepsilon<\varepsilon_0$, namely, we can take the constant M for a sufficiently small $\varepsilon<\varepsilon_0$ so that the inequality (12) holds. Since $L/(1-L|A_0|)>L$, it is evident that M should be greater than L.

In the sequel we choose ε so small that $\varepsilon M<1$ and $l=|A_0|+\varepsilon+\varepsilon^2M<1$.

Let us assume that $\hat{Y}_k(\hat{x})$ is defined in the domain $|\hat{x}|\leqq\hat{\delta}$ and $|\hat{Y}_k(\hat{x})|\leqq \varepsilon M|\hat{x}|$ there.

Then, since $|A\hat{x}+\hat{f}\{\hat{x},\hat{Y}_k(\hat{x})\}|\leqq(|A_0|+\varepsilon+\varepsilon^2M)|\hat{x}|$ by the conditions 1° and 2° on $\hat{f}(\hat{x},\hat{y})$, $\hat{Y}_{k+1}(\hat{x})$ is defined in $|\hat{x}|\leqq\hat{\delta}$ and it follows from the conditions on $\hat{g}(\hat{x},\hat{y})$ and (12) that

$$\begin{aligned}|\hat{Y}_{k+1}(\hat{x})|&\leqq|B_0^{-1}|\,|\hat{g}\{\hat{x},\hat{Y}_k(x)\}|+|B_0^{-1}|\,|\hat{Y}_k[A_0\hat{x}+\hat{f}\{\hat{x},\hat{Y}_k(\hat{x})\}]|\\&<\varepsilon L(|\hat{x}|+\varepsilon M|\hat{x}|)+L\cdot\varepsilon M(|A_0|+\varepsilon+\varepsilon^2M)|\hat{x}|\\&=\varepsilon\{L+L(|A_0|+2\varepsilon)M+\varepsilon^2LM^2\}\,|\hat{x}|\\&\leqq\varepsilon M|\hat{x}|.\end{aligned}$$

Thus we see by induction that $\hat{Y}_k(\hat{x})$ $(k=0,1,2,\cdots)$ are all defined in $|\hat{x}|\leqq\hat{\delta}$ and $|\hat{Y}_k(x)|\leqq\varepsilon M|\hat{x}|$ $(k=0,1,2,\cdots)$ there. This proves that the iteration process (11) is really possible for all $k=0,1,2,\cdots$.

4. Convergence

In this paragraph, we shall show that the sequence $\{Y_k(x)\}$ $(k=0,1,2,\cdots)$ obtained by the iterative process (11) converges uniformly to the limit function $\hat{Y}(\hat{x})$ which satisfies the functional equation of the form (4) obtained from (4) by the linear transformation $x=T\hat{x}$, $y=S\hat{y}$.

First of all it is readily verified by induction that, for any k,

$$|\hat{Y}_k(\hat{x})-\hat{Y}_k(\hat{x}')|\leqq\varepsilon M|\hat{x}-\hat{x}'|. \tag{14}$$

In addition, if ε is sufficiently small,

$$|A_0|+2\varepsilon+2\varepsilon^2M<1$$

and

$$(15)\qquad r=L(|A_0|+2\varepsilon+2\varepsilon^2M)<1,$$

because $|A_0|<1$ and $L|A_0|<1$.

Making use of (14) and (15), let us prove

$$(16)\qquad |\hat{Y}_{k+1}(\hat{x})-\hat{Y}_k(\hat{x})|\leqq\varepsilon Lr^k|\hat{x}|\qquad (k=0,1,2,\dots).$$

For $k=0$, this is evident, because

$$|\hat{Y}_1(\hat{x})|=|-B_0^{-1}\hat{g}(\hat{x},0)|\leqq\varepsilon L|\hat{x}|.$$

If (16) is true up to $k-1$, then, from (14) and (15), follows:

$$\begin{aligned}
|\hat{Y}_{k+1}(\hat{x})-\hat{Y}_k(\hat{x})| &\leqq \varepsilon L|\hat{Y}_k(\hat{x})-\hat{Y}_{k-1}(\hat{x})| \\
&\quad +L|\hat{Y}_k[A_0\hat{x}+\hat{f}\{\hat{x},\hat{Y}_k(\hat{x})\}]-\hat{Y}_k[A_0\hat{x}+\hat{f}\{\hat{x},\hat{Y}_{k-1}(x)\}]| \\
&\quad +L|\hat{Y}_k[A_0\hat{x}+\hat{f}\{\hat{x},\hat{Y}_{k-1}(\hat{x})\}]-\hat{Y}_{k-1}[A_0\hat{x}+\hat{f}\{\hat{x},\hat{Y}_{k-1}(x)\}]| \\
&\leqq \varepsilon L\cdot\varepsilon Lr^{k-1}|\hat{x}|+L\cdot\varepsilon M\cdot\varepsilon\cdot\varepsilon Lr^{k-1}|\hat{x}|+L\cdot\varepsilon Lr^{k-1}[|A_0||\hat{x}|+\varepsilon(|\hat{x}|+\varepsilon M|\hat{x}|)] \\
&= \varepsilon L\cdot r^{k-1}[\varepsilon L+\varepsilon^2LM+L(|A_0|+\varepsilon+\varepsilon^2M)]|\hat{x}| \\
&= \varepsilon L\cdot r^{k-1}L(|A_0|+2\varepsilon+2\varepsilon^2M)|\hat{x}| \\
&= \varepsilon L\cdot r^k|\hat{x}|.
\end{aligned}$$

This proves (16) is valid also for k.

Thus, by induction, we see that (16) is valid for any k $(k=0,1,2,\dots)$.

Then the series

$$\sum_{k=0}^{\infty}\{\hat{Y}_{k+1}(\hat{x})-\hat{Y}_k(\hat{x})\}$$

is absolutely and uniformly convergent on $|\hat{x}|\leqq\hat{\delta}$. This means that the sequence $\{\hat{Y}_k(\hat{x})\}$ $(k=0,1,2,\dots)$ converges uniformly on $|\hat{x}|\leqq\hat{\delta}$. Let $\hat{Y}(\hat{x})=\lim_{k\to\infty}\hat{Y}_k(\hat{x})$, then $\hat{Y}(\hat{x})$ is evidently continuous on $|\hat{x}|\leqq\hat{\delta}$ and satisfies the inequality $|\hat{Y}(\hat{x})|\leqq\varepsilon M|\hat{x}|$.

From its derivation, it is evident that $\hat{Y}(\hat{x})$ satisfy the equation

$$\hat{Y}(\hat{x})=-B_0^{-1}\hat{g}\{\hat{x},\hat{Y}(\hat{x})\}+B_0^{-1}[\hat{Y}A_0\hat{x}+\hat{f}\{\hat{x},\hat{Y}(\hat{x})\}],$$

namely

$$(17)\qquad B_0\hat{Y}(\hat{x})+\hat{g}(\hat{x},\hat{Y}(\hat{x}))=\hat{Y}[A_0\hat{x}+\hat{f}\{\hat{x},\hat{Y}\hat{x})\}].$$

Since $Y_k(x)$ $(k=0,1,2,\dots)$ obtained by the iteration process (7) are equal to $S\hat{Y}_k(T^{-1}x)$ $(k=0,1,2,\dots)$ by (10), the above results imply that the sequence $\{Y_k(x)\}$ $(k=0,1,2,\dots)$ obtained by the iteration process (7) converges to the function $Y(x)=S\hat{Y}(T^{-1}x)$ uniformly on $|\hat{x}|\leqq\hat{\delta}/|T^{-1}|$ and that $Y(x)$ satisfies the functional equation (4).

Lastly let us consider the uniqueness of the solution of the equation (4).

Let $\hat{Y}'(\hat{x})$ be any solution of (17) such that $|\hat{Y}'(\hat{x})| \leqq \varepsilon M|\hat{x}|$. Then evidently

$$|\hat{Y}(\hat{x}) - \hat{Y}'(\hat{x})| \leqq 2\varepsilon M|\hat{x}|,$$

from which follows by induction

$$(18) \qquad |\hat{Y}(\hat{x}) - \hat{Y}'(\hat{x})| \leqq 2\varepsilon M r^k |\hat{x}| \qquad (k = 0, 1, 2, \cdots).$$

Since $0 < r < 1$, this implies $\hat{Y}(\hat{x}) \equiv \hat{Y}'(\hat{x})$, namely the uniqueness of the solution of (17),

Then, since any solution $Y'(x)$ of (4) such that $Y'(x) = 0(|x|)$ as $|x| \to 0$ corresponds to the solution $\hat{Y}'(\hat{x})$ of (17) satisfying $|\hat{Y}'(\hat{x})| \leqq \varepsilon M|\hat{x}|$, it follows that the solution $Y(x)$ of (4) such $Y(x) = 0(|x|)$ as $|x| \to 0$ is unique.

The results obtained above are restated in a

Theorem 1. *For the functional equation* (4), *there exists always one and only one real solution* $Y(x)$ *such that* $Y(x) = 0(|x|)$ *as* $|x| \to 0$.

5. Transformation Π on the invariant manifold

Let V be the manifold defined by

$$y = Y(x),$$

where $Y(x)$ is a unique solution of (14) such that $Y(x) = 0(|x|)$ as $|x| \to 0$. Then, from the formulation (4), it is evident that V is an invariant manifold under the transformation Π given by (3).

Let $(x_k, Y(x_k))$ be the image of a point $(x_0, Y(x_0))$ of V under the transformation Π^k. Then, by (3),

$$x_{k+1} = Ax_k + f\{x_k, Y(x_k)\} \quad (k = 0, 1, 2, \cdots).$$

This can be written in terms of $\hat{x} = T^{-1}x$ defined by (10) as follows:

$$\hat{x}_{k+1} = A_0\hat{x}_k + \hat{f}\{\hat{x}_k, \hat{Y}(\hat{x}_k)\} \quad (k = 0, 1, 2, \cdots),$$

where $\hat{Y}(\hat{x}) = S^{-1}Y(T\hat{x})$. Since $Y(x) = 0(|x|)$ as $|x| \to 0$,

$$(19) \qquad |\hat{Y}(\hat{x})| \leqq \varepsilon M|\hat{x}|$$

provided $|x|$ is sufficiently small. Then, for sufficiently small $\hat{\delta}$, it is valid that

$$(20) \qquad |\hat{x}_{k+1}| \leqq (|A_0| + \varepsilon + \varepsilon^2 M)|\hat{x}_k| = l|\hat{x}_k|$$

whenever $|\hat{x}_k| \leqq \hat{\delta}$. Since ε is chosen so small that $l < 1$, $|\hat{x}_k| \leqq \hat{\delta}$ $(k=0, 1, 2, \cdots)$

if $|\hat{x}_0| \leqq \hat{\delta}$.

Then, from (19) and (20), it readily follows that

$$(\hat{x}_k,\ \hat{Y}(\hat{x}_k)) \to (0,\ 0) \quad \text{as} \quad k \to \infty.$$

This implies $(x_k, Y(x_k))$ tends to the origin on the manifold V as $k\to\infty$ if $|x_0|$ is sufficiently small. In other words, the iteration of the transformation Π converges on the invariant manifold V.

6. Another invariant manifold

Let us assume that $P^{-1}CP$ is of the form

$$P^{-1}CP = \begin{pmatrix} A & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & B \end{pmatrix},$$

where the characteristic roots of A, D and B are respectively less than, equal to and greater than unity in absolute value. In this case we may write the transformation Π as follows:

$$\text{(21)} \qquad \Pi : \begin{cases} x' = Ax + f(x,\ u,\ y), \\ u' = Du + l(x,\ u,\ y), \\ y' = By + g(x,\ u,\ y), \end{cases}$$

where $f(x, u, y)$, $l(x, u, y)$ and $g(x, u, y)$ are the functions satisfying 1° and 2° of 2.

For the transformation Π of the form (21), by our Theorem 1, there exists an invariant manifold $u=F_1(x)$, $y=F_2(x)$ on which the iteration of Π converges.

Assuming det $C\neq 0$, let us consider the inverse transformation Π^{-1} of Π. Then, as is readly seen by the successive approximation, Π^{-1} exists really and is expressed as follows:

$$\text{(22)} \qquad \Pi^{-1} : \begin{cases} x' = A^{-1}x + \tilde{f}(x,\ u,\ y), \\ u' = D^{-1}u + \tilde{l}(x,\ u,\ y), \\ y' = B^{-1}y + \tilde{g}(x,\ u,\ y), \end{cases}$$

where the functions $\tilde{f}(x, u, y)$, $\tilde{l}(x, u, y)$ and $\tilde{g}(x, u, y)$ are those satisfying the assumptions 1° and 2° of 2 like the initial functions $f(x, u, y)$ $l(x, u, y)$ and $g(x, u, y)$. By the assumptions, it is evident that the characteristic roots of A^{-1}, D^{-1} and B^{-1} are respectively greater than, equal to and less than unity in absolute value. Consequently, again, by our theorem 1, we see that, for the transformation Π^{-1}, there exists an invariant manifold $x=G_1(y)$, $u=G_2(y)$ on which the inverse iteration of Π converges. It will be needless to say that the manifold defined by $x=G_1(y)$, $u=G_2(y)$ is also a manifold invariant under the initial transformation Π.

Returning to the initial variables, the above results are summarized into a

Theorem 2. *In the n-dimensional vector space, there is given a real transformation* $\sum$ *of the form* (1), *where*

p characteristic roots of C are less than 1 *in absolute value*;

q characteristic roots of C and greater than 1 *in absolute value*;

$h(z)$ is a real function satisfying the conditions mentioned in 1.

Then, passing through the origin, there exist two manifold V_1 and V_2 invariant under the transformation $\sum$. V_1 is p-dimensional and the iteration of $\sum$ converges on V_1, on the contrary V_2 is q-dimensional and the iteration of $\sum^{-1}$ converges on V_2.

This theorem is our conclusion.

In conclusion, the auther wishes to express his hearty gratitude to Prof. M. Urabe for his kind guidance and constant advice.

Reference

[1] M. Urabe: *Invariant varieties for finite transformation.* J. Sci. Hiroshima Univ. Ser. A., Vol. 16 (1952), 47–55.

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