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On a Space H^f

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Let $f(x)$ be any locally summable and positive-valued function defined almost everywhere on R^n, Euclidean n-space. Let H^f be a Hilbert space obtained by completing the space $\mathscr{S}E$, the linear space of the inverse Fourier transforms of $\mathscr{D}$, with norm $\|\xi\|_f^2 = \int |\hat{\xi}(x)|^2 f(x)dx$, where $\hat{\xi}$ denotes the Fourier transform of $\xi \in \mathscr{S}E$. Under more special conditions on f, the space H^f has been investigated by J. Deny [1] and B. Malgrange [5] in connection with the study of the potential theory and the theory of partial differential equations respectively. In connection with this situation, we say that f is of Deny type (simply type D) if it satisfies the condition:

(D) $$f(x), \frac{1}{f(x)} \in (1+|x|^2)^m \times L^1 \text{ for an integer } m,$$

where L^1 denotes the space of the summable functions on R^n.

We also say that f is of Malgrange type (simple type M) if it satisfies the condition:

(M) $$f(x), \frac{1}{f(x)} \leqq C(1+|x|^2)^m \text{ a.e. for a constant } C \text{ and an integer } m.$$

Actually Malgrange was concerned with the continuous f of type M.

The purpose of our investigation is to characterize these types of f by means of the properties of H^f and its related spaces.

In Section 1 we show that f is of type D if and only if H^f is a normal space of distributions. If μ is a positive measure with which we define the space H^μ in the same way as before, we can show that μ must be of the form $f(x)dx$ when H^μ is a space of distributions.

In the following sections we shall only be concerned with normal H^f. Section 2 begins with the definition of the space $H_{f,\infty} = \bigcap_s H^{f,s}$ (resp. $H'_{f,\infty} = \bigcup_s H^{f,s}$) with the topology of projective limit (resp. of inductive limit). $H^{f,s}$ stands for H^{f_1}, where $f_1(x) = (1+|x|^2)^s f(x)$ and s is a real number. Then $H_{f,\infty}$ will be a reflexive space of type (F) consisting of the distinguished elements of H^f [6], and $H'_{f,\infty}$ the anti-dual of $H_{1/f,\infty}$. We show that f is of type M if and only if $H_{f,\infty} = \mathscr{D}_{L^2}$ or $H'_{f,\infty} = \mathscr{D}'_{L^2}$.

In Section 3 we show that H^f is of local type if and only if, for some integer m, $\dfrac{1}{(1+|x-y|^2)^m}\sqrt{\dfrac{f(y)}{f(x)}}$ is a kernel of a continuous linear application of L_x^2 into L_y^2. The condition is shown to be satisfied if, for a $k(x)$ such that

$k(x) \in (1+|x|^2)^m \times L^1$ for an integer m, the inequality $f(x+y) \leqq k(x)f(y)$ holds almost everywhere for $|y| \geqq c$. On the other hand, $H_{f,\infty}$ and $H'_{f,\infty}$ are of local type without any restriction on f.

Section 4 is devoted to studying convolution $\xi * \eta$ between elements of $H_{f,\infty}$ and $H'_{1/f,\infty}$. We show that f is of type M if and only if any $\xi \in H^f$ is composable with every $\eta \in H^{1/f}$. However it is to be noticed that by means of the Fourier transformation we can as usual define a convolution $\xi \circledast \eta$ in such a way that the operation $\circledast$ is separately continuous on $H^f \times H^{1/f}$ and coincides with the usual convolution $*$ on $\mathscr{D} \times \mathscr{D}$. We show also that f is of type M if and only if $\mathscr{B}H_{f,\infty} \subset H_{f,\infty}$ or $\mathscr{B}H'_{f,\infty} \subset H'_{f,\infty}$. But we could not succeed in giving the conditions on f under which $\mathscr{B}H^f$ is a part of H^f.

1. The Space of $\mathbf{H}^f$. In what follows by $x, y, \cdots$ we denote respectively points $(x_1, x_2, \cdots, x_n)$, $(y_1, y_2, \cdots, y_n)$, $\cdots$ of the n-dimensional Euclidean space R^n. We use the notations $|x| = (x_1^2 + x_2^2 + \cdots + x_n^2)^{1/2}$, $x \cdot y = x_1y_1 + x_2y_2 + \cdots + x_ny_n$, and if $p = (p_1, p_2, \cdots, p_n)$, where the p_j are non-negative integers, we will write $|p| = p_1 + p_2 + \cdots + p_n$, $D^p = \left(\frac{\partial}{\partial x_1}\right)^{p_1} \left(\frac{\partial}{\partial x_2}\right)^{p_2} \cdots \left(\frac{\partial}{\partial x_n}\right)^{p_n}$.

Let $f(x)$ be a locally summable and positive-valued function defined almost everywhere on R^n. Following F. Trèves [13] we denote by $\mathscr{S}E$ the family of the functions whose inverse Fourier transforms lie in $\mathscr{D}$, the space of the indefinitely differentiable functions with compact support in R^n. We consider $\mathscr{S}E$ as a prehilbert space with inner product $(\xi, \xi')_f = \int \hat{\xi}(x)\bar{\hat{\xi}}'(x)f(x)dx$, and hence with norm $\|\xi\|_f = \left\{\int |\hat{\xi}(x)|^2 f(x)dx\right\}^{\frac{1}{2}}$, where $\hat{\xi}$ denotes the Fourier transform of ξ. We shall denote by H^f a completion of $\mathscr{S}E$ with respect to this norm. According to the usual notations, we shall write H^s if $f(x) = (1+|x|^2)^s$ and $H^{f_1, s}$ if $f(x) = (1+|x|^2)^s f_1(x)$. Following L. Schwartz ([9], p. 7) we say that a locally convex space F is a space of distributions if it is algebraically a subspace of $\mathscr{D}'$ and the injection $F \to \mathscr{D}'$ is continuous, and that F is normal if, in addition, it contains $\mathscr{D}$, the injection $\mathscr{D} \to F$ is continuous, and $\mathscr{D}$ is dense in F. In general H^f, as shown by Trèves ([13], p. 184), is not a space of distributions.

We shall first show

PROPOSITION 1. *The space H^f is a space of distributions if and only if the following condition* (A) *is satisfied*:

(A) $\qquad \dfrac{1}{f} \in (1+|x|^2)^m \times L^1$ *for some integer m.*

Before proving the statement we remark that H^f is a space of distributions if and only if the following conditions are satisfied:

(i) The injection of $\mathscr{S}E$ into $\mathscr{D}'$ is continuous, that is, for any $\varphi \in \mathscr{D}$, $\xi \to \langle \xi, \bar{\varphi} \rangle$ is a continuous linear form on $\mathscr{S}E$.

When this condition is satisfied, the linear forms $\xi \to \langle \xi, \bar{\varphi} \rangle$ can be extended to the forms continuous on the whole space H^f.

(ii) If we are given an $\eta \in H^f$ with $\langle \eta, \bar{\varphi} \rangle = 0$ for all $\varphi \in \mathscr{D}$, then $\eta = 0$.

PROOF OF PROPOSITION 1. *Necessity.* For any $\varphi \in \mathscr{D}$ and any $\xi \in \mathscr{S}E$ we have

$$(1) \qquad \langle \xi, \bar{\varphi} \rangle = \int \hat{\xi}(x)\, \bar{\hat{\varphi}}(x) dx = \int \hat{\xi}(x) \sqrt{f(x)}\, \bar{\hat{\varphi}}(x) \sqrt{g(x)}\, dx, \text{ where } g = \frac{1}{f}.$$

Since by (i) the form $\xi \to \langle \xi, \bar{\varphi} \rangle$ is continuous and the set $\{\hat{\xi}\sqrt{f};\ \xi \in \mathscr{S}E\}$ is dense in L^2, it follows that $\hat{\varphi}\sqrt{g} \in L^2$, Then, by the closed graph theorem, the application $\varphi \to \hat{\varphi}\sqrt{g}$ of $\mathscr{D}$ into L^2 becomes continuous. This implies that the application is continuous in $\mathscr{D}_B$, B being the unit ball of R^n with center 0, with the topology induced by $\mathscr{D}_B^k$ for some integer k ($\mathscr{D}_B^k$ is a Banach space of the k-times continuously differentiable functions with support in B). We can take a positive integer l such that an $\alpha \in \mathscr{D}_B^k$ is a parametrix of an iterated Laplacian Δ^l ($\Delta = \sum_j \left(\frac{1}{2\pi i} \frac{\partial}{\partial x_j} \right)^2$):

$$(2) \qquad \delta = \Delta^l \alpha + \beta, \quad \beta \in \mathscr{D}_B.$$

Now we can choose a sequence $\{\alpha_j\}$, $\alpha_j \in \mathscr{D}_B$, such that $\alpha_j \to \alpha$ in $\mathscr{D}_B^k$ as $j \to \infty$. This together with (2) yeilds that $\hat{\alpha}\sqrt{g}$, $\hat{\beta}\sqrt{g} \in L^2$, and $\sqrt{g} = |x|^{2l}\hat{\alpha}\sqrt{g} + \hat{\beta}\sqrt{g} \in (1 + |x|^2)^l \times L^2$, hence it follows that $g(x) \in (1 + |x|^2)^m \times L^1$ for $m = 2l$.

Sufficiency. To complete the proof it is enough to establish the statements (i) and (ii). By the condition (A), $\hat{\varphi}\sqrt{g} \in L^2$ for any $\varphi \in \mathscr{D}$. As for (i), since $\hat{\xi}\sqrt{f} \in L^2$ for any $\xi \in \mathscr{S}E$, the relation (1) shows that the application $\xi \to \langle \xi, \bar{\varphi} \rangle$ is continuous. Let $\{\xi_j\}$ be a sequence from $\mathscr{S}E$ such that $\xi_j \to \eta$ in H^f, η being any given element of H^f. Since $L^2_{fdx} \subset \sqrt{g} \times L^2 \subset (1 + |x|^2)^m \times L^2 \times L^2 \subset (1 + |x|^2)^m \times L^1 \subset \mathscr{S}'$, the injection $L^2_{fdx} \to \mathscr{S}'$ becomes continuous by the closed graph theorem, so that $\{\hat{\xi}_j\}$ converges in both L^2_{fdx} and $\mathscr{S}'$ to the same element which we shall denote by $\hat{\eta}$. Then the relation (1) gives that $\langle \eta, \bar{\varphi} \rangle = \int \hat{\eta}(x) \sqrt{f(x)}\, \bar{\hat{\varphi}}(x) \sqrt{g(x)}\, dx$, Therefore, if $\langle \eta, \bar{\varphi} \rangle = 0$ for all $\varphi \in \mathscr{D}$, then $\eta = 0$ since $\{\hat{\varphi}\sqrt{g};\ \varphi \in \mathscr{D}\}$ is dense in L^2. (ii) is thus established.

REMARK. If H^f is a space of distributions, its elements are characterized as temperate distributions ξ whose Fourier transforms $\hat{\xi}$ lie in L^2_{fdx}.

In general, even if H^f is a space of distributions, it does not contain $\mathscr{D}$.

PROPOSITION 2. *Let H^f be a space of distributions. Then the following three conditions are equivalent*:

(B) $f \in (1 + |x|^2)^m \times L^1$ *for some integer m.*

(i) $\mathscr{D} \subset H^f$.

(ii) H^f *is normal.*

PROOF. Ad (i)→(B). (i) implies that we have $\hat{\varphi}\sqrt{f} \in L^2$ for every $\varphi \in \mathscr{D}$. Then, by the closed graph theorem, the application $\varphi \to \hat{\varphi}\sqrt{f}$ of $\mathscr{D}$ into L^2 becomes continuous. Hence, as shown in the proof of the proposition 1, we can use a parametrix of an iterated Laplacian to conclude that $f \in (1+|x|^2)^m \times L^1$ for some integer m.

Ad (B) → (ii). Let φ be any function of $\mathscr{S}$. Since $\hat{\varphi} \in \mathscr{S}$ and $\hat{\varphi}\sqrt{f} \in \hat{\varphi}(1+|x|^2)^m \times L^2 \subset L^2$, we have $\mathscr{S} \subset H^f$. $\mathscr{S}$ is dense in H^f as an immediate consequence of the definition of H^f. On the other hand, $\mathscr{D}$ is dense in $\mathscr{S}$ and the injection $\mathscr{S} \to H^f$ is continuous, so that $\mathscr{D}$ is dense in H^f, that is, H^f is normal.

The implication (ii)→(i) is almost evident.

Thus the proof is complete.

The following theorem is an immediate consequence of Proposition 1 and Proposition 2.

THEOREM 1. *H^f is a normal space of distributions if and only if the following condition is satisfied*:

(D) $f, \dfrac{1}{f} \in (1+|x|^2)^m \times L^1$ *for some integer* m.

We say that f is *of Deny type* (simply *type D*) if f satisfies the condition (D), which is the same as Deny called Hypothesis (A) in his thesis ([1], p. 119). The condition (D) shows that if H^f is a normal space of distributions then $H^{1/f}$ is so also.

PROPOSITION 3. (i) *If $f(x)$ is of type D, so is $f(x)(1+|x|^2)^s$ for any real s.*

(ii) *If $f_1(x)$, $f_2(x)$ are of type D, so is $f_1^{1-\theta}(x)f_2^{\theta}(x)$ for any $0<\theta<1$.*

PROOF. Ad (i). Setting $h(x)=f(x)(1+|x|^2)^s$, we have $h(x), \dfrac{1}{h(x)} \in (1+|x|^2)^{m+|s|} \times L^1$.

Ad (ii). As we may assume that integer m in (D) is the same for $f_1(x)$ and $f_2(x)$, so we have

$$f_1^{1-\theta}(x)f_2^{\theta}(x) \leqq (1-\theta)f_1(x)+\theta f_2(x) \in (1+|x|^2)^m \times L^1$$

and also

$$\frac{1}{f_1^{1-\theta}(x)} \frac{1}{f_2^{\theta}(x)} \leqq (1-\theta)\frac{1}{f_1(x)} + \theta\frac{1}{f_2(x)} \in (1+|x|^2)^m \times L^1.$$

EXAMPLE 1. If $f(x)=\exp|x|^2$, then H^f is a space of distributions, but not normal.

EXAMPLE 2. If $f(x)=\exp(x_1+x_2+\cdots+x_n)$, then neither H^f nor $H^{1/f}$ is a space of distributions.

PROPOSITION 4. *Let H^{f_1} and H^{f_2} be two spaces of distributions. We have $H^{f_1} \subset H^{f_2}$ if and only if there exists a constant C such that $f_2(x) \leqq Cf_1(x)$ a.e.*

PROOF. By the closed graph theorem the injection $H^{f_1} \to H^{f_2}$ is continuous, so that there is a constant C such that $\int|\hat{\xi}(x)|^2 f_2(x)dx \leqq C\int|\hat{\xi}(x)|^2 f_1(x)dx$ for any $\xi \in H^{f_1}$. Setting $\sigma(x)=\hat{\xi}(x)\sqrt{f_1(x)}$, we have $\int|\sigma(x)|^2 \frac{f_2(x)}{f_1(x)}dx \leqq C\int|\sigma(x)|^2 dx$ for any $\sigma \in L^2$, whence $\frac{f_2(x)}{f_1(x)} \leqq C$ a.e., which concludes the proof.

We can define H^μ for any positive measure μ in the similar way as H^f is defined. We remark that in order that H^μ may be a space of distributions it is necessary for μ to be absolutely continuous with respect to the ordinary Lebesgue measure. For the proof of this fact, we consider a characteristic function of any compact subset K of R^n and a pointwise convergent sequence $\{\hat{\xi}_j\}$, $\xi_j \in \mathscr{S}E$, to χ such that $|\hat{\xi}_j| \leqq 1$ for any j and the supports of $\hat{\xi}_j$ are contained in a fixed compact subset of R^n. Then $\hat{\xi}_j \rightarrow \chi$ in L^2_μ, as $j \rightarrow \infty$. When H^μ is a space of distributions, $\{\xi_j\}$ converges to a distribution T in H^μ, and *a fortiori* in $\mathscr{D}'$. Then for any $\varphi \in \mathscr{D}$ we have

$$<T, \bar{\varphi}> = \lim_j <\xi_j, \varphi> = \lim_j \int \hat{\xi}_j \bar{\varphi}\, dx = \int_K \bar{\varphi} dx.$$

Hence if K is a null set in the Lebesgue measure, then $\int_K \bar{\varphi} dx = 0$, so that $T=0$. This means that $\int|\chi|^2 d\mu = 0$, and therefore $\mu(K)=0$. Thus μ is absolutely continuous with respect to the Lebesgue measure.

2. $H_{f,\infty}$ and $H'_{f,\infty}$. Throughout the following discussions in this paper we shall be concerned only with normal spaces of distributions H^f. Then, as we see in the preceding section, $H^{f,s}$ also is a normal space of distributions for any real s. Let $H'_{f,\infty}$ be the space $\bigcup_s H^{f,s}$ with the topology of the inductive limit of $\{H^{f,s}\}$, and $H_{f,\infty}$ the space $\bigcap_s H^{f,s}$ with the topology of the projective limit of $\{H^{f,s}\}$. Clearly $H_{f,\infty}$ also is a normal space of distributions and of type (F). It is easy to see that any bounded subset of $H_{f,\infty}$ is weakly relatively compact, so that $H_{f,\infty}$ is reflexive and the strong anti-dual $(H_{f,\infty})'$ of $H_{f,\infty}$ is a complete bornological, barrelled space. $H'_{g,\infty}$, where $g=\frac{1}{f}$, consists of the same elements as the anti-dual $(H_{f,\infty})'$. Both $(H_{f,\infty})'$ and $H'_{g,\infty}$ are bornological and their anti-duals coincide with $H_{f,\infty}$. It follows since any bornological space has Mackey topology that $(H_{f,\infty})'=H'_{g,\infty}$ also holds topologically. In a similar way we have $(H_{g,\infty})'=H'_{f,\infty}$.

We first note that $H_{f,\infty} \subset \dot{\mathscr{B}}$. Let m be a positive integer such that $\frac{1}{(1+|x|^2)^m\sqrt{f(x)}} \in L^2$. Consider any element ξ of $H_{f,\infty}$. By definition we have $\xi \in H^{f,2l}$ for any integer l, that is, $\hat{\xi}(x)(1+|x|^2)^{l-m} \in L^1$, so that $(1-\Delta)^{l-m}\xi$ is a continuous function tending to 0 at infinity. This implies that ξ is an element of $\dot{\mathscr{B}}$.

An element $\xi \in H^f$ belongs to $H_{f,\infty}$ if and only if $D^p\xi \in H^f$ for every p. This is clear from the definition of $H_{f,\infty}$. An element $\xi \in H'_{f,\infty}$ belongs to $H_{f,\infty}$ if and only if there exists a bounded subset B of $H'_{f,\infty}$ such that every $D^p\xi$ is absorbed by B, that is, ξ is a distinguished element of $H'_{f,\infty}$ [6]. In fact, necessity is evident. Sufficiency follows from the fact that any bounded subset B is contained in an $H^{f,s}$.

A distribution ξ belongs to $H'_{f,\infty}$ if and only if $\xi*\varphi \in H^f$ (or $H'_{f,\infty}$) for any $\varphi \in \mathscr{D}$. This is shown by means of a parametrix of an iterated Laplacian as in the preceding section.

PROPOSITION 5. *Let $\mathscr{H}$ be a space of distributions contained in $H'_{f,\infty}$. If $\mathscr{H}$ is of type (F) and closed for differentiation, then $\mathscr{H} \subset H_{f,\infty}$. Thus $H_{f,\infty}$ is the maximal one among such $\mathscr{H}$.*

PROOF. Let $\xi \in \mathscr{H}$. As $\mathscr{H}$ is of type (F), there exists a bounded subset B by which each $D^p\xi$ is absorbed. By the closed graph theorem, the injection $\mathscr{H} \to H'_{f,\infty}$ is continuous, so that B also is bounded in $H'_{f,\infty}$, which implies $\xi \in H_{f,\infty}$. The proof is complete.

PROPOSITION 6. *The following conditions are equivalent to each other:*

(i) $H_{f_1,\infty} \subset H_{f_2,\infty}$.

(ii) $H'_{f_1,\infty} \subset H'_{f_2,\infty}$.

(iii) *There exist a constant C and an integer l such that*

$$\frac{f_2}{f_1} \leqq C(1+|x|^2)^l \quad a.e.$$

PROOF. Ad (i)→(ii). For any $\xi \in H'_{f_1,\infty}$, we have that $\xi*\varphi \in H_{f_1,\infty} \subset H_{f_2,\infty} \subset H'_{f_2,\infty}$, whence $\xi \in H'_{f_2,\infty}$.

Ad (ii)→(iii). (ii) implies that $H^{f_1} \subset H^{f_2,s}$ for some s ([2], Théorème A p. 16). Consequently, by Proposition 4, we have (iii).

(iii)→(i) follows from Proposition 4.

Thus the proof is complete.

As an immediate consequence of Proposition 6 we have

COROLLARY. *The following conditions are equivalent to each other:*

(M) $f, \dfrac{1}{f} \leqq C(1+|x|^2)^l$ *a.e. for some constant* C *and an integer* l.

(i) $H_{f,\infty} = \mathscr{D}_{L^2}$.

(ii) $H'_{f,\infty} = \mathscr{D}'_{L^2}$.

(iii) $H_{f,\infty} = H_{1/f,\infty}$.

(iv) $H'_{f,\infty} = H'_{1/f,\infty}$.

We say that f is *of Malgrange type* (simply *type M*) if it satisfies the condition (M). Malgrange called a continuous function of type M "fonction-poids" ([5], p. 284).

3. Spaces of local type. A space of distributions $\mathscr{H}$ is said to be *of local type* if $\mathscr{D}\mathscr{H}\subset\mathscr{H}$. H^f is not necessarily of local type even when f is of type M. Let $\xi\in H^f$. Setting $\sigma=\hat{\xi}\sqrt{f}\in L^2$, we have for any $\alpha\in\mathscr{D}$

$$(1)\qquad \int|\widehat{\alpha\xi}(x)|^2 f(x)dx=\int\Big|\int\hat{\alpha}(y)\,(1+|y|^2)^m\frac{\sigma(x-y)}{(1+|y|^2)^m}\sqrt{\frac{f(x)}{f(x-y)}}dy\Big|^2dx.$$

We first show

PROPOSITION 7. *H^f is of local type if and only if, for some integer m, $\dfrac{1}{(1+|x-y|^2)^m}\sqrt{\dfrac{f(x)}{f(y)}}$ is a kernel of continuous linear application of L^2_y into L^2_x.*

PROOF. *Sufficiency.* Let $\xi\in H^f$ and $\alpha\in\mathscr{D}$. As $\hat{\alpha}(y)\,(1+|y|^2)^m$ is bounded, it follows from (1) that there is a constant C such that

$$\int|\widehat{\alpha\xi}(x)|^2f(x)dx\leqq C\int\Big|\int\frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)}{f(y)}}\;\sigma(y)dy\Big|^2dx<+\infty,$$

since, by hypothesis, the linear operator generated by the kernel $\dfrac{1}{(1+|x-y|^2)^m}\times\sqrt{\dfrac{f(x)}{f(y)}}$ is continuous on L^2.

Necessity. Let $\xi\in H^f$, $\eta\in H^{1/f}$ and $\alpha\in\mathscr{D}$. We first assume that $\hat{\xi}$, $\hat{\eta}$, and $\hat{\alpha}$ are non-negative. As H^f is of local type, we have $\alpha\xi\in H^f$, so that $(\widehat{\alpha\xi})\,\hat{\eta}\in L^1$, i. e. $(\hat{\alpha}*\hat{\xi})\hat{\eta}\in L^1$. Then, by Fubini theorem,

$$\int(\hat{\alpha}*\hat{\xi})\,\hat{\eta}dx=\int\hat{\alpha}(\hat{\xi}^{\check{}}*\hat{\eta})dx,\ \text{where } \hat{\xi}^{\check{}}(x)=\hat{\xi}(-x),$$

which implies that $\hat{\alpha}(\hat{\xi}^{\check{}}*\hat{\eta})\in L^1$, and hence for any $\beta\in\mathscr{D}$

$$|\hat{\beta}|^2(\hat{\xi}^{\check{}}*\hat{\eta})\in L^1$$

For any α, $\beta\in\mathscr{D}$, on account of the inequality $|\hat{\alpha}\,\hat{\beta}|\leqq|\hat{\alpha}|^2+|\hat{\beta}|^2$ it follows from the above relation that

$$\hat{\alpha}\,\hat{\beta}(\hat{\xi}^{\check{}}*\hat{\eta})\in L^1,$$

whence by making use of a parametrix of an iterated Laplacian as in Section 1 we have

$$\hat{\xi}^{\check{}}*\hat{\eta}\in(1+|x|^2)^m\times L^1\quad\text{for some integer } m\geqq 0.$$

Let ξ (resp. η) be any element of H^f (resp. $H^{1/f}$). Then $\mathscr{F}^{-1}(|\hat{\xi}|)\in H^f$ and $\mathscr{F}^{-1}(|\hat{\eta}|)\in H^{1/f}$. It follows that

$$\hat{\xi}^{\check{}}*\bar{\hat{\eta}}\in(1+|x|^2)^m\times L^1\quad\text{for some integer } m\geqq 0,$$

where m may depend on ξ and η. But we can show that m may be chosen independent of ξ and η. In fact, the application $(\hat{\xi},\hat{\eta})\to\hat{\xi}^{\check{}}*\hat{\eta}$ of $L^2_{fdx}\times L^2_{1/fdx}$ into $\mathscr{D}'$ is continuous, each $(1+|x|^2)^m\times L^1$, $m=1,2,\cdots$, is a Banach space, and the injection $(1+|x|^2)^m \times L^1 \to \mathscr{D}'$ is continuous, whence, by a theorem of Yoshinaga-Ogata ([14], p. 16), we can choose m as desired.

By a change of variables we have

$$(2) \qquad \frac{\hat{\xi}(x)\,\bar{\hat{\eta}}(y)}{(1+|x-y|^2)^m} \in L^1_{x,y} \quad \text{for any } \xi \in H^f \text{ and any } \eta \in H^{1/f},$$

from which, by setting $\sigma = \hat{\xi}\sqrt{f} \in L^2$ and $\tau = \hat{\eta}\frac{1}{\sqrt{f}} \in L^2$ we have

$$\iint \frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{f(y)}{f(x)}}\,|\sigma(x)|\,|\bar{\tau}(y)|\,dxdy < +\infty \quad \text{for any } \sigma,\ \tau \in L^2,$$

which concludes the proof.

REMARK. (i) From the proof of Proposition 7 it is clear that for any $\xi \in H^f$, $\eta \in H^{1/f}$ and $\alpha \in \mathscr{D}$, if H^f is of local type, we have

$$(3) \qquad <\alpha\,\xi,\ \bar{\eta}> = \int \hat{\alpha}(\hat{\xi}^{\check{}} * \bar{\hat{\eta}})dx,$$

where $\hat{\xi}^{\check{}} * \bar{\hat{\eta}} \in (1+|x|^2)^m \times L^1$ for an integer m independent of ξ and η. As a consequence we see that if H^f is of local type, then $\mathscr{S}H^f \subset H^f$, and the equation (3) also holds for any $\alpha \in \mathscr{S}$. Consequently we can define multiplicative product $\xi\bar{\eta}$ for any $\xi \in H^f$ and $\eta \in H^{1/f}$ in the sense of [3]. In fact, $\hat{\xi}^{\check{}}$, $\bar{\hat{\eta}}$ have $(\mathscr{S}')$-convolution, since $(\hat{\xi}*\varphi)\bar{\hat{\eta}} \in L^1 \subset \mathscr{D}'_{L^1}$ for any $\varphi \in \mathscr{S}$ ([12], p. 151).

(ii) Owing to the relation (2), H^f is of local type if and only if there exists a positive integer m such that $(1+|x|^2)^{-m} * \hat{\xi} \in L^2_{fdx}$ for any $\hat{\xi} \in L^2_{fdx}$, i.e. $L_{2m}\xi \in H^f$ for any $\xi \in H^f$, where L_{2m} denotes the Fourier transform of $(1+|x|^2)^{-m}$ ([8], p. 116).

(iii) Let $\{\beta_\varepsilon\}_{0<\varepsilon<1}$ be a family of functions of $\mathscr{D}$ such that the support of β_ε is contained in $B_\varepsilon = \{x\,;\ |x| \leqq \varepsilon\}$, $\beta_\varepsilon \geqq 0$, and $\int \beta_\varepsilon(x)dx = 1$. Further we assume that $\beta_\varepsilon \leqq \frac{M}{\varepsilon^n}$, $\left|\frac{\partial\beta_\varepsilon}{\partial x_j}\right| \leqq \frac{M}{\varepsilon^{n+1}}$ for some constant M. Schwartz ([11], p. 28) has shown that the following inequality holds for some constant C:

$$|\hat{\beta}_\varepsilon(x-y) - \hat{\beta}_\varepsilon(x)|(1+|x|^2)^{\frac{1}{2}} \leqq C(1+|y|^2)^{\frac{1}{2}}.$$

Then, using this inequality and noting that the application $\hat{\xi} \to (1+|x|^2)^{-m} * \hat{\xi}$ of L^2_{fdx} into itself is continuous for large m, we can show that Friedrichs' lemma ([11], p. 27) holds: *Let H^f be of local type, then, for any $\xi \in H^f$, $\beta_\varepsilon*(\alpha\xi) - \alpha(\beta_\varepsilon*\xi)$ tends to zero in $H^{f,1}$ as $\varepsilon \to 0$, where α is any element of $\mathscr{S}$.*

It is easy to see from Proposition 7 that $H^{1/f}$ is of local type if so is H^f.

COROLLARY 1. *If H^f is of local type, then so is $H^{f,s}$ for any real s.*

PROOF. Setting $h(x) = (1+|x|^2)^s f(x)$, and using the inequality $(1+|x|^2)^s \leqq C(1+|y|^2)^s\,(1+|x-y|^2)^{|s|}$, C being a constant, we have

$$\sqrt{\frac{h(x)}{h(y)}} \leqq C\sqrt{\frac{f(x)}{f(y)}}(1+|x-y|^2)^{|s|/2},$$

whence we can choose an integer m such that $\frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{h(x)}{h(y)}}$ is a kernel of a continuous linear operator in L^2.

COROLLARY 2. *If H^{f_1} and H^{f_2} is of local type, then H^f, where $f=f_1^{1-\theta}f_2^{\theta}$ and $0<\theta<1$, also is of local type.*

PROOF. It follows from the inequality

$$\sqrt{\frac{f(x)}{f(y)}} \leqq (1-\theta)\sqrt{\frac{f_1(x)}{f_1(y)}} + \theta\sqrt{\frac{f_2(x)}{f_2(y)}}.$$

We shall give a sufficient condition for H^f to be of local type.

PROPOSITION 8. *H^f is of local type if the following condition is satisfied:*

(D′) $\qquad f(x+y) \leqq k(x)f(y)$ *a.e. for* $|y| \geqq c$,

where $k(x) \in (1+|x|^2)^l \times L^1$, l being an integer, and c is a constant.

PROOF. If we put $k(x)=(1+|x|^2)^l h(x)$, then $h \in L^1$. As $f(x+y) \leqq k(x)f(y)$, we have

$$\frac{f(x)}{f(y)} \leqq (1+|x-y|^2)^l h(x-y) \text{ a.e. for } |y| \geqq c.$$

Choose a positive integer m such that $\frac{\sqrt{f(x)}}{(1+|x|^2)^m} \in L^2$ and $\frac{\sqrt{h(x)}}{(1+|x|^2)^{m-l/2}} \in L^1$. Then, by our hypothesis, we have

$$\frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)}{f(y)}} \leqq \frac{\sqrt{h(x-y)}}{(1+|x-y|^2)^{m-l/2}} \text{ a.e. for } |y| \geqq c.$$

Now using the inequality $(1+|x-y|^2)^{-m} \leqq C_1(1+|y|^2)^m(1+|x|^2)^{-m}$, where C_1 is a constant, we have, for any $\sigma \in L^2$,

$$\begin{aligned}
&\int\left|\int\frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)}{f(y)}}\sigma(x)dx\right|^2 dy \\
&=\int_{|y|\leqq c}\left|\int\frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)}{f(y)}}\sigma(x)dx\right|^2 dy \\
&+\int_{|y|\geqq c}\left|\int\frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)}{f(y)}}\sigma(x)dx\right|^2 dy \equiv I+J,
\end{aligned}$$

but

$$I \leqq C_1^2\int_{|y|\leqq c}\frac{(1+|y|^2)^{2m}}{f(y)}dy\left\{\int\frac{\sqrt{f(x)}}{(1+|x|^2)^m}|\sigma(x)|dx\right\}^2$$

$$\leqq C_2 \left\| \frac{\sqrt{f(x)}}{(1+|x|^2)^m} \right\|_{L^2}^2 \|\sigma\|_{L^2}^2$$

and

$$J \leqq \int \left\{ \int \frac{\sqrt{h(x-y)}}{(1+|x-y|^2)^{m-l/2}} |\sigma(x)| \, dx \right\}^2 dy$$

$$\leqq C_3 \left\| \frac{\sqrt{h(x)}}{(1+|x|^2)^{m-l/2}} \right\|_{L^1}^2 \|\sigma\|_{L^2}^2,$$

where C_2, C_3 are some constants. It follows that

$$I+J \leqq C_4 \left(\left\| \frac{\sqrt{f(x)}}{(1+|x|^2)^m} \right\|_{L^2}^2 + \left\| \frac{\sqrt{h(x)}}{(1+|x|^2)^{m-l/2}} \right\|_{L^1}^2 \right) \|\sigma\|_{L^2}^2,$$

where C_4 is a constant. This yields that $\frac{1}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)}{f(y)}}$ is a kernel of a continuous linear application $L_x^2 \to L_y^2$, so that, by Proposition 7, the space H^f is of local type. The proof is complete.

EXAMPLE. If $f(x)=|x|^\lambda$, $0<\lambda<n$, then the space H^f is of local type. Indeed, since the inequalities $|x+y|^\lambda \leqq 2^\lambda(|x|^\lambda+|y|^\lambda) \leqq 2^\lambda |y|^\lambda(1+|x|^\lambda)$ hold for $|y|\geqq 1$, it follows by setting $k(x)=2^\lambda(1+|x|^\lambda)$, that $f(x+y)\leqq k(x)f(y)$ and $k(x)=2^\lambda(1+|x|^\lambda)(1+|x|^2)^{-n}\times(1+|x|^2)^n \in (1+|x|^2)^n \times L^1$, and hence H^f is of local type.

REMARK. Consider the condition (essentially due to Malgrange ([5], p. 289)):

(M′) $\quad f(x+y) \leqq C(1+|x|^2)^m f(y)$ a.e.,

where C is a constant and m is a positive integer.

If f satisfies (M′), the equation (1) gives

$$\|\alpha\xi\|_f^2 \leqq C\|\sigma\|_{L^2}^2 \left\{ \int |\hat{\alpha}(x)|(1+|x|^2)^m \, dx \right\}^2.$$

Let $\alpha_j(x)=\alpha\left(\frac{x}{j}\right)$, and suppose that α is 1 near the origin. $\{\alpha_j(x)\}$ is a sequence of multiplicators. $\hat{\alpha}_j(x)=j^n\hat{\alpha}(jx)$.
Hence

$$\int |\hat{\alpha}_j(x)|(1+|x|^2)^m dx = \int |\hat{\alpha}(x)| \left(1+\left|\frac{x}{j}\right|^2\right)^m dx,$$

whence $\{\|\alpha_j\xi\|_f\}$ is a bounded sequence. Then, by a theorem of Banach-Steinhaus, we see that H^f has the approximation property by truncation, i. e. $\alpha_j\xi \to \xi$ uniformly in H^f when ξ runs through any compact subset of H^f. On the other hand, the approximation property by regularization is possessed by any H^f.

PROPOSITION 9. *$H_{f,\infty}$ and $H'_{f,\infty}$ are of local type.*

To complete the proof of this proposition it is enough to establish the following proposition.

PROPOSITION 10. *There exists a real number s_0 such that $\mathscr{D}H^{f,s}\subset H^{f,s+s_0}$ for every s. But s_0 may depend on f.*

PROOF. First we note that $\mathscr{D}H^f\subset H^{f,s_0}$ if and only if there exists an integer m such that $\frac{(1+|x|^2)^{s_0/2}}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)}{f(y)}}$ is a kernel of a continuous linear application of L^2_y in L^2_x. The proof is very similar to that of Proposition 7 and will not be supplied here.

As H^f is normal, there exists an integer l such that $f, \frac{1}{f}\in(1+|x|^2)^{2l}\times L^1$. Setting $s_0=-4l$, $m=\left|\frac{s}{2}-l\right|$, we have

$$\frac{(1+|x|^2)^{s_0/2}}{(1+|x-y|^2)^m}\sqrt{\frac{f(x)(1+|x|^2)^s}{f(y)(1+|y|^2)^s}}\leqq C\frac{\sqrt{f(x)}}{(1+|x|^2)^l}\frac{1}{(1+|y|^2)^l\sqrt{f(y)}},$$

where C is a constant such that $(1+|x|^2)^{s/2-l}\leqq C(1+|x-y|^2)^{|s/2-l|}(1+|y|^2)^{s/2-l}$. The right hand side of the above inequality is clearly a kernel of a continuous linear operator in L^2, wich concludes the proof.

REMARK. $\mathscr{S}H_{f,\infty}\subset H_{f,\infty}$. $\mathscr{S}H'_{f,\infty}\subset H'_{f,\infty}$. In fact, $\mathscr{D}H^{f,s}\subset H^{f,s+s_0}$ implies $\mathscr{S}H^{f,s}\subset H^{f,s+s_0}$. This can be shown as in Remark (i) after Proposition 7.

PROPOSITION 11. Let $\dot{\mathscr{B}}H^f\subset H^f$. *Then, setting $g=\frac{1}{f}$, we have*

(i) $\xi\bar{\eta}\in\mathscr{D}'_{L^1}$ *for every* $\xi\in H^f$ *and* $\eta\in H^g$,

(ii) $\mathscr{B}H^f\subset H^f$,

(iii) $\mathscr{B}H^{f,s}\subset H^{f,s}$ *for any real* s,

(iv) $\mathscr{B}H^{g,s}\subset H^{g,s}$ *for any real* s.

PROOF. Ad (i). Let ξ (resp. η) be any element of H^f (resp. H^g). For any $\alpha\in\mathscr{D}$, we have

$$<\alpha\xi,\bar{\eta}>=\int(\hat{\alpha}*\hat{\xi})\bar{\hat{\eta}}dx=\int\hat{\alpha}((\hat{\xi})^{\vee}*\bar{\hat{\eta}})dx=<\alpha,\xi\bar{\eta}>.$$

Since the application $\beta\to\beta\xi$ of $\dot{\mathscr{B}}$ into H^f is continuous by the closed graph theorem, the above relations show that $<\alpha,\xi\bar{\eta}>$ is a continuous form of $\alpha\in\mathscr{D}$ even when we impose on $\mathscr{D}$ the topology of $\dot{\mathscr{B}}$. Since $\mathscr{D}'_{L^1}$ is the dual space of $\dot{\mathscr{B}}$, it follows that $\xi\bar{\eta}\in\mathscr{D}'_{L^1}$.

Ad (ii). Let γ be any element of $\mathscr{B}$. Let $\{\alpha_k\}$ be a sequence from $\mathscr{D}$ with $\alpha_k\gamma\to\gamma$ in $\mathscr{B}_c$. Then, as $\alpha_k\gamma\in\mathscr{D}$, we have $<\alpha_k\gamma\xi,\bar{\eta}>=<\alpha_k\gamma,\xi\bar{\eta}>$. Therefore it follows since $\xi\bar{\eta}\in\mathscr{D}'_{L^1}$ that $\{\alpha_k\gamma\xi\}$ converges weakly to a $\xi'\in H^f$ and $<\xi',\bar{\eta}>=<\gamma,\xi\bar{\eta}>$. On the other hand, $\alpha_k\gamma\xi\to\gamma\xi$ in $\mathscr{D}'$, which implies $\xi'=\gamma\xi$. Thus we have $\mathscr{B}H^f \subset H^f$.

Ad (iii). We first consider the case $0<s<1$. For any $\xi \in H^{f,s}$ we put

$$\|\xi\|^*_{f,s}=\left\{\int \frac{\|\xi_a-\xi\|_f^2}{|a|^{n+2s}}\,da\right\}^{\frac{1}{2}},$$

where $\xi_a(x)=\xi(x+a)$.

As in J. Peetre ([7], p.17), we have after some calculations

$$\|\xi\|^*_{f,s}=J(s)\left\{\int |y|^{2s}|\hat{\xi}(y)|^2 f(y)dy\right\}^{\frac{1}{2}},$$

where $J(s)$ is a constant depending only on s. Therefore in $H^{f,s}$ the norm $\|\cdot\|_{f,s}$ is equivalent to $\{\|\cdot\|_f^2+\|\cdot\|^{*2}_{f,s}\}^{\frac{1}{2}}$. Let β be any element of $\mathscr{B}$. Then for any element ξ of $H^{f,s}$ we have

$$\begin{aligned}\|\beta\xi\|^{*2}_{f,s} &= \int \frac{\|\beta_a\xi_a-\beta\xi\|_f^2}{|a|^{n+2s}}\,da \\ &\leqq 2\int \frac{\|\beta_a(\xi_a-\xi)\|_f^2}{|a|^{n+2s}}\,da+2\int \frac{\|(\beta_a-\beta)\xi\|_f^2}{|a|^{n+2s}}\,da \\ &\equiv I_1+I_2.\end{aligned}$$

Now, as the application $(\beta,\xi)\to\beta\xi$ of $\mathscr{B}\times H^f$ into H^f is continuous by the closed graph theorem, there exists a constant C such that

$$\|\beta_a(\xi_a-\xi)\|_f^2 \leqq C\|\xi_a-\xi\|_f^2$$

and

$$\|(\beta_a-\beta)\xi\|_f^2 \leqq C\min(|a|^2, 1)\|\xi\|_f^2,$$

since $\{\beta_a-\beta\}$ is bounded in $\mathscr{B}$ and we can write $\beta_a-\beta=\sum_{i=1}^{n}a_i\gamma_{i,a}$ with bounded $\gamma_{i,a}\in\mathscr{B}$. Hence we have

$$I_1\leqq C\int \frac{\|\xi_a-\xi\|_f^2}{|a|^{n+2s}}\,da=C\|\xi\|^{*2}_{f,s}<+\infty,$$

and

$$I_2\leqq C\left\{\int_{|a|\leqq 1}\frac{da}{|a|^{n+2s-2}}+\int_{|a|\geqq 1}\frac{da}{|a|^{n+2s}}\right\}\|\xi\|_f^2<+\infty.$$

On account of these inequalities we see that $\|\beta\xi\|^*_{f,s}<+\infty$. We also have that $\|\xi\|_f\leqq\|\xi\|_{f,s}<+\infty$. Hence $\|\beta\xi\|_f^2+\|\beta\xi\|^{*2}_{f,s}<+\infty$. From the remark just given with respect to the equivalent norms of $H^{f,s}$, we see that $\beta\xi\in H^{f,s}$ for any $\xi\in H^{f,s}$.

Next consider the case $s>0$. We choose a positive integer N such that $0<\frac{s}{N}<1$. Then repeating the above process N-times we can conclude that

$\beta\xi \in H^{f,s}$ for any $\xi \in H^{f,s}$.

Finally consider the case $s<0$. As H^g is the anti-dual of H^f, then the adjoint application of $\xi \to \beta\xi$ of H^f into H^f yields $\mathscr{B}H^g \subset H^g$. Then, from the preceding discussions, we have $\mathscr{B}H^{g,-s} \subset H^{g,-s}$, and therefore $\mathscr{B}H^{f,s} \subset H^{f,s}$

Ad (iv). From (iii) we have $\mathscr{B}H^{f,s} \subset H^{f,s}$ for any real s. Then, by considering the adjoint application as in the proof of (iii), we see that $\mathscr{B}H^{g,s} \subset H^{g,s}$ for any real s.

The proof is complete.

Remark. The proof of (iii) can also be carried out by the aid of the interpolation theorm (e.g. [4]). As clear from the proof of the case (iii), it suffices to show that $\mathscr{B}H^{f,s} \subset H^{f,s}$ for any positive s. For any temperate distribution ξ, $\xi \in H^{f,1}$ is equivalent to that $\xi, \frac{\partial\xi}{\partial x_1}, \cdots, \frac{\partial\xi}{\partial x_n} \in H^f$. Suppose that $\mathscr{B}H^f \subset H^f$, for any $\beta \in \mathscr{B}$ and any $\xi \in H^{f,1}$ we have $\frac{\partial}{\partial x_j}(\beta\xi) = \frac{\partial\beta}{\partial x_j}\xi + \beta\frac{\partial\xi}{\partial x_j} \in H^f$, so that $\beta\xi \in H^{f,1}$. By repeating this process we see that if $\mathscr{B}H^f \subset H^f$, then $\mathscr{B}H^{f,m} \subset H^{f,m}$ for any positive integer m. Now we can make use of the interpolation theorem cited above to conclude our assertion.

4. Convolution. We shall first recall the defintion of convolution concerning two distributions S, T. We shall say that S, T are composable provided

(1) $$S(\check{T}*\varphi) \in \mathscr{D}'_{L^1} \text{ for every } \varphi \in \mathscr{D}.$$

If this is the case, the convolution $S*T$ is defined by the equation

$$<S*T, \varphi> = \int S(\check{T}*\varphi)dx.$$

This is the usual convolution due to L. Schwartz [9]. Various conditions equivalent to (1) have been discussed by Shiraishi [12]. However, when convolution is considered as an application, another definition is possible. Let $\mathscr{H}$ and $\mathscr{K}$ be normal spaces of distributions and let $\mathscr{L}$ be a space of distributions. We shall follow Schwartz ([10], p.151) in saying that a bilinear application of $\mathscr{H}\times\mathscr{K}$ into $\mathscr{L}$ is a convolution of $\mathscr{H}\times\mathscr{K}$ into $\mathscr{L}$ if the application is separately continuous and coincides with the usual convolution on $\mathscr{D}\times\mathscr{D}$. For our temporary purpose such convolution will be denoted by $\circledast$ and we shall take $\mathscr{L}$ for $\mathscr{D}'$, in the following discussions.

If we are given a subset E of $\mathscr{D}'$, we shall denote by E^* the set of the distributions composable with every element of E. It follows from (1) that E^* is a linear space stable for differentiation, and $\mathscr{B}E^* \subset E^*$. In the following we shall write $g = \frac{1}{f}$.

Proposition 12. (i) $(H^f)^* = (H'_{f,\infty})^* = (H_{f,\infty})^*$. (ii) $(H^f)^* \subset H'_{g,\infty}$.

PROOF. (i) is clear from the fact that S, T are composable if and only if $S*\varphi$, T are composable for any $\varphi \in \mathscr{D}$. As for (ii), let η be any element of $(H^f)^*$. Then by (1) we have $\xi(\check{\eta}*\varphi) \in \mathscr{D}'_{L^1}$ for every $\xi \in H^f$ and every $\varphi \in \mathscr{D}$. Hence $\overline{\check{\eta}*\varphi} \in H^g$ since $\xi \to \xi(\check{\eta}*\varphi)$ is a continuous application of H^f into $\mathscr{D}'_{L^1}$, so that $\check{\eta} \in H'_{g,\infty}$ and in turn $\eta \in H'_{g,\infty}$, as desired.

Now we shall show

THEOREM 2. *f is of type M if and only if any of the following equivalent conditions holds:*

(i) $(H^f)^* \supset H^g$.

(ii) $(H^f)^* = H'_{g,\infty}$. (ii)′ $(H^g)^* = H'_{f,\infty}$.

(iii) $\mathscr{B}H'_{f,\infty} \subset H'_{f,\infty}$. (iii)′ $\mathscr{B}H'_{g,\infty} \subset H'_{g,\infty}$.

(iv) $\mathscr{B}H_{f,\infty} \subset H_{f,\infty}$. (iv)′ $\mathscr{B}H_{g,\infty} \subset H_{g,\infty}$.

(v) $\mathscr{B}H^f \subset H^{f,s_0}$ *for some real* s_0. (v)′ $\mathscr{B}H^g \subset H^{g,s_0}$ *for some real* s_0.

PROOF. Ad (i) → (ii). This follows from the fact that η lies in $H'_{g,\infty}$ if and only if $\eta*\varphi \in H^g$ for any $\varphi \in \mathscr{D}$.

Ad (ii)→(iii)′. This is clear, because, for any $E \subset \mathscr{D}'$, E^* is stable for multiplication by any element of $\mathscr{B}$.

Ad (iii)′⇄(iv). This equivalence is obtained by considering the adjoint application of the multiplications by elements of $\mathscr{B}$.

Ad (iii)′→(v)′. (iii)′ implies that $\mathscr{B}H^g \subset H'_{g,\infty}$. As $\mathscr{B}$, H^g are spaces of type (F) and $H'_{g,\infty}$ is a space of type (LF), so we have $\mathscr{B}H^g \subset H^{g,s_0}$ for some real s_0 [14].

Assume that (v)′ holds. By the closed graph theorem the application $(\beta,\xi) \to \beta\xi$ of $\mathscr{B} \times H^g$ into H^{g,s_0} is continuous, and therefore there exist a constant C_1 and a postitive integer m such that for any $\xi \in H^g$

$$\|\beta\xi\|^2_{g,s_0} \leqq C_1\|\xi\|^2_g \max\|D^p\beta\|^2_{L^\infty},\ |p| \leqq 2m.$$

Consider the set Φ of the functions $\left\{\frac{e^{2\pi i x\cdot t}}{(1+|t|^2)^m}\right\}$, where t is a parameter running through R^n.

Then the set of functions

$$\left\{D^p \frac{e^{2\pi i x\cdot t}}{(1+|t|^2)^m};\, |p| \leqq 2m,\ t \in R^n\right\}$$

is uniformly bounded, whence for a constant C_2

$$\left\| \frac{e^{2\pi i x\cdot t}}{(1+|t|^2)^m}\xi\right\|^2_{g,s_0} \leqq C_2\|\xi\|^2_g$$

for any $\xi \in H^g$ and any $t \in R^n$.

Consequently,

$$\int \frac{|\hat{\xi}(x-t)|^2(1+|x|^2)^{s_0}g(x)}{(1+|t|^2)^{2m}}dx \leqq C_2\int |\hat{\xi}(x)|^2 g(x)dx,$$

which implies for every $t \in R^n$

$$\frac{(1+|x+t|^2)^{s_0}}{(1+|t|^2)^{2m}}\frac{g(x+t)}{g(x)} \leqq C_2 \quad \text{a.e.}$$

Therefore for any x_0 with $g(x_0) \neq 0, \infty$, we have

$$g(x_0+t) \leqq C_2\, g(x_0)\frac{(1+|t|^2)^{2m}}{(1+|x_0+t|^2)^{s_0}} \quad \text{a.e.}$$

If we put $x=x_0+t$, then for some constant C' and a positive integer l'

$$g(x) \leqq C'(1+|x|^2)^{l'} \quad \text{a.e.}$$

As $\mathscr{B}H^g \subset H^{g,s_0}$, then $\mathscr{B}H^{f,-s_0} \subset H^f$. By repeating a similar reasoning as above, we have

$$f(x) \leqq C''(1+|x|^2)^{l''} \quad \text{a.e.,}$$

for some constant C'' and a positive integer l''. Thus we see that f is of type M.

If f is of type M, then $H_{f,\infty} = H_{g,\infty} = \mathscr{D}_{L^2}$, and $H'_{f,\infty} = H'_{g,\infty} = \mathscr{D}'_{L^2}$, so that $(H^f)^* = (\mathscr{D}'_{L^2})^* = \mathscr{D}'_{L^2} = H'_{g,\infty} \supset H^g$.

Now, by definition, f is of type M if and only if g is of type M. Hence the substitution of f by g in the above discusions will complete the proof of the theorem.

Remark. The condition (v) of the theorem implies that $\mathscr{B}H^{f,s} \subset H^{f,s+s_0}$ for every real s. This can be shown by an interpolation theorem as indicated in the remark after Proposition 11. In general, s_0 cannot be chosen to be zero. For, suppose the contrary. Every H^f, f being of type M, would be of local type. However, this is not the case.

Proposition 13. *$(H^f)^* \supset H^f$ if and only if there exist a constant C and a real s_0 such that*

$$f(x) \geqq C(1+|x|^2)^{s_0} \quad \text{a.e.} \tag{2}$$

Proof. *Necessity.* $H^f \subset (H^f)^* \subset H'_{g,\infty}$. Hence $H_{f,\infty} \subset H_{g,\infty}$ by Proposition 5, and we obtain by Proposition 6

$$\frac{g(x)}{f(x)} \leqq C_1(1+|x|^2)^s \quad \text{a.e.}$$

for a constant C_1 and a real s. Thus we have (2).

Sufficiency. (2) implies $H^f \subset H^{s_0} \subset \mathscr{D}'_{L^2}$. Since any two distributions of $\mathscr{D}'_{L^2}$

are composable, so we have $H^f \subset (H^f)^*$, and our proof is complete.

COROLLARY. *$(H^f)^* = H'_{f,\infty}$ if and only if f is of type M.*

PROOF. It is enough to show the "only if" part. By Proposision 13 we have $f \geqq C(1+|x|^2)^{s_0}$ for a real s_0, hence by Proposition 6 $H^f \subset H'_{f,\infty} \subset \mathscr{D}'_{L^2}$, which implies that $H'_{f,\infty} = (H^f)^* \supset (\mathscr{D}'_{L^2})^* = \mathscr{D}'_{L^2}$. Consequently we have $H'_{f,\infty} = \mathscr{D}'_{L^2}$. Then, by the Corollary to Proposition 6, we see that f is of type M.

If we define $\xi \circledast \eta = \mathscr{F}^{-1}(\hat{\xi}\hat{\eta})$, where $\xi \in H'_{f,\infty}$ and $\eta \in H'_{g,\infty}$, then it is not difficult to see that $\circledast$ is a convolution of $H'_{f,\infty} \times H'_{g,\infty}$ into $\mathscr{D}'$. However, as Theorem 2 shows, the application $\circledast$ coincides with the usual convolution $*$ if and only if f is of type M.

Finally we shall conclude this section by stating a sufficient condition for a convolution of $\mathscr{H} \times \mathscr{K}$ into $\mathscr{D}'$ to be well defined, which will also be applied to the case where $\mathscr{H} = H'_{f,\infty}$ and $\mathscr{K} = H'_{g,\infty}$.

PROPOSITION 14. *Let $\mathscr{H}$, $\mathscr{K}$ be normal spaces of distributions. Let $\mathscr{H}$ be barrelled. Assume that the application $(T,\varphi) \to T\check{\ } * \varphi$ of $\mathscr{K} \times \mathscr{D}$ into $\mathscr{H}'$ (the strong dual of $\mathscr{H}$) is hypocontinuous. Then the application $\circledast$ defined by the following relation is a convolution of $\mathscr{H} \times \mathscr{K}$ into $\mathscr{D}'$:*

$$<S \circledast T, \varphi> = <S, T\check{\ } * \varphi>, \ S \in \mathscr{H},\ T \in \mathscr{K} \text{ and } \varphi \in \mathscr{D}.$$

Furthermore if $\mathscr{H}$ possesses the approximation property by truncation, then $S*T$, if it exists, coincides with $S \circledast T$.

PROOF. It is evident that $\circledast$ coincides with $*$ on $\mathscr{D} \times \mathscr{D}$. Let C be any compact disk of $\mathscr{D}$. If $T \to 0$ in $\mathscr{K}$, then $T\check{\ } * C \to 0$ in $\mathscr{H}'$ since the application $(T,\varphi) \to T\check{\ } * \varphi$ is hypocontinuous. Hence $<S, T\check{\ } * C> \to 0$ for any $S \in \mathscr{H}$. If $S \to 0$ in $\mathscr{H}$ and T is a fixed element of $\mathscr{K}$, then $T\check{\ } * C$ is a compact disk of $\mathscr{H}'$, and hence an equicontinuous subset of $\mathscr{H}'$, so that $<S, T\check{\ } * C> \to 0$ as $S \to 0$. Thus we have shown that $\circledast$ is separately continuous.

For the proof of the last part of the statements we use the notations $<\ ,\ >_{\mathscr{K}',\mathscr{K}}$, $<\ ,\ >_{\mathscr{D},\mathscr{D}'}$ to make clear the duality between the spaces of distributions under question. Suppose $S*T$ exists, that is, $S(T\check{\ } * \varphi) \in \mathscr{D}'_{L^1}$ for any $\varphi \in \mathscr{D}$. Let $\{\alpha_k\}$ be a sequence of multiplicators such that $\alpha_k \to 1$ in $\mathscr{B}_c$ and $\alpha_k S \to S$ in $\mathscr{H}$ as $k \to \infty$. Then $<S \circledast T, \varphi>_{\mathscr{D}',\mathscr{D}} = <S, T\check{\ } * \varphi>_{\mathscr{K},\mathscr{K}'} = \lim_k <\alpha_k S, T\check{\ } * \varphi>_{\mathscr{K},\mathscr{K}'} =$

$$\lim_k <S, \alpha_k(T\check{\ } * \varphi)>_{\mathscr{D}',\mathscr{D}} = \lim_k \int \alpha_k S(T\check{\ } * \varphi)dx = \int S(T\check{\ } * \varphi)dx = <S*T, \varphi>_{\mathscr{D}',\mathscr{D}}.$$

Therefore $S \circledast T = S*T$, as desired.

References

[1] J. Deny, *Les potentiels d'énergie fini*, Acta Math., 82 (1950), 107–183.

[2] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., No. 16 (1955).

[3] Y. Hirata and H. Ogata, *On the exchange formula for distributions*. This Journal, 22 (1958), 147–152.

[4] J. L. Lions, *Espaces intermédiaires entre espaces hilbertiens et applications*, Bull. Math. Soc. Sci Math. Phys. R. P. R., 2 (1958), 419–432.
[5] B. Malgrange, *Sur une classe d'opérateurs différentiels hypoelliptiques*, Bull. Soc. Math. France, 85 (1957), 283–306.
[6] K. Miyazaki, *Distinguished elements in a space of distributions*, This Journal, 23 (1960), 527–533.
[7] J. Peetre, *Théorèmes de régularité pour quelques classes d'opérateurs différentiels*, Medd. Lunds Univ. Math. Sem., 16 (1959), 1–122.
[8] L. Schwartz, *Théorie des distributions*, t. I et II, Paris, Hermann, 1950–1951.
[9] ————, *Théorie des distibutions à valeurs vectorielles*, Ann. Inst. Fourier, chap. I, 7 (1957), 1–141.
[10] ————, *Théorie des distributions à valeurs vectorielles*, Ann. Inst. Fourier, chap. II, 8 (1958), 1–209.
[11] ————, *Ecuaciones diferenciales parciales elipticas*, Bogota, Colombia, 1956.
[12] R. Shiraishi, *On the definition of convolutions for distributions*, This Journal, 23 (1957), 19–32.
[13] F. Trèves, *Preblèmes des Cauchy et problèmes mixtes en théorie des distributions*, J. Analyse Math., VII (1959), 105–188.
[14] K. Yoshinaga and H. Ogata, *On convolutions*, This Journal, 22 (1958), 15–24.

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