# The Potential Force Yielding a Periodic Motion with Arbitrary Continuous Half-Periods

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## 1. Introduction

In the previous papers [3, 4], the author has given a method to determine the potential force $g(x)$ so that the period of the periodic solution of the equation

$$(1.1) \qquad \frac{d^2x}{dt^2} + g(x) = 0$$

may be an arbitrary given continuous function of the amplitude of the velocity or an arbitrary given continuously differentiable function of the amplitude.

Let $R$ be the maximum velocity (i.e. the amplitude of the velocity) and let $a$ and $b$ be respectively the positive maximum and negative minimum displacement of $x$. Let $\tau_1/2$ and $\tau_2/2$ be respectively the times required to reach the state of the positive maximum displacement $a$ and the state of the negative minimum displacement $b$ from the equilibrium point $x=0$.

In his paper [1], Z. Opial called the quantities $\tau_1$ and $\tau_2$ respectively the positive half-period and the negative half-period, and discussed the various relations between these half-periods and the potential force $g(x)$.

The half periods $\tau_i$ ($i=1, 2$) are the functions of $R$. Further the positive half-period $\tau_1$ and the negative half-period $\tau_2$ are also respectively the functions of the positive maximum displacement $a$ and the negative minimum displacement $b$.

Opial has proved under very mild conditions that $g(x)$ is uniquely fixed if $\tau_1 = \hat{\tau}_1(a)$ and $\tau_2 = \hat{\tau}_2(b)$ are given. But he has not given a method to determine $g(x)$ for which any solution of (1.1) has the given arbitrary $\hat{\tau}_1(a)$ and $\hat{\tau}_2(b)$.

In the present paper, by means of the techniques used in his previous papers [2, 3, 4], the author will give a method to determine $g(x)$ so that either

1° $\tau_1$ and $\tau_2$ may be respectively arbitrary given continuous functions $\tilde{\tau}_1(R)$ and $\tilde{\tau}_2(R)$, or

2° $\tau_1$ and $\tau_2$ may be respectively arbitrary given continuously differentiable functions $\hat{\tau}_1(a)$ and $\hat{\tau}_2(b)$ whose derivatives fulfill the Lipschitz condition.

The problem of "*tautochronism*" [1] is to determine $g(x)$ so that $\tilde{\tau}_1(R)$ and

$\tilde{\tau}_2(R)$ may be constants.   The answer to this problem is readily derived from our general results.

As is seen in the previous papers [2, 3, 4], the potential force $g(x)$ is characterized by the odd function $S(X)$ and the even function $T(X)^{1)}$.   In the previous papers [3, 4], it has been shown that the continuous behavior of the period determines the even function $T(X)$ but it does not give any affection to the odd function $S(X)^{1)}$.   In the present paper, it will be shown that the odd function $S(X)$ is characterized by the difference of two half-periods.   Evidently the period is a sum of both half-periods.   Thus we can say that the even function $T(X)$ and the odd function $S(X)$ by two of which the potential force $g(x)$ is determined are respectively characterized by the sum and the difference of two half-periods.

## 2.  Preliminary theorems

The works of the present paper are based on the theorems in the paper [3] and the lemma in the paper [4].   So, for the convenience of the readers, they are restated in the present paragraph.

THEOREM A.   *Given the integral equation*

$$(2.1) \qquad \int_0^R \frac{T(X)}{\sqrt{R^2-X^2}}\,dX = f(R) \qquad (R \geqq 0).$$

*If $f(R)$ is continuous, the continuous solution $T(X)$ of (2.1), if it exists, is uniquely determined by $f(R)$ and given by*

$$(2.2) \qquad T(X) = \frac{2}{\pi} \cdot \frac{d}{dX} \int_0^X \frac{f(R)}{\sqrt{X^2-R^2}}\,R\,dR \qquad (X > 0).$$

*Conversely, if the function $T(X)$ defined by (2.2) is continuous, then this function is actually a solution of (2.1) for a continuous $f(R)$.*

*If $f(R) \in C_R^1$ for $R \geqq 0$, then the function $T(X)$ given by*

$$(2.3) \qquad T(X) = \frac{2}{\pi} f(0) + \frac{2}{\pi} X \int_0^X \frac{f'(R)}{\sqrt{X^2-R^2}}\,dR \qquad (X \geqq 0)$$

*which is derived from (2.2) by integration by parts, is the unique solution of (2.1).*

THEOREM B.   *In case $g(x)$ is continuous in the neighborhood of $x = 0$ and differentiable at $x=0$, if any solution of the equation (1.1) near $x=\dot{x}=0$ ($\cdot = d/dt$) oscillates around $x=\dot{x}=0$ with a bounded period, then*

     1°   *the period $\omega (\geqq 0)$ is expressed as*

$$(2.4) \qquad \omega = \Omega(R),$$

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1)   For instance, see Theorem B of the present paper.

*where R is the maximum velocity (namely the amplitude of the velocity)*

$$(2.5) \qquad R = \left| \left[ \frac{dx}{dt} \right]_{x=0} \right|,$$

*and*

$$(2.6) \qquad \Omega(R) \, \epsilon \, C_R, \quad \Omega(0) = \omega_0 > 0 ;$$

$2°$   *the function $g(x)$ satisfies the functional equation*

$$(2.7) \qquad g\left\{ \frac{\omega_0}{2\pi} \int_0^X [1 + S(u) + T(u)] du \right\} = \frac{2\pi}{\omega_0} \cdot \frac{X}{1 + S(X) + T(X)} \, ,$$

*where $S(X)$ is a continuous odd function and $T(X)$ is a continuous even function such that $T(0)=0$ and*

$$(2.8) \qquad T(X) = -\frac{1}{\omega_0} \cdot \frac{d}{dX} \int_0^X \frac{\Omega(R) - \omega_0}{\sqrt{X^2 - R^2}} R \, dR \quad for \quad X > 0.$$

*Conversely, given any function $\Omega(R)$ for which (2.6) holds, if the even function $T(X)$ defined by (2.8) is continuous, then the function $g(x)$ which is determined by the functional equation (2.7) for an arbitrary continuous odd function $S(X)$ and for the continuous even function $T(X)$ defined by (2.8), is continuous in the neighborhood of $x=0$ and is differentiable at $x=0$.   Furthermore, for this $g(x)$, any solution of the equation (1.1) near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with the given period $\omega=\Omega(R)$.*

*In case $\Omega(R) \, \epsilon \, C_R^1$ for $R \geqq 0$, the relation (2.8) can be replaced by*

$$(2.9) \qquad T(X) = \frac{1}{\omega_0} X \int_0^X \frac{\Omega'(R)}{\sqrt{X^2 - R^2}} \, dR \quad for \quad X \geqq 0,$$

*whose right member is continuous. Consequently, for any given $\Omega(R) \, \epsilon \, C_R^1$ with $\Omega(0) = \omega_0 > 0$, there always exists a continuous potential force $g(x)$ which is differentiable at $x=0$ and for which any solution of (1.1) near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with the given period $\omega=\Omega(R)$.*

THEOREM C.   *Given an integral equation*

$$(2.10) \qquad T(X) = \frac{X}{2\pi} \int_0^X \frac{1 + T(R)}{\sqrt{X^2 - R^2}} F\left[ \frac{\omega_0}{2\pi} \int_0^R \{1 + T(u)\} du \right] dR \qquad (\omega_0 > 0),$$

*where $F(a)$ is a given function defined on $I[0, l]$ $(l > 0)$ satisfying the Lipschitz condition:*

$$(2.11) \qquad |F(a') - F(a'')| \leqq L \, |a' - a''| \quad for \quad {}^\forall a', a'' \, \epsilon \, I \qquad (L > 0).$$

*Let*

$$(2.12) \qquad M = \max_{a \in I} |F(a)|.$$

*Then the integral equation* (2.10) *has, on* $J[0, \alpha]$, *one and only one continuous solution* $T(X)$ *such that*

(2.13) $$|T(X)| \leqq K \quad for \quad {}^{\forall}X \in J,$$

*where*

(2.14) $$\alpha = \max_{0 \leqq \kappa \leqq 1} \min \left[ \frac{2\pi l}{\omega_0(1+\kappa)}, \ \frac{4\kappa}{M(1+\kappa)}, \right.$$

$$\left. \frac{\pi^2 M}{2L\omega_0(1+\kappa)} \left\{ \sqrt{1 + \frac{16 kL\omega_0(1+\kappa)}{\pi^2 M^2}} - 1 \right\} \right]$$

$k$ *being a positive constant less than unity and* $K$ *is any value of non-negative* $\kappa \leqq 1$ *for which the minimum of the right member of* (2.14) *equals* $\alpha$.

Theorem A and Theorem B are respectively Theorem 1 and Theorem 2 of [3] and Theorem C is the Lemma of [4] in the case where $S(X) \equiv 0$.

## 3.   The relations between $g(x)$ and $\tilde{\tau}_i(R)$   ($i = 1, 2$)

Corresponding to Theorem B, we have

THEOREM 1.   *In case* $g(x)$ *is continuous in the neighborhood of* $x = 0$ *and differentiable at* $x = 0$, *if any solution of the equation* (1.1) *near* $x = \dot{x} = 0$ *oscillates around* $x = \dot{x} = 0$ *with a bounded period, then*
$1°$   *the half-periods* $\tau_1$ *and* $\tau_2$ *are expressed as*

(3.1) $$\tau_i = \tilde{\tau}_i(R) \qquad (i = 1, 2),$$

*where* $R$ *is the maximum velocity*

$$R = \left| \left[ \frac{dx}{dt} \right]_{x=0} \right|$$

*and*

(3.2) $$\tilde{\tau}_i(R) \in C_R, \quad \tilde{\tau}_i(0) = \tau_0 = \frac{\omega_0}{2} > 0 \qquad (i = 1, 2);$$

$2°$   *the positive maximum displacement* $a$ *and the negative minimum displacement* $b$ *of* $x$ *are connected with* $R$ *as*

(3.3) $$\begin{cases} a = \dfrac{\tau_0}{\pi} \displaystyle\int_0^R [1 + V_1(u)] du, \\[4mm] b = -\dfrac{\tau_0}{\pi} \displaystyle\int_0^R [1 + V_2(u)] du \end{cases}$$

*and the function* $g(x)$ *satisfies the functional equations*

$$(3.4) \quad \begin{cases} g\left\{\dfrac{\tau_0}{\pi}\displaystyle\int_0^X [1+V_1(u)]\,du\right\} = \dfrac{\pi}{\tau_0}\cdot\dfrac{X}{1+V_1(X)}, \\[2mm] g\left\{-\dfrac{\tau_0}{\pi}\displaystyle\int_0^X [1+V_2(u)]\,du\right\} = -\dfrac{\pi}{\tau_0}\cdot\dfrac{X}{1+V_2(X)} \\[2mm] \hspace{5cm} for \quad X \geqq 0, \end{cases}$$

*where $V_1(X)$ and $V_2(X)$ are the continuous functions such that $V_1(0)=V_2(0)=0$ and*

$$(3.5) \qquad V_i(X) = \frac{1}{\tau_0}\cdot\frac{d}{dX}\int_0^X \frac{\tilde{\tau}_i(R)-\tau_0}{\sqrt{X^2-R^2}}R\,dR \quad for \quad X>0$$

$$(i=1, 2);$$

$3°$  *if $\tilde{\tau}_i(R) \in C_R^1$ $(i=1, 2)$, (3.5) can be written as*

$$(3.6) \qquad V_i(X) = \frac{1}{\tau_0}\cdot X\int_0^X \frac{\tilde{\tau}_i'(R)}{\sqrt{X^2-R^2}}\,dR \quad for \quad X>0$$

$$(i=1, 2).$$

*Conversely, given $\tilde{\tau}_i(R) \in C_R$ $(i=1, 2)$ such that $\tilde{\tau}_i(0)=\tau_0\neq0$ $(i=1, 2)$ and either $V_i(X)$ $(i=1, 2)$ determined by (3.5) are continuous or $\tilde{\tau}_i(R) \in C_R^1$ $(i=1, 2)$, there exists a unique continuous potential force $g(x)$ which is differentiable at $x=0$ and for which any solution of (1.1) near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with the given half-periods $\tilde{\tau}_i(R)$ $(i=1, 2)$. In this case, the function $g(x)$ is determined by the functional equations (3.4) for $V_i(X)$ $(i=1, 2)$ determined by (3.5) or (3.6).*

PROOF.   If we write the equation (1.1) in a simultaneous form as

$$(3.7) \qquad \begin{cases} \dfrac{dx}{dt} = y, \\[2mm] \dfrac{dy}{dt} = -g(x), \end{cases}$$

then the periodic solutions of (1.1) are represented by the closed orbits

$$(3.8) \qquad \frac{1}{2}y^2 + G(x) = \text{const.}$$

in the phase plane, where

$$(3.9) \qquad G(x) = \int_0^x g(u)\,du.$$

Hence we have

$$(3.10) \qquad \frac{1}{2}R^2 = G(a) = G(b) \qquad (b<0<a,\ R>0)$$

and

$$(3.11) \quad \begin{cases} \tau_1 = 2 \displaystyle\int_0^a \frac{dx}{\sqrt{R^2 - 2G(x)}}, \\[2mm] \tau_2 = -2 \displaystyle\int_0^b \frac{dx}{\sqrt{R^2 - 2G(x)}}. \end{cases}$$

If we put

$$(3.12) \quad x = \frac{\omega_0}{2\pi} \int_0^X [1 + S(u) + T(u)] du,$$

then

$$\frac{dx}{dX} = \frac{\omega_0}{2\pi} [1 + S(X) + T(X)],$$

from which, by (2.7), follows

$$g(x) \frac{dx}{dX} = X.$$

Consequently, by (3.9) and (3.12), it holds that

$$(3.13) \quad G(x) = \frac{1}{2} X^2.$$

Comparing this with (3.10), from (3.12), we have

$$(3.14) \quad \begin{cases} a = \dfrac{\omega_0}{2\pi} \displaystyle\int_0^R [1 + S(u) + T(u)] du, \\[3mm] b = \dfrac{\omega_0}{2\pi} \displaystyle\int_0^{-R} [1 + S(u) + T(u)] du \\[3mm] \quad = -\dfrac{\omega_0}{2\pi} \displaystyle\int_0^R [1 - S(u) + T(u)] du, \end{cases}$$

because $G'(x) = g(x) \neq 0$ for $x \neq 0$ as is seen from (2.7).

By the substitution (3.12) and (3.14), the formulas (3.11) can be rewritten as follows:

$$(3.15) \quad \begin{cases} \tau_1 = \tilde{\tau}_1(R) = \dfrac{\omega_0}{\pi} \displaystyle\int_0^R \frac{1 + S(X) + T(X)}{\sqrt{R^2 - X^2}} dX, \\[3mm] \tau_2 = \tilde{\tau}_2(R) = \dfrac{\omega_0}{\pi} \displaystyle\int_0^R \frac{1 - S(X) + T(X)}{\sqrt{R^2 - X^2}} dX. \end{cases}$$

Let us put

$$(3.16) \quad \begin{cases} V_1(X) = S(X) + T(X), \\ V_2(X) = -S(X) + T(X), \end{cases}$$

then (3.15) can be written as

$$(3.17) \qquad \tau_i = \tilde{\tau}_i(R) = \frac{\omega_0}{\pi} \int_0^R \frac{1 + V_i(X)}{\sqrt{R^2 - X^2}} dX \qquad (i = 1, 2).$$

This can be rewritten also in the forms

$$\tau_i = \tilde{\tau}_i(R) = \frac{\omega_0}{\pi} \int_0^{\pi/2} [1 + V_i(R \cos \varphi)] d\varphi \qquad (i = 1, 2).$$

From this readily follows (3.2).

By the substitution (3.16), the equalities (3.3) and (3.4) follow respectively from (3.14) and (2.7).

Since $S(X)$, $T(X) \in C_X$ and $S(0) = T(0) = 0$, it is evident that

$$V_1(X), \quad V_2(X) \in C_X, \quad V_1(0) = V_2(0) = 0.$$

By (3.2), the equalities (3.17) can be written as

$$(3.18) \qquad \int_0^R \frac{V_i(X)}{\sqrt{R^2 - X^2}} dX = \frac{\pi}{2\tau_0} [\tilde{\tau}_i(R) - \tau_0] \qquad (i = 1, 2).$$

These are of the form (2.1).  Therefore, applying the formulas (2.2) and (2.3) to (3.18), we have (3.5) and (3.6).

The converse parts of the theorem are proved as follows.

For the continuous functions $V_i(X)$ $(i = 1, 2)$ determined by (3.5) or (3.6), let us determine the function $g(x)$ by (3.4) as follows:

for $x \geq 0$,

$$(3.19) \qquad g(x) = \frac{\pi}{\tau_0} \cdot \frac{X}{1 + V_1(X)}, \qquad x = \frac{\tau_0}{\pi} \int_0^X [1 + V_1(u)] du \qquad (X \geq 0);$$

for $x \leq 0$,

$$(3.20) \qquad g(x) = -\frac{\pi}{\tau_0} \cdot \frac{X}{1 + V_2(X)}, \qquad x = -\frac{\tau_0}{\pi} \int_0^X [1 + V_2(u)] du \qquad (X \geq 0).$$

Then, for either of (3.19) and (3.20), it holds that

$$g(x) \cdot \frac{dx}{dX} = X,$$

from which follows (3.13).  Then, by means of the latter equalities of (3.19) and (3.20), we have (3.3) similarly as (3.14).

For any solution of (1.1) with $g(x)$ determined by (3.19) and (3.20), the half-periods $\tau_i$ $(i = 1, 2)$ are given by (3.11). Then, substituting (3.13), (3.3) and the latter equalities of (3.19) and (3.20) into (3.11), we have

$$(3.21) \qquad \tau_i = \frac{2\tau_0}{\pi} \int_0^R \frac{1 + V_i(X)}{\sqrt{R^2 - X^2}} dX \qquad (i = 1, 2).$$

This is (3.17) themselves.   Thus, for continuous $V_i(X)$ $(i=1, 2)$ determined by (3.5) or (3.6),

$$\tau_i = \tilde{\tau}_i(R) \qquad (i = 1, 2).$$

The uniqueness of $g(x)$ follows readily from the uniqueness of $V_i(X)$ $(i=1, 2)$.

Thus the theorem has been proved completely.

THEOREM 2.   *Assume that the function $g(x)$ satisfies the conditions as follows:*

(i)   $g(x) \in C_x$ *for $x \geq 0$ in the neighborhood of $x=0$;*

(ii)  $g(x) > 0$ *for $x > 0$ and $g(0) = 0$;*

(iii) $g(x)$ *is right differentiable at $x=0$ and its right derivative at $x=0$ does not vanish.*

*Then the positive half-period $\tau_1$ is defined for any solution of (1.1) near $x = \dot{x} = 0$ and $1°$, $2°$ and $3°$ of Theorem 1 are all valid so far as $\tau_1$ and $V_1(X)$ are concerned.*

*Conversely, for any given $\tilde{\tau}_1(R) \in C_R$ such that $\tilde{\tau}_1(0) = \tau_0 \neq 0$ and either $V_1(X)$ determined by (3.5) is continuous or $\tilde{\tau}_1(R) \in C_R^1$, there exists a unique $g(x)$ for $x \geq 0$ which satisfies the conditions (i)-(iii) and for which the positive half-period of any solution of (1.1) is $\tilde{\tau}_1(R)$.*

*If we replace the conditions (i)-(iii) by those as follows:*

(i')   $g(x) \in C_x$ *for $x \leq 0$ in the neighborhood of $x=0$;*

(ii')  $g(x) < 0$ *for $x < 0$ and $g(0) = 0$;*

(iii') $g(x)$ *is left differentiable at $x=0$ and its left derivative at $x=0$ does not vanish,*

*then the similar results stated above hold all for the negative half-period $\tau_2$ and $V_2(X)$ instead of the positive half-period $\tau_1$ and $V_1(X)$.*

PROOF.   When $g(x)$ satisfies (i)-(iii), we construct a function $h(x)$ so that

(3.22)
$$h(x) = \begin{cases} g(x) & \text{for} \quad x \geq 0, \\ -g(-x) & \text{for} \quad x \leq 0. \end{cases}$$

Then $h(x)$ is continuous in the neighborhood of $x=0$ and is differentiable at $x=0$, and furthermore

$$xh(x) > 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad h'(0) \neq 0.$$

Then, by the Lemma[1] of [3], any solution of the equation

(3.23)
$$\frac{d^2x}{dt^2} + h(x) = 0$$

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1)   The Lemma of [3] reads:

If $g(x)$ is continuous in the neighborhood of $x=0$ and differentiable at $x=0$, a necessary and sufficient condition in order that any solution of the equation (1.1) lying near $x = \dot{x} = 0$ may oscillate around $x = \dot{x} = 0$ with a bounded period, is that $xg(x) > 0$ for $x \neq 0$, $g(0) = 0$ and $g'(0) \neq 0$.

near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with a bounded period.   Moreover, by (3.22), for $x\geq0$, any solution of (1.1) is also a solution of (3.23) and vice versa. Thus, by Theorem 1 applied to (3.23), we see that the positive half-period $\tau_1$ is defined for any solution of (1.1) near $x=\dot{x}=0$ and $1°$, $2°$ and $3°$ of Theorem 1 are all valid so far as $\tau_1$ and $V_1(X)$ are concerned.

Conversely, let us suppose there is given $\tilde{\tau}_1(R) \in C_R$ such that $\tilde{\tau}_1(0)=\tau_0\neq 0$ and either $V_1(X)$ determined by (3.5) is continuous or $\tilde{\tau}_1(R) \in C_R^1$. Let us take $\tilde{\tau}_2(R)=\tilde{\tau}_1(R)$.   Then, by Theorem 1 and the Lemma of [3], there exists $g(x)$ which satisfies the conditions:

$$g(x) \in C_x \text{ in the neighborhood of } x=0;$$
$$g(x) \text{ is differentiable at } x=0;$$
$$xg(x)>0 \text{ for } x\neq0 \text{ and } g'(0)\neq0,$$

and for which both half-periods of any solution of (1.1) are $\tilde{\tau}_1(R)$.   Evidently this $g(x)$ satisfies (i)-(iii) for $x\geq0$.

Now, for any $g(x)$ which satisfies (i)-(iii) and for which the positive half-period of any solution of (1.1) is given $\tilde{\tau}_1(R)$, (3.5) or (3.6) holds for $i=1$ by the fact proved in the beginning.   Consequently $V_1(X)$ is uniquely determined by $\tilde{\tau}_1(R)$ only.   Then, by the first of (3.4), $g(x)$ is uniquely determined for $x\geq0$ by $\tilde{\tau}_1(R)$ only.   Thus we see the uniqueness of $g(x)$ which satisfies (i)-(iii) and for which the positive half-period of any solution of (1.1) is given $\tilde{\tau}_1(R)$.

The second half of the theorem can be proved likewise.

THEOREM 3.   *If the positive half-period of any solution of* (1.1) *with* $g(x)$ *satisfying* (i)-(iii) *of Theorem 2 is constant, then, in the neighborhood of* $x=0$,

(3.24) $$g(x)=\left(\frac{\pi}{c}\right)^2 x \quad for \quad x\geq0,$$

*where* $c$ *is a constant to which the positive half-period is equal.*

*The similar equality holds also for* $x\leq0$ *when the negative half-period of any solution of* (1.1) *with* $g(x)$ *satisfying* (i')-(iii') *of Theorem 2 is constant.*

*If the half-periods of any solution of* (1.1) *with* $g(x)$ *satisfying the conditions*:

$$g(x) \in C_x \text{ in the neighborhood of } x=0;$$
$$g(x) \text{ is differentiable at } x=0;$$
$$xg(x)>0 \text{ for } x\neq0,$$

*are both constant, then both half-periods are equal to each other and*

$$g(x)=\left(\frac{\pi}{c}\right)^2 x$$

*in the neighborhood of* $x=0$, *where* $c$ *is a constant to which both half-periods are equal.*

This theorem readily follows from (3.2), (3.4) and (3.5).

The last part of Theorem 3 is our answer to the problem of "*tautochronism*" stated in §1.

## 4.   The relations between $g(x)$ and $(\hat{\tau}_1(a), \hat{\tau}_2(b))$

Making use of Theorem C, we obtain

THEOREM 4.   *In case $g(x)$ is continuous in the neighborhood of $x=0$ and differentiable at $x=0$, if any solution of the equation* (1.1) *near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with a bounded period, then*
   1°   *the half-periods $\tau_1$ and $\tau_2$ are expressed as*

$$(4.1) \qquad\qquad \tau_1 = \hat{\tau}_1(a) \quad and \quad \tau_2 = \hat{\tau}_2(b),$$

*where $a$ and $b$ are respectively the positive maximum and negative minimum displacements of $x$ and*

$$(4.2) \qquad \hat{\tau}_1(a), \quad \hat{\tau}_2(b) \in C, \quad \hat{\tau}_i(0) = \tau_0 = \frac{\omega_0}{2} > 0 \qquad (i=1, 2);$$

   2°   *$a$ and $b$ are connected with the maximum velocity $R$ as* (3.3) *and the function $g(x)$ satisfies the functional equations* (3.4), *where $V_1(X)$ and $V_2(X)$ are the continuous functions such that $V_1(0)=V_2(0)=0$ and*

$$(4.3) \begin{cases} V_1(X) = \dfrac{1}{\tau_0} \cdot \dfrac{d}{dX} \displaystyle\int_0^X \dfrac{\hat{\tau}_1\left[\dfrac{\tau_0}{\pi}\displaystyle\int_0^R \left\{1 + V_1(u)\right\}du\right] - \tau_0}{\sqrt{X^2 - R^2}} RdR \quad for \quad X > 0, \\[4ex] V_2(X) = \dfrac{1}{\tau_0} \cdot \dfrac{d}{dX} \displaystyle\int_0^X \dfrac{\hat{\tau}_2\left[-\dfrac{\tau_0}{\pi}\displaystyle\int_0^R \left\{1 + V_2(u)\right\}du\right] - \tau_0}{\sqrt{X^2 - R^2}} RdR \quad for \quad X > 0; \end{cases}$$

   3°   *if $\hat{\tau}_1(a)$ and $\hat{\tau}_2(b)$ are continuously differentiable,* (4.3) *can be written as*

$$(4.4) \begin{cases} V_1(X) = \dfrac{1}{\pi} X \displaystyle\int_0^X \dfrac{1 + V_1(R)}{\sqrt{X^2 - R^2}} \hat{\tau}_1'\left[\dfrac{\tau_0}{\pi}\displaystyle\int_0^R \left\{1 + V_1(u)\right\}du\right]dR \quad for \quad X > 0, \\[4ex] V_2(X) = -\dfrac{1}{\pi} X \displaystyle\int_0^X \dfrac{1 + V_2(R)}{\sqrt{X^2 - R^2}} \hat{\tau}_2'\left[-\dfrac{\tau_0}{\pi}\displaystyle\int_0^R \left\{1 + V_2(u)\right\}du\right]dR \quad for \quad X > 0. \end{cases}$$

   *Conversely, given $\hat{\tau}_1(a) \in C_a^1$ such that $\hat{\tau}_1(0) = \tau_0 \neq 0$ and $\hat{\tau}_1'(a)$ fulfills the Lipschitz condition, there exists a unique $g(x)$ for $x \geq 0$ which satisfies the conditions* (i)-(iii) *of Theorem 2 and for which the positive half-period is the given $\hat{\tau}_1(a)$.   In this case, $g(x)$ is determined for $x \geq 0$ by the first functional equation of* (3.4) *for $V_1(X)$ determined by the unique solution of the first of* (4.4).

   *Given $\hat{\tau}_2(b) \in C_b^1$ such that $\hat{\tau}_2(0)=\tau_0\neq 0$ and $\hat{\tau}_2'(b)$ fulfills the Lipschitz condition, there exists a unique $g(x)$ for $x \leq 0$ which satisfies the conditions* (i')-(iii') *of Theorem 2 and for which the negative-half period is the given $\hat{\tau}_2(b)$.   In this case, $g(x)$ is determined for $x \leq 0$ by the second functional equation of* (3.4) *for*

$V_2(X)$ *determined by the unique solution of the second of* (4.4).

Given both $\hat{\tau}_1(a) \in C_a^1$ and $\hat{\tau}_2(b) \in C_b^1$ *such that* $\hat{\tau}_1(0) = \hat{\tau}_2(0) = \tau_0 \neq 0$ *and both of* $\hat{\tau}_1'(a)$ *and* $\hat{\tau}_2'(b)$ *fulfill the Lipschitz condition, there exists a unique continuous potential force* $g(x)$ *which is differentiable at* $x = 0$ *and for which any solution of* (1.1) *near* $x = \dot{x} = 0$ *oscillates around* $x = \dot{x} = 0$ *with the given half-periods* $\hat{\tau}_1(a)$ *and* $\hat{\tau}_2(b)$. *In this case, the function* $g(x)$ *is determined by two functional equations of* (3.4) *for* $V_1(X)$ *and* $V_2(X)$ *determined by the unique solutions of* (4.4).

PROOF. The conclusions $1° \sim 3°$ follow readily from $1° \sim 3°$ of Theorem 1 by the substitution (3.3).

The equations (4.4) are evidently of the form of (2.10). Therefore, by Theorem C, each equation of (4.4) has one and only one solution in the neighborhood of $X = 0$ provided $\hat{\tau}_1'(a)$ or $\hat{\tau}_2'(b)$ satisfies the Lipschitz condition respectively.

Then the converse parts of the theorem readily follows from Theorems 1 and 2.

This theorem gives not only the uniqueness of $g(x)$ but also a method to determine $g(x)$ in the case where $\hat{\tau}_1(a)$ and $\hat{\tau}_2(b)$ are both given.

## 5.  Characterization of the odd function $S(X)$ in (2.7)

As is seen in the proof of Theorem 1, two equations of (3.15) are valid for two half-periods, where $S(X)$ and $T(X)$ are the functions connected with $g(x)$ by (2.7).

By (3.2), (3.15) can be written as

$$
(5.1) \qquad \begin{cases} \tau_1 = \tilde{\tau}_1(R) = \dfrac{2\tau_0}{\pi} \displaystyle\int_0^R \dfrac{1 + S(X) + T(X)}{\sqrt{R^2 - X^2}}\, dX, \\[3mm] \tau_2 = \tilde{\tau}_2(R) = \dfrac{2\tau_0}{\pi} \displaystyle\int_0^R \dfrac{1 - S(X) + T(X)}{\sqrt{R^2 - X^2}}\, dX. \end{cases}
$$

From these readily follow

$$
(5.2) \qquad \int_0^R \frac{S(X)}{\sqrt{R^2 - X^2}}\, dX = \frac{\pi}{4\tau_0}\left[\tilde{\tau}_1(R) - \tilde{\tau}_2(R)\right],
$$

$$
(5.3) \qquad \int_0^R \frac{T(X)}{\sqrt{R^2 - X^2}}\, dX = \frac{\pi}{2\omega_0}\left[\Omega(R) - \omega_0\right],
$$

since $\tilde{\tau}_1(R) + \tilde{\tau}_2(R) = \omega = \Omega(R)$. The equations (5.2) and (5.3) are of the form (2.1). Therefore, by Theorem A, we have

$$
(5.4) \qquad S(X) = \frac{1}{2\tau_0} \cdot \frac{d}{dX} \int_0^X \frac{\tilde{\tau}_1(R) - \tilde{\tau}_2(R)}{\sqrt{X^2 - R^2}}\, R\,dR \qquad (X > 0),
$$

$$
(5.5) \qquad T(X) = \frac{1}{\omega_0} \cdot \frac{d}{dX} \int_0^X \frac{\Omega(R) - \omega_0}{\sqrt{X^2 - R^2}}\, R\,dR \qquad (X > 0).
$$

The equation (5.5) is (2.8) itself in Theorem B.

In case $\tilde{\tau}_i(R) \in C_R^1$ $(i=1, 2)$, by Theorem A, (5.4) can be written as follows:

$$(5.6) \qquad S(X) = \frac{1}{2\tau_0} X \int_0^X \frac{\tilde{\tau}_1'(R) - \tilde{\tau}_2'(R)}{\sqrt{X^2 - R^2}} \, dR \qquad (X \geqq 0).$$

From (5.4), it readily follows that $\tilde{\tau}_1(R) = \tilde{\tau}_2(R)$ implies $S(X) \equiv 0$. As is stated in [3], $S(X) \equiv 0$ implies that $g(x)$ determined by (2.7) is odd.

Thus we have

THEOREM 5. *If $g(x)$ is continuous in the neighborhood of $x=0$ and differentiable at $x=0$ and further any solution of (1.1) near $x=\dot{x}=0$ oscillates around $x=\dot{x}=0$ with a bounded period, then $g(x)$ satisfies the functional equation*

$$g\left\{\frac{\omega_0}{2\pi} \int_0^X [1 + S(u) + T(u)] du\right\} = \frac{2\pi}{\omega_0} \cdot \frac{X}{1 + S(X) + T(X)},$$

*where $S(X)$ is a continuous odd function such that (5.4) or (5.6) holds and $T(X)$ is a continuous even function such that $T(0)=0$ and (2.8) or (2.9) holds.*

THEOREM 6. *Under the same assumptions as Theorem 5, if both half-periods of any solution of (1.1) are always equal to each other, then $g(x)$ is odd and vice versa.*

The converse is evident from the symmetry of the closed orbits (3.8).

## References

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