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Function of Generalized Scalar Operators

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Introduction. In the author's previous work [4], generalized scalar and spectral operators were defined and studied on a separated locally convex space E for which $L(E)$ is quasi-complete. The present paper studies some sufficient conditions for a function, especially a polynomial, of two commuting generalized scalar or spectral operators to be again of the same type. In this respect, this paper is a kind of supplement to [4] and we shall use the definitions and results given in [4] without their detailed descriptions.

We shall be especially interested in the case where basic algebras are C^∞ or C_c^∞ (the case considered by Foias [1] on a Banach space) and the results for this case will be stated after corresponding theorems. As a special case, we shall see that sum and product of C^∞-scalar operators are again C^∞-scalar; sum and product of C^∞-spectral operators with compact spectrum are C^∞-spectral, under certain assumptions on commutativity.

Finally, one should remark that the theory can be easily extended to a function or a polynomial of a finite number of commuting generalized scalar or spectral operators.

§1. Φ-proper functions.

Let X be a set and Ψ be an algebra of functions on X containing constants and having a locally convex topology. Given a basic algebra ([4], Def. 1.1) Φ, we consider the following notion:

Definition 1.1. A function f on X will be called *Φ-proper with respect to* (w.r.t.) Ψ if it satisfies the following three conditions:

i) $\varphi\circ f\in\Psi$ for all $\varphi\in\Psi$,

ii) $\varphi\rightarrow\varphi\circ f$ is continuous from Φ into Ψ, and

iii) $1\in\overline{\{\varphi\circ f;\ \varphi\in\Phi\}}$.

We remark that if Ψ is Φ-admissible ([4], Def. 1.4), then any bounded function $f\in\Psi$ is Φ-proper; if, in addition, Φ contains constants, then any function $f\in\Psi$ is Φ-proper.

Proposition 1.1. *Let X be a locally compact space* (resp. *a C^∞-manifold*), *let $\Psi=C^0(X)$ or $C_c^0(X)\oplus C$* (resp. *$C^\infty(X)$ or $C_c^\infty(X)\oplus C$*) *and let $\Phi=C^0$ or C_c^0* (resp. *C^∞ or C_c^∞*). *Then, f is Φ-proper* w.r.t. *Ψ if and only if $f\in\Psi$.*

Proof: "If" part is obvious from the above remark and [4], Example 1.1-1.4. We omit the proof of "only if" part here, since it is not essential in this paper.

Theorem 1.1 and Proposition 2.6 of [4] can be modified by our notion of Φ-proper function as follows:

PROPOSITION 1.2. *Suppose there is a continuous homomorphism V of Ψ into $L(E)$ such that $V(1)=I$. If f is Φ-proper* w.r.t. Ψ, *then U_f defined by $U_f(\varphi)=V(\varphi\circ f)$ is a Φ-spectral representation on E.* ([4], Def. 1.3.) *If, in addition, $f\in\Psi$, then $V(f)$ is Φ-scalar.*

§2. Tensor product of two commuting representations.

Let Φ_1 and Φ_2 be two basic algebras contained in $B(C)$. The complete inductive tensor product $\Phi_1 \hat{\otimes}_i \Phi_2$ (This notation is due to L. Schwartz [5]. Grothendieck [2] denoted it $\Phi_1 \overline{\otimes} \Phi_2$.) of these two algebras can be regarded as a subalgebra of $B(C^2)$, the space of all locally bounded complex valued functions (Borel measurable) on $C^2=R^4$, provided that the topologies of Φ_1 and Φ_2 are stronger than the induced topologies from $B(C)$. Let $\Psi(\Phi_1, \Phi_2)$ be the subalgebra of $B(C^2)$ generated by $\Phi_1 \hat{\otimes}_i \Phi_2$ and C (the constant functions). Then, it is easy to see that

(i) $\Psi(\Phi_1, \Phi_2)=\Phi_1 \hat{\otimes}_i \Phi_2$ if $1\in\Phi_1 \hat{\otimes}_i \Phi_2$,

(ii) $\Psi(\Phi_1, \Phi_2)=(\Phi_1 \hat{\otimes}_i \Phi_2)\oplus C$ if $1\notin\Phi_1 \hat{\otimes}_i \Phi_2$.

In the latter case, we introduce the topology of direct sum in $\Psi(\Phi_1, \Phi_2)$.

We say that a Φ_1-spectral representation U_1 and a Φ_2-spectral representation U_2 are commuting if

$U_1(\varphi_1)U_2(\varphi_2)=U_2(\varphi_2)U_1(\varphi_1)$ for all $\varphi_1\in\Phi_1$ and $\varphi_2\in\Phi_2$.

PROPOSITION 2.1. *If U_1 and U_2 are commuting Φ_1- and Φ_2-spectral representstions respectively, then there is a continuous homomorphism V of $\Psi(\Phi_1, \Phi_2)$ into $L(E)$ such that*

1) $V(\varphi_1\otimes\varphi_2)=U_1(\varphi_1)U_2(\varphi_2)$ for $\varphi_1\in\Phi_1$, $\varphi_2\in\Phi_2$,

2) $V(1)=I$.

PROOF: $V=U_1\otimes U_2$ on $\Phi_1\otimes\Phi_2$ is defined by the equation 1). It is a homomorphism on $\Phi_1\otimes\Phi_2$. Since the mapping $(\varphi_1, \varphi_2)\to V(\varphi_1\otimes\varphi_2)$ is separately continuous from $\Phi_1\times\Phi_2$ into $L(E)$, the mapping $\varphi_1\otimes\varphi_2\to V(\varphi_1\otimes\varphi_2)$ is continuous with respect to the inductive tensor product topology on $\Phi_1\otimes\Phi_2$. (See [2] or [5].) Hence, V can be extended continuously over $\Phi_1\hat{\otimes}_i\Phi_2$. To prove 2), we consider the two cases:

(i) The case $\Psi(\Phi_1, \Phi_2)=\Phi_1\hat{\otimes}_i\Phi_2$.

Choose $\{\varphi_\alpha\}\subseteq\Phi_1$ and $\{\psi_\beta\}\subseteq\Phi_2$ such that $U_1(\varphi_\alpha)\to I$ and $U_2(\psi_\beta)\to I$. For any $x\in E$,

$$V(1)x=V(1)\lim_\alpha U_1(\varphi_\alpha)x$$
$$=V(1)\lim_\alpha U_1(\varphi_\alpha)\lim_\beta U_2(\psi_\beta)x$$

$$= \lim_{\alpha} \lim_{\beta} V(1) U_1(\varphi_\alpha) U_2(\psi_\beta) x$$

$$= \lim_{\alpha} \lim_{\beta} V(1) V(\varphi_\alpha \otimes \psi_\beta) x$$

$$= \lim_{\alpha} \lim_{\beta} V(\varphi_\alpha \otimes \psi_\beta) x$$

$$= \lim_{\alpha} \lim_{\beta} U_1(\varphi_\alpha) U_2(\psi_\beta) x = x.$$

Hence, $V(1) = I$.

(ii) The case $\Psi(\Phi_1, \Phi_2) = (\Phi_1 \hat{\otimes}_i \Phi_2) \oplus C$.

We extend V over $\Psi(\Phi_1, \Phi_2)$ by

$V(\psi + c) = V(\psi) + cI$ for $\psi \in \Phi_1 \hat{\otimes}_i \Phi_2$.

Then, it is easy to see that V is a continuous homomorphism on $\Psi(\Phi_1, \Phi_2)$ and $V(1) = I$. Q. E. D.

THEOREM I. *Let U_1 and U_2 be commuting Φ_1- and Φ_2-spectral representations respectively. If f is Φ-proper* w.r.t. $\Psi(\Phi_1, \Phi_2)$, *then*

$$W_f : W_f(\varphi) = V(\varphi \circ f)$$

is a Φ-spectral representation, where V is the homomorphism defined in the previous proposition. If, in addition, $f \in \Psi(\Phi_1, \Phi_2)$, then $V(f)$ is Φ-scalar.

PROOF: This is an immediate consequence of Proposition 1.2 and the previous proposition.

COROLLARY. *If U_1 and U_2 are commuting C^∞-spectral* (resp. C_c^∞-spectral) *representations, then W_f defined above is a C^∞-spectral representation for any $f \in C^\infty(R^4)$* (resp. $f \in C_c^\infty(R^4) \oplus C$) *and $V(f)$ is C^∞-scalar for such a function f.*

PROOF: Grothendieck [2] (II, p. 84) and L. Schwartz [5] (I, p. 94, II, p. 17) showed that $C^\infty \hat{\otimes}_i C^\infty = C^\infty(R^4)$ and $C_c^\infty \hat{\otimes}_i C_c^\infty = C_c^\infty(R^4)$. Hence $\Psi(C^\infty, C^\infty) = C^\infty(R^4)$ and $\Psi(C_c^\infty, C_c^\infty) = C_c^\infty(R^4) \oplus C$. We know by proposition 1.1. that any function $f \in \Psi$ is C^∞-proper in these cases. Therefore, the corollary follows from the theorem.

REMARK: The above corollary does not hold for C^0-spectral or C_c^0-spectral representations. The example by Kakutani [3] gives an indication of this fact. The difficulty appears in the fact that the topology of $C^0 \hat{\otimes}_i C^0$ is strictly stronger than the topology of $C^0(R^4)$.

§3 Polynomials of two commuting scalar operators.

Let S_i, $i=1, 2$, be Φ_i-scalar operators on E with commuting Φ_i-spectral representations U_i. Let $P(z_1, z_2)$ be a polynomial in two variables. Then $P(S_1, S_2)$ is formally given as an element of $L(E)$. Is it scalar again? The answer is partially given by the following proposition.

PROPOSITION 3.1.

(i) *If* $sp(S_i)$ *are compact* $(i=1, 2)$, *then* $P(S_1, S_2)$ *is* Φ*-scalar whenever* $\Psi(\Phi_1, \Phi_2)$ *is* Φ*-admissible.*

(ii) *Suppose both* Φ_1 *and* Φ_2 *contain polynomials, so that* $S_i=U_i(z)$ $(i=1, 2)$, *and suppose* $\Psi(\Phi_1, \Phi_2)$ *is* Φ*-admissible and* Φ *contains constants. Then,* $P(S_1, S_2)$ *is* Φ*-scalar.*

PROOF: (i) We can choose $\varphi_i \in \Phi_i$ $(i=1, 2)$ such that $\varphi_i=1$ on a neighborhood of $sp(S_i)$. Let $f(z_1, z_2)=P(z_1\varphi_1(z_1), z_2\varphi_2(z_2))$. Then f is bounded and $f\in \Phi_1\otimes\Phi_2$, so that f is Φ-proper w.r.t. $\Psi(\Phi_1, \Phi_2)$. Hence,

$$V(f)=P(U_i(z_1\varphi_1(z_1)), \quad U_2(z_2\varphi_2(z_2)))=P(S_1, S_2)$$

is Φ-scalar by Theorem I.

(ii) Under our assumptions, $P\in\Phi_1\otimes\Phi_2$. Since Φ contains constants, P is Φ-proper w.r.t. $\Psi(\Phi_1, \Phi_2)$ (see the remark after Def. 1.1). Hence, again by Theorem I, $V(P)=P(U_1(z_1), U_2(z_2))=P(S_1, S_2)$ is Φ-scalar.

COROLLARY. (i) *Let* S_1 *and* S_2 *be* C^∞*-scalar operators with commuting* C^∞*-spectral representations. Then,* $P(S_1, S_2)$ *is* C^∞*-scalar for any polynomial P.*

(ii) *Let* S_1 *and* S_2 *be* C_c^∞*-scalar operators with commuting* C_c^∞*-spectral representations. If* $sp(S_i)$ *are compact, then* $P(S_1, S_2)$ *is* C^∞*-scalar for any polynomial P.* (cf. Foias [1], Theorem 4)

REMARK. In the case $sp(S_i)$ $(i=1, 2)$ are compact, we can define (uniquely) $f(S_1, S_2)$ for any function $f(z_1, z_2)$ in two variables, holomorphic in a neighborhood of $sp(S_1)\times sp(S_2)$. (Waelbroeck [6]) Here, we may assume that $f\in\Psi(C_c^\infty, C_c^\infty)$, so that $f(S_1, S_2)$ is C^∞-scalar.

§4 Polynomials of two commuting spectral operators.

For generalized spectral operators, the following theorem is an easy consequence of the previous section.

THEOREM II. *Let* T_i *be* Φ_i*-spectral operators with* Φ_i*-spectral representations* U_i $(i=1, 2)$. *Suppose that* T_1, T_2, $U_1(\varphi_1)$ *and* $U_2(\varphi_2)$ *belong to a same commutative subalgebra of* $L(E)$ *and suppose* $\Psi(\Phi_1, \Phi_2)$ *is* Φ*-admissible.*

If $sp(T_i)$, $i=1, 2$, *are compact, then* $P(T_1, T_2)$ *is* Φ*-spectral for any polynomial P.*

PROOF: Let $i=1$ or 2. If $sp(T_i)$ is compact, then $T_i=S_i+Q_i$, where $S_i=U(z\varphi_i)$ and Q_i is quasi-nilpotent on E. Then, T_i, S_i, Q_i $(i=1, 2)$ commute each other, so that

$$P(T_1, T_2)=P(S_1, S_2)+R_1(S_1, S_2, Q_1, Q_2)Q_1+R_2(S_1, S_2, Q_1, Q_2)Q_2,$$

where R_1 and R_2 are polynomials.

By Proposition 3.1, $P(S_1, S_2)$ is Φ-scalar and its spectrum is compact. Since the quasi-nilpotent operators form an ideal in $L_r(E)$ (the algebra of all elements of $L(E)$ with compact spectrum), $R_1Q_1 + R_2Q_2$ is again quasi-nilpotent. Hence, by Th. 4.2 of [4], $P(T_1, T_2)$ is a Φ-spectral operator.

REMARK. This proof can not be applied to the case where the $sp(T_i)$ are not compact, due to the following fact: "Let Q be a quasi-nilpotent operator. If $S \in L(E)$ has non-compact spectrum, then SQ is not necessarily quasi-nilpotent even if S and Q commute." (cf. Appendix).

If, however, Q is nilpotent, then SQ is again nipotent whenever S and Q commute. Therefore, the following proposition is an immediate consequence of Proposition 3.1, (ii):

PROPOSITION 4.1. *Let T_i be as in the previous theorem except that $sp(T_i)$ may not be compact. Suppose Φ_i contains polynomials, Φ contains constants and $T_i = U_i(z) + Q_i$ with nilpotent operators $Q_i (i=1, 2)$, then $P(T_1, T_2)$ is Φ-spectral.*

COROLLARY TO THEOREM I. *Let $T_i (i=1, 2)$ be regular C^∞-spectral operators with C^∞-spectral representations U_i such that T_1, T_2, $U_1(\varphi_1)$, $U_2(\varphi_2)$; $\varphi_1, \varphi_2 \in C^\infty$ belong to a same commutative subalgebra of $L(E)$. Then $P(T_1, T_2)$ is C^∞-spectral for any polynomial P.*

REMARK: The corresponding statement in C_c^∞ to Proposition 4.1 is a triviality, since, in this case, T_i are C_c^∞-scalar. (See [1].)

Appendix. *An example of a quasi-nilpotent operator Q and a non-regular operator S which are commutative but SQ is not quasi-nilpotent.*

Let us consider the space

$$E = \{f(x, y) \in C^\infty([0,1] \times R);\ (\partial^k f/\partial x^k)(0, y) = 0,\ k = 0, 1, \cdots,\ f(., y) \in S_y(R)\}.$$

Here, $S_y(R)$ is the space of rapidly decreasing functions in y. The space E is Frèchet with a countable number of norms $p_{k,m,q}$: $(k, m, q = 0, 1, \cdots)$

$$p_{k,m,q}(f) = \sup_{x\in[0,1], y\in R} |y^k (\partial^{m+q} f/\partial x^m \partial y^q)(x, y)|.$$

Let

$$Sf(x, y) = yf(x, y), \quad Qf(x, y) = \int_0^x f(t, y)\,dt.$$

It is easy to see that $S, Q \in L(E)$, Q is quasi-nilpotent and $SQ = QS$. Now,

$$(SQ)^n f(x, y) = y^n \int_0^x \frac{(x-t)^n}{n!} f(t, y)\,dt.$$

Taking the function $f(x, y) = \exp\left(-\dfrac{\sqrt{1+y^2}}{x}\right) \in E$, let us compute $a_n = [p_{0,0,0}((SQ)^n f(x, y))]^{1/n}$. If SQ were quasi-nilpotent, then $a_n \to 0$ $(n \to \infty)$. We shall show this is not the case.

$$
\begin{aligned}
a_n &= \sup_y |y| \frac{1}{(n!)^{1/n}} \left(\int_0^1 (1-t)^n f(t, y)\, dt \right)^{1/n} \\
&\geqq \sup_y |y| \frac{1}{(n!)^{1/n}} \left(\int_{1/3}^{2/3} (1-t)^n \exp\left(- \frac{\sqrt{1+y^2}}{t} \right) dt \right)^{1/n} \\
&\geqq K \sup_y \frac{1}{(n!)^{1/n}} |y| \exp(-3|y|/2n) \\
&\geqq K_1 \frac{n}{(n!)^{1/n}} \quad (\text{taking } y = n) \\
&\rightarrow eK_1 \ (n \rightarrow \infty).
\end{aligned}
$$

Hence, SQ cannot be quasi-nilpotent.

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