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Bifurcation of a Periodic Solution of van der Pol's Equation with the Harmonic Forcing Term

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1. Introduction.

In this paper, we are concerned with van der Pol's equation with harmonic forcing term

$$\frac{d^2x}{dt^2} - \varepsilon(1-x^2)\frac{dx}{dt} + x = \varepsilon E \sin \omega t \quad (E \neq 0), \tag{1.1}$$

where ω is a number close to unity. Here, as is readily seen, we may assume $\varepsilon > 0$ and $E > 0$ without loss of generality. By the substitution of the time variable, the equation (1.1) can be written in the form as follows:

$$\frac{d^2x}{dt^2} + x = \varepsilon\left\{-Ax + (1-x^2)\frac{dx}{dt} + E\sin t\right\}, \tag{1.2}$$

which is rewritten in a simultaneous form as follows:

$$\begin{cases} \dfrac{dx}{dt} = y, \\ \dfrac{dy}{dt} = -x + \varepsilon\{E\sin t - Ax + (1-x^2)y\}. \end{cases} \tag{1.3}$$

In (1.2) or (1.3), notice that, besides ε, the quantities A and E are also supposed to be the parameters.

By the linear transformation

$$\begin{cases} x = \xi\cos t + \eta\sin t, \\ y = -\xi\sin t + \eta\cos t, \end{cases} \tag{1.4}$$

the system (1.3) is transformed to the system of the form as follows:

$$\begin{cases} \dfrac{d\xi}{dt} = \varepsilon f(\xi, \eta, t), \\ \dfrac{d\eta}{dt} = \varepsilon g(\xi, \eta, t), \end{cases} \tag{1.5}$$

where

$$(1.6)\quad \begin{cases} f(\xi,\eta,t)=\left(-\frac{1}{2}E+\frac{1}{2}\xi+\frac{1}{2}A\eta-\frac{1}{8}\xi^3-\frac{1}{8}\xi\eta^2\right) \\ \quad +\left(\frac{1}{2}E-\frac{1}{2}\xi-\frac{1}{2}A\eta+\frac{1}{2}\xi\eta^2\right)\cos 2t \\ \quad +\left(\frac{1}{2}A\xi-\frac{1}{2}\eta-\frac{1}{4}\xi^2\eta+\frac{1}{4}\eta^3\right)\sin 2t \\ \quad +\left(\frac{1}{8}\xi^3-\frac{3}{8}\xi\eta^2\right)\cos 4t+\left(\frac{3}{8}\xi^2\eta-\frac{1}{8}\eta^3\right)\sin 4t, \\ g(\xi,\eta,t)=\left(-\frac{1}{2}A\xi+\frac{1}{2}\eta-\frac{1}{8}\xi^2\eta-\frac{1}{8}\eta^3\right) \\ \quad +\left(-\frac{1}{2}A\xi+\frac{1}{2}\eta-\frac{1}{2}\xi^2\eta\right)\cos 2t \\ \quad +\left(\frac{1}{2}E-\frac{1}{2}\xi-\frac{1}{2}A\eta+\frac{1}{4}\xi^3-\frac{1}{4}\xi\eta^2\right)\sin 2t \\ \quad +\left(-\frac{3}{8}\xi^2\eta+\frac{1}{8}\eta^3\right)\cos 4t+\left(\frac{1}{8}\xi^3-\frac{3}{8}\xi\eta^2\right)\sin 4t. \end{cases}$$

Let

$$(1.7)\quad \begin{cases} \xi=\xi(u,v,t,\varepsilon), \\ \eta=\eta(u,v,t,\varepsilon) \end{cases}$$

be the solution of (1.5) such that

$$(1.8)\quad \begin{cases} \xi(u,v,0,\varepsilon)=u, \\ \eta(u,v,0,\varepsilon)=v. \end{cases}$$

Then, in the finite interval of t containing $t=0$, for sufficiently small $|\varepsilon|$, $\xi(u,v,t,\varepsilon)$ and $\eta(u,v,t,\varepsilon)$ are written as follows:

$$(1.9)\quad \begin{cases} \xi(u,v,t,\varepsilon)=\xi_0(u,v,t)+\varepsilon\xi_1(u,v,t)+\varepsilon^2\xi_2(u,v,t)+\varepsilon^3\xi_3(u,v,t)+o(\varepsilon^3), \\ \eta(u,v,t,\varepsilon)=\eta_0(u,v,t)+\varepsilon\eta_1(u,v,t)+\varepsilon^2\eta_2(u,v,t)+\varepsilon^3\eta_3(u,v,t)+o(\varepsilon^3). \end{cases}$$

But, by the initial conditions,

$$\begin{cases} \xi_0(u,v,0)=u, \quad \xi_1(u,v,0)=\xi_2(u,v,0)=\xi_3(u,v,0)=0, \\ \eta_0(u,v,0)=v, \quad \eta_1(u,v,0)=\eta_2(u,v,0)=\eta_3(u,v,0)=0. \end{cases}$$

Therefore, substituting (1.9) into (1.5), we have:

$$(1.10)\quad \begin{cases} \xi_0(u,v,t)=u, \\ \eta_0(u,v,t)=v \end{cases}$$

and

$$(1.11)\qquad \begin{aligned} \xi_1(u, v, 2\pi) &= \int_0^{2\pi} f(u, v, t)dt = -\frac{\pi}{4}(4E - 4u - 4Av + u^3 + uv^2), \\ \eta_1(u, v, 2\pi) &= \int_0^{2\pi} g(u, v, t)dt = -\frac{\pi}{4}(4Au - 4v + u^2v + v^3). \end{aligned}$$

Now, let us consider the equations

$$(1.12)\qquad \begin{cases} 4E - 4u - 4Av + u^3 + uv^2 = 0, \\ 4Au - 4v + u^2v + v^3 = 0. \end{cases}$$

By the assumption $E \neq 0$, these equations can be solved as follows:

$$(1.13)\qquad u = \frac{1}{4E}(4-\rho)\rho, \quad v = \frac{A}{E}\rho,$$

where $\rho (= u^2 + v^2)$ is a root of the equation

$$(1.14)\qquad \rho^3 - 8\rho^2 + 16(1 + A^2)\rho - 16E^2 = 0.$$

Evidently the Jacobian of the left member of (1.12) with respect to (u, v) is

$$(1.15)\qquad \begin{aligned} J_0 &= 3(u^2 + v^2)^2 - 16(u^2 + v^2) + 16(1 + A^2) \\ &= 3\rho^2 - 16\rho + 16(1 + A^2). \end{aligned}$$

If the equation (1.14) has a non-negative root for which the Jacobian J_0 does not vanish, then, by the well-known theorem, there exists one and only one periodic solution of (1.2) whose stability is decided by the sign of the real parts of the characteristic roots of the matrix

$$(1.16)\qquad \begin{pmatrix} 3u^2 + v^2 - 4 & 2uv - 4A \\ 2uv + 4A & u^2 + 3v^2 - 4 \end{pmatrix}.$$

But, in the case where the equation (1.14) has a non-negative root which satisfies the equation

$$(1.17)\qquad 3\rho^2 - 16\rho + 16(1 + A^2) = 0$$

at the same time, the existence of a periodic solution of (1.2) is not guaranteed by the ordinary method.

When $\rho = 2$, the characteristic roots of the matrix (1.16) are pure imaginary for $|A| > \frac{1}{2}$ as is seen later (§4). This implies that the stability of the periodic solution is not decided by the above rule even if the existence is certified.

In the present paper we shall study such a bifurcation problem making use of the idea of the author's previous paper [3].

2. The results obtained.

Let Ω_1, Ω_2, Ω_3 and Ω_4 be the domains in the (A, ρ)-plane such that

$$\Omega_1 = \{(A, \rho) : 3\rho^2 - 16\rho + 16(1 + A^2) > 0 \text{ and } \rho > 2\},$$
$$\Omega_2 = \{(A, \rho) : 3\rho^2 - 16\rho + 16(1 + A^2) < 0\},$$
$$\Omega_3 = \{(A, \rho) : 3\rho^2 - 16\rho + 16(1 + A^2) > 0 \text{ and } 2 > \rho > 0\},$$
$$\Omega_4 = \{(A, \rho) : \rho \leqq 0\}.$$

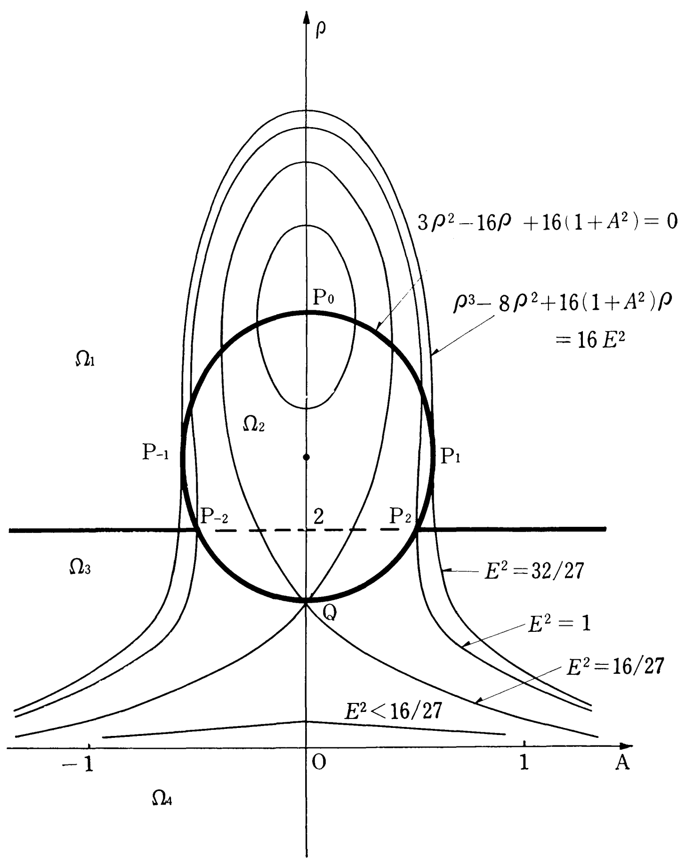
Then, as is known already [1, 2],

for $(A, \rho) \in \Omega_1$, *there exists one and only one periodic solution of* (1.2) *which is stable or unstable according as* $\varepsilon > 0$ *or* < 0;

for $(A, \rho) \in \Omega_2$, *there exists one and only one periodic solution of* (1.2) *which is semi-stable*;

for $(A, \rho) \in \Omega_3$, *there exists one and only one periodic solution of* (1.2) *which is unstable or stable according as* $\varepsilon > 0$ *or* < 0;

for $(A, \rho) \in \Omega_4$, *there exists no periodic solution.*



Our results obtained in the present paper are connected with the distribution of periodic solutions of (1.2) for the point (A, ρ) which belongs to the boundaries of the domains Ω_1 and Ω_2.

Since the boundary of Ω_2 is an ellipse (1.17), any point of it can be expressed as

$$(2.1)\qquad \begin{cases} A=\dfrac{1}{\sqrt{3}}\sin 2\varphi, \\ \rho=\dfrac{4}{3}(2+\cos 2\varphi). \end{cases} \qquad (-\pi<2\varphi\leqq\pi)$$

Let us divide the boundary of Ω_2 into several parts according to the values of φ as follows:

$$\widehat{QP_{-2}}=\left\{(A,\rho):-\frac{\pi}{2}<\varphi<-\frac{\pi}{3}\right\},$$

$$P_{-2}=\left(-\frac{1}{2},2\right),$$

$$\widehat{P_{-2}P_{-1}}=\left\{(A,\rho):-\frac{\pi}{3}<\varphi<-\frac{\pi}{4}\right\},$$

$$P_{-1}=\left(-\frac{1}{\sqrt{3}},\frac{8}{3}\right),$$

$$\widehat{P_{-1}P_0}=\left\{(A,\rho):-\frac{\pi}{4}<\varphi<0\right\},$$

$$P_0=(0,4),$$

$$\widehat{P_0P_1}=\left\{(A,\rho):0<\varphi<\frac{\pi}{4}\right\},$$

$$P_1=\left(\frac{1}{\sqrt{3}},\frac{8}{3}\right),$$

$$\widehat{P_1P_2}=\left\{(A,\rho):\frac{\pi}{4}<\varphi<\frac{\pi}{3}\right\},$$

$$P_2=\left(\frac{1}{2},2\right),$$

$$\widehat{P_2Q}=\left\{(A,\rho):\frac{\pi}{3}<\varphi<\frac{\pi}{2}\right\},$$

$$Q=\left(0,\frac{4}{3}\right).$$

Further let L_+ *and* L_- be the half lines as follows:

$$L_+=\left\{(A.\rho):A>\frac{1}{2},\rho=2\right\},$$

$$L_- = \left\{(A, \rho) : A < -\frac{1}{2}, \rho = 2\right\}.$$

Then, our results obtained in the present paper are as follows:

for $(A, \rho) \in \widehat{QP_{-2}} \cup P_{-2}$, *there exist two periodic solutions of* (1.2) *one of which is semi-stable and the other is unstable*;

for $(A, \rho) \in \widehat{P_{-2}P_{-1}} \cup \widehat{P_0P_1}$, *there exist two periodic solutions of* (1.2) *one of which is stable and the other is semi-stable*;

for $(A, \rho) \in P_{-1} \cup P_1 \cup L_+$, *there exists one and only one periodic solution of* (1.2) *which is stable*;

for $(A, \rho) \in L_-$, *there exists one and only one periodic solution of* (1.2) *which is unstable*;

for $(A, \rho) \in \widehat{P_{-1}P_0} \cup P_0 \cup \widehat{P_1P_2} \cup P_2 \cup \widehat{P_2Q} \cup Q$, *there exists no periodic solution.*

In case $\varepsilon < 0$, *the same results hold replacing* A *by* $-A$ *and interchanging the terminologies "stable" and "unstable" with each other.*

		$\varepsilon > 0$		$\varepsilon < 0$		
regions	φ	nos. of periodic solutions	stability	nos. of periodic solutions	stability	A, ρ
Ω_1		1	stable	1	unstable	
Ω_2		1	semi-stable	1	semi-stable	
Ω_3		1	unstable	1	stable	
Ω_4		0		0		
$\widehat{QP_{-2}}$	$-\frac{\pi}{2} < \varphi < -\frac{\pi}{3}$	2	semi-stable unstable	0		
P_{-2}	$\varphi = -\frac{\pi}{3}$	2	semi-stable unstable	0		$A = -1/2$ $\rho = 2$
$\widehat{P_{-2}P_{-1}}$	$-\frac{\pi}{3} < \varphi < -\frac{\pi}{4}$	2	semi-stable stable	0		
P_{-1}	$\varphi = -\frac{\pi}{4}$	1	stable	1	unstable	$A = -1/\sqrt{3}$ $\rho = 8/3$
$\widehat{P_{-1}P_0}$	$-\frac{\pi}{4} < \varphi < 0$	0		2	semi-stable unstable	
P_0	$\varphi = 0$	0		0		$A = 0$ $\rho = 4$
$\widehat{P_0P_1}$	$0 < \varphi < \frac{\pi}{4}$	2	semi-stable stable	0		
P_1	$\varphi = \frac{\pi}{4}$	1	stable	1	unstable	$A = 1/\sqrt{3}$ $\rho = 8/3$
$\widehat{P_1P_2}$	$\frac{\pi}{4} < \varphi < \frac{\pi}{3}$	0		2	semi-stable unstable	
P_2	$\varphi = \frac{\pi}{3}$	0		2	semi-stable stable	$A = 1/2$ $\rho = 2$
$\widehat{P_2Q}$	$\frac{\pi}{3} < \varphi < \frac{\pi}{2}$	0		2	semi-stable stable	
Q	$\varphi = \frac{\pi}{2}$	0		0		$A = 0$ $\rho = 4/3$
L_+		1	stable	1	stable	
L_-		1	unstable	1	unstable	

To make clear the distribution of periodic solutions of (1.2), we show the results stated above including the older ones in a table in the preceding page.

3. The proof for the case where the Jacobian J_0 vanishes.

In this paragraph, we study the case where the equation (1.14) has a non-negative root which satisfies the equation (1.17) at the same time.

As was shown in the preceding paragraph, A, and ρ satisfying (1.17) are expressed as (2.1). Therefore, substituting (2.1) into (1.14), we have

$$E=\kappa\frac{4}{3\sqrt{3}}(2+\cos 2\varphi)\sin\varphi, \tag{3.1}$$

where $\kappa=+1$ or -1 according as $\varphi\geqq 0$ or $\varphi<0$. Substituting (2.1) and (3.1) into (1.13), we see that, in the case in question, the solution of the equation (1.12) is given by

$$\begin{cases} u=\dfrac{2}{\sqrt{3}}\kappa\sin\varphi, \\ v=2\kappa\cos\varphi. \end{cases} \tag{3.2}$$

Let us transform the variables (ξ, η) to $(\bar{\xi}, \bar{\eta})$ by

$$\begin{cases} \xi=\bar{\xi}+\dfrac{2}{\sqrt{3}}\kappa\sin\varphi, \\ \eta=\bar{\eta}+2\kappa\cos\varphi \end{cases} \tag{3.3}$$

and put

$$\begin{cases} \bar{f}(\bar{\xi}, \bar{\eta}, t)=f\left(\bar{\xi}+\dfrac{2}{\sqrt{3}}\kappa\sin\varphi,\ \bar{\eta}+2\kappa\cos\varphi,\ t\right), \\ \bar{g}(\bar{\xi}, \bar{\eta}, t)=g\left(\bar{\xi}+\dfrac{2}{\sqrt{3}}\kappa\sin\varphi,\ \bar{\eta}+2\kappa\cos\varphi,\ t\right). \end{cases}$$

These functions can be expanded with respect to $(\bar{\xi}, \bar{\eta})$ as follows:

$$\begin{cases} \bar{f}(\bar{\xi}, \bar{\eta}, t)=\kappa\left\{-\dfrac{1}{3\sqrt{3}}\sin\varphi(1-10\cos^2\varphi)\cos 2t+\cos\varphi\cos 2\varphi\sin 2t\right. \\ \qquad\left.+\dfrac{1}{3\sqrt{3}}\sin\varphi(1-10\cos^2\varphi)\cos 4t-\cos\varphi\cos 2\varphi\sin 4t\right\} \\ \qquad+\left\{-\dfrac{1}{2}(1-4\cos^2\varphi)\cos 2t-\dfrac{1}{2\sqrt{3}}\sin 2\varphi\sin 2t\right. \\ \qquad\left.+\dfrac{1}{2}(1-4\cos^2\varphi)\cos 4t+\dfrac{\sqrt{3}}{2}\sin 2\varphi\sin 4t\right\}\bar{\xi} \\ \qquad+\left\{\dfrac{\sqrt{3}}{2}\sin 2\varphi\cos 2t-\dfrac{5}{6}(1-4\cos^2\varphi)\sin 2t\right. \end{cases}$$

$$
(3.4)\quad \left\{
\begin{aligned}
&-\frac{\sqrt{3}}{2}\sin 2\varphi\cos 4t+\frac{1}{2}(1-4\cos^2\varphi)\sin 4t\Big\}\bar{\eta}\\
&+\kappa\Big\{-\frac{\sqrt{3}}{4}\sin\varphi-\frac{1}{2}\cos\varphi\sin 2t+\frac{\sqrt{3}}{4}\sin\varphi\cos 4t\\
&+\frac{3}{4}\cos\varphi\sin 4t\Big\}\xi^2\\
&+\kappa\Big\{-\frac{1}{2}\cos\varphi+2\cos\varphi\cos 2t-\frac{1}{\sqrt{3}}\sin\varphi\sin 2t\\
&-\frac{3}{2}\cos\varphi\cos 4t+\frac{\sqrt{3}}{2}\sin\varphi\sin 4t\Big\}\xi\bar{\eta}\\
&+\kappa\Big\{-\frac{1}{4\sqrt{3}}\sin\varphi+\frac{1}{\sqrt{3}}\sin\varphi\cos 2t+\frac{3}{2}\cos\varphi\sin 2t\\
&-\frac{\sqrt{3}}{4}\sin\varphi\cos 4t-\frac{3}{4}\cos\varphi\sin 4t\Big\}\bar{\eta}^2+[\xi,\bar{\eta};t]_3,\\
\bar{g}(\xi,\bar{\eta},t)=\;&\kappa\Big\{\cos\varphi\cos 2\varphi\cdot\cos 2t+\frac{1}{3\sqrt{3}}\sin\varphi(1-10\cos^2\varphi)\sin 2t\\
&+\cos\varphi\cos 2\varphi\cos 4t+\frac{1}{3\sqrt{3}}\sin\varphi(1-10\cos^2\varphi)\sin 4t\Big\}\\
&+\Big\{-\frac{1}{\sqrt{3}}\sin 2\varphi-\frac{5}{2\sqrt{3}}\sin 2\varphi\cos 2t+\frac{1}{2}(1-4\cos^2\varphi)\sin 2t\\
&-\frac{\sqrt{3}}{2}\sin 2\varphi\cos 4t+\frac{1}{2}(1-4\cos^2\varphi)\sin 4t\Big\}\xi\\
&+\Big\{\frac{1}{3}(1-4\cos^2\varphi)-\frac{1}{6}(1-4\cos^2\varphi)\cos 2t-\frac{\sqrt{3}}{2}\sin 2\varphi\sin 2t\\
&-\frac{1}{2}(1-4\cos^2\varphi)\cos 4t-\frac{\sqrt{3}}{2}\sin 2\varphi\sin 4t\Big\}\bar{\eta}\\
&+\kappa\Big\{-\frac{1}{4}\cos\varphi-\cos\varphi\cos 2t+\frac{\sqrt{3}}{2}\sin\varphi\sin 2t\\
&-\frac{3}{4}\cos\varphi\cos 4t+\frac{\sqrt{3}}{4}\sin\varphi\sin 4t\Big\}\xi^2\\
&+\kappa\Big\{-\frac{1}{2\sqrt{3}}\sin\varphi-\frac{2}{\sqrt{3}}\sin\varphi\cos 2t-\cos\varphi\sin 2t\\
&-\frac{\sqrt{3}}{2}\sin\varphi\cos 4t-\frac{3}{2}\cos\varphi\sin 4t\Big\}\xi\bar{\eta}\\
&+\kappa\Big\{-\frac{3}{4}\cos\varphi-\frac{1}{2\sqrt{3}}\sin\varphi\sin 2t+\frac{3}{4}\cos\varphi\cos 4t\\
&-\frac{\sqrt{3}}{4}\sin\varphi\sin 4t\Big\}\bar{\eta}^2+[\xi,\bar{\eta};t]_3,
\end{aligned}\right.
$$

where $[\xi, \bar{\eta}; t]_3$ stands for the terms of the 3rd and higher order in $(\xi, \bar{\eta})$

whose coefficients are periodic functions of t.

By the substitution (3.3), the system (1.5) is transformed to the system

$$(3.5)\qquad \begin{cases} \dfrac{d\bar{\xi}}{dt} = \varepsilon \bar{f}(\bar{\xi}, \bar{\eta}, t), \\ \dfrac{d\bar{\eta}}{dt} = \varepsilon \bar{g}(\bar{\xi}, \bar{\eta}, t). \end{cases}$$

Evidently the solution

$$(3.6)\qquad \begin{cases} \bar{\xi} = \bar{\xi}(\bar{u}, \bar{v}, t, \varepsilon), \\ \bar{\eta} = \bar{\eta}(\bar{u}, \bar{v}, t, \varepsilon) \end{cases}$$

of (3.5) such that

$$\begin{cases} \bar{\xi}(\bar{u}, \bar{v}, 0, \varepsilon) = \bar{u}, \\ \bar{\eta}(\bar{u}, \bar{v}, 0, \varepsilon) = \bar{v} \end{cases}$$

is expanded with respect to ε in the finite interval of t containing $t=0$ as follows:

$$(3.7)\qquad \begin{cases} \bar{\xi}(\bar{u}, \bar{v}, t, \varepsilon) = \bar{\xi}_0(\bar{u}, \bar{v}, t) + \varepsilon\bar{\xi}_1(\bar{u}, \bar{v}, t) + \cdots, \\ \bar{\eta}(\bar{u}, \bar{v}, t, \varepsilon) = \bar{\eta}_0(\bar{u}, \bar{v}, t) + \varepsilon\bar{\eta}_1(\bar{u}, \bar{v}, t) + \cdots. \end{cases}$$

From (3.3), it is evident that the solution (3.6) of (3.5) is related with the solution (1.7) of (1.5) as follows:

$$\begin{cases} \bar{\xi}(\bar{u}, \bar{v}, t, \varepsilon) = \xi\left(\bar{u} + \dfrac{2}{\sqrt{3}}\kappa \sin\varphi, \bar{v} + 2\kappa\cos\varphi, t, \varepsilon\right) - \dfrac{2}{\sqrt{3}}\kappa\sin\varphi, \\ \bar{\eta}(\bar{u}, \bar{v}, t, \varepsilon) = \eta\left(\bar{u} + \dfrac{2}{\sqrt{3}}\kappa \sin\varphi, \bar{v} + 2\kappa\cos\varphi, t, \varepsilon\right) - 2\kappa\cos\varphi. \end{cases}$$

If we substitute (3.7) into (3.5) and make use of (3.4), it is seen after elementary calculations that

$$\begin{cases} \bar{\xi}_0(\bar{u}, \bar{v}, t) = \bar{u}, \quad \bar{\eta}_0(\bar{u}, \bar{v}, t) = \bar{v}; \\ \bar{\xi}_1(\bar{u}, \bar{v}, 2\pi) = \displaystyle\int_0^{2\pi} \bar{f}(\bar{u}, \bar{v}, t)dt \\ \qquad = -\kappa\pi\left[\dfrac{\sqrt{3}}{2}\sin\varphi\cdot\bar{u}^2 + \cos\varphi\cdot\bar{u}\bar{v} + \dfrac{1}{2\sqrt{3}}\sin\varphi\cdot\bar{v}^2\right] + [\bar{u}, \bar{v}]_3, \\ \bar{\eta}_1(\bar{u}, \bar{v}, 2\pi) = \displaystyle\int_0^{2\pi} \bar{g}(\bar{u}, \bar{v}, t)dt \\ \qquad = -\dfrac{2\pi}{\sqrt{3}}\sin 2\varphi\cdot\bar{u} + \dfrac{2\pi}{3}(1-4\cos^2\varphi)\bar{v} \\ \qquad\quad -\kappa\pi\left[\dfrac{1}{2}\cos\varphi\cdot\bar{u}^2 + \dfrac{1}{\sqrt{3}}\sin\varphi\cdot\bar{u}\bar{v} + \dfrac{3}{2}\cos\varphi\cdot\bar{v}^2\right] + [\bar{u}, \bar{v}]_3; \end{cases}$$

$$(3.8)\quad \begin{cases} \bar{\xi}_2(\bar{u}, \bar{v}, 2\pi) = \displaystyle\int_0^{2\pi} \left\{ \frac{\partial \bar{f}}{\partial \bar{\xi}}(\bar{u}, \bar{v}, t)\bar{\xi}_1(\bar{u}, \bar{v}, t) + \frac{\partial \bar{f}}{\partial \bar{\eta}}(\bar{u}, \bar{v}, t)\bar{\eta}_1(\bar{u}, \bar{v}, t) \right\} dt \\ \quad = -\kappa \dfrac{\pi}{36} \cos\varphi(1 + 2\cos^2\varphi)^2 + \dfrac{\pi}{4\sqrt{3}} \sin 2\varphi \cdot \bar{u} \\ \quad + \dfrac{\pi}{24}(-3 + 28\cos^2\varphi - 52\cos^4\varphi)\bar{v} + [\bar{u}, \bar{v}]_2, \\ \bar{\eta}_2(\bar{u}, \bar{v}, 2\pi) = \displaystyle\int_0^{2\pi} \left\{ \frac{\partial \bar{g}}{\partial \bar{\xi}}(\bar{u}, \bar{v}, t)\bar{\xi}_1(\bar{u}, \bar{v}, t) + \frac{\partial \bar{g}}{\partial \bar{\eta}}(\bar{u}, \bar{v}, t)\bar{\eta}_1(\bar{u}, \bar{v}, t) \right\} dt \\ \quad = \kappa \dfrac{\pi}{36\sqrt{3}} \sin\varphi(7 - 44\cos^2\varphi + 172\cos^4\varphi) \\ \quad + \left\{ \dfrac{\pi}{72}(17 - 100\cos^2\varphi + 236\cos^4\varphi) \right. \\ \quad \left. - \dfrac{2\pi^2}{3\sqrt{3}}(1 - 4\cos^2\varphi)\sin 2\varphi \right\} \bar{u} \\ \quad + \left\{ \dfrac{\pi}{4\sqrt{3}} \sin 2\varphi(-1 + 16\cos^2\varphi) + \dfrac{2\pi^2}{9}(1 - 4\cos^2\varphi)^2 \right\} \bar{v} \\ \quad + [\bar{u}, \bar{v}]_2. \end{cases}$$

From these, it holds for a periodic solution of (3.5) that

$$(3.9)\quad -\kappa\pi\left[\frac{\sqrt{3}}{2}\sin\varphi\cdot\bar{u}^2 + \cos\varphi\cdot\bar{u}\bar{v} + \frac{1}{2\sqrt{3}}\sin\varphi\cdot\bar{v}^2\right] + \cdots$$

$$+\varepsilon\left[-\kappa\frac{\pi}{36}\cos\varphi(1 + 2\cos^2\varphi)^2 + \frac{\pi}{4\sqrt{3}}\sin 2\varphi\cdot\bar{u}\right.$$

$$\left.+\frac{\pi}{24}(-3 + 28\cos^2\varphi - 52\cos^4\varphi)\bar{v} + \cdots\right] + \cdots = 0,$$

$$(3.10)\quad -\frac{2\pi}{\sqrt{3}}\sin 2\varphi\cdot\bar{u} + \frac{2\pi}{3}(1 - 4\cos^2\varphi)\bar{v}$$

$$-\kappa\pi\left[\frac{1}{2}\cos\varphi\cdot\bar{u}^2 + \frac{1}{\sqrt{3}}\sin\varphi\cdot\bar{u}\bar{v} + \frac{3}{2}\cos\varphi\cdot\bar{v}^2\right] + \cdots$$

$$+\varepsilon\left[\kappa\frac{\pi}{36\sqrt{3}}\sin\varphi(7 - 44\cos^2\varphi + 172\cos^4\varphi)\right.$$

$$+\left\{\frac{\pi}{72}(17 - 100\cos^2\varphi + 236\cos^4\varphi) - \frac{2\pi^2}{3\sqrt{3}}(1 - 4\cos^2\varphi)\sin 2\varphi\right\}\bar{u}$$

$$+\left\{\frac{\pi}{4\sqrt{3}}\sin 2\varphi(-1 + 16\cos^2\varphi) + \frac{2\pi^2}{9}(1 - 4\cos^2\varphi)^2\right\}\bar{v}$$

$$\left.+\cdots\right] + \cdots = 0.$$

Case I. The case where $\varphi \neq \frac{\pi}{2}, \pm\frac{\pi}{4}$.

In this case, $\sin 2\varphi \neq 0$, because $\varphi=0$ is excluded by the assumption $E\neq 0$. Therefore the equation (3.10) can be solved with respect to $\bar{u}$ as follows:

$$\bar{u}=\frac{1-4\cos^2\varphi}{2\sqrt{3}\sin\varphi\cos\varphi}\bar{v}+\kappa\frac{7-44\cos^2\varphi+172\cos^4\varphi}{144\cos\varphi}\varepsilon+[\bar{v},\varepsilon]_2.$$

If this is substituted for $\bar{u}$ in (3.9), the resulting equation is of the form as follows:

$$\kappa\pi\frac{(1+2\cos^2\varphi)\cos 2\varphi}{8\sqrt{3}\sin\varphi\cos^2\varphi}\bar{v}^2-\kappa\frac{\pi}{36}\cos\varphi(1+2\cos^2\varphi)^2\varepsilon+\cdots=0.$$

Here the coefficients of $\bar{v}^2$ and ε do not vanish due to the condition on φ. Therefore, by Weierstrass's preparation theorem, we see that,

for φ such that $0<\varphi<\frac{\pi}{4}$ or $-\frac{\pi}{2}<\varphi<-\frac{\pi}{4}$, the equations (3.9) and (3.10) have the two real solutions

$$(3.11)\quad \begin{cases} \bar{u}=\tilde{u}(\varepsilon^{1/2})=\pm\dfrac{3^{-1/4}}{6}\dfrac{1-4\cos^2\varphi}{\sin\varphi}\sqrt{(1+2\cos^2\varphi)\tan 2\varphi}\cdot\varepsilon^{1/2}+o(\varepsilon^{1/2}), \\ \bar{v}=\tilde{v}(\varepsilon^{1/2})=\pm 3^{-3/4}\cos\varphi\sqrt{(1+2\cos^2\varphi)\tan 2\varphi}\cdot\varepsilon^{1/2}+o(\varepsilon^{1/2}); \end{cases}$$

for φ such that $\frac{\pi}{4}<\varphi<\frac{\pi}{2}$ or $-\frac{\pi}{4}<\varphi<0$, the equations (3.9) and (3.10) have no real solution.

This implies that the equation (1.2) has two periodic solutions (with period 2π) for A and E corresponding to φ (by (2.1) and (3.1)) such that $0<\varphi<\frac{\pi}{4}$ or $-\frac{\pi}{2}<\varphi<-\frac{\pi}{4}$ and that the equation (1.2) has no periodic solution for A and E corresponding to φ such that $\frac{\pi}{4}<\varphi<\frac{\pi}{2}$ or $-\frac{\pi}{4}<\varphi<0$. It is needless to say that the periodic solution of (1.2) with period 2π corresponds to the periodic solution of the initial equation (1.1) with period $\frac{2\pi}{\omega}$.

Now let us investigate the stability of the periodic solutions obtained just above. Let us consider the transformation

$$(3.12)\quad \begin{cases} r'=\bar{\xi}(\tilde{u}+r,\tilde{v}+s,2\pi,\varepsilon)-\tilde{u}, \\ s'=\bar{\eta}(\tilde{u}+r,\tilde{v}+s,2\pi,\varepsilon)-\tilde{v}. \end{cases}$$

By (3.8) and (3.11), the matrix A of the coefficients of the linear parts in the right members of (3.12) is of the form as follows:

$$(3.13)\quad A=E+\varepsilon A_1+\varepsilon^{3/2}A_2+O(\varepsilon^2),$$

where

$$\begin{cases} A_1 = \begin{pmatrix} 0 & 0 \\ -\dfrac{2\pi}{\sqrt{3}}\sin 2\varphi & \dfrac{2\pi}{3}(1-4\cos^2\varphi) \end{pmatrix}, \\ A_2 = \pm\dfrac{\kappa\pi}{2}\sqrt{D}\begin{pmatrix} 3^{-3/4}\cos 2\varphi & 3^{-1/4}\dfrac{\cos\varphi\cos 2\varphi}{\sin\varphi} \\ 3^{-1/4}\dfrac{\cos\varphi\cos 2\varphi}{\sin\varphi} & -\dfrac{3^{-3/4}}{3}(1+14\cos^2\varphi) \end{pmatrix} \end{cases}$$

$$(D = (1+2\cos^2\varphi)\tan 2\varphi).$$

The matrix A of the form (3.13) can be written in the exponential form as follows:

$$A = \exp(\varepsilon B),$$

where

$$B = A_1 + \varepsilon^{1/2}A_2 + O(\varepsilon).$$

The characteristic roots of the matrix B are found after elementary calculations as follows:

in case $\varphi \neq -\frac{\pi}{3}$ [1),]

$$\frac{2\pi}{3}(1-4\cos^2\varphi) + O(\varepsilon^{1/2}),$$

$$\varepsilon^{1/2}\left\{\pm\kappa\pi\frac{3^{-3/4}}{2}\sqrt{D}\frac{\cos 2\varphi(1+2\cos^2\varphi)}{1-4\cos^2\varphi} + O(\varepsilon^{1/2})\right\};$$

in case $\varphi = -\frac{\pi}{3}$,

$$\varepsilon^{1/4}\left[\pm 2^{-5/4}\pi i + \frac{\pi}{2\sqrt{2}}\varepsilon^{1/4} + o(\varepsilon^{1/4})\right]$$

for the upper sign of the double signs in (3.11),

$$\varepsilon^{1/4}[\pm 2^{-5/4}\pi + O(\varepsilon^{1/4})]$$

for the lower sign of the double signs in (3.11).

From this, the stability of the periodic solutions of (1.2) determined by the initial conditions (3.11) is concluded as follows:

for φ such that $0<\varphi<\frac{\pi}{4}$ or $-\frac{\pi}{3}<\varphi<-\frac{\pi}{4}$, one solution is semi-stable and the other is stable;

---

1) $\varphi = \frac{\pi}{3}$ is excluded, because there exists no periodic solution of (1.2) for this value of φ.

for φ such that $-\frac{\pi}{2}<\varphi\leqq-\frac{\pi}{3}$, one solution is semi-stable and the other is unstable.

Case II. The case where $\varphi=\pm\frac{\pi}{4}$.

In this case, it is found after elementary calculations that

$$(3.14)\quad \begin{cases} \xi_1(\bar{u},\bar{v},2\pi)=-\pi\left(\frac{3\sqrt{2}}{4\sqrt{3}}\bar{u}^2\pm\frac{\sqrt{2}}{2}\bar{u}\bar{v}+\frac{\sqrt{2}}{4\sqrt{3}}\bar{v}^2\right)-\frac{\pi}{4}(\bar{u}^3+\bar{u}\bar{v}^2), \\ \bar{\eta}_1(\bar{u},\bar{v},2\pi)=\mp\frac{2\pi}{\sqrt{3}}\bar{u}-\frac{2\pi}{3}\bar{v}\mp\pi\left(\frac{\sqrt{2}}{4}\bar{u}^2\pm\frac{\sqrt{2}}{2\sqrt{3}}\bar{u}\bar{v}+\frac{3\sqrt{2}}{4}\bar{v}^2\right) \\ \qquad\qquad -\frac{\pi}{4}(\bar{u}^2\bar{v}+\bar{v}^3). \end{cases}$$

Then, in the present case, the equations (3.9) and (3.10) become:

$$(3.15)\quad -\pi\left(\frac{3\sqrt{2}}{4\sqrt{3}}\bar{u}^2\pm\frac{\sqrt{2}}{2}\bar{u}\bar{v}+\frac{\sqrt{2}}{4\sqrt{3}}\bar{v}^2\right)-\frac{\pi}{4}(\bar{u}^3+\bar{u}\bar{v}^2)$$
$$+\varepsilon\left\{\mp\frac{\pi}{9\sqrt{2}}\pm\frac{\pi}{4\sqrt{3}}\bar{u}-\frac{\pi}{12}\bar{v}+[\bar{u},\bar{v}]_2\right\}+O(\varepsilon^2)=0,$$

$$(3.16)\quad \mp\frac{2\pi}{\sqrt{3}}\bar{u}-\frac{2\pi}{3}\bar{v}\mp\pi\left(\frac{\sqrt{2}}{4}\bar{u}^2\pm\frac{\sqrt{2}}{2\sqrt{3}}\bar{u}\bar{v}+\frac{3\sqrt{2}}{4}\bar{v}^2\right)-\frac{\pi}{4}(\bar{u}^2\bar{v}+\bar{v}^3)$$
$$+\varepsilon\left\{\frac{7\pi}{9\sqrt{6}}+\left(\frac{13\pi}{36}\pm\frac{2\pi^2}{3\sqrt{3}}\right)\bar{u}+\left(\pm\frac{7\pi}{4\sqrt{3}}+\frac{2\pi^2}{9}\right)\bar{v}+[\bar{u},\bar{v}]_2\right\}$$
$$+O(\varepsilon^2)=0.$$

The equation (3.16) can be uniquely solved with respect to $\bar{u}$ as follows:

$$(3.17)\quad \bar{u}=\left(\mp\frac{1}{\sqrt{3}}\bar{v}-\frac{\sqrt{2}}{\sqrt{3}}\bar{v}^2\pm\frac{1}{2\sqrt{3}}\bar{v}^3+\cdots\right)+\varepsilon\left(\pm\frac{7}{18\sqrt{2}}+\cdots\right)+O(\varepsilon^2).$$

The substitution of this into (3.15) yields

$$\pm\frac{\pi}{3\sqrt{3}}\bar{v}^3+O(\bar{v}^5)+\varepsilon\left(\mp\frac{\pi}{9\sqrt{2}}-\frac{1}{6}\pi\bar{v}+O(\bar{v}^2)\right)+O(\varepsilon^2)=0.$$

This can be uniquely solved with respect to $\bar{v}$ as follows:

$$(3.18)\quad \bar{v}=\tilde{v}(\varepsilon^{1/3})=6^{-1/6}\varepsilon^{1/3}\pm\frac{\sqrt{3}}{6}\cdot 6^{1/6}\varepsilon^{2/3}+O(\varepsilon),$$

from which, by (3.17), follows

$$(3.19)\quad \bar{u}=\tilde{u}(\varepsilon^{1/3})=\mp\frac{1}{\sqrt{3}}\cdot 6^{-1/6}\varepsilon^{1/3}-\frac{1}{2}\cdot 6^{-1/6}\varepsilon^{2/3}+O(\varepsilon).$$

The values given by (3.18) and (3.19) are evidenly the unique solution of the equations (3.15) and (3.16). This implies that, for $\varphi=\frac{\pi}{4}$ and $\varphi=-\frac{\pi}{4}$, there exists respectively one and only one periodic solution of the given equation (1.2).

In the present case, by (3.14), (3.18) and (3.19), the matrix A of the coefficients of the linear parts in the right members of (3.12) is of the form as follows:

$$A=E+\varepsilon A_1+\varepsilon^{4/3}A_2+\varepsilon^{5/3}A_3+O(\varepsilon^2), \tag{3.20}$$

where

$$A_1=\begin{pmatrix} 0 & 0 \\ \mp\frac{2\pi}{\sqrt{3}} & -\frac{2\pi}{3}\end{pmatrix},$$

$$A_2=\begin{pmatrix} 0 & 0 \\ 0 & \mp\frac{8\pi}{3\sqrt{2}}\cdot 6^{-1/6}\end{pmatrix},$$

$$A_3=\begin{pmatrix} \frac{\pi}{2}\cdot 6^{-1/3} & \pm\frac{\pi}{2\sqrt{2}}\cdot 6^{1/6} \\ \pm\frac{\pi}{2\sqrt{2}}\cdot 6^{1/6} & -\frac{11\pi}{6}\cdot 6^{-1/3}\end{pmatrix}.$$

The matrix A of the form (3.20) can be written in the exponential form as follows:

$$A=\exp(\varepsilon B), \tag{3.21}$$

where

$$B=A_1+\varepsilon^{1/3}A_2+\varepsilon^{2/3}A_3+O(\varepsilon).$$

As is readily found, the characteristic roots of the matrix B are

$$-\frac{2\pi}{3}+O(\varepsilon^{1/3}) \quad \text{and} \quad \varepsilon^{2/3}(-6^{-1/3}\pi+O(\varepsilon^{1/3})).$$

This says that the periodic solutions of (1.2) determined by the initial conditions (3.18) and (3.19) are both stable.

Case III. The case where $\varphi=\frac{\pi}{2}$.

In this case, it is found after the elementary calculations that

$$\xi_1(\bar{u},\bar{v},2\pi)=-\frac{\sqrt{3}\pi}{2}\bar{u}^2-\frac{\pi}{2\sqrt{3}}\bar{v}^2-\frac{\pi}{4}\bar{u}^3-\frac{\pi}{4}\bar{u}\bar{v}^2,$$

$$(3.22)\quad \begin{cases} \bar{\eta}_1(\bar{u}, \bar{v}, 2\pi) = \dfrac{2\pi}{3}\bar{v} - \dfrac{\pi}{\sqrt{3}}\bar{u}\bar{v} - \dfrac{\pi}{4}\bar{u}^2\bar{v} - \dfrac{\pi}{4}\bar{v}^3, \\ \bar{\xi}_2(\bar{u}, \bar{v}, 2\pi) = -\dfrac{\pi}{8}\bar{v} + [\bar{u}, \bar{v}]_2, \\ \bar{\eta}_2(\bar{u}, \bar{v}, 2\pi) = \dfrac{7\pi}{36\sqrt{3}} + \dfrac{17\pi}{72}\bar{u} + \dfrac{2\pi^2}{9}\bar{v} + [\bar{u}, \bar{v}]_2, \\ \bar{\xi}_3(\bar{u}, \bar{v}, 2\pi) = -\dfrac{\pi}{36\sqrt{3}} + [\bar{u}, \bar{v}]_1. \end{cases}$$

Then, in the present case, the equation (3.9) and (3.10) become

$$(3.23)\quad -\frac{\sqrt{3}\pi}{2}\bar{u}^2 - \frac{\pi}{2\sqrt{3}}\bar{v}^2 + [\bar{u}, \bar{v}]_3 + \varepsilon\left(-\frac{\pi}{8}\bar{v} + [\bar{u}, \bar{v}]_2\right)$$

$$+\varepsilon^2\left(-\frac{\pi}{36\sqrt{3}} + [\bar{u}, \bar{v}]_1\right) + O(\varepsilon^3) = 0,$$

$$(3.24)\quad \frac{2\pi}{3}\bar{v} + [\bar{u}, \bar{v}]_2 + \varepsilon\left(\frac{7\pi}{36\sqrt{3}} + [\bar{u}, \bar{v}]_1\right) + O(\varepsilon^2) = 0.$$

The equation (3.24) implies

$$(3.25)\quad \bar{v} = -\frac{7}{24\sqrt{3}}\varepsilon + [\bar{u}, \varepsilon]_2.$$

The substitution of this into (3.23) yields

$$-\frac{\sqrt{3}\pi}{2}\bar{u}^2 - \frac{19\pi}{3456\sqrt{3}}\varepsilon^2 + [\bar{u}, \varepsilon]_3 = 0.$$

This implies there is no real solution of (3.23) and (3.24), or in other words, there is no periodic solution of (1.2) for $\varphi = \frac{\pi}{2}$.

4. The proof for the case where $\rho = 2$ and $|A| > \frac{1}{2}$.

Put $\rho = 2$ in (1.14), then we have

$$(4.1)\quad 2E^2 - 4A^2 = 1.$$

The E and A satisfying (4.1) can be expressed as follows:

$$(4.2)\quad \begin{cases} E = \dfrac{1}{\sqrt{2}}\sec\varphi, \\ A = \dfrac{1}{2}\tan\varphi, \end{cases}$$

by the assumption $|A|>\frac{1}{2}$, we may suppose $\frac{\pi}{4}<\varphi<\frac{\pi}{2}$ or $-\frac{\pi}{4}>\varphi>-\frac{\pi}{2}$.

Putting $\rho=2$ in (1.13), from (4.2), we get the parametric representation of the solution of the equations (1.12) as follows:

$$\begin{cases} u=\sqrt{2}\cos\varphi, \\ v=\sqrt{2}\sin\varphi. \end{cases} \tag{4.3}$$

As in the preceding paragraph, let us transform the variables (ξ, η) to $(\bar{\xi}, \bar{\eta})$ by

$$\begin{cases} \xi=\bar{\xi}+\sqrt{2}\cos\varphi, \\ \eta=\bar{\eta}+\sqrt{2}\sin\varphi \end{cases} \tag{4.4}$$

and put

$$\begin{cases} \bar{f}(\bar{\xi}, \bar{\eta}, t)=f(\bar{\xi}+\sqrt{2}\cos\varphi, \bar{\eta}+\sqrt{2}\sin\varphi, t), \\ \bar{g}(\bar{\xi}, \bar{\eta}, t)=g(\bar{\xi}+\sqrt{2}\cos\varphi, \bar{\eta}+\sqrt{2}\sin\varphi, t). \end{cases}$$

These functions can be expanded with respect to $(\bar{\xi}, \bar{\eta})$ as follows:

$$\bar{f}(\bar{\xi}, \bar{\eta}, t)=\Big\{\frac{1}{2\sqrt{2}}\cos\varphi(3-4\cos^2\varphi)\cos 2t+\frac{1}{2\sqrt{2}}\sin\varphi(1-4\cos^2\varphi)\sin 2t$$

$$-\frac{1}{2\sqrt{2}}\cos\varphi(3-4\cos^2\varphi)\cos 4t-\frac{1}{2\sqrt{2}}\sin\varphi(1-4\cos^2\varphi)\sin 4t\Big\}$$

$$+\Big\{-\frac{1}{4}\cos 2\varphi-\frac{1}{2}\cos 2\varphi\cos 2t+\frac{1}{4}\tan\varphi(1-4\cos^2\varphi)\sin 2t$$

$$+\frac{3}{4}\cos 2\varphi\cos 4t+\frac{3}{4}\sin 2\varphi\sin 4t\Big\}\bar{\xi}$$

$$+\Big\{-\frac{1}{4}\tan\varphi\cos 2\varphi-\frac{1}{4}\tan\varphi(1-8\cos^2\varphi)\cos 2t-\cos 2\varphi\sin 2t$$

$$-\frac{3}{4}\sin 2\varphi\cos 4t+\frac{3}{4}\cos 2\varphi\sin 4t\Big\}\bar{\eta}$$

$$+\Big\{-\frac{3}{4\sqrt{2}}\cos\varphi-\frac{1}{2\sqrt{2}}\sin\varphi\sin 2t+\frac{3}{4\sqrt{2}}\cos\varphi\cos 4t$$

$$+\frac{3}{4\sqrt{2}}\sin\varphi\sin 4t\Big\}\bar{\xi}^2$$

$$+\Big\{-\frac{1}{2\sqrt{2}}\sin\varphi+\sqrt{2}\sin\varphi\cos 2t-\frac{1}{\sqrt{2}}\cos\varphi\sin 2t-\frac{3}{2\sqrt{2}}\sin\varphi\cos 4t$$

$$+\frac{3}{2\sqrt{2}}\cos\varphi\sin 4t\Big\}\,\bar{\xi}\bar{\eta}$$

$$+\Big\{-\frac{1}{4\sqrt{2}}\cos\varphi+\frac{1}{\sqrt{2}}\cos\varphi\cos 2t+\frac{3}{2\sqrt{2}}\sin\varphi\sin 2t-\frac{3}{4\sqrt{2}}\cos\varphi\cos 4t$$

$$-\frac{3}{4\sqrt{2}}\sin\varphi\sin 4t\Big\}\,\bar{\eta}^2+[\bar{\xi},\bar{\eta};t]_3,$$

$$\bar{g}(\bar{\xi},\bar{\eta},t)=\Big\{\frac{1}{2\sqrt{2}}\sin\varphi(1-4\cos^2\varphi)\cos 2t-\frac{1}{2\sqrt{2}}\cos\varphi(3-4\cos^2\varphi)\sin 2t$$

$$+\frac{1}{2\sqrt{2}}\sin\varphi(1-4\cos^2\varphi)\cos 4t-\frac{1}{2\sqrt{2}}\cos\varphi(3-4\cos^2\varphi)\sin 4t\Big\}$$

$$+\Big\{-\frac{1}{4}\tan\varphi(1+2\cos^2\varphi)-\frac{1}{4}\tan\varphi(1+8\cos^2\varphi)\cos 2t+\cos 2\varphi\sin 2t$$

$$-\frac{3}{4}\sin 2\varphi\cos 4t+\frac{3}{4}\cos 2\varphi\sin 4t\Big\}\,\bar{\xi}$$

$$+\Big\{\frac{1}{4}\cos 2\varphi-\frac{1}{2}\cos 2\varphi\cos 2t-\frac{1}{4}\tan\varphi(1+4\cos^2\varphi)\sin 2t$$

$$-\frac{3}{4}\cos 2\varphi\cos 4t-\frac{3}{4}\sin 2\varphi\sin 4t\Big\}\,\bar{\eta}$$

$$+\Big\{-\frac{1}{4\sqrt{2}}\sin\varphi-\frac{1}{\sqrt{2}}\sin\varphi\cos 2t+\frac{3}{2\sqrt{2}}\cos\varphi\sin 2t-\frac{3}{4\sqrt{2}}\sin\varphi\cos 4t$$

$$+\frac{3}{4\sqrt{2}}\cos\varphi\sin 4t\Big\}\,\bar{\xi}^2$$

$$+\Big\{-\frac{1}{2\sqrt{2}}\cos\varphi-\sqrt{2}\cos\varphi\cos 2t-\frac{1}{\sqrt{2}}\sin\varphi\sin 2t-\frac{3}{2\sqrt{2}}\cos\varphi\cos 4t$$

$$-\frac{3}{2\sqrt{2}}\sin\varphi\sin 4t\Big\}\,\bar{\xi}\bar{\eta}$$

$$+\Big\{-\frac{3}{4\sqrt{2}}\sin\varphi-\frac{1}{2\sqrt{2}}\cos\varphi\sin 2t+\frac{3}{4\sqrt{2}}\sin\varphi\cos 4t$$

$$-\frac{3}{4\sqrt{2}}\cos\varphi\sin 4t\Big\}\,\bar{\eta}^2+[\bar{\xi},\bar{\eta};t]_3.$$

By the substitution (4.4), the system (1.5) is transformed to the system of the same form as (3.5). Let us write the solution of the transformed system in the same notations as in the preceding paragraph. Then, in the present case, for a periodic solution of the transformed system, we have the equations as follows:

$$
(4.5)\quad \begin{cases}
-\frac{\pi}{2}\cos 2\varphi\cdot\bar{u}-\frac{\pi}{2}\tan\varphi\cos 2\varphi\cdot\bar{v}-\frac{3\pi}{2\sqrt{2}}\cos\varphi\cdot\bar{u}^2-\frac{\pi}{\sqrt{2}}\sin\varphi\cdot\bar{u}\bar{v} \\
\quad -\frac{\pi}{2\sqrt{2}}\cos\varphi\cdot\bar{v}^2+[\bar{u},\bar{v}]_3 \\
\quad +\varepsilon\Big[-\frac{\pi}{16\sqrt{2}}\sin\varphi(3-4\cos^2\varphi)^2 \\
\quad +\left\{-\frac{\pi}{16}\tan\varphi(7-34\cos^2\varphi+32\cos^4\varphi)+\frac{\pi^2}{8}\frac{\cos 2\varphi}{\cos^2\varphi}\right\}\bar{u} \\
\quad -\frac{\pi}{32}\frac{1}{\cos^2\varphi}(6-15\cos^2\varphi-4\cos^4\varphi+16\cos^6\varphi)\bar{v}+[\bar{u},\bar{v}]_2\Big] \\
\quad +O(\varepsilon^2)=0, \\
-\frac{\pi}{2}\tan\varphi(1+2\cos^2\varphi)\bar{u}+\frac{\pi}{2}\cos 2\varphi\cdot\bar{v}-\frac{\pi}{2\sqrt{2}}\sin\varphi\cdot\bar{u}^2 \\
\quad -\frac{\pi}{\sqrt{2}}\cos\varphi\cdot\bar{u}\bar{v}-\frac{3\pi}{2\sqrt{2}}\sin\varphi\cdot\bar{v}^2+[\bar{u},\bar{v}]_3 \\
\quad +\varepsilon\Big[-\frac{\pi}{16\sqrt{2}}\frac{1}{\cos\varphi}(1-13\cos^2\varphi+24\cos^4\varphi-16\cos^6\varphi) \\
\quad -\frac{\pi}{32}\frac{1}{\cos^2\varphi}(2-25\cos^2\varphi+76\cos^4\varphi-80\cos^6\varphi)\bar{u} \\
\quad +\left\{\frac{\pi}{16}\tan\varphi(7-2\cos^2\varphi)+\frac{\pi^2}{8}\frac{\cos 2\varphi}{\cos^2\varphi}\right\}\bar{v}+[\bar{u},\bar{v}]_2\Big] \\
\quad +O(\varepsilon^2)=0.
\end{cases}
$$

Since the Jacobian of the left members of the equations (4.5) with respect to $(\bar{u}, \bar{v})$ does not vanish for the value of φ under condition, the (4.5) can be solved uniquely with respect to $(\bar{u}, \bar{v})$ as follows:

$$
(4.6)\quad \begin{cases}
\tilde{u}=-\frac{1}{8\sqrt{2}}\sin\varphi(1-4\cos^2\varphi)\varepsilon+o(\varepsilon), \\
\tilde{v}=-\frac{1}{4\sqrt{2}}\frac{\cos\varphi}{\cos 2\varphi}(5-15\cos^2\varphi+12\cos^4\varphi)\varepsilon+o(\varepsilon).
\end{cases}
$$

The unique existence of the solution of the equations (4.5) implies that there exists one and only one periodic solution of (1.2) for any (A, ρ) lying on L_+ or L_-.

The stability of the periodic solution corresponding to the initial value $(\tilde{u}, \tilde{v})$ given by (4.6) is decided by the signs of the real parts of the characteristic roots of the matrix

$$A = \begin{pmatrix} \dfrac{\partial \bar{\xi}}{\partial \bar{u}}(\tilde{u}, \tilde{v}, 2\pi, \varepsilon) & \dfrac{\partial \bar{\xi}}{\partial \bar{v}}(\tilde{u}, \tilde{v}, 2\pi, \varepsilon) \\ \dfrac{\partial \bar{\eta}}{\partial \bar{u}}(\tilde{u}, \tilde{v}, 2\pi, \varepsilon) & \dfrac{\partial \bar{\eta}}{\partial \bar{v}}(\tilde{u}, \tilde{v}, 2\pi, \varepsilon) \end{pmatrix}.$$

By (4.6), A is of the form

$$A = E + \varepsilon A_1 + \varepsilon^2 A_2 + o(\varepsilon^2).$$

Consequently A can be written in the exponential form as follows:

$$A = \exp(\varepsilon B),$$

where

$$B = A_1 + \varepsilon\left(A_2 - \frac{1}{2} A_1^2\right) + o(\varepsilon).$$

Now, if we write B as

$$B = \begin{pmatrix} a_1 + \varepsilon a_2 + o(\varepsilon) & b_1 + \varepsilon b_2 + o(\varepsilon) \\ c_1 + \varepsilon c_2 + o(\varepsilon) & d_1 + \varepsilon d_2 + o(\varepsilon) \end{pmatrix},$$

it is readily seen from (4.5) that $a_1 + d_1 = 0$ and $a_1 d_1 - b_1 c_1 > 0$. Hence it is seen that the real parts of the characteristic roots of the matrix B are of the form as follows:

$$\varepsilon \frac{a_2 + d_2}{2} + o(\varepsilon).$$

The quantities a_2 and d_2 can be calculated using (4.5) and (4.6), and there is found

$$a_2 + d_2 = \frac{\pi}{4} \frac{\sin\varphi \cos\varphi}{\cos 2\varphi}. \qquad (4.7)$$

This is negative for $\frac{\pi}{4} < \varphi < \frac{\pi}{2}$ and is positive for $-\frac{\pi}{4} > \varphi > -\frac{\pi}{2}$. Hence the periodic solution corresponding to the initial value given by (4.6) is stable for (A, ρ) lying on L_+ and is unstable for (A, ρ) lying on L_- (cf. (4.2)).

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