J. Sci. Hiroshima Univ. Ser. A-I
27 (1963), 61-72

Groupoid and Cohomology with Values in a Sheaf of Groupoids

Shigeru Izuho

(Received March 5, 1963)

A. Grothendieck developed in [1]* systematically the notion of fibre space on a topological space X with structure sheaf $\mathfrak{G}$, where $\mathfrak{G}$ is any sheaf of groups, and the notion of 1-cohomology set $H^1(X, \mathfrak{G})$ of X with values in $\mathfrak{G}$. He showed the important relation between the elements of $H^1(X, \mathfrak{G})$ and the classes of fibre spaces on X with structure sheaf $\mathfrak{G}$. And he obtained the exact sequence of the cohomology sets (dim. 0 and 1) of X with values in sheaves of groups on X.

A. Haefliger introduced in [2] the cohomology sets $H^0(X, \mathfrak{P})$ and $H^1(X, \mathfrak{P})$ of X with values in a sheaf of groupoids $\mathfrak{P}$ on X. And under the assumption that $\mathfrak{P}$ is transitive, he proved that there exists a one-to-one correspondence between $H^1(X, \mathfrak{P})$ and $H^1(X, \mathfrak{G}^f)$, where $\mathfrak{G}^f$ is the sheaf of groups associated to an element f of $Z^1(\mathfrak{U}, \mathfrak{P})$.

In this paper, it is shown that we can obtain an exact sequence of the cohomology sets (dim. 0 and 1) of X with values in sheaves of groupoids on X. We deal with the inverse problem of the relation between $\mathfrak{P}$ and $\mathfrak{G}^f$ which was shown by A. Haefliger: When a sheaf of groups $\mathfrak{G}$ on X is given, we may introduce a sheaf of transitive groupoids $\mathfrak{P}$ on X and an element f of $Z^1(\mathfrak{U}, \mathfrak{P})$ such that we have a one-to-one correspondence $H^1(X, \mathfrak{P}) \rightarrow H^1(X, \mathfrak{G}^f)$ where the latter set can be identified with $H^1(X, \mathfrak{G})$. And when the sheaf of transitive groupoids $\mathfrak{P}$ has a unit section, it is shown that there exists a sheaf of groups $\mathfrak{G}$, which is simpler than $\mathfrak{G}^f$, such that $H^1(X, \mathfrak{P})$ corresponds one-one to $H^1(X, \mathfrak{G})$.

In the first half part, we prepare ourselves for treating the above-mentioned problems. In §1, we prove that two systems of axioms in the definition of groupoid—those in A. Haefliger's paper [2] and in P. Dedecker's [3] —are equivalent, and in §§2-3 we introduce the concept of normal subgroupoid and quotient groupoid. In §4, we refer to a representation of groupoid.

In the last half part, we solve the above-mentioned problems, applying the results in §§1-4 and introducing the concept of groupoid extension of a sheaf of groups.

§1. Axioms of groupoid.[2][3] A groupoid is a set Π which has a composition law $(x, y) \rightarrow xy$, defined for some pairs of elements x, y ($\in \Pi$) and

* The numbers in brackets refer to References at the end of this paper.

satisfying some axioms we shall state below. At first we state a definition.

Definition. An element e of Π is called a *right unit* (resp. *left unit*), if $xe=x$ (resp. $ex=x$) whenever xe (resp. ex) is defined. And an element e of Π is called a *unit*, if $xe=x$ and $ey=y$ whenever xe and ey are defined.

Our axioms are as follows:

(G1) *If for x, y, z ($\in \Pi$) one of $(xy)z$ or $x(yz)$ is defined, then the other is defined and two are equal*: $(xy)z=x(yz)$.

(G2) *For any $x(\in \Pi)$, there exist a right unit $e_x(\in \Pi)$ and a left unit ${}_xe(\in \Pi)$ such that $xe_x=x$ and ${}_xex=x$.*

(G3) *For any $x(\in \Pi)$, there exists an inverse element $x^{-1}(\in \Pi)$ such that $x^{-1}x=e_x$ and $xx^{-1}={}_xe$.*

Proposition 1. *The system of axioms* (G1), (G2) *and* (G3) *in the definition of groupoid is equivalent to that of axioms* (G1) *and* (G2′) *and* (G3′) *which follow*:

(G2′) *For any $x(\in \Pi)$, there exist units $e_x(\in \Pi)$ and ${}_xe(\in \Pi)$ such that $xe_x=x$ and ${}_xex=x$.*

(G3′) *For any $x(\in \Pi)$, there exists an inverse element $x^{-1}(\in \Pi)$ such that $x^{-1}x=e_x$.*

Proof of (G1), (G2), (G3)$\Rightarrow$(G1), (G2′), (G3′). Suppose that e is a right unit and that ey is defined. Then $z=ey\Rightarrow e_z=(z^{-1}e)y\Rightarrow e_z=z^{-1}y\Rightarrow z=(zz^{-1})y\Rightarrow z={}_zey\Rightarrow z=y$. Therefore e is also a left unit, so that it is a unit. Similarly if e is a left unit, then it is also a right unit, so that it is a unit.

Proof of (G1), (G2′), (G3′)$\Rightarrow$(G1), (G2), (G3). We must only prove $xx^{-1}={}_xe$. For this, at first we prove the uniqueness of e_x and ${}_xe$ by (G1), (G2′) and (G3′).

Suppose $xe_x=x\bar{e}_x=x$, then x^{-1} can be multiplied from the left, and $(x^{-1}x)e_x=(x^{-1}x)\bar{e}_x$ by (G1). Hence $e_xe_x=e_x\bar{e}_x$, so we have $e_x=\bar{e}_x$. Next suppose ${}_xex={}_x\bar{e}x=x$, then x^{-1} can be multiplied from the left, and $(x^{-1}{}_xe)x=(x^{-1}{}_x\bar{e})x$ by (G1). Hence $x^{-1}{}_xe$ and $x^{-1}{}_x\bar{e}$ are defined and both are equal to x^{-1}, so we have ${}_xe={}_x\bar{e}=e_{x^{-1}}$ by the uniqueness of $e_{x^{-1}}$.

Now, we have $x^{-1}x=e_x$ by (G3′), so that $(xx^{-1})x=x$ by (G1) and (G2′). Hence we have $xx^{-1}={}_xe$ by the uniqueness of ${}_xe$.

Groupoid has the following properties ((1)—(7)):

(1) For any $x(\in \Pi)$, e_x and ${}_xe$ are uniquely defined. Therefore, when B is the subset of Π consisting of units of elements of Π, $x\to e_x$ and $x\to {}_xe$, two mappings from Π onto B, are defined. We denote these mappings by a and b respectively. That is, $a(x)=e_x$ and $b(x)={}_xe$.

(2) $e\in B\Rightarrow a(e)=b(e)=e$. Hence $B=a(\Pi)=b(\Pi)$. $a(xy)=a(y)$, $b(xy)=b(x)$ and $a(x^{-1})=b(x)$.

(3) xy is defined $\Leftrightarrow a(x)=b(y)$.

(4) xy and yz are defined$\Rightarrow(xy)z$ and $x(yz)$ are defined.

(5) $yx=zx$ (or $xy=xz$)$\Rightarrow y=z$. By this, for any $x(\in \Pi)$ x^{-1} is uniquely defined.

(6) $(x^{-1})^{-1}=x$ and $(xy)^{-1}=y^{-1}x^{-1}$.

Definition. If for $e, e'(\in B)$ there exists $x(\in \Pi)$ such that $e'=xex^{-1}$, then we say e and e' are *mutually transitive* in Π. And if any two elements of B are mutually transitive in Π, then Π is called a *transitive groupoid.*

(7) Let $\Pi_{ef}=\{x\in\Pi: b(x)=e, a(x)=f;\ e, f\in B\}$, then Π_{ee} is a group with e as the unit. When Π is transitive, there exist $y, z(\in \Pi)$ such that $e'=yey^{-1}$ and $f'=zfz^{-1}$, and we have a one-to-one correspondence $x\in\Pi_{ef}\rightarrow yxz^{-1}\in\Pi_{e'f'}$. In particuler, $x\in\Pi_{ee}\rightarrow yxy^{-1}\in\Pi_{ff}$ is an isomorphism, and such an isomorphism is determined modulo inner automorphisms of $\Pi_{e'e'}$.

§2. Subgroupoid of Π[2].

Definition. When a subset Π' of Π is itself a groupoid under the composition law induced from that of Π, it is called a *subgroupoid* of Π. When a subset Π' of Π contains all elements of Π having the same unit as x if Π' contains $x(\in\Pi)$, it is a subgroupoid of Π and is called a *complete subgroupoid* of Π.

Any groupoid Π is the union of disjoint complete transitive subgroupoids. This fact can be shown as follows: When $e, e'(\in B)$ are mutually transitive in Π, we say that these are equivalent. This relation is an equivalence relation, and so B can be classified by this relation. Let $B=\bigcup_\lambda B_\lambda$, where B_λ are classes under this relation. Then $\Pi_\lambda=a^{-1}(B_\lambda)$ is a complete transitive subgroupoid of Π, and $\Pi=\bigcup_\lambda \Pi_\lambda$.

§3. Homomorphism.[2] Normal subgroupoid and quotient groupoid.

Definition. Suppose that Π and Π' are two groupoids, and that $B=a(\Pi)$ and $B'=a'(\Pi')$, where a' is the right unit mapping in Π'. Let φ be a mapping from Π into Π', and if $x, y(\in\Pi)$ are composable, then let $\varphi(x), \varphi(y)$ $(\in \Pi')$ be composable and let $\varphi(xy)=\varphi(x)\varphi(y)$. Then φ is called a *homomorphism* from Π into Π'. If φ is one-to-one, then it is called an *isomorphism.*

It is clear that $\varphi(a(x))=a'(\varphi(x))$, $\varphi(b(x))=b'(\varphi(x))$, where b' is the left unit mapping in Π', and it is also clear that $\varphi(x^{-1})=\varphi(x)^{-1}$.

Let Π_0 be a subgroupoid of Π. If for $x, \bar{x}(\in\Pi)$ there exists $x_0(\in\Pi_0)$ such that $\bar{x}=x_0x$, then we say $\bar{x}$ is *equivalent* to x with respect to Π_0. In order that this relation be an equivalence relation, it is necessary and sufficient that $B\subset\Pi_0$. In this case we denote by Π/Π_0 the quotient set of Π relative to this equivalence relation and by $[x]$ the equivalence class containing x.

Let us consider the conditions on Π_0 in order that Π/Π_0 be a groupoid under a natural composition law. The composition law we wish to define as follows: $[x][y]$ is defined if and only if these cotain the composable elements $\bar{x}$ and $\bar{y}$ respectively, and $[x][y]=[\bar{x}\bar{y}]$. Then let us consider the condition on Π_0 in order that this definition be adequate. Now if $\bar{\bar{x}}(\in[x])$ and $\bar{\bar{y}}(\in[y])$ are other composable elements, then $\bar{\bar{x}}\bar{\bar{y}}$ must be equivalent to $\bar{x}\bar{y}$. For this, it is necessary and sufficient that $z\Pi_0z^{-1}\subset\Pi_0$ for any $z(\in\Pi)$. That is, if zx_0z^{-1} is defined for $x_0(\in\Pi_0)$ and $z(\in\Pi)$, then $zx_0z^{-1}\in\Pi_0$. In this case we have $[x][y]=[\bar{x}\bar{y}]=[\bar{\bar{x}}\bar{\bar{y}}]$. Next, let us consider the conditions on Π_0 in order that Π/Π_0 satisfy the axioms of groupoid (G1), (G2) and (G3) under the above composition law.

(G1) in Π/Π_0 follows from (G1) in Π.

Next we consider (G2). By the definition we have $[x][a(x)]=[x]$, hence it is necessary that $[y][a(x)]=[y]$ for any $[y](\in\Pi/\Pi_0)$ such that $[y][a(x)]$ is defined. Since $x_0a(x)=x_0\in[a(x)](x_0\in\Pi_0)$, for any $[y]$ such that $a(y)=b(x_0)$, $[y][a(x)]$ is defined and equal to $[yx_0](\in\Pi/\Pi_0)$. Since it must be $[yx_0]=[y]$, it is necessary that there exists $z_0(\in\Pi_0)$ such that $yx_0=z_0y$. Hence we have $b(x_0)=a(x_0)=a(y)$. Therefore it is necessary that $b(x_0)=a(x_0)$ for any $x_0(\in\Pi_0)$. This condition is clearly sufficient in order that there exist a right unit for any element of Π/Π_0. Similarly this condition is necessary and sufficient in order that there exist a left unit for any element of Π/Π_0.

(G3) is satisfied in Π/Π_0, and $[x]^{-1}=[x^{-1}]$.

Now, when Π_0 satisfied the condition that $b(x_0)=a(x_0)$ for any $x_0(\in\Pi_0)$, the composition law of Π/Π_0 becomes the following: If $[x][y]$ is defined, then any elements of $[x]$ and $[y]$ are composable, and $[x][y]=[xy]$.

When Π_0 satisfies the conditions $B\subset\Pi_0$, $z\Pi_0z^{-1}\subset\Pi_0$ for any $z(\in\Pi)$ and $b(x_0)=a(x_0)$ for any $x_0(\in\Pi_0)$, we have $\Pi_0=\bigcup_{e\in B}N_e(N_e\subset\Pi_{ee})$, where N_e has the following properties:

(1) N_e is a normal subgroup of Π_{ee}.

(2) When $e, e'(\in B)$ are mutually transitive in Π, $N_{e'}$ is isomorphic to N_e. In fact, in this case $e'=zez^{-1}$ $(z\in\Pi)$, and $x_0\in N_e\to zx_0z^{-1}\in N_{e'}$ is an isomorphism mentioned above. And such an isomorphism is defined uniquely modulo inner automorphisms of $\Pi_{e'e'}$.

So, we have the following proposition:

Proposition 2. *In order that Π/Π_0 be a groupoid, it is necessary and sufficient that there holds $\Pi_0=\bigcup_{\lambda}(\bigcup_{e\in B_\lambda}N_e)$, where N_e is a normal subgroup of Π_{ee} and for all $e(\in B_\lambda)$ N_e are isomorphic to one another. In this case $\varphi: x\in\Pi\to[x]\in\Pi/\Pi_0$ is a homomorphism from Π onto Π/Π_0, and if $\varphi(x)\varphi(y)$ is defined, then xy is also defined. And when B' is the subset consisting of units of Π/Π_0, the restriction of φ to B, $\varphi/B: B\to B'$ is one-to-one.*

Definition. $\Pi_0=\bigcup_{\lambda}(\bigcup_{e\in B_\lambda}N_e)$ in the above Proposition is called a *normal subgroupoid* of Π.

In connection with this we have the following proposition as in a group.

Proposition 3. *Suppose that φ is a homomorphism from Π onto Π', and that φ/B is one-to-one mapping from B onto B'. Let $\Pi_0=\varphi^{-1}(B')$, then Π_0 is a normal subgroupoid of Π, and Π/Π_0 is isomorphic to Π'.*

§4. Representation of groupoid.[4] Suppose that Π is a transitive groupoid and that $B=a(\Pi)$. Let $\Pi_{ef}=\{x\in\Pi: b(x)=e,\ a(x)=f;\ e, f\in B\}$, then Π_{ee} is a group with e as the unit, and all Π_{ee} are isomorphic to one another. We put $G_e=\Pi_{ee}$. And let us consider a product set $B\times G_e\times B=\Pi'$. This becomes a transitive groupoid under the following composition law and is isomorphic to Π.

We define that $(p, g, q), (r, g', s)\ (\in\Pi')$ are *composable* if and only if $q=r$, and that $(p, g, q)\,(q, g', s)=(p, gg', s)$. Let a' (resp. b') be the right (resp. left) unit mapping in Π', then $a'(p, g, q)=(q, e, q)$, $b'(p, g, q)=(p, e, p)$ and $(p, g, q)^{-1}=(q, g^{-1}, p)$. Identifying (p, e, p) with $p(\in B)$, $a'(\Pi')$ can be identified with B. Π' is evidently transitive. Hence Π' is a transitive groupoid with B as the set of units.

For any $f(\in B)$, suppose that $x_{ef}(\in\Pi_{ef})$ is fixed such that $x_{ee}=e$ and that $x_{fe}=x_{ef}^{-1}$. Then $g(=x_{ep}xx_{qe})$ is an element of G_e for any $x(\in\Pi_{pq}\subset\Pi)$, and so h: $x\in\Pi_{pq}\subset\Pi\rightarrow(p, g, q)\in\Pi'$ is an isomorphism from Π onto Π'. Hence Π' is a representation of Π.

In general, when $\Pi=\bigcup\limits_{\lambda}\Pi_\lambda$ where Π_λ is transitive, Π can be represented by $\bigcup\limits_{\lambda}B_\lambda\times G_e\times B_\lambda$ where e is a fixed element of B_λ. When $\Pi_0(=\bigcup\limits_{\lambda}(\bigcup\limits_{e\in B\lambda}N_e))$ is a normal subgroupoid of Π, since the set consisting of units of Π/Π_0 is one-to-one onto B, Π/Π_0 is represented by $\bigcup\limits_{\lambda}B_\lambda\times G_e/N_e\times B_\lambda$.

§5. Cohomology with values in a sheaf of groupoids on a topological space.[1][2]

Definition. Let P, Q and R be sets of elements, and let R have a subset D whose elements are called neuter. Let us consider a sequence:

$$P\xrightarrow{u}Q\xrightarrow{v}R, \tag{5.1}$$

where u and v are mappings. When $v^{-1}(D)=u(P)$, we say that the sequence (5.1) is *exact* as usual. Further suppose that Q has neuter elements which form a subset C of Q, and that $v(C)\subset D$ and that a mapping p: $Q\rightarrow C$ is defined such that the restriction of p to C is identity. And suppose that P is a groupoid of operators on Q, and that $B=a(P)$ where a is the right unit mapping in P, and that $u(B)\subset C$ and $pu=ua$. The fact that P is a groupoid of operators on Q means the following: $x(\in P)$ can operate on $y(\in Q)$ if and only if $u(a(x))=p(y)$, and $x\cdot y\in Q$, satisfying the following properties:

(α) $p(x\cdot y)=u(b(x))$, where b is the left unit mapping in P.

(β) When $x_1, x_2 \in P, y \in Q$, and x_1x_2 is defined, and x_1x_2 can operate on y, there holds $(x_1x_2)\cdot y = x_1\cdot(x_2\cdot y)$.

(γ) When $e(\in B)$ can operate on $y(\in Q)$, $e\cdot y = y$.

When the sequence (5.1) satisfies the following conditions, we say that it is *strongly exact.*

(i) *For any* $x(\in P)$, $u(x) = x\cdot c$ *or* $u(x) = x^{-1}\cdot c$ *where* $c = u(a(x))$ $(\in C)$ *or* $c = u(b(x))$ $(\in C)$ *respectively.*

(ii) *For* $y_1, y_2(\in Q)$,
$v(y_1) = v(y_2) \Leftrightarrow$ *there exists* $x(\in P)$ *such that* $y_2 = x\cdot y_1$.

(iii) v/C: $C \to D$ *is onto.*

LEMMA 1. *If the sequence* (5.1) *is strongly exact, then it is exact.*

PROOF. For any $x(\in P)$, $u(x) = x\cdot c$ or $u(x) = x^{-1}\cdot c$ by (i). Hence $v(u(x)) = v(x\cdot c)$ or $v(x^{-1}\cdot c) = v(c) \in D$ by (ii).

Conversely suppose $y \in v^{-1}(D)$ and $v(y) = d(\in D)$. By (iii) there exists $c(\in C)$ such that $v(c) = d$. Therefore, there exists $x(\in P)$ such that $y = x\cdot c = u(x)$ or $u(x^{-1})$ by (ii) and (i).

LEMMA 2. *If P, Q and R are all groupoids, $C = a'(Q)$, $p = a'$, $D = a''(R)$ (where a' and a'' are the right unit mappings in Q and R resp.), u and v are homomorphisms of groupoids, and if v/C: $C \to D$ is one-to-one, then P can be a groupoid of operators on Q and we have the following: If the sequence* (5.1) *is exact, then it is strongly exact under the above operation.*

PROOF. P can be a groupoid of operators on Q as follows: We say that $x(\in P)$ can operate on $y(\in Q)$ if and only if $yu(x^{-1})$ is defined, and in this case we shall define $x\cdot y = yu(x^{-1})$. Now $yu(x^{-1})$ is defined if and only if $a'(y) = b'(u(x^{-1})) = u(b(x^{-1})) = u(a(x))$, where b' is the left unit mapping in Q. In this case $a'(x\cdot y) = a'(yu(x^{-1})) = a'(u(x^{-1})) = u(a(x^{-1})) = u(b(x))$, hence this operation satisfies (α), and clearly it satisfies (β) and (γ).

Now suppose the sequence (5.1) is exact, then we can prove its strong exactness as follows:

Proof of (i). Let x be any element of P and let $c = u(b(x))$. Then $x^{-1}\cdot c = cu(x) = u(b(x))u(x) = b'(u(x))u(x) = u(x)$.

Proof of (ii). Let $y_1, y_2 \in Q$ and $v(y_1) = v(y_2)$. Then $v(y_1^{-1}) = v(y_2)^{-1}$, hence $a''(v(y_1^{-1})) = a''(v(y_2)^{-1}) = b''(v(y_2))$, hence $v(a'(y_1^{-1})) = v(b'(y_2))$. Therefore, since v/C: $C \to D$ is one-to-one, $a'(y_1^{-1}) = b'(y_2)$. Hence $y_1^{-1}y_2$ is defined, and $v(y_1^{-1}y_2) = v(y_1^{-1})v(y_2) = a''(v(y_2)) \in D$. Hence from the exactness of the sequence (5.1) there exists $x^{-1} \in P$ such that $y_1^{-1}y_2 = u(x^{-1})$. Hence $y_2 = y_1u(x^{-1}) = x\cdot y_1$. Conversely let $y_2 = x\cdot y_1$, then $v(y_2) = v(y_1u(x^{-1})) = v(y_1)v((u(x^{-1})) = v(y_1)$ from the exactness of the sequence (5.1).

(iii) is contained in our given assumption.

Suppose that X is a topological space, $\mathfrak{P}$ is a sheaf of groupoids on X, $\mathfrak{p}$ is the projection mapping from $\mathfrak{P}$ onto X, $\mathfrak{B}$ is the subsheaf consisting of units of $\mathfrak{P}$, $\mathfrak{N}$ is a subsheaf of normal subgroupoids of $\mathfrak{P}$, and that $\mathfrak{n}(=\mathfrak{p}/\mathfrak{N})$ is the restriction of $\mathfrak{p}$ to $\mathfrak{N}$. The fact that $\mathfrak{N}$ is a subsheaf of normal subgroupoids of $\mathfrak{P}$, means that $\mathfrak{N}_x(=\mathfrak{n}^{-1}(x))$ is a normal subgroupoid (we have defined in §3) of $\mathfrak{P}_x(=\mathfrak{p}^{-1}(x))$ for any $x(\in X)$. Let $\mathfrak{H}=\mathfrak{P}/\mathfrak{N}$, then this is a sheaf of groupoids on X. And $\mathfrak{B}\to\mathfrak{N}\xrightarrow{i}\mathfrak{P}\xrightarrow{j}\mathfrak{H}\to\mathfrak{B}$ is exact. Hereafter, we shall denote the unit mappings in $\mathfrak{N}$, $\mathfrak{P}$ and $\mathfrak{H}$ by the same notations a and b, and use the following notations as usual.

$C^0(\mathfrak{U}, \mathfrak{P})$: set of 0-cochains in $\mathfrak{P}$ over an open covering $\mathfrak{U}(=(U_i)_{i\in I})$ of X.

$Z^1(\mathfrak{U}, \mathfrak{P})$: set of 1-cocycles in $\mathfrak{P}$ over $\mathfrak{U}$, etc.

We have an analogous proposition as in a sheaf of groups.

Proposition 4. (5.2) $H^0(X, \mathfrak{B})\xrightarrow{i_0'}H^0(X, \mathfrak{N})\xrightarrow{i_0}H^0(X, \mathfrak{P})\xrightarrow{j_0}H^0(X, \mathfrak{H})\xrightarrow{\delta_0}H^1(X, \mathfrak{N})\xrightarrow{i_1}H^1(X, \mathfrak{P})\xrightarrow{j_1}H^1(X, \mathfrak{H})$ *is exact, and is strongly exact till* $H^1(X, \mathfrak{P})$.

Proof. We define that the product p_1p_2 of $p_1, p_2(\in H^0(X, \mathfrak{P}))$ is defined if and only if $p_1(x)p_2(x)$ is defined for any $x(\in X)$. Then $H^0(X, \mathfrak{P})$ is a groupoid. Similarly $H^0(X, \mathfrak{B})$, $H^0(X, \mathfrak{N})$ and $H^0(X, \mathfrak{H})$ are also groupoids, and i_0', i_0 and j_0 are homomorphisms of groupoids. Let the neuter elements of $H^0(X, \mathfrak{B})$, $H^0(X, \mathfrak{N})$, $H^0(X, \mathfrak{P})$ and $H^0(X, \mathfrak{H})$ be unit sections of $\mathfrak{B}$, $\mathfrak{N}$, $\mathfrak{P}$ and $\mathfrak{H}$ respectively, then these satisfy the condition in Lemma 2, $v/C: C\to D$ is one-to-one. Evidently the sequence (5.2) is exact till $H^0(X, \mathfrak{H})$, hence by Lemma 2 this is strongly exact till $H^0(X, \mathfrak{H})$.

Definition of δ_0. Let h be any element of $H^0(X, \mathfrak{H})$ and let $\mathfrak{U}(=(U_i)_{i\in I})$ be a sufficiently fine open covering of X. There exists $p_i(\in H^0(U_i, \mathfrak{P}))$ such that $j_0(p_i)=h/U_i$ for any $i(\in I)$. Then, there exists $n_{ij}(\in H^0(U_{ij}, \mathfrak{N}))$ such that $p_i=n_{ij}p_j$ on $U_{ij}(=U_i\cap U_j)$. Since $n_{ij}=p_ip_j^{-1}$, we have $(n_{ij})\in Z^1(\mathfrak{U}, \mathfrak{N})$. We define $\delta_0(h)$ is the element of $H^1(X, \mathfrak{N})$ which is represented by (n_{ij}). Evidently this does not depend on the choice of (p_i).

Proof of the strong exactness at $H^0(X, \mathfrak{H})$. Since $H^0(X, \mathfrak{P})$ and $H^0(X, \mathfrak{H})$ are groupoids, $H^0(X, \mathfrak{P})$ can be a groupoid of operators on $H^0(X, \mathfrak{H})$ as stated generally in the proof of Lemma 2. That is, $p(\in H^0(X, \mathfrak{P}))$ can operate on $h(\in H^0(X, \mathfrak{H}))$ if and only if $hj_0(p^{-1})$ is defined, and we have defined that $p\cdot h=hj_0(p^{-1})$.

Proof of (i). Let p be any of $H^0(X, \mathfrak{P})$, then $p^{-1}\cdot j_0(b(p))=j_0(b(p))j_0(p)=b(j_0(p))j_0(p)=j_0(p)$.

Proof of (ii). Let $h_1, h_2\in H^0(X, \mathfrak{H})$, and we assume $h_2=p\cdot h_1$ where $p\in H^0(X, \mathfrak{P})$. Let $j_0(p_{1i})=h_1/U_i$, then $j_0(p_{1i}p^{-1})=h_2/U_i$, therefore $\delta_0(h_1)=\delta_0(h_2)$. Conversely we assume $\delta_0(h_1)=\delta_0(h_2)$, and let $j_0(p_{1i})=h_1/U_i$ and let $j_0(p_{2i})=h_2/U_i$, then $p_{1i}p_{1j}^{-1}=n_ip_{2i}p_{2j}^{-1}n_j^{-1}$ on U_{ij} where $(n_i)\in C^0(\mathfrak{U}, \mathfrak{N})$. Therefore $p_{2i}^{-1}n_i^{-1}p_{1i}=p_{2j}^{-1}n_j^{-1}p_{1j}$ on U_{ij}, hence $(p_{2i}^{-1}n_i^{-1}p_{1i})$ defines an element of $H^0(X, \mathfrak{P})$, we shall put this as p. Then we have $n_ip_{2i}=p_{1i}p^{-1}$ on U_i. Hence $j_0(n_ip_{2i})=j_0(p_{1i})j_0(p^{-1})$ on U_i, that is $h_2/U_i=p\cdot h_1/U_i$ for any U_i, so that we have $h_2=p\cdot h_1$.

Proof of (iii). Let a neuter element e of $H^1(X, \mathfrak{N})$ be represented by (n_{ij}) $(\in Z^1(\mathfrak{U}, \mathfrak{N}))$, where n_{ij} is a unit section over U_{ij}. Then $n_{ii}=n_{ij}=n_{jj}$ on U_{ij}.

Therefore (n_{ii}) defines a neuter element n of $H^0(X, \mathfrak{P})$, and $\delta_0(j_0(n))=e$. Thus δ_0 maps the set of neuter elements of $H^0(X, \mathfrak{H})$ onto the set of those of $H^1(X, \mathfrak{N})$.

Proof of the strong exactness at $H^1(X, \mathfrak{N})$. $H^0(X, \mathfrak{H})$ can be a groupoid of operators on $H^1(X, \mathfrak{N})$ as follows: Suppose that $h \in H^0(X, \mathfrak{H})$, $n \in H^1(X, \mathfrak{N})$ and that n is represented by (n_{ij}) $(\in Z^1(\mathfrak{U}, \mathfrak{N}))$. In this case $a(n)$ is the neuter element of $H^1(X, \mathfrak{N})$ which is represented by $(n'_{ij})=(n_{ii})=(n_{jj})$ $(\in Z^1(\mathfrak{U}, \mathfrak{N}))$. If and only if $\delta_0(a(h))=a(n)$, h can operate on n, and we define $h\cdot n$ as follows. We replace $\mathfrak{U}$ by a finer open covering when necessary, and we denote it by the same notation $\mathfrak{U}$. Let $j_0(p_i)=h/U_i$, where $p_i \in H^0(U_i, \mathfrak{P})$, then $j_0(a(p_i))=a(j_0(p_i))$ $=a(h)/U_i$. Therefore $\delta_0(a(h))$ is the element of $H^1(X, \mathfrak{N})$, which is represented by $(a(p_i)a(p_j)^{-1})$ $(\in Z^1(\mathfrak{U}, \mathfrak{N}))$. Therefore h can operate on n if and only if $a(p_i)a(p_j)^{-1}=n_in'_{ij}n_j^{-1}$ on U_{ij}, where $(n_i) \in C^0(\mathfrak{U}, \mathfrak{N})$. In this case $a(p_j)=b(n_i)=a(n_i)=b(n'_{ij})=n_{ii}=b(n_{ij})$, hence $p_in_{ij}p_j^{-1}$ can be defined, and $(p_in_{ij}p_j^{-1})=((p_in_{ij}p_i^{-1})(p_ip_j^{-1})) \in Z^1(\mathfrak{U}, \mathfrak{N})$, thus we define that $h\cdot n$ is the element of $H^1(X, \mathfrak{N})$ which is represented by $(p_in_{ij}p_j^{-1})$ $(\in Z^1(\mathfrak{U}, \mathfrak{N}))$. It does not depend on the choice of (p_i) and representatives of n. And this operation satisfies the conditions of operation (α), (β) and (γ).

Proof of (i). Let $h \in H^0(X, \mathfrak{H})$ and let $j_0(p_i)=h/U_i$. Then $\delta_0(h)$ is the element of $H^1(X, \mathfrak{N})$ which is represented by $(p_ip_j^{-1})$ $(\in Z^1(\mathfrak{U}, \mathfrak{N}))$, and this is equal to $h\cdot(a(h))$ by the definition.

Proof of (ii). We assume $n_2=h\cdot n_1$ for $n_1, n_2(\in H^1(X, \mathfrak{N}))$ and $h(\in H^0(X, \mathfrak{H}))$, and suppose that (n_{1ij}), (n_{2ij}) $(\in Z^1(\mathfrak{U}, \mathfrak{N}))$ are representatives of n_1, n_2 respectively and that $j_0(p_i)=h/U_i$. Then $n_{2ij}=n_ip_in_{1ij}p_j^{-1}n_j^{-1}$ on U_{ij} where $(n_i) \in C^0(\mathfrak{U}, \mathfrak{N})$, so that we have $i_1(n_1)=i_1(n_2)$. Conversely, we assume that $i_1(n_1)=i_1(n_2)$. Then, $n_{2ij}=p_in_{1ij}p_j^{-1}$ on U_{ij} where $(p_i) \in C^0(\mathfrak{U}, \mathfrak{P})$, hence $n_{2ij}p_j=(p_in_{1ij}p_i^{-1})p_i$ on U_{ij}, so we have $j_0(p_j)=j_0(p_i)$ on U_{ij}. Therefore, $(j_0(p_i))$ defines an element h of $H^0(X, \mathfrak{H})$, and $n_2=h\cdot n_1$.

In this case, it is clear that (iii) is satisfied.

Finally the exactness at $H^1(X, \mathfrak{P})$ is clear.

§6. Groupoid extension. Suppose that $\mathfrak{P}$ is a sheaf of transitive groupoids on X, and that $f=(f_{ij}) \in Z^1(\mathfrak{U}, \mathfrak{P})$. Let $\mathfrak{G}_i^f$ be a subsheaf of $\mathfrak{P}/U_i$ which consists of elements g_i such that $a(g_i)=b(g_i) \in f_{ii}(U_i)$, and let $\Gamma=\bigcup_{i\in I}\mathfrak{G}_i^f$. When $(i, g_i) \in \mathfrak{G}_i^f$ and $(j, g_j) \in \mathfrak{G}_j^f$, it is said that these are equivalent if and only if $g_i=f_{ij}g_jf_{ji}$ on U_{ij}. The quotient sheaf of Γ relative to the above equivalence relation is a sheaf of groups on X, and denoted by $\mathfrak{G}^f$. Then as shown by A. Haefliger[2] there exists a one-to-one correspondence $H^1(X, \mathfrak{P})\rightarrow H^1(X, \mathfrak{G}^f)$.

In this section we consider the inverse of the above relation. That is to say, when at first a sheaf of groups $\mathfrak{G}$ on X is given, we want to induce a sheaf of transitive groupoids $\mathfrak{P}$ on X and $f(\in Z(\mathfrak{U}, \mathfrak{P}))$ such that we have the above one-to-one correspondence $H^1(X, \mathfrak{P})\rightarrow H^1(X, \mathfrak{G}^f)$ where we can identify $H^1(X, \mathfrak{G}^f)$ with $H^1(X, \mathfrak{G})$.

When G is any group and B is any set, $\Pi = B\times G\times B$ becomes a transitive groupoid as in §4, and B can be identified with the set of units of Π. We shall call this Π *groupoid extension* of G by B.

Suppose $\mathfrak{G}$ is a given sheaf of groups on X, B is any topological space with discrete topology, and also suppose $\mathfrak{B}=X\times B$. Then $\mathfrak{B}$ is a trivial sheaf on X.

Let α: $\mathfrak{G}\rightarrow X$ and β: $\mathfrak{B}\rightarrow X$ be projection mappings, and let $\mathfrak{P}=(\mathfrak{B}, \mathfrak{G}, \mathfrak{B})$ be the set of triples $z=(p, g, q)$, where $p, q \in \mathfrak{B}$ and $g \in \mathfrak{G}$ such that $\beta(p)=\alpha(g) =\beta(q)$, and let $\mathfrak{p}$: $\mathfrak{P}\rightarrow X$ such that $\mathfrak{p}(z)=\beta(p)$, be the projection mapping from $\mathfrak{P}$ onto X. And let W_p, V_g and W_q be the open sets containing p, g and q in $\mathfrak{B}$, $\mathfrak{G}$ and $\mathfrak{B}$ respectively which are homeomorphic to an open set U of $X(\mathfrak{p}(z) \in U)$. We shall define the topology in $\mathfrak{P}$ such that $\{(W_p, V_g, W_q)\}$ is the fundamental system of the open sets. Then $\mathfrak{P}$ is a sheaf on X. In each $\mathfrak{P}_x(=\mathfrak{p}^{-1}(x))$ $(x \in X)$, we introduce a composition law as mentioned in §4. In this way $\mathfrak{P}$ becomes a sheaf of transitive groupoids on X. And $\mathfrak{B}$ can be regarded as the subsheaf of units of $\mathfrak{P}$.

Let $t(\in H^1(X, \mathfrak{P}))$ be represented by $(h_{ij})=((p_{ij}, g_{ij}, q_{ij}))$ $(\in Z^1(\mathfrak{U}, \mathfrak{P}))$. Since $h_{ii}=b(h_{ij})$ and $h_{jj}=a(h_{ij})$ on U_{ij}, there hold $(p_{ii}, g_{ii}, q_{ii})=b(p_{ij}, g_{ij}, q_{ij})=(p_{ij}, e, p_{ij})$ and $(p_{jj}, g_{jj}, q_{jj})=a(p_{ij}, g_{ij}, q_{ij})=(q_{ij}, e, q_{ij})$ on U_{ij}, where e is the unit section of $\mathfrak{G}$ over U_{ij}. Hence $p_{ij}=p_{ii}=q_{ii}$ and $q_{ij}=p_{jj}=q_{jj}$ on U_{ij}. We put $p_{ii}=q_{ii}=p_i$, then h_{ij} becomes the form (p_i, g_{ij}, p_j). So, from the relation $h_{ij}h_{jk}=h_{ik}$ on U_{ijk} $(=U_i\cap U_j\cap U_k)$, we have $g_{ij}g_{jk}=g_{ik}$ on U_{ijk}. Hence $(g_{ij}) \in Z^1(\mathfrak{U}, \mathfrak{G})$, by this the element of $H^1(X, \mathfrak{G})$ is determined and it does not depend on representatives of t. Conversely, when $l(\in H^1(X, \mathfrak{G}))$ is represented by (g_{ij}) $(Z^1(\mathfrak{U}, \mathfrak{G}))$, let (p_i) be any element of $C^0(\mathfrak{U}, \mathfrak{B})$, then $(h_{ij})=((p_i, g_{ij}, p_j)) \in Z^1(\mathfrak{U}, \mathfrak{P})$. By this the element of $H^1(X, \mathfrak{P})$ is determined and it does not depend on the choice of (p_i) and representatives of l. So we have a one-to-one correspondence $H^1(X, \mathfrak{P})\rightarrow H^1(X, \mathfrak{G})$. This is no other than the following correspondence.

Let c be the neuter element of $H^1(X, \mathfrak{G})$ represented by (e_{ij}) $(\in Z^1(\mathfrak{U}, \mathfrak{G}))$, where e_{ij} is the unit section of $\mathfrak{G}$ over U_{ij}, and let (p_i) be any of $C^0(\mathfrak{U}, \mathfrak{B})$. Then $f=(f_{ij})=((p_i, e_{ij}, p_j)) \in Z^1(\mathfrak{U}, \mathfrak{P})$. Let $\mathfrak{G}^f$ be the sheaf of groups on X associated to f such that we have stated at first in this section, then $\mathfrak{G}^f$ is isomorphic to $\mathfrak{G}$. It results from the following: Since $f_{ii}=(p_i, e_{ii}, p_i)$, $\mathfrak{G}^f_i$ consists of elements such as $(i, (p_i, g_i, p_i))$ where $g_i \in \mathfrak{G}$. And the equivalence relation is as follows: $(i, (p_i, g_i, p_i))$ is equivalent to $(j, (p_j, g_j, p_j))$, if and only if $(p_i, g_i, p_i)=(p_i, e_{ij}, p_j)$ $(p_j, g_j, p_j)(p_j, e_{ji}, p_i)=(p_i, g_j, p_i)$ on U_{ij}, that is $g_i=g_j$ on U_{ij}. Therefore $[(i, (p_i, g_i, p_i))] \in \mathfrak{G}^f$ (the equivalence class containing $(i, (p_i, g_i, p_i))\rightarrow g_i \in \mathfrak{G}$ is an isomorphism.

Therefore $H^1(X, \mathfrak{G}^f)$ is one-to-one onto $H^1(X, \mathfrak{G})$. Now, the one-to-one correspondence φ: $H^1(X, \mathfrak{P})\rightarrow H^1(X, \mathfrak{G}^f)$ stated at first in this section is as follows: Let $(h_{ij})=((p_i, g_{ij}, p_j))$ $(\in Z^1(\mathfrak{U}, \mathfrak{P}))$ be a representative of any $s(\in H^1(X, \mathfrak{P}))$. Then $\varphi(s)$ is the element of $H^1(X, \mathfrak{G}^f)$ which is represented by $([(i, h_{ij}f_{ji})])$ $(\in Z^1(\mathfrak{U}, \mathfrak{G}^f))$, where $h_{ij}f_{ji}=(p_i, g_{ij}, p_j)(p_j, e_{ji}, p_i)=(p_i, g_{ij}, p_i)$.

Therefore $H^1(X, \mathfrak{P}) \to H^1(X, \mathfrak{G})$ stated before is no other than the product $H^1(X, \mathfrak{P}) \to H^1(X, \mathfrak{G}^f) \to H^1(X, \mathfrak{G})$. Thus we have the following:

Proposition 5. *Suppose $\mathfrak{G}$ is a sheaf of groups on X. Then there exists a sheaf of transitive groupoids $\mathfrak{P}$ on X such that we have a one-to-one correspondence $H^1(X, \mathfrak{P}) \to H^1(X, \mathfrak{G})$. Let $f = ((p_i, e_{ij}, p_j)) \in Z^1(\mathfrak{U}, \mathfrak{P})$ where e_{ij} is the unit section of $\mathfrak{G}$ over U_{ij}, then $\mathfrak{G}$ is isomorphic to $\mathfrak{G}^f$ and $H^1(X, \mathfrak{G}) \to H^1(X, \mathfrak{G}^f)$ is one-to-one. And the product $H^1(X, \mathfrak{P}) \to H^1(X, \mathfrak{G}) \to H^1(X, \mathfrak{G}^f)$ is no other than the one-to-one correspondence $H^1(X, \mathfrak{P}) \to H^1(X, \mathfrak{G}^f)$ stated at first in this section.*

§7. Suppose $\mathfrak{P}$ is a sheaf of transitive groupoids on X, $\mathfrak{p}$ is its projection mapping, $\mathfrak{B}$ is the subsheaf of $\mathfrak{P}$ that consists of units of $\mathfrak{P}$ and that $\mathfrak{B}$ has a section e over X. And let us consider the sheaf of groups $\mathfrak{G}$ on X which is the subsheaf of $\mathfrak{P}$ consisting of $g(\in \mathfrak{P}_x = \mathfrak{p}^{-1}(x))$ for any $x(\in X)$ such that $a(g) = b(g) = e(x)$.

Next let $\mathfrak{P}' (= (\mathfrak{B}, \mathfrak{G}, \mathfrak{B}))$ be the set of triples (p, g, q), where $p, q \in \mathfrak{B}_x = \mathfrak{p}^{-1}(x) \cap \mathfrak{B}$ and $g \in \mathfrak{G}_x = \mathfrak{p}^{-1}(x) \cap \mathfrak{G}$. Then $\mathfrak{P}'$ is a sheaf of transitive groupoids on X as in the last section.

Let us consider the one-to-one correspondences: $H^1(X, \mathfrak{P}) \xrightarrow{\varphi} H^1(X, \mathfrak{P}') \xrightarrow{\varphi'} H^1(X, \mathfrak{G})$. Let s be any element of $H^1(X, \mathfrak{P})$, and let $f = (f_{ij})$ $(\in Z^1(\mathfrak{U}, \mathfrak{P}))$ be its representative. We replace $\mathfrak{U}$ by a finer open covering of X when necessary, but here we denote it by the same notation $\mathfrak{U}$. Since $\mathfrak{P}$ is transitive, for any $i \in I$ there exists a section $z_{ef_{ii}}$ of $\mathfrak{P}$ over U_i such that $b(z_{ef_{ii}}(U_i)) = e(U_i)$ and $a(z_{ef_{ii}}(U_i)) = f_{ii}(U_i)$. We denote $z_{ef_{ii}}^{-1}$ by $z_{f_{ii}e}$, then $f'_{ij} = (f_{ii}, z_{ef_{ii}} f_{ij} z_{f_{jj}e}, f_{jj})$ is a section of $\mathfrak{P}'$ over U_{ij}, and $f'_{ij} f'_{jk} = f'_{ij}$ on U_{ijk}. Hence (f'_{ij}) represents an element $s'(\in H^1(X, \mathfrak{P}'))$. Since s' does not depend on representatives of s and the choice of $z_{ef_{ii}}$, we define that $\varphi(s) = s'$.

Put $g_{ij} = z_{ef_{ii}} f_{ij} z_{f_{jj}e}$, then g_{ij} is a section of $\mathfrak{G}$ over U_{ij}, and $g_{ij} g_{jk} = g_{ik}$ on U_{ijk}. Hence $(g_{ij}) \in Z^1(\mathfrak{U}, \mathfrak{G})$. The element of $H^1(X, \mathfrak{G})$ represented by (g_{ij}) is denoted by t. Since t does not depend on representatives of s', we define that $\varphi'(s') = t$. It is clear that φ and φ' are both one-to-one. The neuter element of $H^1(X, \mathfrak{P})$ is mapped by φ on the neuter element of $H^1(X, \mathfrak{P}')$ and it is mapped by φ' on the neuter element of $H^1(X, \mathfrak{G})$.

Thus we have the following:

Proposition 6. *Suppose that $\mathfrak{P}$ is a sheaf of transitive groupoids on a topological space X, $\mathfrak{B}$ is the subsheaf of $\mathfrak{P}$ consisting of units of $\mathfrak{P}$ and that $\mathfrak{B}$ has a section e over X. And suppose $\mathfrak{G}$ is the sheaf of groups on X, which consists of $g(\in \mathfrak{P})$ such that $a(g) = b(g) = e$, and let $\mathfrak{P}' = (\mathfrak{B}, \mathfrak{G}, \mathfrak{B})$.*

Then there exist one-to-one correspondences: $H^1(X, \mathfrak{P}) \xrightarrow{\varphi} H^1(X, \mathfrak{P}') \xrightarrow{\varphi'} H^1(X, \mathfrak{G})$, in which the neuter element of $H^1(X, \mathfrak{P})$ corresponds to the neuter elements of $H^1(X, \mathfrak{P}')$ and $H^1(X, \mathfrak{G})$.

We consider a geometrical meaning of this Proposition. Let (E, p) be a fibre space on $\mathfrak{B}$, where p is its projection mapping. And suppose $\mathfrak{P}$ is a sheaf of transitive groupoids on X which operates on (E, p). That is, $z(\in \mathfrak{P})$ can operate on $y(\in (E, p))$ if and only if $a(z)=p(y)$, and $z\cdot y \in (E, p)$, satisfying the following:

(α) $p(z\cdot y)=b(z)$.

(β) When $z_1, z_2 \in \mathfrak{P}$, $y \in (E, p)$, z_1z_2 is defined and z_1z_2 can operate on y, then $(z_1z_2)\cdot y=z_1\cdot(z_2\cdot y)$.

(γ) When $e(\in \mathfrak{B})$ can operate on $y(\in (E, p))$, $e\cdot y=y$.

Suppose $s \in H^1(X, \mathfrak{P})$ and that $f=(f_{ij})(\in Z^1(\mathfrak{U}, \mathfrak{P}))$ is its representative. Let $E_i^f=p^{-1}(f_{ii}(U_i))$ and let an element of E_i^f be denoted by (i, y_i). In $\sum=\bigcup_{i\in I} E_i^f$, when $x \in U_{ij}$, $(i, y_i) \in E_i^f$, $(j, y_j) \in E_j^f$, $p(y_i)=f_{ii}(x)$ and $p(y_j)=f_{jj}(x)$, we shall say that (i, y_i) is equivalent to (j, y_j) if $y_i=f_{ij}(x)\cdot y_j$. E^f is the quotient space of $\sum$ relative to the above equivalence relation. The element of E^f represented by (i, y_i), is denoted by $[(i, y_i)]$. Let $p^f: E^f \to X$ such that $p^f([(i, y_i)])=\mathfrak{p}(p(y_i))$ be the projection mapping from E^f onto X. Then (E^f, p^f) is a fibre space on X, which is locally homeomorphic to (E, p). When $f'=(f'_{ij})(\in Z^1(\mathfrak{U}, \mathfrak{P}))$ is another representative of s, $(E^{f'}, p^{f'})$ is isomorphic to (E^f, p^f). Thus, as shown in [2], we have a one-to-one corespondence between the elements of $H^1(X, \mathfrak{P})$ and the classes of fibre spaces (E^f, p^f) which are isomorphic to one another.

Suppose that $t=\varphi'\varphi(s)(\in H^1(X, \mathfrak{G}))$, $g=(g_{ij})(\in Z^1(\mathfrak{U}, \mathfrak{G}))$ is its representative, and that $E_e=p^{-1}(e(X))(\subset E)$. Then E_e is a fibre space on $e(X)$ and we define $p_e=\mathfrak{p}p$. Thus (E_e, p_e) is a fibre space on X with projection mapping p_e. Further $\mathfrak{G}$ becomes a sheaf of groups of operators on (E_e, p_e) as follows: $h(\in \mathfrak{G})$ can operate on $y_e(\in E_e)$ if and only if $\mathfrak{p}(h)=p_e(y_e)$. In fact, in this case if we put $\mathfrak{p}(h)=p_e(y_e)=x(\in X)$, then $a(h)=e(x)=p(y_e)$, hence h can operate on y_e by means of operation of $\mathfrak{P}$ on (E, p). And since $p(h.y_e)=b(h)=e(x)$, $h\cdot y \in E_e$ and $p_e(h\cdot y_e)=x$. Thus, as shown in [1] we can define (E_e^g, p_e^g) which is fibre space on X. That is, let $E_{ei}^g=E_e/U_i$ and $\sum_e=\bigcup_{i\in I} E_{ei}^g$, then E_e^g is the quotient space of $\sum_e$ relative to the equivalence relation such that $y_{ei}(\in E_{ei}^g)$ is equivalent to $y_{ej}(\in E_{ei}^g)$ if $y_{ei}=g_{ij}\cdot y_{ej}$. And as shown in [1], we have a one-to-one correspondence between the elements of $H^1(X, \mathfrak{G})$ and the classes of fibre spaces (E_e^g, p_e^g) which are isomorphic to one another.

In connection with this we have the following:

Proposition 7. *(E^f, p^f) is isomorphic to (E_e^g, p_e^g).*

Proof. We shall denote by $[y_{ei}]$ the element of E_e^g which is represented by y_{ei}. Let $[(i, y_i)]$ be any element of E^f, and if (j, y_j) is another representative of $[(i, y_i)]$, then $y_i=f_{ij}\cdot y_j$. Then, $y_{ei}=z_{ef_{ii}}\cdot y_i \in E_{ei}^g$, $y_{ej}=z_{ef_{jj}}\cdot y_j \in E_{ej}^g$ and $y_{ei}=z_{ef_{ii}}\cdot y_i=z_{ef_{ii}}\cdot(f_{ij}\cdot y_j)=(z_{ef_{ii}}f_{ij}z_{f_{jj}e})\cdot(z_{ef_{jj}}\cdot y_j)=g_{ij}\cdot y_{ej}$. Thus, clearly h: $[(i, y_i)] \in E^f \to [y_{ei}] \in E_e^g$ is a homeomorphic mapping and $p_e^g h=p^f$, hence h is an isomorphism from E^f onto E_e^g.

Thus we have the following commutative diagram:

$$\begin{array}{ccc} H^1(X, \mathfrak{P}) & \xrightarrow[1:1]{\varphi'\varphi} & H^1(X, \mathfrak{G}) \\ \downarrow {\scriptstyle 1:1} & & \downarrow {\scriptstyle 1:1} \\ (\text{classes of } E^f) & \xrightarrow[1:1]{} & (\text{classes of } E^g_e). \end{array}$$

This is a geometrical meaning of $\varphi'\varphi$.

In conclusion, I wish to express my thanks to Prof. K. Morinaga for his encouragement and kind guidance.

References

[1] A. Grothendieck, *A general theory of fibre spaces with structure sheaf*, University of Kansas, Report no. 4 (1955).

[2] A. Haefliger, *Structures feuilletées et cohomologie à valeur dans un faisceau de groupoïdes*, Comment. Math. Helv., **32** (1957–58), 248–329.

[3] P. Dedecker, *Sur la cohomologie non abelienne* I (*dimension deux*), Canadian J. of Math. vol. **2**, no. 2 (1960), 231–251.

[4] R. H. Bruch, *A suvey of binary systems*, Berlin, (1958).

Faculty of General Education,
Hiroshima University