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On the Theory of Differentials on Algebraic Varieties

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Introduction

In our previous paper [7](1) we have treated a theory of differentials in commutative rings. In this paper we shall discuss some applications of the foregoing results to problems in algebraic geometry. Let X be a variety and x a point on X and let $\mathcal{O}_x$ be the local ring of x on X. We shall call $D_k(\mathcal{O}_x)=\Omega_x$ the module of local differentials(2) at x, k being the universal domain of our algebraic geometry. In a natural way we can introduce on the set-theoretic union $\Omega=\bigcup_{x\in X}\Omega_x$ a suitable topology in such a way that Ω turns out to be an algebraic coherent sheaf on X. If x is a simple point of X, Ω_x is a free module over $\mathcal{O}_x$ and hence has no torsion. Then we can identify Ω_x with a submodule composed of the differentials of the function field K of X over k. Hence if X is a non-singular variety the sheaf introduced above is identical with the sheaf of germs of regular differentials of degree 1. On the other hand if x is a singular point of x, Ω_x may have torsion in general and some new phenomena take place when we treat the variety with singularities. Although we have no intention to treat the torsion problem here we will present an example to indicate the difference(3). In §2 we deal with the adjoint map associated with a morphism f of a variety Y into X. There we shall introduce two local adjoint maps denoted by f^* and f^{**} respectively. It is one of the purposes of this work to give foundations on the theory of differentials on algebraic varieties based on the theory of local differentials. Hence some known results will be presented with an entirely new proof. The contents of §3 is running along this line, and the existence of invariant differentials on group varieties will be proved within the scope of our method. Though most of the results in this paragraph are not new our formulation is helpful for further discussion. In §4 we shall prove an exact sequence related with the module of local differentials, and it is useful when we discuss the injection of a subvariety Y into the ambiant variety X. Thanks to the exact sequence given in §4 we can prove that the non-existence of non-trivial section of $H^0(X, \mathscr{P}\Omega)$ and $H^0(X, \mathscr{P}/\mathscr{P}^2)$ will be sufficient conditions for ι^* to be a monomorphism, where $\mathscr{P}$ is the sheaf of ideals defined by the subvariety Y and ι is the injec-

(1) The number in the bracket refers to the bibliography at the end of the paper.
(2) For the notion of the module of differentials in commutative rings the readers are expected to refer the article [7].
(3) See the Example in §1.

tion $Y \rightarrow X$ (§5). In §6 we shall discuss some cases where we have $H^0(X, \mathscr{P}\Omega) = \{0\}$ and $H^0(X, \mathscr{P}/\mathscr{P}^2) = \{0\}$, and in §7 we shall show under what conditions the adjoint map ι^* will be a monomorphism. It will be interesting to see that ι^* will be a monomorphism for an irreducible hyperplane section Y of X if every differential form of the first kind is closed. Since we can admit that the latter property of the differential forms of the first kind holds in the classical case, this result may be considered as an alternative proof for the injectivity of ι^* in this case. §8 is devoted to the discussion of the case where Y is a curve of an abelian variety X and ι is the injection of Y into X. Under these circumstances we can give several formulations which are equivalent to the fact that ι^* is a monomorphism. In particular if Y is a generic 1-section of X, ι^* is always a monomorphism and hence Y generates X separably[(4)]. If we denote by λ the linear extension of ι to the Jacobian variety J of Y, λ is seen to be a separable homomorphism of J onto X. In the case where dim $X=2$ we can prove the above result for any curve Y which generates X. It is plausible that even when dim $X>2$ the similar result will hold, but it still remains unsolved. In §9 we discuss the case where the morphism f is a covering map of a variety Y onto X. In this case we also get a new type of exact sequence on the sheaves of local differentials. But the geometric interpretation of these cohomology groups are not adequate, so they are of no great use except the case of dimension 1. Nevertheless the results in the case of dimension 1 encourage us to some extent, because Hurwitz's genus formula can be derived naturally from the exact sequence of the associated cohomology groups.

Notations and Terminologies: We shall denote by k the universal domain of our algebraic geometry. Then any entities such as varieties, functions,$\cdots$, etc. are supposed to be defined over some subfield of k. But since in the most part of the paper we do not use the notion of a "generic point" at all no mention will be made of their field of definitions unless it becomes necessary. By a generic r-section of X^n we mean the intersection of X with $(n-r)$-independent generic hyperplanes in the ambient projective space with respect to the smallest field of definition for X. Let X be a variety and let x be a point of X, then the local ring of x on X will be the subring of the function field $k(X)$ of X composed of functions regular at x. All rings which will appear in this paper are assumed to be commutative and contain 1. Let R and S be rings and assume that S is an R-algebra. Then the module of differentials in S over R will be denoted by $D_R(S)$. The differential operator will be denoted by d_R^S, but the superscript or the subscript or both will often be omitted if it is clear from the context.

(4) Cf. §8 for the notion "generate separably".

§1. The sheaf of local differentials.

Let X be a variety and let x be a point of X. Let $\mathcal{O}_x$ be the local ring of x on X and let $\mathfrak{m}_x$ be the maximal ideal of $\mathcal{O}_x$. The union $\bigcup_{x \in X} \mathcal{O}_x$, forms the sheaf of local rings on X which will be denoted by $\mathcal{O}$. Let $\Omega_x = D_k(\mathcal{O}_x)$ be the module of k-differentials of $\mathcal{O}_x$ over k. (We shall call it the module of *local differentials* at x). The differential operator $d_k^{\mathcal{O}_x}$ will be denoted simply by d_x. Let Ω be the union of Ω_x, i.e. $\Omega = \bigcup_{x \in X} \Omega_x$, where x ranges over all points x on X. We can define a topology in Ω in the following way. Let x be a point on X and let $f_1, \cdots, f_n, g_1, \cdots, g_n$ be a set of functions regular at x. Then there exists an open set U containing x such that f_i's and g_i's are regular in U. Then $d_y f_i$'s are the well-defined elements of Ω_y for any point y in U. Let s be a function on U with the value in Ω defined by

$$s(y) = \sum_i g_i d_y f_i.$$

Then we can define a unique topology in Ω in which the set

$$\{s(y), y \in U\}$$

forms a complete system of neighbourhoods of Ω. Thus Ω turns out to be an algebraic sheaf on V.

Proposition 1. *Ω is an algebraic coherent sheaf.*

Proof. Since the problem is of local character we can assume that X is an affine variety with the affine coordinate ring A. In this case $\Omega_x = \mathcal{O}_x \otimes_A D_k(A)$ for any x in X. On the other hand $D_k(A)$ is a finite A-module, hence Ω is a coherent sheaf on X by Prop. 3, p. 241 in [10]. q.e.d.

Let U be an open subset of X. An element ω of $\Gamma(U, \Omega)$ will be called *a differential (regular)* in U.

Remark. Let $\mathfrak{D}$ be the module of derivations of K over k with the value in K. Then $\mathfrak{D}$ is isomorphic to $\mathrm{Hom}_K(D_k(K), K)$ (Cf. [1] Exposé 13). Since $\mathfrak{D}$ is a finite dimensional vector space over K, $D_k(K)$ is isomorphic to $\mathrm{Hom}_K(\mathfrak{D}, K)$. Thus $D_k(K)$ is no other than the differentials introduced in [11]. On the other hand $D_k(K) = K \otimes \Omega_x$ by [7]. Hence if Ω_x is a torsion free $\mathcal{O}_x$-module we can identify Ω_x as a submodule of $D_k(K)$. But in general Ω_x has torsion at a singular point x, hence we can not consider Ω_x as a submodule of $D_k(K)$. (See example 1)

We shall denote by α_x the homomorphism of Ω_x into $k_x \otimes_{\mathcal{O}_x} \Omega_x$ defined by $(\alpha_x(t)) = 1 \otimes t$, where $k_x = \mathcal{O}_x / \mathfrak{m}_x$.

Proposition 2. *Let ω be a differential on X. Assume that $\alpha_x(\omega(x)) = 0$ for any point x on X. Then $\omega(x) = 0$ at any simple point of V.*

PROOF. Let x be a simple point of X and let U be an open neighborhood of x not containing the singular points of X such that there exist r functions $t_1, \cdots, t_r$ satisfying the condition(5): *For any point y in U, $t_1 - t_1(y), \cdots, t_r - t_r(y)$ form a regular system of parameters of O_y.* Then there exist functions $g_1, \cdots, g_r$ regular in some open subset U' of U containing x such that

$$\omega(y) = \sum_i g_i d_y t_i, \ y \in U'.$$

Since y is a simple point of V, Ω_y is a free module over $\mathcal{O}_y$ and $d_y t_i$'s form a free base of Ω_y (§4 of [7]). The assumption of the proposition implies that g_i's are contained in $\mathfrak{m}_y$ for any y in U'. But this is impossible unless g_i's are identically zero. In particular $\omega(x) = 0$. q.e.d.

It is worthwhile to note that we can not say more under the assumption of Prop. 2. In the following example we shall show the existence of a non-trivial regular differential ω on V such that $\alpha_x(\omega(x)) = 0$ at any point x of V.

Example. Let X be a curve in an affine 2-spaces defined by the equation $T^2 = U^3$, and let us denote by t and u the coordinate functions on X. The function of X with the values in Ω defined by $\omega(x) = u d_x t$ is clearly an element of $\Gamma(X, \Omega)$. Now assume that the characteristic of k is 3. Let x be a point of X different from $x_0 = (0, 0)$. Then the function t is a unit in $\mathcal{O}_x$. Hence $\omega(x) = (u/2t) 2t d_x t = 0$. We shall show in the next place that $u d_0 t \neq 0$. (to avoid the confusion we shall write $d_0, \Omega_0, \cdots$ instead of $d_{x_0}, \Omega_{x_0}, \cdots$) Let $A = k[t, u]$, then $D_k(A) = (ADT + ADU)/2TdT$ (Theorem 2 in [7]). Since $D_k(\mathcal{O}_0) = \Omega_0 = \mathcal{O}_0 \otimes D_k(A)$, $u d_0 t$ can be zero if and only if there exists a function f in A such that $f(o) \neq 0$ and $fudt = 0$ in $D_k(A)$. This is equivalent to saying that fu is contained in the ideal (tA), or u is contained in $t\mathcal{O}_0$. But this is impossible because x_0 is a singular point of X and k is algebraically closed.

We can find such a cynical example even in the case where the characteristic of the universal domain is 0. In the above Example assume that the characteristic is zero. Then the differential ω defined by $\omega(x) = 2u d_x t - 3t d_x u$ offers such an example. The verification will be left to the readers.

§2. Morphism and its adjoint map.

Let

$$f \colon Y \to X$$

be a morphism (regular rational map) of an irreducible variety Y into an irreducible variety X. Let y be a point of Y and let us put $x = f(y)$. We shall denote by $\mathcal{O}^Y$ and $\mathcal{O}^X$ the sheaves of local rings on Y and X respectively. Then there exists a ring homomorphism h of $\mathcal{O}_x^X$ into $\mathcal{O}_y^Y$

(5) Cf. Cor. 1 of Prop. 1 in [6].

(1) $$h\colon \mathcal{O}_x \to \mathcal{O}_y$$

such that $h(1)=1$. (We shall frequently take off the superscript Y and X to denote the ambiant varieties if it is clearly seen by the subscript denoting the points). Thus $\mathcal{O}_y$ is an $\mathcal{O}_x$-algebra. From this we have an exact sequence (Cf. [1] or [7])

(2) $$\mathcal{O}_y \otimes \Omega_x \xrightarrow{\varphi} \Omega_y \xrightarrow{\tau} D_x(\mathcal{O}_x) \longrightarrow 0$$

where Ω_y and Ω_x stand for Ω_y^Y and Ω_x^X respectively and $D_x(\mathcal{O}_y)=D_{\mathcal{O}_x}(\mathcal{O}_y)$. Let β be a homomorphism of Ω_x into $\mathcal{O}_y \otimes \Omega_x$ defined by

$$\beta(d_x t) = 1 \otimes d_x t.$$

The combined homomorphism ($\mathcal{O}_x$-homomorphism)

$$f^*_{y,x} = \varphi \circ \beta$$

gives a map $\Omega_x \to \Omega_y$, which we shall call the *local adjoint map associated with* f.

We shall define another kind of adjoint map in the following way. Since h is a morphism we have

$$h^{-1}(\mathfrak{m}_y) = (h^{-1}(h(\mathcal{O}_x) \cap \mathfrak{m}_y)) = \mathfrak{m}_x.$$

Hence the residue field $\mathcal{O}_x/\mathfrak{m}_x = k_x$ can be identified with $\mathcal{O}_y/\mathfrak{m}_y$. From this we can define a homomorphism $f^{**}_{y\cdot x}$ of $k_x \otimes \Omega_x$ into $k_y \otimes \Omega_y$ such that

(3) $$\alpha_y f^*_{y,x} = f^{**}_{y\cdot x}\alpha_x.$$

We shall call $f^{**}_{y\cdot x}$ also a *local adjoint map.*

The following Proposition is seen immediately from the definitions.

Proposition 3. *Let $g\colon Z \to Y$ and $f\colon Y \to X$ be two morphisms. Then the map $h=f\circ g$ gives a morphism of Z into X. Moreover if x, y and z are points of X, Y and Z such that $y=g(z)$ and $x=f(y)$, then we have*

$$h^*_{z,x} = g^*_{z,y} f^*_{y,x}$$
$$h^{**}_{z\cdot x} = g^{**}_{z\cdot y} f^{**}_{y,x}.$$

Let U be an open subset of X and let ω be a differential on U. Then $V = f^{-1}(U)$ is an open subset of Y. Let y be a point in V and let $x=f(y)$. Then

$$f^*_{y,x}\omega(x)$$

is an element of Ω_y. We shall show that the map

(4) $$y \to f^*_{y,x}\omega(x)$$

is a section on V with coefficients in Ω^Y. To see this it is sufficient to show that the above defined map (4) is continuous in y. Since ω is a section in U, there exists an open set U_1 containing x such that

$$\omega(x') = \sum_j g_i d_{x'} t_i$$

for any x' in U_1, where g_i's and t_i's are regular functions in U_1. Hence

$$f^*_{y,x}\omega(x) = \sum_i (g_i \circ f) d_y (t_i \circ f)$$

is seen also to be a section in $f^{-1}(U_1)$, i.e. the correspondence (4) is a continuous function of y. Since y is an arbitrary point of V we see that the function (4) is continuous everywhere in V. We shall denote this section by

$$f^*\omega.$$

In particular if ω is a regular differential on X, i.e. an element of $H^0(X, \Omega^X)$, $f^*\omega$ is a well-defined element of $H^0(Y, \Omega^Y)$. We shall call the above defined map f^* *the adjoint map of f.*

Remark. If the morphism is a constant map, i.e. f sends Y into a point of X, then h is a natural homomorphism $\mathcal{O}_x \to \mathcal{O}_x/\mathfrak{m}_x$. Hence the homomorphism τ in (2) is an isomorphism and φ is the zero map. Thus f^* gives a zero map.

Let Y be an irreducible subvariety of X and let f be the injection of Y into X. In this case the homomorphism h in (1) is the natural homomorphism

$$h\colon \mathcal{O}^X_y \to \mathcal{O}^X_y/\mathscr{P}_y = \mathcal{O}^Y_y$$

where $\mathscr{P}_y$ is the ideal of Y in $\mathcal{O}^X_y$. In this case the associated exact sequence is given by

$$(\mathscr{P}_y/\mathscr{P}^2_y) \to \mathcal{O}^Y_y \otimes \Omega^X_y \to \Omega^Y_y \to 0.$$

We shall use this fact in §3 and some detailed investigation will be seen in §5.

§3. Invariant differentials on group varieties.

We shall apply our method to the existence proof of the invariant differential forms on group varieties.

Let G be a group variety and let $x, y, \cdots$ be points on G. The group multiplication will be denoted as usual by xy. Let $g_{(a)}$ be the left translation of G onto itself defined by

$$g_{(a)}(x) = a^{-1}x.$$

Then $g_{(a)}$ send the point a into the neutral point e. Let w_e be an arbitrary element of Ω_e. Then using the local adjoint map $g^*_{(a)}$ we can associate an element of Ω_a by the rule

$$g^*_{(a)a,e}(w_e).$$

In the following we shall put $g^*_a = g^*_{(a)a,e}$ for the sake of simplicity. It should be noted that g^*_a is a local adjoint map. It will cause no confusion since $g_{(a)}$ is a $1-1$ map.

PROPOSITION 4. *Let a be an arbitrary point of G and w_e be an element of Ω_e. Then there exists a differential s in an open set U containing a such that*

$$\alpha_a(s(a)) = 1 \otimes g^*_a(w_e) = \alpha_a(g^*_a(w_e)). \tag{5}$$

PROOF. Let $\tilde{G} = G \times G$ be the product of G and let φ be the map of $\tilde{G}$ onto G defined by

$$\varphi(x, y) = y^{-1}x.$$

Let ι be the injection of $G \times a$ into G, and let ψ be the map of G into $G \times a$ defined by

$$\psi(x) = x \times a.$$

Then we can divide $g_{(a)}$ into the three steps

$$g_{(a)} = \varphi \circ \iota \circ \psi.$$

Hence by Prop. 3 we have

$$g^*_{(a)} = \psi^* \circ \iota^* \circ \varphi^*.$$

We shall investigate the behaviour of each step in detail. For the sake of simplicity we shall use the following notations. The differential operator in $\mathcal{O}^G_x$ will be denoted simply by d_x. Let us denote by a^* the point $a \times a$ in $\tilde{G}$. We have to consider two differential operators $d^{\tilde{G}}_{a*}$ and $d^{G\times a}_{a^*}$ in $\mathcal{O}^{\tilde{G}}_{a*}$ and $\mathcal{O}^{G\times a}_{a^*}$ respectively. The former will be denoted by $\tilde{d}_{a^*}$ and the latter by d_{a^*}. Since the adjoint maps are linear it will be sufficient to treat the case where w_e is of the form $w_e = f d_e g$ where f and g are elements of $\mathcal{O}_e$. Then we have

$$\varphi^*_{a^*,e}(w_e) = (f \circ \varphi) d_{a^*}(g \circ \varphi).$$

The functions $f \circ \varphi$, $g \circ \varphi$ are regular in the neighborhood of Δ_G, then $\varphi^*_{a^*,e}(w_e)$ is a section in the neighborhood of a^* with value in $\Omega^{\tilde{G}}$. Let $t_1, \cdots, t_n$ ($n = \dim G$) be a set of regular system of parameters in $\mathcal{O}^G_a$. Then $t_i - t_i(b)$ $(i = 1, \cdots, n)$ are a regular system of parameters at any point b in a neighborhood of a. We shall denote by $t_i \otimes 1$ and $1 \otimes t_i$ the functions on G regular in the neighborhood of a^* defined by

$$(t_1 \otimes 1)(x \times x') = t_i(x)$$
$$(1 \otimes t_i)(x \times x') = t_i(x') \quad (i = 1, 2, \cdots, n)$$

respectively. Then $t_i \otimes 1$ and $1 \otimes t_i$ $(i = 1, \cdots, n)$ form a regular system of parameters of $\mathcal{O}^{G}_{a^*}$. Using these functions we can express $\varphi^*_{a^*,e}(w_e)$ uniquely in a neighborhood of a^* in the following way.

$$\varphi^*_{a^*,e}(w_e)=f'\sum_i u_i\tilde{d}_{a^*}(t_i\otimes 1)+f'\sum_j v_j\tilde{d}_{a^*}(1\otimes t_i)$$

with u_j's and v_i's are all in $\mathcal{O}^{\tilde{G}}_{a^*}$ and $f'=f\circ\varphi$. Let $\mathscr{P}$ be the ideal in $\mathcal{O}^{\tilde{G}}_{a^*}$ defining the subvariety $G\times a$. Then $\mathscr{P}$ is generated by $(1\otimes t_1,\cdots, 1\otimes t_n)$. Hence

$$\iota^*_{a^*,a^*}\circ\varphi^*_{a^*,e}(w_e)=\bar{f}'\sum_i \bar{u}_i d_{a^*}t_i$$

where $\bar{f}'$, $\bar{u}_i$'s are the function on G such that

$$\bar{u}_i(x)=u_i(x, a),\ \bar{f}'(x)=f'(x, a)=f(a^{-1}x).$$

(Cf. §3) Hence

$$\begin{aligned} g^*_a(w_e) &= \psi^*_{a,a^*}\iota^*_{a^*,a^*}\varphi^*_{a^*,e}(w_e) \\ &= \bar{f}'\sum_i \bar{u}_i d_a t_i \end{aligned}$$

and

$$1\otimes g^*_a(w_e)=f(e)\sum_{i=1}^{n} u_i(a, a)(1\otimes d_a t_i) \quad \text{in } \mathcal{O}_a/\mathfrak{m}_a\otimes\Omega_a.$$

Now we shall define a differential s in some open set U containing the point a by

$$s(b)=f(e)\sum_i u_i(x, x)d_b t_i,\ b\in U.$$

Since $u_i(x, y)$ is regular in the neighborhood of the point $a\times a$ the function $u_i(x, x)$ is regular in the neighborhood of the point $a\big(u_i(x, x)$ is the class of $u_i(x, y)$ modulo the ideal $I(\Delta_G)$ defining the diagonal Δ_G in $G\times G\big)$. Hence $s(b)$ gives actually a differential regular in a neighborhood of a. Moreover

$$\alpha_b\big(s(b)\big)=f(e)\sum_i u_i(b, b)(1\otimes d_b t_i)=1\otimes g^*_b(w_e)$$

for any b in a neighborhood U of a. Thus the proposition is proved completely. q.e.d.

Let a be a point of G and let s_a be a differential on U satisfying the condition of Prop. 4 with the fixed element w_e in Ω_e. s_a is defined in some open set U of G containing a. If $U\neq G$, take a point b outside of U and construct s_b in a similar way. Let U' be the open set in which s_b is defined. Then for a common point c of U and U' we have $\alpha_c\big(s_a(c)\big)=1\otimes g^*_c(w_e)=\alpha_c\big(s_b(c)\big)$. Since the intersection of U and U' is again an open set of X, the above fact implies that $s_a=s_b$ in $U\cap U'$ by Prop. 2. In this way we can get an element ω of $H^0(G, \Omega)$ such that

$$\alpha_x(\omega(x)) = 1 \otimes g_x^*(w_e).$$

The differential thus constructed will be said to be the *differential associated with* w_e.

We shall show in the next place that the differential associated with w_e is left invariant on G. Let a be an arbitrary point on G. To prove the invariance of the differential ω it is necessary and sufficient to prove that

$$(6) \qquad (g_{(a)}^*(\omega))(x) = \omega(x)$$

for any point x and a on G. By Prop. 2, the proof of the relation (6) is reduced to the proof of

$$(7) \qquad \alpha_x(g_{(a)}^*(\omega))(x) = \alpha_x \omega(x).$$

Since $g_{(a)}$ transform x into $a^{-1}x$ we have

$$(g_{(a)}^*(\omega))(x) = g_a^*(\omega(a^{-1}x)).$$

Hence the left hand side of (7) is equal to

$$\begin{aligned} \alpha_x g_a^*(\omega(a^{-1}x)) &= g_a^{**}\alpha_{a^{-1}x}(\omega(a^{-1}x)) && [\text{by } (3)] \\ &= g_a^{**}\alpha_{a^{-1}x} g_{a^{-1}x}^*(w_e) && [\text{by } (5)] \\ &= g_a^{**} g_{a^{-1}x}^{**}\alpha_e(w_e) = g_x^{**}\alpha_e(w_e). \end{aligned}$$

Since

$$g_{(a)}^{**} g_{(b)}^{**} = g_{(ab)}^{**}.$$

Using again the formula (3) and (5) we get

$$g_x^{**}\alpha_e(w_e) = \alpha_x g_x^*(w_e) = \alpha_x(\omega(x)).$$

Thus the relation (7) is established and thereby we get the following:

THEOREM 1. *Let G be a group variety and let e be the neutral element of G. Let w_e be an element of Ω_e. Then there exists a unique left invariant differential ω such that*

$$\alpha_e(\omega(e)) = \alpha_e(w_e).$$

The existence is proved above. The uniqueness is contained in the following Proposition which states much more than the uniqueness.

PROPOSITION 5. *Let ω be a left invariant differential form on the group variety G. Assume that $\alpha_a\omega(a) = 0$ for a point a on G. Then $\omega = 0$ identically.*

PROOF. Let b be an arbitrary point of G, and let $t = ba^{-1}$. Using the formula (3) we get

$$\alpha_b\,\omega(b) = \alpha_b\big((g^*{}_{(t)}\omega)\,(b)\big) = \alpha_b g_t^*\big(\omega(a)\big)$$
$$= g_{(t)}^{**}\big(\alpha_a\omega(a)\big) = 0.$$

Since this holds for any point b, ω must be identically zero by Prop. 2.

THEOREM 2. *Let n be the dimension of the group variety G. Then there exist n left invariant differentials $\omega_1, \cdots, \omega_n$ on G. Moreover for any point x on G, $\omega_1(x), \cdots, \omega_n(x)$ form a free base of Ω_x over $\mathcal{O}_x$.*

PROOF. Let $dt_1, \cdots, dt_n$ be a free base of Ω_e, and let $\omega_1, \cdots, \omega_n$ be invariant differentials associated with $dt_1, \cdots, dt_n$ respectively. Then the Prop. 5 implies that $\omega_1, \cdots, \omega_n$ are linearly independent over k since $\alpha_e(\omega_i) = \alpha_e(dt_i)$ $(1 \leqq i \leqq n)$ are linearly independent over k. In the similar way we see that any left invariant differentials are linearly dependent on ω_i's over k. The last assertion can be seen easily if we remember the following fact: Ω_x is a free module over $\mathcal{O}_x$ with the base $\omega_1, \cdots, \omega_n$ if $\big(\alpha_x\omega_1(x), \cdots, \alpha_x\omega_n(x)\big)$ is a base of the vector space $(\mathcal{O}_x/\mathfrak{m}_x)\otimes\Omega_x$. q.e.d.

Let K be the function field of G over k. Then $D_k(K) = K\otimes\Omega_x$. From this we can deduce, using the Theorem 2, the following

COROLLARY 1. *The left invariant differential forms on G form a base of $D_k(K)$ over K, where K is the function field of G over k.*

COROLLARY 2. *Let G be a group variety. Then $H^0(G, \Omega)$ is a free module of rank n over $\Gamma(G, \mathcal{O})$.*

PROOF. Let ω be an element of $H^0(G, \Omega)$. Then by Cor. 1 we can write ω in the form

$$\omega = \sum_i f_i\omega_i$$

where f_i's are elements of the function field K. Since $\omega_1(x), \cdots, \omega_n(x)$ form a free base of Ω_x for any point x on G, $\omega(x)$ can be an element of Ω_x if, and only if, the function f's are all contained in $\mathcal{O}_x$, i.e. f_i are elements of $\Gamma(G, \mathcal{O})$.

A variety V is called *quasi-complete* if there exists no non-constant function which is regular everywhere on V. Then we have

COROLLARY 3. *Let G be a quasi-complete group variety. Then the differential forms of the first kind on G form a vector space of dimension $n(=\dim G)$ over k.*

On this occasion we will propose here a problem:

Does there exist a quasi-complete group variety which is not an abelian variety?

PROPOSITION 6. *Let G be a group variety and let $g_{(a)}$ be the left translation of G defined by $g_{(a)} = a^{-1}x$. Let Y be an irreducible subvariety of Y and let Y_a be* *the transform of Y by $g_{(a)}$. We shall denote by ι and ι_a the injection of Y and Y_a into G respectively. Then if $\iota^*(\omega)=0$ for a left invariant differential form ω, we also have $\iota_a^*(\omega)=0$.*

PROOF. Let t_a be the restriction of $g_{(a)}$ on Y, then t_a is a biregular birational transformation of Y onto Y_a. Since $g_{(a)}\circ\iota=\iota_a\circ t_a$, we have $\iota^* g^*_{(a)}=t_a^*\iota_a^*$. Since ω is a left invariant differential on G the assumption implies that $t_a^*\iota_a^*(\omega)=0$. On the other hand t_a^* is an isomorphism, hence $\iota_a^*(\omega)=0$.

§4. An exact sequence on the differentials in local rings.

In Exposé 17 of [1], it is proved that if (R, M) is a local ring containing a field k such that the residue field R/M is separable over k, then the following sequence is exact:

$$0\to(M/M^2)\to(R/M)\otimes_R D_k(R)\to D_k(R/M)\to 0 \tag{8}$$

We shall give a generalization of this result which is necessary for further investigation.

THEOREM 3. *Let O be a local ring containing a field k and let $\mathfrak{P}$ be a prime ideal of O such that the quotient field of $O/\mathfrak{P}$ is a separable extension of k. Assume that $\mathfrak{P}^2$ is $\mathfrak{P}$-primary. Then the following sequence is exact:*

$$0\to\mathfrak{P}/\mathfrak{P}^2\to(O/\mathfrak{P})\otimes_O D_k(O)\to D_k(O/\mathfrak{P})\to 0 \tag{9}$$

PROOF. In the first place we shall remind that the sequence

$$\mathfrak{P}/\mathfrak{P}^2\xrightarrow{\rho}(O/\mathfrak{P})\otimes_O D_k(O)\to D_k(O/\mathfrak{P})\to 0 \tag{10}$$

is known to be exact ([7], Prop. 9). Hence to prove the assertion it is sufficient to show that map ρ is a monomorphism.

Let $R=\mathcal{O}_{\mathfrak{P}}$ and let $M=\mathfrak{P}\mathfrak{O}_{\mathfrak{P}}$. Then by (8) we have an exact sequence

$$0\to(M/M^2)\to(R/M)\otimes_R D_k(R)\to D_k(R/M)\to 0 \tag{11}$$

since R/M is, as the quotient field of $O/\mathfrak{P}$, separable over k. On the other hand $D_k(R)=R\otimes_O D_k(O)$ (cf. [7]). Hence

$$\begin{aligned}(R/M)\otimes_R D_k(R)&=(R/M)\otimes_R(R\otimes_O D_k(O)=(R/M)\otimes_O D_k(O)\\&=(R/M)\otimes_O\big((O/\mathfrak{P})\otimes_O D_k(O)\big).\end{aligned} \tag{12}$$

We have also

$$D_k(R/M)=(R/M)\otimes_{O/\mathfrak{P}}D_k(O/\mathfrak{P})=(R/M)\otimes_O D_k(O/\mathfrak{P}) \tag{13}$$

since $D_k(O/\mathfrak{P})$ is annihilated by $\mathfrak{P}$. In the last place we can see that

$$(M/M^2)=(R/M)\otimes_O(\mathfrak{P}/\mathfrak{P}^2). \tag{14}$$

In fact it is easy to see that $M/M^2 = R \otimes_O (\mathfrak{P}/\mathfrak{P}^2)$. Applying $\otimes_O(\mathfrak{P}/\mathfrak{P}^2)$ to the sequence

$$0 \to M \to R \to R/M \to 0$$

we see immediately the relation $R \otimes_O (\mathfrak{P}/\mathfrak{P}^2) = (R/M) \otimes_O (\mathfrak{P}/\mathfrak{P}^2)$. Using the formulas (9)–(14) we get a commutative diagram of the exact sequences

$$\begin{tikzcd}
0 \arrow[r] & (R/M)\otimes_O(\mathfrak{P}/\mathfrak{P}^2) \arrow[r] & (R/M)\otimes_O[(O/\mathfrak{P})\otimes_O D_k(O)] \arrow[r] & (R/M)\otimes_O D_k(O/\mathfrak{P}) \arrow[r] & 0 \\
& \mathfrak{P}/\mathfrak{P}^2 \arrow[u, "\psi"] \arrow[r, "\rho"] & (O/\mathfrak{P})\otimes_O D_k(O) \arrow[u] \arrow[r] & D_k(O/\mathfrak{P}) \arrow[u] \arrow[r] & 0.
\end{tikzcd}$$

Hence to prove the assertion it is sufficient to show that the homomorphism given by

$$\psi(\bar{a}) = 1 \otimes \bar{a}$$

is a monomorphism, where $\bar{a}$ is the class of an element a in $\mathfrak{P}$ modulo $\mathfrak{P}^2$. Now assume that $\psi(\bar{a}) = 0$. Then $a \in M^2$ and there exists an element b in O not contained in $\mathfrak{P}$ such that $ab \in \mathfrak{P}^2$. Since $\mathfrak{P}^2$ is $\mathfrak{P}$-primary and b is not contained in $\mathfrak{P}$, we must have $a \in \mathfrak{P}^2$, i.e. $\bar{a} = 0$.

We shall give some important case in which $\mathfrak{P}^2$ becomes a $\mathfrak{P}$-primary ideal.

PROPOSITION 7. *Let A be a noetherian ring and let $\mathfrak{P}$ be a principal prime ideal generated by a non-zero divisor a. Then $\mathfrak{P}^n$* is $\mathfrak{P}$-primary.

This proposition is not new. But the following simple proof will be of some interest(6).

PROOF. Let $\mathfrak{P}'$ be a prime divisor of (a^n). Then for some r we have (a^n): $(r) = \mathfrak{P}'$. Since a^n is in $\mathfrak{P}'$, a is also in $\mathfrak{P}'$, hence $ar = a^n s$ for some s. Then $r = a^{n-1}s$ and $\mathfrak{P}' = (a^n)\colon (r) = (a^n)\colon (a^{n-1}s) = (a)\colon (s)$. From this we set easily $\mathfrak{P}' = \mathfrak{P}$, i.e. (a^n) has only one prime divisor $\mathfrak{P}$. Hence (a^n) must be a $\mathfrak{P}$-primary ideal.

PROPOSITION 8. *Let A be a regular local ring and let $\mathfrak{P}$ be a prime ideal of A such that $A/\mathfrak{P}$ is also a regular local ring. Then $\mathfrak{P}^2$ is a $\mathfrak{P}$-primary ideal.*

This is proved in Theorem 3 of [9]. But we shall present here a simple proof for the sake of convenience.

PROOF. *Let* $n = \dim A$, $d = \operatorname{rank} \mathfrak{P}$. Then the assumption implies that $\mathfrak{P}$ is generated by d-elements $(u_1, \cdots, u_d)$ such that u_i's form a subset of a regular system of parameters of A. We shall use the induction on d. The case $d=1$ is treated in Prop. 6. Assume that Prop. is valid for a prime ideal of rank $<d$. Now assume that $\mathfrak{P}^2$ is not $\mathfrak{P}$-primary. Then there exist elements a, b such that

(6) This device is due to H. SATO.

$$ab \equiv 0(\bmod\ \mathfrak{P}^2), \quad a \not\equiv 0(\mathfrak{P}), \quad b \not\equiv 0(\mathfrak{P}^2).$$

Passing to the quotient ring $A/(u_1)$ we see that

$$b \equiv 0(\bmod\ \mathfrak{P}^2 + (u_1)).$$

Let c be an element of A such that $b \equiv cu_1$ $(\mathfrak{P}^2)$. Then $c \not\equiv 0$ $(\mathfrak{P})$. Hence if we put $ac=d$, $du_1 \equiv 0(\mathfrak{P}^2)$, $d \not\equiv 0(\mathfrak{P})$. If we denote the residue class modulo (u_2) by $^-$, we get $\bar{d}\bar{u}_1 \equiv 0(\bar{\mathfrak{P}}^2)$. Since $\bar{d} \not\equiv 0(\bar{\mathfrak{P}})$ and $\bar{u}_1 \not\equiv 0(\bar{\mathfrak{P}}^2)$ the above relation contradicts the induction hypothesis that $\bar{\mathfrak{P}}^2$ is $\bar{\mathfrak{P}}$-primary. Thus the proof is complete.

q.e.d.

In a similar way it is easily seen that $\mathfrak{P}^n$ is $\mathfrak{P}$-primary, but we do not go further.

§5. Morphism associated with injection.

Let X be an irreducible variety and let Y be subvariety of X. Let ι be the injection of Y into X. The homomorphism h of $\mathcal{O}_x^X$ onto $\mathcal{O}_x^Y$ (x is a point on Y) associated with ι is given by the natural homomorphism of $\mathcal{O}_x^X$ onto $\mathcal{O}_x^Y = \mathcal{O}_x^X/\mathscr{P}_x$, where $\mathscr{P}_x$ is the ideal of $\mathcal{O}_x^X$ defining the subvariety Y. Now assume that either one of the following conditions hold:

(A) *Y is a non-singular subvariety and Y does not meet any singular point of X.*

(B) *Y is a subvariety of codimension* 1 *and everywhere locally principal.*

In the case (A), $\mathcal{O}_x^X$ and $\mathcal{O}_x^Y$ are regular local rings, and in the case (B), $\mathscr{P}_x$ is a principal prime ideal for any point x on X. Hence we can apply Th. 3 and we get an exact sequence

$$(*) \qquad 0 \to \mathscr{P}_x/\mathscr{P}_x^2 \to \mathcal{O}_x^Y \otimes_{\mathcal{O}_x} \Omega_x^X \to \Omega_x^Y \to 0.$$

If we extend the sheaf $\mathcal{O}^Y$, Ω^Y, outside of Y by assigning 0 for the stalk over point x not belonging to Y (the extended sheaf will be denoted by the same letter), we get an exact sequence of sheaves:

$$0 \to (\mathscr{P}/\mathscr{P}^2)^{(7)} \to \mathcal{O}^Y \otimes_{\mathcal{O}_x} \Omega^X \to \Omega^Y \to 0.$$

Hence we obtain the associated exact sequence of cohomology groups

$$0 \to H^0(X, \mathscr{P}/\mathscr{P}^2) \to H^0(X, \mathcal{O}^Y \otimes \Omega^X) \to H^0(Y, \Omega^Y).$$

Thus we get the following:

PROPOSITION 9. *Under the assumption* (A) *or* (B) *the homomorphism* φ: $H^0(X, \mathcal{O}^Y \otimes \Omega^X) \to H^0(Y, \Omega^Y)$ *is injective if, and only if,* $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$.

In some cases the condition (A) can be weakened to the following:

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(7) It is well known that the sheaf $\mathscr{P}/\mathscr{P}^2$ is interpreted as the sheaf of germs of regular sections of the vector bundle E, which is the dual of the normal bundle of Y.

(A′) *Y contains only a finite number of singular points and the intersection of Y with the singular locus of X is composed of a finite number of points.*

We shall denote by $\mathscr{F}$ the kernel of $\mathcal{O}^Y \otimes \Omega^X \to \Omega^Y$, then we have two exact sequences

$$0 \to \mathscr{F} \to \mathcal{O}^Y \otimes \Omega^X \to \Omega^Y \to 0$$

$$0 \to \mathscr{K} \to \mathscr{P}/\mathscr{P}^2 \to \mathscr{F} \to 0.$$

Under the assumption (A′) the support of $\mathscr{K}$ is a finite set of points on Y, hence $H^1(X, \mathscr{K})=0$. Then we have $H^0(X, \mathscr{F})=0$ if $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$.

PROPOSITION 10. *Under the assumption* (A′), *the homomorphism* φ *is a monomorphism provided* $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$.

In particular if Y is a simple curve of a surface X, Y satisfies clearly the condition (A′) and hence Prop. 10 is applicable.

In the next place we shall investigate the homomorphism

$$\psi: H^0(X, \Omega^X) \to H^0(X, \mathcal{O}^Y \otimes \Omega^X).$$

From the exact sequence

$$0 \to \mathscr{P} \to \mathcal{O}^X \to \mathcal{O}^Y \to 0$$

we get immediately the exact sequences:

$$0 \to \mathscr{P}\Omega^X \to \Omega^X \to \mathcal{O}^Y \otimes \Omega^X \to 0$$

and

$$0 \to H^0(X, \mathscr{P}\Omega^X) \to H^0(X, \Omega^X) \to H^0(X, \mathcal{O}^Y \otimes \Omega^X).$$

From this we get the following:

PROPOSITION 11. *The homomorphism* ψ *is injective if, and only if,* $H^0(X, \mathscr{P}\Omega^X)=0$.

The combined homomorphism $\varphi \circ \psi$ is no other than the adjoint map ι^*. Hence we get the following

THEOREM 4. *Let* ι *be the injection of a subvariety* Y *into* X, *and let* $\mathscr{P}$ *be the sheaf of ideals determined by* Y. *Then* ι^* *is a monomorphism if we have* $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$ *and* $H^0(X, \mathscr{P}\Omega^X)=0$.

It is obvious that the condition $H^0(X, \mathscr{P}\Omega^X)=0$ is necessary for ι^* to be monomorphic. On the contrary, the first condition $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$ does not necessarily follow from monomorphisms of ι^*[(8)].

EXAMPLE. Let X be a product of a projective straight line D and a non-

(8) The important case where $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$ becomes a sufficient condition will be seen in Theorem 7 in §8.

singular curve Γ of genus $g(\geqq 1)$. Let Y be a subvariety $P_0\times\Gamma$. Then ι^* is a monomorphism. But $H^0(X, \mathscr{P}/\mathscr{P}^2)$ is isomorphic to k.

We shall show in the following paragraph some cases in which one of the above mentioned cohomology groups vanishes.

§6. Vanishing of $H^0(X, \mathscr{P}/\mathscr{P}^2)$ and $H^0(X, \mathscr{P}\Omega)$.

PROPOSITION 12. *Let X^n be a non-singular projective variety and let Y be an irreducible subvariety of X. Assume that the class (Y^2) contains a cycle of positive degree*, i.e., *for any divisor Y' linearly equivalent to Y such that Y and Y' intersect properly an X, the cycle $Y\cdot Y'$ is of positive degree. Then we have $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$, where $\mathscr{P}$ is the sheaf of ideals defined by Y.*

PROOF. For a divisor Z on X we shall denote by $\mathscr{L}(Z)$ the sheaf of germs of rational functions g on X such that $(g)+Z>0$. Let $\mathscr{P}$ be the sheaf of ideals defined by Y, then $\mathscr{P}$ is isomorphic to $\mathscr{L}(-Y)$ and $\mathscr{P}^2$ is isomorphic to $\mathscr{L}(-2Y)$. Let f be a function on X such that $(f)=Y'-Y$ and Y' does not contain any singular subvariety of Y. Multiplying by f, we have an isomorphism of sheaves

$$\mathscr{L}(-Y)/\mathscr{L}(-2Y)=\mathscr{L}(Y')/\mathscr{L}(-Y'-Y).$$

The latter is isomorphic to the sheaf $\mathscr{L}(-Y'\cdot Y)$ on Y. Hence $H^0(X, \mathscr{P}/\mathscr{P}^2)$ $=H^0(Y, \mathscr{L}(-Y'\cdot Y))$. Since $Y\cdot Y'$ is a Y-divisor of positive degree $H^0(Y, \mathscr{L}(-Y'\cdot Y))$ cannot have non-trivial section, proving the assertion. q.e.d.

The Proposition 12 can be generalized to an arbitrary projective variety X and an irreducible subvariety Y which is everywhere locally principal on X. To prove this generalized result it becomes necessary to develop some clumsy preparations to make it clear what is meant by the intersection $Y\cdot Y'$ and $H^0(Y, \mathscr{L}(-Y\cdot Y))$ and so on. Since we don't make any use of generalization of this type we shall not go further into this direction.

COROLLARY 1. *Let X be a non-singular surface and let Y be its subvariety, then $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$ if we have $(Y^2)>0$, where $\mathscr{P}$ is the sheaf of ideals defined by Y.*

COROLLARY 2. *Let X be a projective variety having no singularity of codimension 1 and let Y be a generic hyperplane section of X. Then $H^0(X, \mathscr{P}/\mathscr{P}^2)$ $=0$.*

COROLLARY 3. *Let X be an abelian variety and let Y be a subvariety of codimension 1 on X such that Y generates X. Then we have $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$.*

PROOF. Let $S_1,\cdots, S_t$ be singular subvarieties of codimension 1 on Y. Let a be a point on X such that $Y_a\not\supset S_j$ $(j=1, \cdots, t)$. Since Y is a generating subvariety of X, we have $Y\cap Y_a\neq\phi$[9] and $Y\cdot Y_a$ is a well-defined positive divisor on Y. Let Y' be a divisor linearly equivalent to Y and such that $Y\not\supset S_i$ $(i=1, \cdots, t)$. Since $\deg(Y\cdot Y')=\deg(Y\cdot Y_a)>0$, we can apply the proposition to our case and we get the corollary. q.e.d.

When codimension of Y is greater than 1 we have the following

PROPOSITION 13. *Let $X^n(n\geqq 2)$ be a projective variety and let W^{n-1} be an irreducible subvariety of X. Let Y be a non-singular subvariety of W such that Y does not meet any singular point of X and of W. We shall denote by $\mathscr{W}$ and $\mathscr{P}$ the sheaves of ideals in $\mathcal{O}$ defined by W and Y respectively and let $\mathscr{P}'=\mathscr{P}/\mathscr{W}$ be the quotient sheaf. Assume that*

(i) $H^0(W, \mathscr{P}'/\mathscr{P}'^2)=0$

(ii) *The line bundle B defined by the divisor class $-W\cdot Y$ on Y has no non-trivial section.*

Then we have $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$.

PROOF. Since the sequence

$$0\rightarrow\mathscr{P}^2\cup\mathscr{W}/\mathscr{P}^2\rightarrow\mathscr{P}/\mathscr{P}^2\rightarrow\mathscr{P}'/\mathscr{P}'^2\rightarrow 0$$

is exact and we have $H^0(X, \mathscr{P}'/\mathscr{P}'^2)=H^0(W, \mathscr{P}'/\mathscr{P}'^2)=0$, it suffices to prove that $H^0(X, \mathscr{P}^2\cup\mathscr{W}/\mathscr{P}^2)=0$. From the homomorphism theorem we have an isomorphism

$$\mathscr{P}^2\cup\mathscr{W}/\mathscr{P}^2=\mathscr{W}/\mathscr{P}^2\cap\mathscr{W}$$

as $\mathcal{O}$-modules. We shall prove that we have $H^0(X, \mathscr{W}/\mathscr{P}^2\cap\mathscr{W})=0$. Let $\{U_i, i\in I\}$ be an affine open covering of X and φ_i be defining equations for W in U_i, $i\in I$. If we put $a_{ij}=\varphi_i/\varphi_j$ we have $a_{ij}\in H^0(U_i\cap U_j, \mathcal{O}^*)$ where $\mathcal{O}^*$ is the sheaf of multiplicative group of units in $\mathcal{O}$. Let s be a section of $H^0(X, \mathscr{W}/\mathscr{P}^2\cap\mathscr{W})$. Without loss of generalities we can assume that s is given by the local section s_i of $\Gamma(U_i, \mathscr{W})$. Hence s_i can be written as

$$s_i=a_i\varphi_i,\ a_i\in\Gamma(U_i, \mathcal{O}).$$

Since we have

$$\begin{aligned} s_i-s_j&=a_i\varphi_i-a_i\varphi_j\\ &=a_ia_{ij}\varphi_j-a_j\varphi_j\\ &=(a_ia_{ij}-a_j)\varphi_j. \end{aligned}$$

It is seen that the functions $(a_ia_{ij}-a_j)\varphi_j$ are contained in $\mathscr{P}^2\cap\mathscr{W}$ in $U_i\cap U_j$. Since any point y of $Y\cap U_i\cap U_j$ is simple on W, $\mathcal{O}_y/(\varphi_j)$ is a regular local ring of rank $n-1$, hence φ_j is not contained in $\mathfrak{m}_y^2$, and a fortiori, $\varphi_j\notin\mathscr{P}_y^2$. From

(9) Cf. Prop. 6 of [12].

Prop. 8, $\mathscr{P}_y^2$ is $\mathscr{P}_y$-primary, hence $a_i a_{ij} - a_j \in \mathscr{P}_y$. It implies that $a_i a_{ij} - a_j$ is an element of $H^0(U_i \cap U_j, \mathscr{P})$. Taking the trace of the functions and denoting by $^-$ the trace on Y, we have

$$\bar{a}_i \bar{a}_{ij} = \bar{a}_j.$$

It implies that the set of functions $\bar{a}_i$ regular in $U_i \cap Y$ defines a section of a line bundle defined by the class of divisor $-W \cdot Y$ on Y. By assumption, there is no non-trivial section in this line bundle. Hence $\bar{a}_i = 0$, i.e. $a_i \in \Gamma(U_i, \mathscr{P})$, and $s_i = a_i \varphi_i \in \mathscr{P}^2 \cap \mathscr{W}$, i.e., $s = 0$. q.e.d.

REMARK. If $n \leqq 1$, the condition (i) implies nothing.

COROLLARY 1. *Using the notations as in Theorem* 5 *if* dim $Y=1$, *the condition* (ii) *can be replaced by the condition*

$$\text{(ii)}^* \quad I(W \cdot Y) > 0.$$

COROLLARY 2. *Let X^n be a non-singular variety and let*

$$X^n = W_n \supset W_{n-1} \supset \cdots \supset W_i = Y$$

be a sequence of non-singular subvarieties W_j of dimension j ($i \leqq j \leqq n$), and assume that the line bundle B_j defined by the divisor class $-W_j$ on W_{j+1} induces on Y a line bundle B_i' such that B_i' has no non-trivial section for $j = i$, i+1, $\cdots$, *$n-1$. Then we have $H^0(X, \mathscr{P}/\mathscr{P}^2) = 0$ where $\mathscr{P}$ is the sheaf of ideals defined by Y.*

This is the immediate consequence of Propositions 12 and 13.

COROLLARY 3. *Let X be a non-singular projective variety and let Y be a generic* 1*-section of X, and let $\mathscr{P}$ be the sheaf of ideals defined by Y. Then we have $H^0(X, \mathscr{P}/\mathscr{P}^2) = 0$.*

We shall turn our attention to the cohomology group $H^0(X, \mathscr{P}\Omega)$.

PROPOSITION 14. *Let X be an abelian variety. Then $H^0(X, \mathscr{P}\Omega) = 0$ for any subvariety Y of X.*

This is an immediate consequence of Prop. 5.

PROPOSITION 15. *Let X be a non-singular projective variety belonging to the projective space of dimension N and let Y be an irreducible hyperplane section of X. Let q be the dimension of $H^0(X, \Omega)$ over k. Then if $q \leqq N$, we must have $H^0(X, \mathscr{P}\Omega) = 0$.*

PROOF. Assume that $H^0(X, \mathscr{P}\Omega) \neq 0$, and let ω be a non-zero section. Then the divisor (ω) contains Y. Let φ be an element of $L(Y) = H^0(X, \mathscr{L}(Y))$. Then $(\varphi\omega) = (\varphi) + Y + [(w) - Y] > 0$, and $\varphi\omega$ is also a section of $H^0(X, \Omega)$. Since

X is non-singular $\varphi\omega$ is also of the first kind ([3]). Thus we get at least $N+1$ linearly independent elements belonging to $H^0(X, \Omega)$ contradicting our hypotheses $q \leqq N$.

COROLLARY 1. *Let X^n be a non-singular projective variety and let Y be an irreducible hypersurface section of sufficiently high order m, then we have $H^0(X, \mathscr{P}\Omega)=\{0\}$.*

COROLLARY 2. *Let X be as in Cor. 1 and assume that $q=\dim_k H^0(X, \Omega) \leqq n+2$. Then Corollary 1 holds for $m=1$.*

PROOF. Let N be the dimension of the ambiant projective space S of X. Assume that $N \leqq n+1$, then $q=0$ by [8]. In this case we have nothing to prove. In the case $N \geqq n+2$ it is the immediate consequence of the preceding Proposition.

It will be of some interest to point out that the vanishing of cohomology group $H^0(X, \mathscr{P}\Omega)$ is a natural consequence of the closedness of the differential form of the first kind. This is proved already in §4 of [5], but for the sake of completeness we shall write down here.

PROPOSITION 16. *Let X be a non-singular variety and let Y be its generic hyperplane section, and let $\mathscr{P}$ be the sheaf of ideals defined by Y. Then if every differential form of the first kind is closed we have $H^0(X, \mathscr{P}\Omega)=0$.*

PROOF. Let ω be a non-zero element of $H^0(X, \mathscr{P}\Omega)$ and let φ be a function on X such that $(\varphi)+Y>0$. Then $(\varphi\omega)=(\varphi)+(\omega)>(\varphi)+Y>0$ and $\varphi\omega$ is also a differential form of the first kind ([3]). By our assumption we have therefore $d(\varphi\omega)=d\varphi\wedge\omega=0$, hence there exists a function f on X such that $\omega = fd\varphi$. This is impossible since we can choose a function φ such that the divisor of $d\varphi$ is strictly negative (Lemma 3 of [5]).

We shall denote by $\mathfrak{K}$ the canonical divisor of X. We shall say a divisor Y is non-special if $\dim|\mathfrak{K}-Y|=-1$.

PROPOSITION 17. *Let X^n, Y^{n-1} and $\mathscr{P}$ be as before and assume that $\dim_K(H^0(X, \Omega)\otimes K)=n$ and Y is non-special. Then we have $H^0(X, \mathscr{P}\Omega)=0$, where K is the function field $k(X)$ of X.*

PROOF. Assume that there exists a non-trivial section ω of $H^0(X, \mathscr{P}\Omega)$. Let ω_i $(i=1, \cdots, n-1)$ be elements of $H^0(X, \Omega)$ such that $\omega, \omega_1, \cdots, \omega_{n-1}$ are linearly independent over K. Then $\omega\wedge\omega_1\wedge\cdots\wedge\omega_{n-1}$ is non-trivial differential of the first kind on X. Moreover we have $\mathfrak{K}\sim(\omega\wedge\omega_1\wedge\cdots\wedge\omega_{n-1})>(\omega)>Y$, by Prop. 4 of [4]. Hence $\dim|\mathfrak{K}-Y|\geqq 0$, and we arrive at a contradiction. q.e.d.

§7. The injectivity of the adjoint map ι^*.

Let X be a projective variety and let Y be its subvariety. We shall denote by ι the injection $Y \to X$. As a result of the preceding paragraphs we can state the several cases in which the adjoint map ι^* associated with ι is a monomorphism.

THEOREM 5. *The adjoint map ι^* is a monomorphism in the following cases:*

(I) *X is an abelian variety and Y is a generating subvariety of X of codimension* 1.

(II) *X^n is an abelian variety and Y^r is a generic r-section of $A (1 \leq r \leq n-1)$.*

(III) *X^n is a non-singular projective variety and Y^{n-1} is a generic hypersurface section of a sufficiently high order.*

(IV) *X^n is a non-singular projective variety such that $\dim_K(H^0(X, \Omega)\otimes K) \geqq n$ and Y^{n-1} is a subvariety of X which is non-special such that (Y^2) contains a cycle of positive degree, where $K=k(X)$ is the function field of X oven k.*

(V) *X^n is a non-singular projective variety such that $q=\dim_k H^0(X, \Omega) \leqq n+2$ and Y is a generic hyperplane section.*

PROOF. Owing to Theorem 4 we can see the results in the following way: The case (I) follows from Cor. 3 of Prop. 12 and Prop. 14. The case (II) comes from Cor. 2 of Prop. 13 and Prop. 14. The case (III) is a direct consequence of Prop. 12 and Cor. 1 of Prop. 15. The case (IV) follows from Prop. 12 and Prop. 17. The case (V) is an immediate consequence of Prop. 12 and Cor. 2 of Prop. 15.

THEOREM 6. *Let X be a non-singular projective variety and let Y be an irreducible hyperplane section of X. Assume that every differential form of the first kind on X is closed, then the adjoint map ι^* is a monomorphism.*

This is an immediate consequence of Prop. 12 and Prop. 16.

In the case where $\dim X=2$ we can assert a little more.

THEOREM 6′. *Let X be a normal surface in a projective space and let Y be an irreducible hyperplane section of X which does not contain any singular point of X. Let ι be the injection of Y into X. Then ι^* will be a monomorphism if every differential form of the first kind on X is closed.*

PROOF. In the proof of PROP. 16 we conclude that $\varphi\omega$ is a differential of the first kind since $(\varphi\omega) > 0$. It is this part of the proof which cannot be applied directly to the present case since X may contain a singular point(10). But when X is of dimension 2 we can proceed as follows. By a result of Zariski [14], we see that there exists a non-singular surface X' such that X' is birationally equivalent to X and the birational transformation $f: X \to X'$ is antiregular. Let x' be an arbitrary point of X'. Then since ω is of the first kind $\omega(x')$ is an element of $\Omega_{x'}$, where $\Omega_{x'}=D_k(\mathcal{O}_{x'}^{X'})$ in our previous notations. Now

(10) When X contains a singular point a differential form such that $(\omega) > 0$ is not necessarily of the 1st kind, Cf. [2].

assume that x' is not contained in the total transform $f\{Y\}$ of Y. Then φ is also a regular function in the neighborhood of x'. Hence $\varphi\omega(x')$ is still an element of $\Omega_{x'}$. Now assume that x' is contained in $f\{Y\}$. Then the point $y=f^{-1}(x')$ is a uniquely determined point of Y. By our assumption y is a simple point of X, hence $\varphi\omega$ is an element of Ω_y. Otherwise $\varphi\omega$ will have a polar divisor passing through y. Since the local ring $\mathcal{O}_y$ of y is dominated by the local ring $\mathcal{O}_{x'}$ of x', $\varphi\omega$ is also contained in $\Omega_{x'}$. Thus $\varphi\omega$ is a differential everywhere regular on X', i.e. a differential form of the first kind of X.

According to the recent work of Hironaka we know the following:

Let X be a projective variety of any dimension. Then if the universal domain is the complex number field, there exists a non-singular projective variety X' birationally equivalent to X and an anti-regular transformation $f: X \to X'$ which is regular at any simple point of X.

If we use this results we can generalize Theorem 6′ in the following

THEOREM 6″. *When the universal domain is the complex number field* Theorem 6′ *holds for a normal variety X of any dimension.*

§8. Theory on abelian varieties.

Let A be an abelian variety and let Y be a subvariety of A. We shall denote as before the sheaf of ideal determined by Y by the letter $\mathscr{P}$. Let ι and ι^* be, as before, injection of Y into A and its adjoint map respectively. In the first place we shall prove:

THEOREM 7. *Under the same notations and assumptions as above, assume that Y is a subvariety satisfying the condition (A), (A') or (B) in §5. Then the adjoint map ι^* is a monomorphism if and only if $H^0(A, \mathscr{P}/\mathscr{P}^2)=0$.*

PROOF. "If" part of the Theorem is an immediate consequence of Prop. 14 and Theorem 4. As we can see from the commutative diagram below

$$\begin{tikzcd}
 & 0 \arrow[d] & \\
 & H^0(X, \Omega^X) \arrow[d, "\psi"] \arrow[dr, "\iota^*"] & \\
0 \arrow[r] H^0(X, \mathscr{P}/\mathscr{P}^2) \arrow[r] & H^0(X, \mathcal{O}^Y\otimes\Omega^X) \arrow[r, "\varphi"] & H^0(Y, \Omega^Y)
\end{tikzcd}$$

"only if" part of the Theorem is obtained if we show that ψ in the diagram is an epimorphism (hence an isomorphism). Let s be an element of $H^0(X, \mathcal{O}^Y\otimes\Omega^X)$. Let $\omega_1, \cdots, \omega_n$ be invariant differential forms on A. Then $\omega_1, \cdots, \omega_n$ form a base of $D_k(K)$ over K where K is the function field of A over k(Cor. 1 of Th. 2). From this we see that there is a suitable open covering $\{U_\alpha\}$ of X and the functions $f_{i\alpha}$ on Y regular in $Y\cap U_\alpha$ such that

$$s(x)=\sum_{i=1}^{n} f_{i\alpha}\otimes\omega_i,$$

for any point x in $Y\cap U_\alpha$. For any point x in $Y\cap U_\alpha\cap U_\beta$, we have

$$\sum_{i=1}^{n} f_{i\alpha}\otimes\omega_i=\sum_{i=1}^{n} f_{i\beta}\otimes\omega_i.$$

On the other hand $1\otimes\omega_i$ $(i=1, \cdots, n)$ form a free base of $\mathcal{O}_x^Y\otimes\Omega_x$ for any point x in X (Th. 2), hence we must have $f_{i\alpha}=f_{i\beta}$ in $Y\cap U_\alpha\cap U_\beta$. This means that $f_{i\alpha}=f_{i\beta}$ is a constant function c_i on Y and $s=\psi(\sum_{i=1}^{n} c_i\omega_i)$. Since s is an arbitrary element of $H^0(X, \mathcal{O}^Y\otimes\Omega)$ the above result shows that ψ is an epimorphism. q.e.d.

Corollary. *Let A be an abelian variety and let Y, Z be irreducible subvarieties of A such that Y is a subvariety of Z and they satisfy the condition* (A) *or* (B) *in* §5. *Let $\mathscr{P}$ and $\mathscr{P}_1$ be respectively the sheaves of ideals corresponding to Y and Z respectively. Then if $H^0(X, \mathscr{P}/\mathscr{P}^2)=0$, we must also have $H^0(A, \mathscr{P}_1/\mathscr{P}_1^2)=0$.*

Definition 1. *Let V^n be a variety and let P be a simple point of V. Let $\Gamma_1, \Gamma_2, \cdots, \Gamma_n$ be n-curves such that P is a simple point of each Γ_i $(i=1, \cdots, n)$. Let O be the local ring of P in V, $\mathfrak{m}$ its maximal ideal of O and let $\mathfrak{P}_i$ be the ideal of Γ_i in O. We shall say that n curves $\Gamma_1, \cdots, \Gamma_n$ are transversal to each other at P if $\bigcap_{i=1}^{n}(\mathscr{P}_i+\mathfrak{m}^2)$ is contained in $\mathfrak{m}^2$.*

In the case where $n=2$, our definition coincides with the ordinary definition of transversality.

Definition 2. *Let A^n be an abelian variety and let Γ be a curve on A. Assume that there exist n simple points $a_1, \cdots, a_n$ of Γ such that if we put $a=a_1+\cdots+a_n$, $b_i=a-a_i$, the n curves Γ_{b_i} $(i=1, \cdots, n)$ are transversal to each other at the point a. In this case we shall say that Γ is a favourable curve.*

Proposition 18. *Let A be an abelian variety and let Γ be a curve on A. Let ι be the injection of Γ into A. Then the adjoint map ι^* is a monomorphism provided Γ is a favourable curve.*

Proof. Assume that Γ is a favourable curve. Then by the definition there are n simple points $a_1, \cdots, a_n$ on Γ such that $\Gamma_1, \cdots, \Gamma_n$ are transversal to each other at $a=a_1+\cdots+a_n$ where $\Gamma_\nu=\Gamma_{b_\nu}$, $b_\nu=a-a_\nu (\nu=1, \cdots, n)$. Now assume that ι^* is not a monomorphism, then there exists an invariant differential ω on A such that $\iota^*(\omega)=0$. Then if we denote by ι_ν the injection of Γ into A we have also $\iota_\nu^*(\omega)=0$ for $\nu=1, 2, \cdots, n$. Let O be the local ring of the point a in A and let $\mathfrak{P}_\nu$ be the ideal in O defined by Γ_ν $(\nu=1, \cdots, n)$. We shall denote by Ω the module of k-differentials in O and by $\mathfrak{m}$ the maximal ideal of O. Let

$V(\mathscr{P}_\nu)$ be the subspace of $(O/\mathfrak{m})\otimes\Omega$ spanned by elements of the form $1\otimes dx$, $x\in\mathscr{P}_\nu$. Now assume that $\iota_\nu^*(\omega)=0$, then we can easily see that $\alpha_a(\omega)\in V(\mathscr{P}_\nu)$. Since this relation holds for any $\nu=1, 2, \dots, n$, we see that $\alpha_a(\omega)\in\bigcap_{\nu=1}^{n} V(\mathscr{P}_\nu)$. Since ω is an invariant form $\alpha_a(\omega)\neq 0$ (Prop. 5), and we can write $\alpha_a(\omega)$ in the form $1\otimes d\xi$, where ξ is an element of $\mathfrak{m}$ different from 0. On the other hand $1\otimes d\xi$ is in $V(\mathscr{P}_\nu)$, hence there exists an element η_ν in $\mathscr{P}_\nu$ such that $1\otimes d\xi=1\otimes d\eta_\nu$, i.e. $1\otimes d(\xi-\eta_\nu)=0$. As we know $(O/\mathfrak{m})\otimes\Omega$ is isomorphic to $\mathfrak{m}/\mathfrak{m}^2$, hence $1\otimes d(\xi-\eta_\nu)=0$ is equivalent to saying that $(\xi-\eta_\nu)\in\mathfrak{m}^2$. Thus we see that ξ is contained in $\bigcap_\nu(\mathscr{P}_\nu+\mathfrak{m}^2)$. Moreover $1\otimes d\xi\neq 0$, $\xi\notin\mathfrak{m}^2$, thus the above relation implies that $\bigcap_\nu(\mathscr{P}_\nu+\mathfrak{m}^2)$ is not contained in $\mathfrak{m}^2$, i.e. Γ is not a favourable curve. q.e.d.

Let A^n be an abelian variety and let Γ be a curve on A, and ι be the injection of Γ into A. Let φ be a function on $\Gamma\times\cdots\times\Gamma$ with values in A defined by $\varphi(P_1\times\cdots\times P_n)=\sum_{\nu=1}^{n}\iota(P_\nu)$. Then as is easily seen φ is a morphism of $\Gamma\times\cdots\times\Gamma$ into A. If φ is a morphism onto A, we usually say that Γ generates A. If, moreover, we have an additional condition that φ is a separable map, we shall say that Γ *generates A separably.*

Let f be a morphism of U^n onto V^n and let v be a simple point of V such that $f^{-1}(v)$ consist of a finite number of points of U. Let u be one of the points in $f^{-1}(U)$ and let S and R be the local rings of u and v on U and V respectively. If the local ring S is unramified over R, i.e. the maximal ideal M of R generates the maximal ideal N of S, we shall say that f is unramified at the point u. If f is unramified at any point in $f^{-1}(u)$, we shall say that f is *unramified over v.*

LEMMA 1. *Let f be a morphism of a variety U^n onto V^n and let v be a simple point of V such that f is unramified over v. Let $\Gamma_1, \dots, \Gamma_n$ be curves on U passing through a point u in $f^{-1}(v)$ such that they are transversal to each other at u. Then if v is a simple point of every one of the curves $f(\Gamma_i)$, then they are also transversal at the point v.*

PROOF. Let (S, N) and (R, M) be local rings of u and v on U and V respectively and let $\mathfrak{P}_i$ be the ideal of Γ_i in S. We shall put $\mathfrak{p}_i=\mathfrak{P}_i\cap R$. Then $\mathfrak{p}_i$ is the ideal of $f(\Gamma_i)$ in R. Now assume that $\bigcap_{i=1}^{n}(\mathfrak{p}_i+M^2)\not\subseteq M^2$, then there exists an element x in $\bigcap_{i=1}^{n}(\mathfrak{p}_i+M^2)$ not contained in M^2. This element x can be a member of regular system of parameters of M. Since S is unramified over R, x cannot be contained in N^2. Thus we see that $\bigcap_{i=1}^{n}(\mathfrak{P}_i+N^2)\not\subset N^2$, contradicting the hypothesis. q.e.d.

PROPOSITION 19. *If a curve Γ on A generates A separably, then Γ is a* *favourable curve.*

PROOF. Let φ be a map from $\Gamma\times\cdots\times\Gamma$ onto A defined by $\varphi(P_1\times\cdots\times P_n)=\sum_{i=1}^{n}\iota(P_i)$. Since φ is a separable map there exist n simple points $P_1, P_2, \cdots, P_n$ of Γ such that φ is unramified at the points $x=\sum_{i=1}^{n}(P_i)$. Let Γ_i be the curve $P_1\times\cdots\times P_{i-1}\times\Gamma\times P_{i+1}\times\cdots\times P_n$ on $\Gamma\times\cdots\times\Gamma$. Then $\varphi(\Gamma_i)=\Gamma_{\sum_{j\neq i}\iota(P_j)}$ contains the point a $=\varphi(P_1\times\cdots\times P_n)$ as a simple point. Moreover $\Gamma_1, \cdots, \Gamma_n$ are transversal at the point $P_1\times\cdots\times P_n$. Hence the proposition follows from the Lemma. q.e.d.

PROPOSITION 20. *Let Γ be an irreducible curve of an abelian variety A^n and let ι be the injection of Γ into A. Assume that the adjoint map ι^* is a monomorphism, then the following holds: let r be any positive integer $\leqq n$ and let φ_r be a rational map from $\Gamma_r=\underbrace{\Gamma\times\cdots\times\Gamma}_{r}$ into A defined by $\varphi_r(a_1\times\cdots\times a_r)=a_1+\cdots+a_r$. Then φ is always a separably algebraic map. In particular Γ generates A separably.*

PROOF. In the first place we shall remark that Γ generates A. In fact if Γ does not generate A there exists an abelian subvariety B such that $B\supset\Gamma_{-a}$, where a is an arbitrary point of Γ [12]. Since ι_B^* is not clearly a monomorphism $\iota_{\Gamma_{-a}}^*$ cannot be a monomorphism, hence neither ι_Γ^* is a monomorphism by Prop. 6.

After we know Γ generates A, it is easily seen that the degree of the map φ_r is finite for any $r\leqq n$. Let $a_1, \cdots, a_{r-1}$ be arbitrary $(r-1)$ points of Γ and let α be the injection of Γ into Γ_r defined by $\alpha(u)=u\times a_1\times\cdots\times a_{r-1}$. Let j be the injection of $W_r=\varphi_r(\Gamma_r)$ into A. If we put $\hat{\Gamma}=\Gamma_{a_1+\cdots\cdots+a_{r-1}}$, we have $\iota_{\hat{\Gamma}}\circ t=j\circ\varphi_r\circ\alpha$ where t is a biregular transformation $\Gamma\to\hat{\Gamma}$. Taking the adjoint of each map we have $t^*\circ\iota_{\hat{\Gamma}}^*=\alpha^*\circ\varphi_r^*\circ j^*$. Since t^* is an isomorphism and $\iota_{\hat{\Gamma}}^*$ is a monomorphism $\varphi_r^*\circ j^*$ must be a monomorphism. Let K be the function field of $W=\varphi_r(\Gamma_r)$, then we shall show that $\mathrm{Im}(j^*)$ contains a basis of $D_k(K)$. In fact let x be a simple point of W and let O be the local ring of the point x on A. Let $t_1, \cdots, t_n$ be a regular system of parameters of O such that the ideal $\mathfrak{p}$ of W in O is generated by $t_{r+1}, \cdots, t_n$. By Theorem 1 there exist invariant differential forms ω_i $(i=1, \cdots, r)$ such that $1\otimes\omega_i=1\otimes dt_i$ (in $(O/\mathfrak{m})\otimes D_k(O)$, where $\mathfrak{m}$ is the maximal ideal of O). This means that if we represent ω_i in the form

$$\omega_i=a_{i1}dt_1+\cdots+a_{ir}dt_r+a_{ir+1}dt_{r+1}+\cdots+a_{in}dt_n$$

we must have $a_{ii}\equiv 1$ (mod. $\mathfrak{m}$) and $a_{ij}\in\mathfrak{m}$ if $i\neq j$. Denoting by $^-$ the trace of the functions on W we see that

$$j^*(\omega_i)=\bar{a}_{i1}d\bar{t}_1+\cdots+\bar{a}_{ir}d\bar{t}_r \qquad (i=1, \cdots, r)$$

and $\det|\bar{a}_{ij}|$ is a unit in $\bar{O}=O/\mathfrak{p}$ because except the terms on principal diagonal, every term is contained in $\mathfrak{m}/\mathfrak{p}$. Moreover $\bar{t}_1, \cdots, \bar{t}_r$ are a set of separating transcendence basis of K over k (Cor. 2 of Prop. 17 [7]), hence $d\bar{t}_1, \cdots, d\bar{t}_r$ as well as $j^*(\omega_1), \cdots, j^*(\omega_r)$ form a basis of $D_k(K)$ over K. On the other hand φ_r^* is monomorphic on $\mathrm{Im}(j^*)$ and $D_k(\bar{O})$ has no torsion. Hence the monomorphism φ_r^* of $D_k(\bar{O})$ into $D_k(L)$, where L is the function field of Γ_r, can be extended uniquely to the monomorphism of $L\otimes_K D_k(K)=L\otimes_K(K\otimes_{\bar{O}} D_k(\bar{O}))$ into $D_k(L)$. As we remarked above $D_k(K)$ and $D_k(L)$ have the same dimension r and both of them are regular extension of dimension r. Hence the extended monomorphism φ_r^* must be surjective, and we have $D_K(L)=0$. It then follows that L is separably algebraic over K since L is finitely generated over K.

The preceding results will be unified in the

THEOREM 8. *Let Γ be a curve on an abelian variety A and let ι be the injection of Γ into A. Then the following conditions are equivalent:*

(a) *ι^* is a monomorphism.*

(b) *Γ is a favourable curve.*

(c) *Γ generates A separably*(11).

COROLLARY 1. *Let A be an abelian variety of dimension 2, and let Γ be a curve of genus $\geqq 2$. Then Γ generates A separably.*

PROOF. Let ι be the injection of Γ into A. Then Γ generates A since genus of Γ is not less than 2. This implies that $H^0(A, \mathscr{P}/\mathscr{P}^2)=0$ by Cor. of Prop. 12, where $\mathscr{P}$ is the sheaf of ideals defined by Γ. This is equivalent to saying that ι^* is injective by Th. 7. Then the assertion follows from Th. 8.

COROLLARY 2. *Let Γ be a non-singular curve and let J be its Jacobian variety. Let φ be a canonical map of Γ into J, then φ^* is a monomorphism.*

As is known Γ generates J separably [12], hence the corollary follows immediately from Th. 8.

In the rest of this paragraph we shall discuss the separability property of the linear extension λ of the injection $\iota\colon \Gamma\to X$, for a generating curve Γ of X. For this purpose we need several Lemmas.

LEMMA 2. *Let A^n be an abelian variety and let W be a subvariety of A. Then the rational map φ of $A\times W$ onto A defined by $\varphi(x\times w)=x+w$ is a regular map.*

PROOF. Let x, w be independent generic points of A and W over a common field of definition F for A and W. Since $F(x, w)=F(x+w, w)$ and $\dim_{F(w)}(x+w)=n$, we see that $F(x+w)$ and $F(w)$ are linearly disjoint over k. Hence the join $F(x+w, w)=F(x, w)$ is a regular extension of $F(x+w)$.

(11) Equivalence of (a) and (c) is stated in [13] without proof.

LEMMA 3. *Let U_1, U_2 be varieties defined over F such that there exist separable rational maps φ_i of U_i into A $(i=1, 2)$ defined over a field F. Assume that at least one of φ_i is surjective on A, then the map Φ of $U_1 \times U_2$ on A defined by $\Phi(u_1, u_2)=\varphi_1(u_1)+\varphi_2(u_2)$ is a separable map.*

PROOF. Let u_1, u_2 be independent generic points of U_1, U_2 over F. Then by assumptions $F(u_i)$ is a separable extension of $F(\varphi_i(u_i))$, hence $F(u_1, u_2)$ is a separable extension of $F(\varphi_1(u_1), \varphi_2(u_2))$. On the other hand by Lemma 2, $F(\varphi_1(u_1), \varphi_2(u_2))$ is a separable extension of $F(\varphi_1(u_1)+\varphi_2(u_2))=F(\Phi(u_1, u_2))$. Hence $F(u_1, u_2)$ is also a separable extension of $F(\Phi(u_1+u_2))$. q.e.d.

COROLLARY. *Let $A_n=A\times \cdots \times A$ be n product of an abelian variety A, and let W be a subvariety of A. Let φ be a map of $A_n \times W$ onto A defined by $\varphi(x_1 \times x_2 \times \cdots \times x_n \times w)=x_1+x_2+\cdots+x_n+w$,* where $x_i \in A$ and $w \in W$. Then φ is a separable map.

PROOF. Induction on n. The case $n=1$ is proved in Lemma 1. Since $A_n \times W=A_{n-1}\times A\times W$ and the map ψ_1 of A_{n-1} onto A defined by $\psi_1(x_1 \times \cdots \times x_{n-1}) =\sum_{i=1}^{n-1} x_i$ is regular as well as the map ψ_2 of $A\times W$ onto A defined by $\psi_2(x_n \times w) =x_n+w$. Hence by Lemma 3 the map φ is also a separable map.

THEOREM 9. *Let A_n be an abelian variety and let C be a curve on A and let ι be the injection $C\rightarrow A$. Assume that the adjoint map ι^* is a monomorphism, then the linear extension λ of ι is a separable homomorphism of the Jacobian variety J onto A.*

PROOF. Let g be the genus of C and let $g=an+b$ where a is an integer $\geqq 1$ and b is an integer such that $0\leqq b<n$. Let $\Gamma_n=\underbrace{C\times \cdots \times C}_{n}$ and let φ be a canonical function of C into J. The map ψ of Γ_n onto A defined by $\psi(x_1 \times \cdots \times x_n)=x_1+\cdots+x_n$ is a separable map since ι^* is a monomorphism. The map Φ of $\underbrace{\Gamma_n \times \cdots \times \Gamma_n}_{a} \times \Gamma_b$ onto A defined by $\Phi(y_1 \times \cdots \times y_a \times z)=y_1+\cdots+y_a+z$, where y's are points of Γ_n and z is a point of Γ_b, is decomposed as $\Gamma_n \times \cdots \times \Gamma_n \times \Gamma_b \rightarrow A\times \cdots \times A\times W\rightarrow A$. Each map is separable by Prop. 20 and Cor. of Lemma 3, hence Φ itself must be a separable map.

§9. Morphism associated with a covering map.

Let X^* and X be normal varieties and let π be a morphism of X^* onto X. Assume that X^* and X have the same dimension and π has no fundamental curve, i.e., for any curve C on X^* the image of C by π is also of one dimensional. Moreover we shall assume that X is a non-singular variety and π is a separable map, i.e., if $L^*(L)$ is the function field of $X^*(X)$ over k, then L^* is a separably algebraic extension of L. Let x^* be a point on X^* and let $x=$

$\pi(x^*)$. As before we shall denote by $\mathcal{O}^*_{x^*}(\mathcal{O}_x)$ the local ring of $x^*(x)$ on $X^*(X)$. Since X is assumed to be non-singular, $\mathcal{O}_x$ is a regular local ring and its residue field coincides with k. Let $D_k(\mathcal{O}^*_{x^*})=\Omega^*_{x^*}(D_k(\mathcal{O}_x)=\Omega_x)$ be the module of k-differentials in $\mathcal{O}^*_{x^*}(\mathcal{O}_x)$. Then $D_k(\mathcal{O}_x)$ is a free module by Theorem 3 in [7]. Using Prop. 3 in [7] we see that the sequence

$$0\to\mathcal{O}^*_{x^*}\otimes_{\mathcal{O}_x}D_k(\mathcal{O}_x)\to D_k(\mathcal{O}^*_{x^*})\to D_{\mathcal{O}_x}(\mathcal{O}^*_{x^*})\to 0 \tag{15}$$

is exact. We shall denote as before by $\mathcal{O}^*(\mathcal{O})$ the sheaf of local rings on $X^*(X)$, and by $\Omega^*(\Omega)$ the sheaves of local differentials on $X^*(X)$ whose stalk is given by $\Omega^*_{x^*}(\Omega_x)$. Then from (15) we can deduce an exact sequence of algebraic coherent sheaves on X^*:

$$0\to\mathcal{O}^*\otimes\Omega\to\Omega^*\to D_{\mathcal{O}}(\mathcal{O}^*)\to 0. \tag{16}$$

It will be noted that the support of $D_{\mathcal{O}}(\mathcal{O}^*)$ is contained in the different divisor (Cf. §5 of [7]).

THEOREM 7. *Assume that X^* is an unramified covering of a non-singular variety X. Then we have an isomorphism of the sheaves*

$$\Omega^*\cong\mathcal{O}^*\otimes_{\mathcal{O}}\Omega.$$

Let $\chi(X, \mathcal{F})$ be the Euler characteristic of X with coefficients in the sheaf $\mathcal{F}$, i.e., $\chi(X, \mathcal{F})=\sum_{q=0}^{\infty}(-1)^q\dim_k H^q(X, \mathcal{F})$. From the exact sequence (16) we get

$$\chi(X^*, \Omega^*)=\chi(X^*, D_{\mathcal{O}}(\mathcal{O}))+\chi(X^*, \mathcal{O}^*\otimes\Omega). \tag{17}$$

We shall show that the *relation* (17) *is nothing other than Hurwitz's genus formula when X and X^* are non-singular curves. Let* $\mathfrak{k}(\mathfrak{k}^*)$ be the canonical divisor on $X(X^*)$, and let ω be an element of $D_k(L)$ such that $(\omega)=\mathfrak{k}$. By our assumption Ω_x is a free module, hence it has no torsion element. Then Ω_x can be identified with the submodule of $L\otimes D_k(\mathcal{O}_x)=D_k(L)$. We shall denote by $\mathcal{L}(\mathfrak{k})$ the sheaf of germs of rational functions f on X such that $(f)+\mathfrak{k}>0$. Then we have a canonical isomorphism ϕ of $\mathcal{L}(\mathfrak{k})$ onto $D_k(\mathcal{O})$ defined by

$$\phi(f)=f\omega.$$

In the similar way we can see easily that the sheaf $\mathcal{O}^*\otimes\Omega$ is canonically isomorphic to the sheaf $\mathcal{L}(\pi^{-1}(\mathfrak{k}))$ on X^*, where $\pi^{-1}(\mathfrak{k})$ is defined by

$$\pi^{-1}(\mathfrak{k})=pr_{X^*}[(X^*\times\mathfrak{k})\cdot\Gamma_\pi]$$

in which Γ_π is the graph of the morphism π.

From these considerations we can derive the following evaluation:

$$\chi(X^*, \Omega^*)=(X^*, \mathcal{L}(\mathfrak{k}^*)) \tag{18}$$

$$= \dim H^0(X^*, \mathscr{L}(\mathfrak{k}^*)) - H^1(X^*, \mathscr{L}(\mathfrak{k}^*))$$
$$= g^* - 1$$

(19)
$$\chi(X^*, \mathcal{O}^* \otimes D_k(\mathcal{O})) = \chi(X^*, \mathscr{L}(\pi^{-1}(\mathfrak{k}))$$
$$= n(2g-2) - g^* + 1$$

where g and g^* are genera of X and X^* respectively.

In the next place we shall determine the value of $\chi(X^*, D_{\mathcal{O}}(\mathcal{O}^*))$. Since the support of $D_{\mathcal{O}}(\mathcal{O}^*)$ is a finite set of points, $\chi(V^*, D_{\mathcal{O}}(\mathcal{O}^*))$ is equal to the dimension of $H^0(X^*, D_{\mathcal{O}}(\mathcal{O}^*))$ over k, i.e.,

(20)
$$\chi(X^*, D_{\mathcal{O}}(\mathcal{O}^*)) = \sum_x \dim_k D_{\mathcal{O}_x}(\mathcal{O}^*_{x^*})$$

where the sum $\sum$ is extended over all the branch points x for the covering X^*/X. We shall calculate the dimension of $D_{\mathcal{O}_x}(\mathcal{O}^*_{x^*})$ in the following.

Let S be a domain containing a ring R and assume that S is a discrete valuation ring of a field E. Assume that R contains a field k such that the residue field of S is a finite separable extension of k. Let us denote by t a prime element of S and let $\mathscr{D}$ be the d-different of S/R, i.e., the annihilator of the module $D_R(S)$ of R-differentials in S. Let λ be an integer such that $\mathscr{D} = (t^\lambda)$. Then we have the

PROPOSITION 19. *Assume that $D_k(S)$ is a finite module. Then the dimension of $D_R(S)$ over k is equal to $\lambda[(S/tS): k]$.*

PROOF. By the lemma of Godement (Cf. Exposé 17 of [1]), we have an exact sequence

$$0 \to \mathfrak{m}/\mathfrak{m}^2 \to (S/\mathfrak{m}) \otimes D_k(S) \to 0$$

where $\mathfrak{m}$ is the maximal ideal of S. Since $\mathfrak{m}$ is a principal ideal (t) and $D_k(S)$ is a finite module over the local ring S we see that $D_k(S)$ is generated by a single element $d_k^S t$. Hence $D_R(S)$ is also generated by a single element dt (d stands for d_R^S). By assigning dt to 1 we have an isomorphism of $S/t^\lambda S$ onto $D_R(S)$. Let $a_1, \cdots, a_s$ be elements of S such that their residue class modulo $\mathfrak{m}$ form a base of $S/(t)$ over k. Then it is a straightforward verification to see that $a_i t^\mu$ ($i=1, \cdots, s$; $\mu=0, 1, \cdots, \lambda-1$) form a base of $S/t^\lambda S$ as a k-vector space, and thereby the theorem is proved.

Let $\mathfrak{D}_x$ be the d-different of $\mathcal{O}^*_{x^*}$ over $\mathcal{O}_x$ and λ_x be an integer defined by $\mathfrak{D}_x = (t_x^{*\lambda_x})$, where t_x^* is a prime element of $\mathcal{O}^*_{x^*}$. Then λ_x is equal to the differential index defined in [6]. Since the different divisor is given by $\sum_{x^*} \lambda_{x^*} x^*$, we see that

(21)
$$\dim_k H^0(V^*, D_{\mathcal{O}}(\mathcal{O}^*)) = \sum_{x^*} \dim_k D_{\mathcal{O}_{x^*}}(\mathcal{O}^*_{x^*}) = \sum \lambda_{x^*}$$

is equal to the degree d of different divisors.

Combining (17)–(21) we get the final result

$$2g^* - 2 = n(2g - 2) + d.$$

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