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A Remark to the Construction of Riemann Surfaces by Welding

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1. Let S_1 and S_2 be bordered Riemann surfaces and let h be an orientation-reversing homeomorphism from (a part of) the border of S_1 onto (a part of) that of S_2. The union $S_1 \cup S_2$ becomes canonically a surface if we identify the points corresponding under h and introduce the factor topology. Suppose that the surface so obtained carries a structure of Riemann surface which is equivalent to the original structures in the interiors of S_1 and S_2. Then we shall say that *a Riemann surface is obtained by welding S_1 and S_2 together by h.* For example, it is easily seen that if h is real analytic then the welding is possible and the resulting Riemann surface is determined uniquely.

The problem of deciding the possibility and the uniqueness of welding has been discussed by many authors [1-9]. Among them Pfluger [6] treated the case where h is real analytic except at a point, and considered the welding across the whole border. The purpose of the present note is to give an answer to the question given at the end of his paper.

2. A part of the problem of welding may be localized (see, e.g., [5], p. 39) and be reduced to studies of the welding of halfdisks as follows:

Definition. Let $f(x)$ be a monotone increasing continuous function on $-1 \leqq x \leqq 1$ such that $f(-1)=-1$, $f(1)=1$. Let $D^+=\{z;\ |z|<1, \operatorname{Im} z \geqq 0\}$ and $D^-=\{z;\ |z|<1, \operatorname{Im} z \leqq 0\}$. If there exist conformal mappings $\zeta=\varphi^+(z)$ and $\zeta=\varphi^-(z)$ of the interiors of D^+ and D^-, respectively, which are homeomorphic on D^+ and D^-, and are such that $\varphi^+ \circ f(x)=\varphi^-(x)$ on $-1 \leqq x \leqq 1$, then it is said that *D^+ and D^- are welded together by f.* If, in addition, for any other $\varphi_1^+(z)$ and $\varphi_1^-(z)$, there exists a conformal mapping ψ such that $\varphi_1^+=\psi \circ \varphi^+$ and $\varphi_1^-=\psi \circ \varphi^-$, then it is said that D^+ and D^- are welded together *uniquely* by f.

From now on, let us *assume that f is real analytic on $-1 \leqq x<0$ and $0<x \leqq 1$*, and ask the possibility of the welding throughout $-1 \leqq x \leqq 1$. Note that in this situation the welding is unique whenever it is possible.

Pfluger [6] considered a pair of curves γ^+ and γ^- with the following properties: They are rectifiable, $\gamma^+ \subset D^+-\{f(0)\}$, $\gamma^- \subset D^- - \{0\}$, their end points are on $[-1, 0) \cup (0, 1]$, f(end point of γ^-)=(initial point of γ^+), f(initial point of γ^-)=(end point of γ^+). Let $\gamma=\gamma^+ \cup \gamma^-$ and consider the family $\{\gamma\}$ consisting of all the γ. Denote its extremal length by $\lambda\{\gamma\}$.

At the end of his paper quoted above, he stated that $\lambda\{\gamma\}=0$ is a necessary condition for the possibility of the welding and asked if the condition is sufficient.

We remark first that the words necessary and sufficient are to be exchanged, since it is rather simple to show that $\lambda\{\gamma\}=0$ is a sufficient condition for the possiblity of the welding. For the sake of completeness we shall give its proof (Theorem 1). After this exchange, we then solve Pfluger's problem. The answer is negative (Theorem 2).

3. THEOREM 1. *If $\lambda\{\gamma\}=0$ then the welding is possible.*

PROOF. Since f is real analytic on $-1\leqq x<0$, $0<x\leqq 1$, the welding is possible there. Thus we can find conformal mappings $\zeta=\varphi^+(z)$ and $\zeta=\varphi^-(z)$ of the interiors of D^+ and D^-, respectively, which are homeomorphic on $D^+-\{f(0)\}$ and $D^- - \{0\}$ and are such that $\varphi^+\circ f(x)=\varphi^-(x)$ on $-1\leqq x<0$ and $0<x\leqq 1$. The domain $\Delta=\varphi^+(D^+-\{f(0)\})\cup\varphi^-(D^- -\{0\})$ is doubly connected. If it is shown that the boundary component of Δ being the image of $x=0$ reduces to a point, then we see immediately that these φ^+ and φ^- satisfy the requirements in the DEFINITION. To that end, it suffices to show that the family $\{\gamma^*\}$ of curves in Δ separating the boundary components has the vanishing extremal length. The standard argument implies that $\lambda\{\gamma\}\geqq\lambda\{\gamma^*\}$ so that $\lambda\{\gamma\}=0$ is a sufficient condition for the possibility of the welding.

REMARK. On taking a suitable subfamily of $\{\gamma\}$, we can get the following criterion (cf. [5], p. 44): *The welding is possible if f satisfies*

$$\int_0 \frac{\min\left(\dfrac{1}{t}, \dfrac{f'(t)}{f(t)-f(0)}, \dfrac{f'(-t)}{f(0)-f(-t)}\right)}{\pi^2+\left(\log\dfrac{f(t)-f(0)}{f(0)-f(-t)}\right)^2}\,dt=\infty.$$

4. THEOREM 2. *There is an f with $\lambda\{\gamma\}>0$ which permits the welding.*

For the proof, let us consider in the $\zeta=\xi+i\eta$-plane the curve

$$\eta=F(\xi)=\begin{cases}0 & -1\leqq\xi\leqq 0\\ \xi^p\sin\dfrac{1}{\xi} & 0<\xi\leqq 1\end{cases}\qquad (p>0).$$

Map the interior of D^+ onto $\{\zeta;\ |\xi|<1,\ F(\xi)<\eta<1\}$ by $\zeta=\varphi^+(z)$ and the interior of D^- onto $\{\zeta;\ |\xi|<1,\ -1<\eta<F(\xi)\}$ by $\zeta=\varphi^-(z)$, where we require them to have the boundary correspondence $\varphi^\pm(-1)=-1$, $\varphi^\pm(0)=0$, $\varphi^\pm(1)=1+i\sin 1$. Define f by

$$f(x)=(\varphi^+)^{-1}\circ\varphi^-(x).$$

It is a function on $-1\leqq x\leqq 1$ which, of course, permits the welding. The remaining part of the proof of THEOREM 2 is contained in the first half of the following:

THEOREM 3. *$\lambda\{\gamma\}>0$ if $0<p<1$ and $\lambda\{\gamma\}=0$ if $1\leqq p$.*

PROOF. Suppose $0<p<1$. The extremal length of $\{\gamma\}$ is equal to that of $\{\gamma'\}$, the image of $\{\gamma\}$ under $\zeta=\varphi^{\pm}(z)$. Each γ' crosses the curve $\eta=F(\xi)$ $(0<\xi\leqq 1)$ at a single point (it may be tangent at other points, though). If the point of intersection is on the arc $\eta=F(\xi)$ $(1/2n\pi\leqq\xi\leqq 1/2(n-1)\pi)$, then γ' contains a subarc in Q_n connecting its upper side with the lower; here Q_n is the quadrilateral bounded by the following four arcs:

$$\begin{array}{ll} \eta=F(\xi) & (1/2(n+3/4)\pi\leqq\xi\leqq 1/2(n+1/4)\pi) \\ \eta=F(\xi) & (1/2(n-5/4)\pi\leqq\xi\leqq 1/2(n-7/4)\pi) \\ \eta=\xi^p & (1/2(n+3/4)\pi\leqq\xi\leqq 1/2(n-5/4)\pi) \\ \eta=-\xi^p & (1/2(n+1/4)\pi\leqq\xi\leqq 1/2(n-7/4)\pi). \end{array}$$

Therefore, γ' contains also a subarc in the quadrilateral

$$Q_n^*=\{\zeta;\, 1/2(n+3/4)\pi<\xi<1/2(n-7/4)\pi,\ |\eta|<(1/2(n+4/3)\pi)^p\}$$

connecting its upper side with the lower.

To evaluate $\lambda\{\gamma'\}$ we let

$$\rho_n(\zeta)=\begin{cases} \dfrac{(2(n+3/4)\pi)^p}{2} & \zeta\in Q_n^* \\ 0 & \text{otherwise,} \end{cases}$$

and consider

$$\rho(\zeta)=\max_{n=1,2,\cdots}\rho_n(\zeta).$$

Since a point in the ζ-plane is covered at most three Q_n^*'s, ρ is actually equal to max $(\rho_n, \rho_{n+1}, \rho_{n+2})$ in Q_n^*, so that it is an admissible density function. The above topological consideration implies that $\int_{\gamma'}\rho ds\geqq 1$ for every γ'. Besides

$$\iint_{|\zeta|<\infty}\rho^2 d\xi d\eta\leqq\sum_n\iint_{Q_n^*}\rho^2 d\xi d\eta\leqq\sum_n\iint_{Q_n^*}\frac{(2(n+2+3/4)\pi)^{2p}}{4}\,d\xi d\eta$$

$$\sim\frac{5}{2^{3-p}}\cdot\frac{1}{\pi^{1-p}}\sum_n\frac{1}{n^{2-p}}<\infty.$$

We conclude that $\lambda\{\gamma'\}=\lambda\{\gamma\}>0$.

Next let $1\leqq p$, and let $\{\gamma'\}$ be the image of $\{\gamma\}$ under $\zeta=\varphi^{\pm}(z)$. For $t(0<t<1/2)$, let γ_t' be the curve which is the union of $\gamma_t''=\{\zeta;\, \mathrm{Re}\,\zeta=t,\ |\mathrm{Im}\,\zeta|\leqq t^p\}$ and $\gamma_t'''=\{\zeta;\, |\zeta|=\sqrt{t^2+t^{2p}},\ \tan^{-1}t^{p-1}\leqq\arg\zeta\leqq 2\pi-\tan^{-1}t^{p-1}\}$. Since $\lambda\{\gamma\}=\lambda\{\gamma'\}\leqq\lambda\{\gamma_t'\}$, it suffices to show that $\lambda\{\gamma_t'\}=0$.

On using usual notations, we have

$$L_\rho\{\gamma_t'\}^2 \leqq \left(\int_{\gamma_t'} \rho ds\right)^2 \leqq \left(\int_{\gamma_t'} ds\right)\left(\int_{\gamma_t'} \rho^2 ds\right) \leqq 2\pi r\left(\int_{\gamma_t''} \rho^2 d\eta + \int_{\gamma_t'''} \rho^2 r d\theta\right)$$

where $r=\sqrt{t^2+t^{2p}}$. Divide it by r and integrate it with respect to t over $0<t<1/2$. The assumption $p\geqq 1$ shows $dr/dt \geqq 1/\sqrt{2}$, so that we have, on putting $\Delta''=\bigcup \gamma_t''$, $\Delta'''=\bigcup \gamma_t'''$, that

$$L_\rho\{\gamma_t'\}^2 \int_0^{1/2} \frac{dt}{\sqrt{t^2+t^{2p}}} \leqq 2\pi \iint_{\Delta''} \rho^2 d\xi d\eta + 2\sqrt{2}\pi \iint_{\Delta'''} \rho^2 r dr d\theta \leqq 2\sqrt{2}\pi \iint_{|\zeta|<1/2} \rho^2 d\xi d\eta.$$

By $\displaystyle\int_0^{1/2} \frac{dt}{\sqrt{t^2+t^{2p}}} = \infty$, $L_\rho\{\gamma_t'\}=0$ for any square integrable ρ, i.e., $\lambda\{\gamma_t'\}=0$.

References

[1] C. Blanc, Les surfaces de Riemann des fonctions méromorphes. Comment. Math. Helv., **9** (1937), 193–216, 335–368.

[2] ——— Les demi-surfaces de Riemann. Ibid. **11** (1939), 130–150.

[3] R. Courant, Dirichlet Principle. New York, 1950. (p. 69ff).

[4] J. A. Jenkins, On a type problem. Canad. J. Math., **11** (1959), 427–431.

[5] K. Oikawa, Welding of polygons and the type of Riemann surfaces. Kōdai Math. Sem. Rep., **13** (1961), 37–52.

[6] A. Pfluger, Über die Konstruktion Riemannscher Flächen durch Verheftung. J. Indian Math. Soc., **24** (1960), 401–412.

[7] L. I. Volkoviskiĭ, On the problem of type of simply-connected Riemann surfaces. Mat. Sbor. **18 (60)** (1946), 185–211. (Russian)

[8] ——— Studies in the problem of type of simply-connected Riemann surfaces. Trudi Mat. Inst. B. A. Steklova, **34** (1950). (Russian)

[9] ——— Quasiconformal mapping and the problem of conformal pasting. Ukrain. Mat. Z. **3** (1951), 39–51. (Russian)

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