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On θ-convolutions of vector valued distributions

Risai Shiraishi

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Introduction

In our previous paper [4] collaborated with Y. Hirata, we have introduced the notion of ι-convolution of vector valued distributions, which is a natural extension of the notion of the usual convolution of scalar valued distributions. On the other hand, in the Schwartz theory of convolution [11] [12], the problem concerning convolution has been worked out from a different standpoint. For instance, let $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$ be three normal spaces of distributions on R^N, N-dimensional Euclidean space. Let $\cup: \mathscr{H}\times\mathscr{K}\rightarrow\mathscr{L}$ be a bilinear map which is hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$. Let E, F, G be three Banach spaces and $\theta: E\times F\rightarrow G$ be a continuous bilinear map. He asked the question whether it would be possible to define a unique bilinear map $\cup_\theta: \mathscr{H}(E)\times\mathscr{K}(F)\rightarrow\mathscr{L}(G)$ in such a way that the map satisfies the following conditions:

(a) $\cup_\theta$ is hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{K}(F)$.

(b) For decomposed elements, that is, for elements of the type $S\otimes e$ and $T\otimes f$ of $\mathscr{H}(E)$ and $\mathscr{K}(F)$ we have

$$(S\otimes e)\cup_\theta(T\otimes f)=(S\cup T)\otimes\theta(e, f).$$

Under certain plausible conditions imposed on $\mathscr{H}$, $\mathscr{K}$ and $\mathscr{L}$, the problem has been settled definitely. Consider the convolution map $*: \mathscr{H}\times\mathscr{K}\rightarrow\mathscr{L}$ which is, by definition [12], a separately continuous map coinciding on $\mathscr{D}\times\mathscr{D}$ with the usual convolution. Suppose that $*$ is hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$. Now if we take the map $*$ for the map $\cup$ above, then the problem just described turns out to be the one concerning the convolution. Although his theory sheds a new light on the basic operations of vector valued distributions, there remains something to be desired as to the convolution maps:

(1) Since the map $*$ need not agree with the usual convolution (the example is given in [14]), $S*T$ may have only relative meanings and *a fortiori* the same for $\vec{S}*_\theta\vec{T}$.

(2) Even if the $*$ agrees with the usual convolution, $\vec{S}*_\theta\vec{T}$, considered as convolution of two vector valued distributions, has no intrinsic meanings, but may depend on $\mathscr{H}(E)$ and $\mathscr{K}(F)$ in which $\vec{S}$ and $\vec{T}$ are contained respectively.

Centering around these two points, especially (2), the present paper is devoted to the investigation of the convolution map $*_\theta$.

Most normal spaces of distributions $\mathscr{H}$ referred to as examples in Schwartz [11] [12] satisfy the further conditions: $\mathscr{H}$ is closed under the formation of multiplication by any $\beta \in \dot{\mathscr{B}}$ and the linear endomorphism $S \to \beta S$ of $\mathscr{H}$ is uniformly continuous with respect to β in any bounded subset of $\dot{\mathscr{B}}$. $\mathscr{H}$ equipped with this property is referred to as a *$\dot{\mathscr{B}}$-normal* space in this paper.

In Section 1, where we are concerned with some preliminary discussions on locally convex spaces and convolutions, we show that any continuous linear map of a $\dot{\mathscr{B}}$-normal space of distributions into $\mathscr{D}'$ is of the form $S \to S*T$ (in the usual sense) when the map agrees on $\mathscr{D}$ with the usual convolution. Such a convolution map is called *strict* in order to be distinguished from the one in the sense of Schwartz [12]. Section 2 deals with θ-convolution of two vector valued distributions which will be defined after the model of ι-convolution of our previous paper [4]. Several equivalent conditions on $*_\theta$-composability are also discussed here. θ-convolution of $\vec{S} \in \mathscr{D}'(E)$ and $\vec{T} \in \mathscr{D}'(F)$ will be denoted by $\vec{S} *_\theta \vec{T}$, where E, F, G are locally convex spaces and G is assumed quasi-complete and $\theta: E \times F \to G$ is a separately continuous bilinear map. Evidently the meanings of $\vec{S} *_\theta \vec{T}$ just defined is entirely different from the one considered before as a solution of a general problem. In Section 3 a brief discussion is devoted to the subject on θ-$\mathscr{S}'$-convolution of two vector valued tempered distributions after the model of the theory of $\mathscr{S}'$-convolution [13] and the treatments in the preceding section.

In Section 4 we shall consider a convolution map of a space of vector valued distributions into another one. We show that if u is a continuous linear map of $\mathscr{H}(E)$ into $\mathscr{D}'(G)$ and agrees on $\mathscr{D} \otimes E$ with the θ-convolution by a $\vec{T} \in \mathscr{D}'(F)$, then $u(\vec{S}) = \vec{S} *_\theta \vec{T}$ whenever $\mathscr{H}$ is $\dot{\mathscr{B}}$-normal and θ is hypocontinuous with respect to the compact disks of E. As an application of the result just obtained we can show that if $\mathscr{H}$ is $\dot{\mathscr{B}}$-normal and u is a continuous linear map of $\mathscr{H}(E)$ into $\mathscr{D}'(G)$ commuting with any translation on $\mathscr{D} \otimes E$, then $u(\vec{S}) = \vec{S} *_\theta \vec{T}$, for a $\vec{T} \in \mathscr{D}'(\mathfrak{L}_b(E; G))$, where $\mathfrak{L}_b(E; G)$ is assumed sequentially complete and θ the usual bilinear map of $E \times \mathfrak{L}_b(E; G)$ into G. A special case thereof has been shown by Lions [5] in connection with the theory of semigroup distribution which is characterized as a Green operator of a partial differential equation.

The sections 5 and 6 deal with θ-convolution map between spaces of vector valued distributions. Here the two basic cases (elementary and general) concerning the θ-convolution maps as developed in Schwartz [12] are considered again from our standpoint described above. In this paper if any of the maps $*: \mathscr{H} \times \mathscr{K} \to \mathscr{L}$ and $\theta: E \times F \to G$ is continuous, the case is referred to as "elementary" in a sense somewhat different from Schwartz's [12], where "elementary convolution" is only confined to the case where θ is continuous. Section 5 deals with the elementary cases. For instance, if $\mathscr{H}$, $\mathscr{K}$, $\mathscr{L}$ are normal spaces of distributions, where the convolution map of $\mathscr{H}\times\mathscr{K}$ into $\mathscr{L}$ is defined and is hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$, and if E, F, G are locally convex spaces, where a continuous bilinear map θ of $E\times F$ into G is defined, then any $\vec{S}\in\mathscr{H}(E)$ and $\vec{T}\in\mathscr{K}(F)$ are $*_\theta$-composable and the map $(\vec{S},\vec{T})\to\vec{S}*_\theta\vec{T}\in\mathscr{L}(G)$ is hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{K}(F)$, whenever $\mathscr{H}$, $\mathscr{K}$ are quasi-complete, $\mathscr{L}$ and G are complete, and $\mathscr{H}$ is $\dot{\mathscr{B}}$-normal and $\mathscr{H}$, $\mathscr{H}'_c$ are nuclear. When θ is only assumed to be separately continuous, a more complicated formulation of the behaviors of θ-convolution maps is needed as in Schwartz [12] (see Prop. 38, p. 159). Section 6 is devoted to this treatment which brings us a similar conclution as Schwartz's result just cited. It is to be noticed that although in our treatment we have assumed $\dot{\mathscr{B}}$-normal spaces of distributions according to the need, it is often possible to deduce more general conclusions than Schwartz [12].

The final section is devoted to some examples of θ-convolution maps. For instance, if θ is separately continuous the strict θ-convolution map of $\mathcal{O}'_C(E)\times\mathcal{O}'_C(F)$ into $\mathcal{O}'_C(G)$ is well defined and the exchange formula also holds: $\mathcal{F}(\vec{S}*_\theta\vec{T})=[\mathcal{F}(\vec{S})\mathcal{F}(\vec{T})]_\theta$, where $\mathcal{F}$ stands for Fourier transform. As an application we can show that the multiplicative product of any two distributions of $\mathcal{S}(E)$ and $\mathcal{S}(F)$ is well defined as an element of $\mathcal{S}(G)$. The section is closed with an indication of several examples in which the convolution map $*$: $\mathscr{H}\times\mathscr{K}\to\mathscr{L}$ is continuous.

§ 1. Preliminaries

Let E be a locally convex Hausdorff topological vector space. We will often refer to such a space as LCS. A subset A of E is called *quasi-closed* if any point of E adherent to a bounded subset of A belongs to A. The *strict closure* of A in E is the smallest quasi-closed subset containing A. A is *strictly dense* if its strict closure is E ([12], p. 198). As usual we denote by E'_σ and E'_c respectively the dual E' equipped with the weak topology $\sigma(E', E)$ and the topology of uniform convergence on the compact disks of E.

Proposition 1. *Let E, F and G be three LCSs. Suppose E is barrelled. Let u: $E\to F$ and i: $F\to G$ be linear maps such that i is a continuous injection and $i\circ u$ is continuous. If ${}^ti(G')$ is strictly dense in F'_σ, then u is continuous.*

Proof. We have only to show that $<u(x), f'>$ is a continuous form on E for every $f'\in F'$. For, then u will be weakly continuous and hence become continuous under the Mackey topologies of E and F ([1], p. 104). Since E is barrelled, the topology of E coincides with the Mackey topology ([1], p. 70). Therefore it follows that u is continuous.

Let A' be the set of $f'\in F'$ such that $x\to<u(x), f'>$ is a continuous form on E. Owing to a Banach-Steinhaus theorem ([1], p. 27) it is easy to see that A' is quasi-closed in F'. On the other hand, A' contains ${}^ti(G')$, the strict closure of which is F'. Therefore $A'=F'$. Thus the proof is completed.

Let us denote by $\mathcal{D}$ the space of all C^∞-functions with compact supports on R^N, N-dimensional Euclidean space. It is provided with its usual topology ([8], Chap. I, p. 24). We denote by $\mathcal{D}'$ the strong dual of $\mathcal{D}$. $\mathcal{D}'$ is the space of distributions on R^N. A space of distributions $\mathcal{H}$ is, by definition, an LCS contained in $\mathcal{D}'$ as a linear subspace with a finer topology: that is to say, the injection $i: \mathcal{H}\to\mathcal{D}'$ is continuous. $\mathcal{H}$ is called *normal* ([11], p.7), if $\mathcal{D}$ is contained in $\mathcal{H}$ with a finer topology and is dense in $\mathcal{H}$. We say that a space of distributions $\mathcal{H}$ has a *c-property* if any linear map $u: E\to\mathcal{H}$, E being barrelled, is continuous whenever $i\circ u$ is continuous ([13], p. 22). As an immediate consequence of Proposition 1 we have

COROLLARY. *$\mathcal{H}$ has a c-property if $\mathcal{D}$ is strictly dense in $\mathcal{H}'_\sigma$.*

$\mathcal{H}$ is called *permitted* ([15], p. 18) provided it is a normal space of distributions with the following properties: $\alpha_k(S*\rho_k)\to S$ and $(\alpha_k S)*\rho_k\to S$ in $\mathcal{H}$ for any $S\in\mathcal{H}$ as $k\to\infty$, where $\{\alpha_k\}$ is a sequence of multiplicators and $\{\rho_k\}$ is a sequence of regularizations.

If $\mathcal{H}$ is a permitted space or has the approximation properties by regularization and truncation ([11], p. 7), then it is easy to verify that $\mathcal{D}$ is strictly dense in $\mathcal{H}'_\sigma$, whence $\mathcal{H}$ has a c-property ([15], p. 18, [13], p. 21).

The *ε-product* $L\varepsilon M$ of two LCSs L and M is, by definition ([11], p. 18), the set of bilinear forms on $L'_c\times M'_c$ hypocontinuous with respect to the equicontinuous subsets of L' and M'. $L\varepsilon M$ is a linear space, on which we put the topology of uniform convergence on the products of equicontinuous subsets of L' and M'. $\mathfrak{L}_\varepsilon(L'_c; M)$ is the space of continuous linear maps of L'_c into M with the topology of uniform convergence on the equicontinuous subsets of L'. To any ξ of $L\varepsilon M$, we can associate an element $\tilde{\xi}$ of $\mathfrak{L}_\varepsilon(L'_c; M)$ and its transposed map ${}^t\tilde{\xi}\in\mathfrak{L}_\varepsilon(M'_c; L)$ by the formulas:

$$\xi(l', m') = <\tilde{\xi}(l'), m'> = <l', {}^t\tilde{\xi}(m')>.$$

The correspondences $\xi\to\tilde{\xi}\to{}^t\tilde{\xi}$ give rise to the algebraic and topological isomorphism between $L\varepsilon M$, $\mathfrak{L}_\varepsilon(L'_c; M)$ and $\mathfrak{L}_\varepsilon(M'_c; L)$ ([11], p. 34). Hence we can identify $L\varepsilon M$ with $\mathfrak{L}_\varepsilon(L'_c; M)$ or with $\mathfrak{L}_\varepsilon(M'_c; L)$. We shall write simply ξ instead of $\tilde{\xi}$ and ${}^t\tilde{\xi}$ without explicit statement when no confusion arises. For any space of distributions $\mathcal{H}$ we write $\mathcal{H}(E)$ instead of $\mathcal{H}\varepsilon E$, a space of E-valued distributions. $\vec{T}\in\mathcal{H}(E)$ is a continuous linear map of $\mathcal{H}'_c$ into E.

PROPOSITION 2. *Let E be a barrelled space. Let $i: L_1\to M_1$ and $j: L_2\to M_2$ be continuous injections, where L_1, L_2, M_1 and M_2 are LCSs. Let $u: E\to L_1\varepsilon L_2$ be a linear map such that $(i\otimes j)\circ u$ is continuous. Then u is continuous when ${}^ti(M'_1)$ and ${}^tj(M'_2)$ are strictly dense in $(L'_1)_\sigma$ and $(L'_2)_\sigma$ respectively.*

PROOF. Let A' be the set of elements $(l'_1, l'_2)\in L'_1\times L'_2$ such that $x\to u(x)(l'_1, l'_2)$ are continuous forms on E. The Banach-Steinhaus theorem shows that

A' is quasi-closed in $(L_1')_\sigma \times (L_2')_\sigma$. Since ${}^t i(M_1')$ and ${}^t j(M_2')$ are strictly dense in $(L_1')_\sigma$ and $(L_2')_\sigma$ respectively, we see that $A' = L_1' \times L_2'$. Any element of the dual space $(L_1 \varepsilon L_2)'$ of $L_1 \varepsilon L_2$ is an element of an equicontinuous subset of $(L_1 \varepsilon L_2)'$, which is the weak convex closure of a subset $B' \otimes C'$, where B' and C' are equicontinuous subsets of L_1' and L_2' respectively. Using again the Banach-Steinhaus theorem, we see that u is weakly continuous. This proves that u is continuous. The proof is completed.

As an immediate consequence of Proposition 2 we have

COROLLARY. *Let $\mathcal{H}$ be a space of distributions such that $\mathcal{D}$ is strictly dense in $\mathcal{H}_\sigma'$. Let E and F be two LCSs, where E is barrelled. Then a linear map u: $E \to \mathcal{H}(F)$ is continuous whenever $(i \otimes I) \circ u$: $E \to \mathcal{D}'(F)$ is continuous.*

Let E and F be two LCSs. A linear continuous map $u: E \to F$ is called *nuclear* ([3], Chap. I, p. 80) if it can be written as

$$u = \sum_\nu \lambda_\nu (e_\nu' \otimes f_\nu)$$

with the e_ν''s contained in an equicontinuous subset of E' and the f_ν's contained in a compact disk of F and $\sum |\lambda_\nu| < \infty$.

Let F be an LCS and U any disked neighbourhood of 0 in F. We can associate a seminorm p with U. This seminorm gives a certain equivalence relation in F: $x \sim y$ if and only if $p(x-y)=0$. We put on F the coarsest topology under which the seminorm p is continuous. F_U denotes the quotient space under the equivalence relation defined with the help of the seminorm p. Let $\hat{F}_U$ be the completion of F_U. $\hat{F}_U$ is a Banach space. A continuous linear map $u: E \to F$ is called *subnuclear* if, for any U, the induced map $E \to \hat{F}_U$ derived from u is nuclear ([12], p. 54). E is called *nuclear* if the identical map is subnuclear ([3], Chap. II, p. 34). This is equivalent to saying that E is nuclear if and only if any continuous linear map of E into any Banach space is nuclear.

Let $\mathcal{H}$ be a normal space of distributions. Let $\mathcal{K}$ be a space of distributions. A continuous linear map $u: \mathcal{H} \to \mathcal{K}$ is called a *convolution map* of $\mathcal{H}$ into $\mathcal{K}$ if there exists a distribution T such that $u(\phi) = \phi * T$ for any $\phi \in \mathcal{D}$. Here the convolution $S*T$ in the usual sense need not be defined for any $S \in \mathcal{H}$ ([14], p. 18), and therefore we shall write $S *_1 T$ to denote $u(S)$. We shall say that a continuous linear map u is a *strict convolution map* of $\mathcal{H}$ into $\mathcal{K}$, if we can write $u(S) = S*T$ for any $S \in \mathcal{H}$. We have discussed the various equivalent definitions of the usual convolution in [13] (p. 24).

Let $\mathcal{B}$ be the space of C^∞-functions defined on R^N, each of which is bounded with its derivatives of every order. We denote by $\dot{\mathcal{B}}$ the closure of $\mathcal{D}$ in $\mathcal{B}$. $\dot{\mathcal{B}}$ is a normal space of distributions of type (**F**). The strong dual of $\dot{\mathcal{B}}$ is the space $\mathcal{D}_{L^1}'$ of summable distributions. It is bornological and barrelled ([11], p. 126). A normal space of distributions $\mathcal{H}$ is called *$\dot{\mathcal{B}}$-normal* if $\mathcal{H}$ is stable under multiplication by any element of $\dot{\mathcal{B}}$, and if the maps $S \to \alpha S$ of $\mathcal{H}$

into itself are uniformly continuous when α runs through any bounded subset of $\dot{\mathcal{B}}$, that is, if for any neighbourhood $\mathcal{V}$ of 0 in $\mathcal{H}$ and any bounded subset B of $\dot{\mathcal{B}}$, there exisis a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{H}$ such that $B\mathcal{U}\subset\mathcal{V}$. If a normal space of distributions $\mathcal{H}$ is barrelled and possesses the c-property, then $\mathcal{H}$ is $\dot{\mathcal{B}}$-normal when $\dot{\mathcal{B}}\mathcal{H}\subset\mathcal{H}$.

Let $\mathcal{H}$, $\mathcal{L}$ be spaces of distributions, $\mathcal{H}$ being normal. $T\in\mathcal{D}'$ is said to be a *multiplicator* of $\mathcal{H}$ into $\mathcal{L}$, if there exists a continuous linear map $[T]$ of $\mathcal{H}$ into $\mathcal{L}$ which coincides with the multiplication by T on $\mathcal{D}\subset\mathcal{H}$ ([11], p. 69). We write $[T]S=TS$.

If $\mathcal{H}$ is $\dot{\mathcal{B}}$-normal and T is any element of $\mathcal{H}'$, T is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'_{L^1}$ and $<S, T>_{\mathcal{H},\mathcal{H}'}=\int STdx$. In fact, the linear map $S\rightarrow<\alpha S, T>_{\mathcal{H},\mathcal{H}'}$ is uniformly continuous with respect to α in any bounded subset of $\dot{\mathcal{B}}$. Hence the linear map $\phi\rightarrow\phi T$ of $\mathcal{D}$ into $\mathcal{D}'_{L^1}$ is continuous when we put on $\mathcal{D}$ the relative topology of $\mathcal{H}$. This proves that T is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'_{L^1}$. Since $<\phi, T>_{\mathcal{H},\mathcal{H}'}=\int\phi Tdx$, the continuity of the multiplication implies that $<S, T>_{\mathcal{H},\mathcal{H}'}=\int STdx$.

We shall denote by $\mathcal{H}^*$ the set of distributions composable with any element of $\mathcal{H}$, that is, of distributions T such that $S*T$ is defined for any $S\in\mathcal{H}$. Then we have

PROPOSITION 3. *Let $\mathcal{H}$ be a $\dot{\mathcal{B}}$-normal space of distributions. Let u: $\mathcal{H}\rightarrow\mathcal{D}'$ be a continuous linear map commutative with any translation τ_h on $\mathcal{D}\subset\mathcal{H}$. Then u is a strict convolution map.*

Moreover if $\mathcal{H}$ is barrelled, then, for any $T\in\mathcal{H}^*$, the linear map $S\rightarrow S*T$ of $\mathcal{H}$ into $\mathcal{D}'$ is a strict convolution map.

PROOF. The restriction of u to $\mathcal{D}$ is commutative with any translation τ_h. Then according to a theorem of Schwartz ([8], Chap. II, p. 18) there exists a unique distribution T such that for every $\phi\in\mathcal{D}$

$$u(\phi)=T*\phi.$$

Consider the transposed map ${}^tu$ which maps $\mathcal{D}$ into $\mathcal{H}'$. Then for any ϕ, $\psi\in\mathcal{D}$ we have

$$\begin{aligned}<{}^tu(\psi), \phi> &= <\psi, u(\phi)>\\ &=<\psi, T*\phi> = <\check{T}*\psi, \phi>,\end{aligned}$$

where $\vee$ denotes symmetrization. Hence $\check{T}*\psi\in\mathcal{H}'$ for every $\psi\in\mathcal{D}$. $\mathcal{H}$ being $\dot{\mathcal{B}}$-normal, any element of $\mathcal{H}'$ is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'_{L^1}$. Therefore $S(\check{T}*\psi)\in\mathcal{D}'_{L^1}$ for every $S\in\mathcal{H}$, so that $S*T$ is defined and

$$<S*T, \psi> = \int S(\check{T}*\psi)dx = <S, \check{T}*\psi>$$

$$= <S, {}^t u(\psi)> = <u(S), \psi>.$$

Consequently we have $u(S)=S*T$.

Now let $\mathscr{H}$ be barrelled. If $T \in \mathscr{H}^*$, then $S*T$ exists and $S(\check{T}*\psi) \in \mathscr{D}'_{L^1}$ for every $S \in \mathscr{H}$ and $\psi \in \mathscr{D}$. The bilinear map $(S, \psi) \rightarrow S(\check{T}*\psi)$ of $\mathscr{H} \times \mathscr{D}$ into $\mathscr{D}'_{L^1}$ is separately continuous owing to Corollary to Proposition 1 since $\mathscr{D}'_{L^1}$ is a permitted space. $\mathscr{D}$ is also barrelled. Hence the bilinear map is hypocontinuous, so is also the bilinear form $(S, \psi) \rightarrow \int S(\check{T}*\psi)dx = <S*T, \psi>$. This proves that the linear map $S \rightarrow S*T$ of $\mathscr{H}$ into $\mathscr{D}'$ is continuous. The proof is completed.

Let S be a distribution and $\vec{T}$ be a G-valued distribution, G being a quasi-complete LCS. We say that S and $\vec{T}$ are composable if we have for any $\phi \in \mathscr{D}$

$$(1) \qquad S(\check{\vec{T}}*\phi) \in \mathscr{D}'_{L^1}(G).$$

Then, owing to Corollary to Proposition 2, the linear map $\phi \rightarrow S(\check{\vec{T}}*\phi)$ of $\mathscr{D}$ into $\mathscr{D}'_{L^1}(G)$ is continuous. The convolution $S*\vec{T} \in \mathscr{D}'(G)$ is defined as follows:

$$<S*\vec{T}, \phi> = \int S(\check{\vec{T}}*\phi)dx.$$

Proposition 4. *Let $\mathscr{H}$ be $\dot{\mathscr{B}}$-normal and G a quasi-complete LCS. Let u be a continuous linear map of $\mathscr{H}$ into $\mathscr{D}'(G)$ such that the restriction of u to $\mathscr{D} \subset \mathscr{H}$ is commutative with any translation. Then there exists a unique $\vec{T} \in \mathscr{D}'(G)$ composable with any $S \in \mathscr{H}$, and $u(S)=S*\vec{T}$,*

Proof. For any $g' \in G'$, the linear map $S \rightarrow <u(S), g'>$ satisfies the requisites of the preceding proposition. Therefore there exists a unique distribution $T_{g'}$ depending on g' such that S and $T_{g'}$ are composable and $<u(S), g'> = S*T_{g'}$.

Next we show that there is a $\vec{T} \in \mathscr{D}'(G)$ such that $<\vec{T}, g'> = T_{g'}$. Let $\{\rho_n\}$ be a sequence of regularizations. Put $\vec{T}_n = u(\rho_n)$. Then

$$(2) \qquad \phi \cdot <\vec{T}_n, g'> = \phi \cdot (\rho_n * T_{g'}) = \check{\rho}_n \cdot (\check{\phi} * T_{g'}) \quad \text{for} \quad \phi \in \mathscr{D}, g' \in G'.$$

Let A' be an equicontinuous disk in G'. The bilinear map $(\phi, g') \rightarrow \check{\phi}*T_{g'}$ of $\mathscr{D} \times G'_{A'}$ into $\mathscr{E}$ is separately continuous. Since $\mathscr{D}$ and $G'_{A'}$ are barrelled spaces, the map is hypocontinuous. $\{\check{\phi}*T_{g'}\}$ is bounded in $\mathscr{E}$ when ϕ and g' lie in a compact subset C of $\mathscr{D}$ and A' respectively. It follows from (2) that $\{\phi \cdot <\vec{T}_n, g'>\}$ converges uniformly to $\phi \cdot T_{g'}$ as $n \rightarrow \infty$ when ϕ and g' run through C and A' respectively. This shows that $\{\vec{T}_n\}$ is a Cauchy sequence in $\mathscr{D}'(G)$, which is known to be quasi-complete ([11], p. 29). Putting $\vec{T} = \lim_{n \rightarrow \infty} \vec{T}_n$, $<\vec{T}, g'> = T_{g'}$. Now we shall prove that $S(\check{\vec{T}}*\phi) \in \mathscr{D}'_{L^1}(G)$ for every $\phi \in \mathscr{D}$. Clearly we have $\psi(\check{\vec{T}}*\phi) \in \mathscr{E}'(G) \subset \mathscr{D}'_{L^1}(G)$ for any $\psi \in \mathscr{D}$. The map $\psi \rightarrow \psi(\check{\vec{T}}*\phi)$ of $\mathscr{D}$ into $\mathscr{D}'_{L^1}(G)$ is continuous when we put on $\mathscr{D}$ the relative topology of $\mathscr{H}$. In fact, let β run through a bounded subset B of $\dot{\mathcal{B}}$. Then we have

$$(3) \qquad \int <\beta\psi(\check{\vec{T}}*\phi), g'> dx = \int \beta\psi(\check{T}_{g'}*\phi)dx$$
$$= \int \phi(T_{g'}*\beta\psi)dx$$
$$= \phi\cdot <u(\beta\psi), g'>.$$

Let $\psi \to 0$ in $\mathcal{D}$ with the relative topology of $\mathcal{H}$. $\{\beta\psi\}$ converges uniformly in $\mathcal{H}$ to 0 when β runs through B. Since u is continuous, it follows from (3) that $\{\int <\beta\psi(\check{\vec{T}}*\phi), g'> dx\}$ converges uniformly to 0 when ϕ and g' run through C and A' respectively. Hence we see that $\{\psi(\check{\vec{T}}*\phi)\}$ converges to 0 in $\mathcal{D}'_{L^1}(G)$ when $\psi \to 0$ in $\mathcal{D}$ with the relative topology of $\mathcal{H}$. For any $S \in \mathcal{H}$, we choose $\{\psi\} \subset \mathcal{D}$ such that $\{\psi\}$ converges to S in $\mathcal{H}$. Then $\{\psi(\check{\vec{T}}*\phi)\}$ converges to $S(\check{\vec{T}}*\phi)$ in $\mathcal{D}'_{L^1}(\hat{G})$, $\hat{G}$ being the completion of G. On the other hand, choose a sequence of multiplicators $\{\alpha_n\}$, $\alpha_n \in \mathcal{D}$, where $0 \leqq \alpha_n \leqq 1$, $\alpha_n \to 1$ in $\mathcal{E}$ and $\{\alpha_n\}$ is bounded in $\dot{\mathcal{B}}$. Since for each n we have $\alpha_n S(\check{\vec{T}}*\phi)\} \in \mathcal{D}'_{L^1}(G)$ and $\mathcal{D}'_{L^1}(G)$ is quasi-complete, the sequence $\{\alpha_n S(\check{\vec{T}}*\phi)\}$ converges to $S(\check{\vec{T}}*\phi)$ in $\mathcal{D}'_{L^1}(G)$, and *a fortiori* $S(\check{\vec{T}}*\phi) \in \mathcal{D}'_{L^1}(G)$. Accordingly

$$\phi\cdot <u(S), g'> = \phi\cdot(S* <\vec{T}, g'>) = \phi\cdot <S*\vec{T}, g'>.$$

Consequently we have $u(S) = S*\vec{T}$ for every $S \in \mathcal{H}$.

Finally we show the uniqueness of $\vec{T}$. Suppose $u(S) = S*\vec{T}'$ for every $S \in \mathcal{H}$. Then we have $u(\phi) = \phi*\vec{T} = \phi*\vec{T}'$ and hence $\phi*(\vec{T}-\vec{T}') = 0$ for every $\phi \in \mathcal{D}$. This implies $\vec{T} = \vec{T}'$.

Thus the proof is completed.

§ 2. θ-convolution of two vector valued distributions

Let E, F be two LCSs. Consider the tensor product $E \otimes F$. Then for any LCS G there exists a biunique linear correspondence between the bilinear maps u of $E \times F$ into G and the linear maps $\tilde{u}$ of $E \otimes F$ into G by the definition $u(e, f) = \tilde{u}(e \otimes f)$. The map η: $E \times F \to E \otimes F$ defined by $\eta(e, f) = e \otimes f$ is a bilinear map, the canonical bilinear map of $E \times F$ into $E \otimes F$. Now we put on $E \otimes F$ a unique locally convex topology ι in such a way that the separately continuous u corresponds precisely to the continuous $\tilde{u}$ under the topology ι. The tensor product $E \otimes F$ equipped with this topology ι is denoted by $E \otimes_\iota F$, the inductive tensor product of E and F. $E \bar{\otimes}_\iota F$ denotes its quasi-completion. From now on we suppose that G is quasi-complete. Let θ be a separately continuous bilinear map of $E \times F$ into G. Let $\bar{\eta}$ stand for the canonical bilinear map of $E \times F$ into $E \bar{\otimes}_\iota F$. Then there exists a unique continuous linear map $\bar{\theta}$ of $E \bar{\otimes}_\iota F$ into G such that $\theta = \bar{\theta} \circ \bar{\eta}$.

Let $\mathcal{H}$ be a normal space of distributions. A vector valued distribution $\vec{S} \in \mathcal{H}'_c(E)$ is called β_0*-bounded* and is denoted by $\vec{S} \in \mathcal{H}'_c(E; \beta_0)$ ([12], p. 54) if

$\vec{S}$ maps a neighbourhood of 0 in $\mathscr{H}$ into a bounded completing subset, a subset of E contained in an absolutely convex bounded subset B of E such that E_B is a Banach space. $\vec{S} \in \mathscr{H}'_c(E)$ is called *locally β_0-bounded* if any multiplicative product $\phi\vec{S}$ is β_0-bounded for $\phi \in \mathscr{D}$. A subset $\mathfrak{B}$ of $\mathscr{H}'_c(E)$ is *β_0-equibounded* ([12], p. 54) if there exists a neighbourhood $\mathscr{U}$ of 0 in $\mathscr{H}$ such that $\bigcup_{\vec{S}\in\mathfrak{B}} \mathscr{U}\cdot\vec{S}$ is a bounded completing subset of E. A *locally β_0-equibounded* subset of $\mathscr{H}'_c(E)$ will be defined in an obvious way. It is easy to see that if $\mathscr{H}$ is nuclear, $\vec{S}$ is β_0-bounded if and only if it is a nuclear map of $\mathscr{H}$ into E.

Consider two vector valued distributions $\vec{S} \in \mathscr{D}'(E)$ and $\vec{T} \in \mathscr{D}'(F)$. The regularization $\check{\vec{T}}*\phi$, $\phi \in \mathscr{D}$, is locally β_0-bounded in $\mathscr{E}(F)$ ([12], p. 172). The multiplicative product $[\vec{S}(\check{\vec{T}}*\phi)]_\iota \in \mathscr{D}'(E\otimes_\iota F)$ is well defined ([12], p. 134). The G-valued distribution $[\vec{S}(\check{\vec{T}}*\phi)]_\theta$ is then defined as $(I\otimes\bar{\theta})([\vec{S}(\check{\vec{T}}*\phi)]_\iota)$.

In a similar way we define the tensor product $\vec{S}\otimes_\theta\vec{T} \in \mathscr{D}'(G)$ of $\vec{S}$ and $\vec{T}$: $\vec{S}\otimes_\theta\vec{T}=(I\otimes\bar{\theta})(\vec{S}\otimes\otimes_\iota\vec{T})$. Here we note that the bilinear map $(\vec{S}, \vec{T})\to\vec{S}\otimes_\theta\vec{T}$ is separately continuous since the map $(\vec{S}, \check{\vec{T}})\to\vec{S}\otimes\otimes_\iota\vec{T}$ is separately continuous ([12], p. 146) and the map $I\otimes\bar{\theta}$ is continuous.

For our later purpose we need the following lemmas.

Lemma 1. *If $\vec{S}$ lies in a bounded subset of $\mathscr{D}'(E)$ and if $\vec{T}$ lies in a locally β_0-equibounded subset of $\mathscr{E}(F)$, then $[\vec{S}\vec{T}]_\theta$ also lies in a bounded subset of $\mathscr{D}'(G)$.*

Proof. Since $\mathscr{D}$ is barrelled, it is sufficient to show that for any $\phi \in \mathscr{D}$ the set $\{\phi\cdot[\vec{S}\vec{T}]_\theta\}$ is bounded. But $\phi\cdot[ST]_\theta=\vec{S}\cdot_\theta(\phi\vec{T})$. We can therefore apply Proposition 10 of Schwartz [12] (p. 58) to conclude our statement, completing the proof.

As an immediate consequence of the preceding lemma we have

Lemma 2. *The linear map $\phi\to[\vec{S}(\check{\vec{T}}*\phi)]_\theta$ of $\mathscr{D}$ into $\mathscr{D}'(G)$ is continuous.*

Proof. $\mathscr{D}$ is bornological, and if ϕ lies in a bounded subset of $\mathscr{D}$, the set $\{\check{\vec{T}}*\phi\}$ is locally β_0-equibounded in $\mathscr{E}(F)$ ([12], p. 149). Therefore it follows from Lemma 1 that the map $\phi\to[\vec{S}(\check{\vec{T}}*\phi)]_\theta$ is continuous.

The lemma can also be proved as follows. The map $\phi\to(\vec{S}_x\otimes_\theta\vec{T}_y)\,\phi(\hat{x}+\hat{y})$ of $\mathscr{D}$ into $\mathscr{D}'_{x,y}(G)$ is continuous. It is easy to verify that $(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(\hat{x}+\hat{y}) \in (\mathscr{D}'_{L^1})_y(\mathscr{D}'_x(G))$. Using Corollary to Proposition 2 we see that $\phi\to(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(\hat{x}+\hat{y})$ is a continuous map of $\mathscr{D}$ into $(\mathscr{D}'_{L^1})_y(\mathscr{D}'_x(G))$. Hence $\phi\to[\vec{S}(\check{\vec{T}}*\phi)]_\theta=\int(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(\hat{x}+\hat{y})dy$ is continuous.

Lemma 3. *The linear map $U\to[(\vec{S}*\phi)(\check{\vec{T}}*U)]_\theta$ of $\mathscr{E}'$ into $\mathscr{D}'(G)$ is continuous.*

Proof. $\mathscr{E}'$ is bornological, and if U lies in a bounded subset of $\mathscr{E}'$, the set $\{\check{\vec{T}}*U\}$ is bounded in $\mathscr{D}'(F)$. Therefore it follows from Lemma 1 that the map $U\to[(\vec{S}*\phi)(\check{\vec{T}}*U)]_\theta$ is continuous.

Now we shall turn to the discussion of θ-convolution of two vector valued distributions. In our previous paper [4], the notion of ι-convolution of such distributions was introduced after the pattern of the theory of classical convolution. θ-convolution, to which the present section will devote itself, will also be treated along the same line.

$\vec{S} \in \mathcal{D}'(E)$ and $\vec{T} \in \mathcal{D}'(F)$ are called $*_\theta$-*composable* if

$$(*) \qquad (\vec{S}_x \otimes_\theta \vec{T}_y)\phi(\hat{x}+\hat{y}) \in (\mathcal{D}'_{L^1})_{x,y}(G) \quad \text{for every} \quad \phi \in \mathcal{D}.$$

Then owing to Corollary to Proposition 2 the map $\phi \rightarrow (\vec{S}_x \otimes_\theta \vec{T}_y)\phi(\hat{x}+\hat{y})$ of $\mathcal{D}$ into $(\mathcal{D}'_{L^1})_{x,y}(G)$ is continuous. The θ-convolution $\vec{S}*_\theta\vec{T} \in \mathcal{D}'(G)$ is defined as follows:

$$\phi \cdot (\vec{S}*_\theta \vec{T}) = \iint (\vec{S}_x \otimes_\theta \vec{T}_y)\phi(x+y)dx\, dy.$$

PROPOSITION 5. *Each of the following conditions is equivalent to* $(*)$.

(i) $\delta(\hat{z}-\hat{x}-\hat{y})(\vec{S}_x \otimes_\theta \vec{T}_y) \in (\mathcal{D}'_z(\mathcal{D}'_{L^1})_{x,y})(G)$, *where* δ *denotes a Dirac measure at* 0 *in* R^N;

(ii) $[(\check{\vec{S}}*\phi)\vec{T}]_\theta \in \mathcal{D}'_{L^1}(G)$ *for every* $\phi \in \mathcal{D}$;

(iii) $[\vec{S}(\check{\vec{T}}*\phi)]_\theta \in \mathcal{D}'_{L^1}(G)$ *for every* $\phi \in \mathcal{D}$;

(iv) $\vec{S}(\hat{x}-\hat{y})\otimes_\theta \vec{T}(\hat{y})$ *is partially summable with respect to* y, *that is,* $\vec{S}(\hat{x}-\hat{y})\otimes_\theta \vec{T}(\hat{y}) \in (\mathcal{D}'_{L^1})_y(\mathcal{D}'_x(G))$;

(v) $\vec{S}(\hat{y})\otimes_\theta \vec{T}(\hat{x}-\hat{y})$ *is partially summable with respect to* y;

(vi) $[(\check{\vec{S}}*\phi)(\check{\vec{T}}*\psi)]_\theta \in L^1 \varepsilon G$ *for every* $\phi, \psi \in \mathcal{D}$.

PROOF. $(*)\rightarrow$(ii): From $(*)$ it follows that

$$[(\check{\vec{S}}*\phi)\vec{T}]_\theta = \int (\vec{S}_x \otimes_\theta \vec{T}_y)\phi(x+\hat{y})dx \in \mathcal{D}'_{L^1}(G).$$

(ii)$\rightarrow$(iv): For every $\phi \in \mathcal{D}$, we have

$$\begin{aligned}(1) \qquad <\vec{S}(\hat{x}-\hat{y})\otimes_\theta \vec{T}(\hat{y}), \phi(\hat{x})> &= \int [\vec{S}(x-\hat{y})\otimes_\theta T(\hat{y})]\phi(x)dx \\ &= (I\otimes\bar{\theta})\{\int [\vec{S}(x-\hat{y})\otimes\otimes_\iota \vec{T}(\hat{y})]\phi(x)dx\} \\ &= (I\otimes\bar{\theta})\,[(\check{\vec{S}}*\phi)\vec{T}]_\iota \\ &= [(\check{\vec{S}}*\phi)\vec{T}]_\theta.\end{aligned}$$

If (ii) holds, it follows from Lemma 2 and Corollary to Proposition 2 that the linear map $\phi \rightarrow [(\check{\vec{S}}*\phi)\vec{T}]_\theta$ of $\mathcal{D}$ into $\mathcal{D}'_{L^1}(G)$ is continuous, and therefore by the relation (1) we see that (ii) implies (iv).

(iv)$\rightarrow(*)$: By (iv) we have for every $\phi \in \mathcal{D}$

$$\begin{aligned}(\vec{S}(\hat{x}-\hat{y})\otimes_\theta \vec{T}(\hat{y}))\phi(\hat{x}) &\in (\mathcal{E}'_x(\mathcal{D}'_{L^1})_y)(G)\\ &= (\mathcal{E}'_x\hat{\otimes}_\pi(\mathcal{D}'_{L^1})_y)(G)\\ &\subset ((\mathcal{D}'_{L^1})_x\hat{\otimes}_\pi(\mathcal{D}'_{L^1})_y)(G)\\ &= (\mathcal{D}'_{L^1})_{x,y}(G).\end{aligned}$$

Consequently, by change of variables we can conclude that

$$(\vec{S}(\hat{x})\otimes_\theta \vec{T}(\hat{y}))\phi(\hat{x}+\hat{y}) \in (\mathcal{D}'_{L^1})_{x,y}(G).$$

(ii)→(vi): Owing to Lemma 3 the bilinear map $(R, \phi)\to R*[(\check{\vec{S}}*\phi)\vec{T}]_\theta$ of $\mathcal{E}'\times\mathcal{D}$ into $\mathcal{D}'_{L^1}(G)$ is hypocontinuous. This together with the relation: $[(\check{\vec{S}}*\phi)(\tau_y\vec{T})]_\theta=\tau_y\delta*[(\check{\vec{S}}*\tau_{-y}\phi)\vec{T}]_\theta$ shows that $y\to[(\check{\vec{S}}*\phi)(\tau_y\vec{T})]_\theta$ is a continuous $\mathcal{D}'_{L^1}(G)$-valued function. Then for any $\psi\in\mathcal{D}$ we have

$$\begin{aligned}[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta &= [(\check{\vec{S}}*\phi)\int(\tau_y T)\psi(y)dy]_\theta\\ &= \int[(\check{\vec{S}}*\phi)(\tau_y\vec{T})]_\theta\psi(y)dy \in \mathcal{D}'_{L^1}(G).\end{aligned}$$

Now since the map $(\phi, \psi)\to[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta$ of $\mathcal{D}\times\mathcal{D}$ into $\mathcal{D}'_{L^1}(G)$ is hypocontinuous, the derivation formula may be applied.

$$(2)\qquad \begin{aligned}&D^p[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta\\ &\quad=\sum_{r\leqq p}\frac{p!}{r!(p-r)!}[(\check{\vec{S}}*D^r\phi)(\vec{T}*D^{p-r}\psi)]_\theta.\end{aligned}$$

Miyazaki has proved that the pair $\mathcal{D}'_{L^1}(G)$ and $\mathcal{D}_{L^1}(G)$ is distinguished in his sense ([6], p. 532): $\mathcal{D}_{L^1}(G)$ is the set of vector valued distributions $\vec{R}\in\mathcal{D}'_{L^1}(G)$ for which there exists for any equicontinuous subset $A'\subset G'$ a sequence of positive numbers λ_p such that the set $\{<\lambda_p D^p\vec{R}, g'>\}_{g'\in A', p}$ is bounded in $\mathcal{D}'_{L^1}$. Now we put $\vec{R}=[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta$ and show that $\vec{R}$ satisfies the above condition. The map $(\phi, \psi)\to\vec{R}$ is hypocontinuous and there exist for any fixed ϕ and ψ two sequences of positive numbers μ_p and ν_p such that the sets $\{\mu_p D^p\phi\}$ and $\{\nu_p D^p\psi\}$ are bounded in $\mathcal{D}$. Therefore it follows from the derivation formula (2) that there exists a sequence of positive numbers λ_p stipulated above. This shows that $\vec{R}\in\mathcal{D}_{L^1}(G)$ and therefore $\vec{R}\in L^1\varepsilon G$, as desired.

(vi)→(ii): Let $\{\rho_n\}$ be a sequence of regularizations. Owing to Lemma 3, since $\mathcal{D}'_{L^1}(G)$ is quasi-complete ([10], p. 29), it is sufficient to show that $\{[(\check{\vec{S}}*\phi)(\vec{T}*\rho_n)]_\theta\}_{1\leqq n<\infty}$ forms a Cauchy sequence in $\mathcal{D}'_{L^1}(G)$. To this end let A' be any equicontinuous disk of G'. Let U be the closed unit cube with center 0 in R^N. The trilinear map $(\phi, \psi, g')\to<[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta, g'>$ of $\mathcal{D}_U\times\mathcal{D}_U\times G'_{A'}$ into $\mathcal{D}'_{L^1}$ is separately continuous and hence continuous, because $\mathcal{D}_U$ is a space of type (**F**) and $G'_{A'}$ is a Banach space. Therefore we can find a positive integer l such that the map is continuous in the topology induced by that of $\mathcal{D}^l_U\times\mathcal{D}^l_U\times G'_{A'}$. We can take a positive integer k such that a $\xi\in\mathcal{D}^l_{\frac{1}{2}U}$ is a parametrix of an iterated Laplacian Δ^k ([8], p. 47):

$$\delta = \Delta^k \xi + \eta, \quad \eta \in \mathcal{D}_{\frac{1}{2}U}.$$

Consequently we can write

$$[(\check{\vec{S}} * \phi)(\vec{T} * \rho_n)]_\theta = [(\check{\vec{S}} * \phi)(\vec{T} * \Delta^k \xi * \rho_n)]_\theta + [(\check{\vec{S}} * \phi)(\vec{T} * \eta * \rho_n)]_\theta.$$

Since $\xi \in \mathcal{D}^l_{\frac{1}{2}U}$, we can choose a sequence $\{\xi_j\}$, $\xi_j \in \mathcal{D}_{\frac{1}{2}U}$, tending to ξ in $\mathcal{D}^l_{\frac{1}{2}U}$ as $j \to \infty$. Passing to the limit as $j, n \to \infty$, $\{\xi_j * \rho_n\}$ and $\{\eta * \rho_n\}$ tend to ξ and η in $\mathcal{D}^l_U$ respectively. Therefore $\{<[(\check{\vec{S}} * \phi)(\vec{T} * \xi_j * \rho_n)]_\theta, g'>\}$ and $\{<[(\check{\vec{S}} * \phi)(\vec{T} * \eta * \rho_n)]_\theta, g'>\}$ converge uniformly in $\mathcal{D}'_{L^1}$ to $<[(\check{\vec{S}} * \phi)(\vec{T} * \xi)]_\theta, g'>$ and $<[(\check{\vec{S}} * \phi)(\vec{T} * \eta)]_\theta, g'>$ respectively when g' runs through A' and ϕ lies in a neighbourhood of 0 in $\mathcal{D}_U$. On the other hand, we have

$$<[(\check{\vec{S}} * \phi)\left(\vec{T} * \frac{\partial}{\partial x_i} \xi_j * \rho_n\right)]_\theta, g'>$$

$$= <\frac{\partial}{\partial x_i}[(\check{\vec{S}} * \phi)(\vec{T} * \xi_j * \rho_n)]_\theta, g'>$$

$$- <[\left(\check{\vec{S}} * \frac{\partial}{\partial x_i} \phi\right)(\vec{T} * \xi_j * \rho_n)]_\theta, g'>.$$

Consequently $\{<[(\check{\vec{S}} * \phi)\left(\vec{T} * \frac{\partial}{\partial x_i} \xi_j * \rho_n\right)]_\theta, g'>\}$ converges uniformly in $\mathcal{D}'_{L^1}$ to $<[(\check{\vec{S}} * \phi)\left(\vec{T} * \frac{\partial}{\partial x_i} \xi\right)]_\theta, g'>$ when g' runs through A' and ϕ lies in a neighbourhood of 0 in $\mathcal{D}_U$. Repeating this process, we see that $\{[(\check{\vec{S}} * \phi)(\vec{T} * \rho_n)]_\theta\}_{1 \leq n < \infty}$ is a Cauchy sequence in $\mathcal{D}'_{L^1}(G)$, as desired.

The implications $(*) \to$(iii)$\to$(v)$\to (*)$ may be proved just as in the cases $(*) \to$(ii)$\to$(iv)$\to (*)$. The proof is omitted.

$(*) \rightleftarrows$(i): We have for every $\phi \in \mathcal{D}$

(3) $$<\phi(\hat{z}), \delta(\hat{z} - \hat{x} - \hat{y})(\vec{S}_x \otimes_\theta \vec{T}_y)> = (\vec{S}_x \otimes_\theta \vec{T}_y)\phi(\hat{x} + \hat{y}).$$

Consequently (i) implies $(*)$.

Conversely, if $(*)$ holds, the linear map $\phi \to (\vec{S}_x \otimes_\theta \vec{T}_y)\phi(\hat{x} + \hat{y})$ of $\mathcal{D}$ into $(\mathcal{D}'_{L^1})_{x,y}(G)$ is continuous owing to Corollary to Proposition 2, and therefore the relation (3) shows that (i) holds.

Thus the proof of the Proposition 5 is completed.

PROPOSITION 6. *If $\vec{S}$, $\vec{T}$ are $*_\theta$-composable, then*

(i) $$\vec{S} *_\theta \vec{T} = \int \vec{S}(\hat{x} - y) \otimes_\theta \vec{T}(y) dy = \int \vec{S}(y) \otimes_\theta \vec{T}(\hat{x} - y) dy;$$

(ii) $$\phi \cdot (\vec{S} *_\theta \vec{T}) = \int [(\check{\vec{S}} * \phi)\vec{T}]_\theta dx = \int [\vec{S}(\check{\vec{T}} * \phi)]_\theta dx \quad \textit{for every} \quad \phi \in \mathcal{D};$$

(iii) $(\phi*\check{\psi})\cdot(\vec{S}*_\theta\vec{T})=\int[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta dx$ *for every* $\phi, \psi \in \mathcal{D}$.

PROOF. Owing to the preceding proposition, the equivalent conditions (i)～(vi) in Proposition 5 hold.

(i) and (ii): From the condition (iv) in Proposition 5, we have for any $\phi \in \mathcal{D}$

$$<\phi(\hat{x}), \int \vec{S}(\hat{x}-y)\otimes_\theta \vec{T}(y)dy> = \iint (\vec{S}(x-y)\otimes_\theta \vec{T}(y))\phi(x)dx\,dy$$
$$= \iint (\vec{S}(x)\otimes_\theta \vec{T}(y))\phi(x+y)dx\,dy$$
$$= \phi\cdot(\vec{S}*_\theta\vec{T}).$$

Hence $\int \vec{S}(\hat{x}-y)\otimes_\theta \vec{T}(y)dy=\vec{S}*_\theta\vec{T}$, and similarly $\int \vec{S}(y)\otimes_\theta \vec{T}(\hat{x}-y)dy=\vec{S}*_\theta\vec{T}$.

Next, by the definition of θ-convolution, we have

$$\phi\cdot(\vec{S}*_\theta\vec{T})=\iint(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(x+y)dx\,dy$$
$$=\int\{\int(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(x+y)dx\}\,dy$$
$$=\int[(\check{\vec{S}}*\phi)\vec{T}]_\theta dy.$$

Similarly we have $\phi\cdot(\vec{S}*_\theta\vec{T})=\int[\vec{S}(\check{\vec{T}}*\phi)]_\theta dx$.

(iii): Let ψ be any element of $\mathcal{D}$. Now consider the map $\phi\to[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta$ of $\mathcal{D}$ into $L^1\varepsilon G$, which is continuous owing to Corollary to Proposition 2. Then there exists a distribution $\vec{K}_\psi \in \mathcal{D}'(G)$ such that $\int[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta dx = <\phi, \vec{K}_\psi>$, where $\psi\to\vec{K}_\psi$ is a linear map of $\mathcal{D}$ into $\mathcal{D}'(G)$. From this equation it follows that $\psi\to\vec{K}_\psi$ is continuous and is commutative with any translation. Owing to Proposition 4 there exists a distribution $\vec{K}\in\mathcal{D}'(G)$ such that $\vec{K}_\psi=\vec{K}*\psi$ for every $\psi\in\mathcal{D}$, whence $\int[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta dx=\phi\cdot(\vec{K}*\psi)$. Let $\{\rho_n\}$ be a sequence of regularizations. $\{\int[(\check{\vec{S}}*\phi)(\vec{T}*\rho_n)]_\theta dx\}$ converges to $\int[(\check{\vec{S}}*\phi)\vec{T}]_\theta dx$ as $n\to\infty$ (see the proof of (vi)$\to$(ii) in the preceding proposition). On the other hand $\{\phi\cdot(\vec{K}*\rho_n)\}$ converges to $\phi\cdot\vec{K}$. Hence $\int[(\check{\vec{S}}*\phi)\vec{T}]_\theta dx=\phi\cdot\vec{K}$. It follows from (ii) proved above that $\vec{S}*_\theta\vec{T}=\vec{K}$. This ensures us to conclude that

$$\int[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta dx=\phi\cdot(\vec{S}*_\theta\vec{T}*\psi)=(\phi*\check{\psi})\cdot(\vec{S}*_\theta\vec{T}).$$

Thus the proof is completed.

REMARK 1. Let $\vec{S}$, $\vec{T}$ be $*_\theta$-composable. Let U be any element of $\mathcal{E}'$. Then $[\vec{S}(\check{\vec{T}}*(\check{U}*\phi))]_\theta \in \mathcal{D}'_{L^1}(G)$ for every $\phi \in \mathcal{D}$. Hence $\vec{S}$, $\vec{T}*U$ are $*_\theta$-composable. Now

$$\begin{aligned}\phi\cdot(\vec{S}*_\theta(\vec{T}*U)) &= \int[\vec{S}(\check{\vec{T}}*\check{U}*\phi)]_\theta dx\\ &= (\check{U}*\phi)\cdot(\vec{S}*_\theta\vec{T})\\ &= \phi\cdot(U*(\vec{S}*_\theta\vec{T})).\end{aligned}$$

By symmetry we have

$$U*(\vec{S}*_\theta\vec{T}) = (U*\vec{S})*_\theta\vec{T} = \vec{S}*_\theta(U*\vec{T}).$$

In particular, if we take U as $D^p\delta$, then

$$D^p(\vec{S}*_\theta\vec{T}) = D^p\vec{S}*_\theta\vec{T} = \vec{S}*_\theta D^p\vec{T}.$$

REMARK 2. Let E be one-dimensional, then $\mathcal{D}'(E)$ may be identified with $\mathcal{D}'$. Thus we may consider the convolution of a scalar valued distribution and a vector valued distribution as a special case of θ-convolution. Therefore $S \in \mathcal{D}'$ and $\vec{T} \in \mathcal{D}'(G)$ are composable in our sense when any one of the following equivalent conditions holds:

(i) $(S_x\otimes\vec{T}_y)\phi(\hat{x}+\hat{y}) \in (\mathcal{D}'_{L^1})_{x,y}(G)$ for every $\phi \in \mathcal{D}$;

(ii) $S(\check{\vec{T}}*\phi) \in \mathcal{D}'_{L^1}(G)$ for every $\phi \in \mathcal{D}$;

(iii) $(\check{S}*\phi)\vec{T} \in \mathcal{D}'_{L^1}(G)$ for every $\phi \in \mathcal{D}$;

(iv) $S(\hat{x}-\hat{y})\otimes\vec{T}(\hat{y})$ is partially summable with respect to y;

(v) $S(\hat{y})\otimes\vec{T}(\hat{x}-\hat{y})$ is partially summable with respect to y;

(vi) $(\check{S}*\phi)(\vec{T}*\psi) \in L^1\varepsilon G$ for every $\phi, \psi \in \mathcal{D}$.

REMARK 3. Let $\vec{S}=S\otimes e \in \mathcal{D}'(E)$ and let $\vec{T} \in \mathcal{D}'(F)$. If S is composable with $\vec{T}$, then $S\otimes e$ is $*_\theta$-composable with $\vec{T}$, and $(S\otimes e)*_\theta\vec{T} = (I\otimes\tilde{\theta}(e))(S*\vec{T})$, where $\tilde{\theta}(e)\colon f\to\theta(e,f)$ is a continuous linear map of F into G. Indeed, we have

$$\begin{aligned}[(S\otimes e)(\check{\vec{T}}*\phi)]_\theta &= S(I\otimes\tilde{\theta}(e))(\check{\vec{T}}*\phi)\\ &= (I\otimes\tilde{\theta}(e))[S(\check{\vec{T}}*\phi)] \in \mathcal{D}'_{L^1}(G).\end{aligned}$$

Consequently $S\otimes e$ is $*_\theta$-composable with $\vec{T}$. Moreover we have

$$\begin{aligned}\phi\cdot\{(S\otimes e)*_\theta\vec{T}\} &= \int[(S\otimes e)(\check{\vec{T}}*\phi)]_\theta dx\\ &= \int(I\otimes\tilde{\theta}(e))[S(\check{\vec{T}}*\phi)]dx\\ &= \tilde{\theta}(e)\left(\int S(\check{\vec{T}}*\phi)dx\right)\\ &= \tilde{\theta}(e)(\phi\cdot(S*\vec{T}))\end{aligned}$$

$$= \phi\cdot\big(I\otimes\tilde{\theta}(e)\big)\,(S*\vec{T}),$$

from which it follows that $(S\otimes e)*_\theta\vec{T}=\big(I\otimes\tilde{\theta}(e)\big)(S*\vec{T})$, as desired.

§ 3. θ-$\mathscr{S}'$-convolution of two vector valued distributions

Let $\mathscr{S}'$ be the space of tempered distributions, the strong dual of the space $\mathscr{S}$ of rapidly decreasing C^∞-functions defined on R^N. Two tempered distributions S and T were called *$\mathscr{S}'$-composable* ([13], p. 26) provided that

$$(S_x\otimes T_y)\phi(\hat{x}+\hat{y})\in(\mathscr{D}'_{L^1})_{x,y}\quad\text{for every}\quad \phi\in\mathscr{S}.$$

Then the $\mathscr{S}'$-convolution $S*_{\mathscr{S}'}T$ was defined as

$$<S*_{\mathscr{S}'}T,\,\phi>=\iint(S_x\otimes T_y)\phi(x+y)dx\,dy.$$

We can extend this kind of convolution to the vector valued distributions along the same line as in our previous paper [4].

Let E, F, G be three LCSs, G being quasi-complete. Suppose the bilinear map θ: $E\times F\to G$ is separately continuous. $\vec{S}\in\mathscr{S}'(E)$ and $\vec{T}\in\mathscr{S}'(F)$ are called *$*_\theta$-$\mathscr{S}'$-composable* if we have

$$(*)_{\mathscr{S}'}\qquad(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(\hat{x}+\hat{y})\in(\mathscr{D}'_{L^1})_{x,y}(G)\quad\text{for every}\quad\phi\in\mathscr{S}.$$

Then, owing to Corollary to Proposition 2, since $\mathscr{S}$ is barrelled, we can see that the map $\phi\to(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(\hat{x}+\hat{y})$ of $\mathscr{S}$ into $(\mathscr{D}'_{L^1})_{x,y}(G)$ is continuous. Thus when the condition $(*)_{\mathscr{S}'}$ is satisfied, we shall define the *θ-$\mathscr{S}'$-convolution* $\vec{S}*_{\theta,\mathscr{S}'}\vec{T}\in\mathscr{S}'(G)$ as follows:

$$\phi\cdot(\vec{S}*_{\theta,\mathscr{S}'}\vec{T})=\iint(\vec{S}_x\otimes_\theta\vec{T}_y)\phi(x+y)dx\,dy$$

for every $\phi\in\mathscr{S}$.

Comparing the condition $(*)_{\mathscr{S}'}$ with the condition $(*)$ of the preceding section, we see that the θ-$\mathscr{S}'$-convolution, if defined, coincides with the θ-convolution. First we shall show the following lemma guaranteeing that $[(\check{\vec{S}}*\phi)\vec{T}]_\theta$ makes sense for every $\vec{S}\in\mathscr{S}'(E)$, $\vec{T}\in\mathscr{S}'(F)$, $\phi\in\mathscr{S}$.

Lemma 4. *Let $\vec{S}\in\mathscr{S}'(E)$ and let B be a bounded subset of $\mathscr{S}$. Then the set $\{\vec{S}*\phi\}_{\phi\in B}$ is β_0-equibounded in $\mathscr{E}(E)$.*

Proof. $\mathscr{E}'$ and $\mathscr{S}'$ are the spaces of type (**DF**) and the bilinear map $(S,T)\to S*T$ of $\mathscr{E}'\times\mathscr{S}'$ into $\mathscr{S}'$ is hypocontinuous. Therefore it follows from a theorem of Grothendieck ([2], p. 64) that the map is continuous. Then we can find two neighbourhoods $\mathcal{U}$, $\mathcal{V}$ of 0 in $\mathscr{E}'$ and $\mathscr{S}'$ respectively such that $\check{\mathcal{U}}*\mathcal{V}\subset\check{B}^0$, whence $\mathcal{U}*\check{B}\subset\mathcal{V}^0$, which implies that

$$\mathcal{U}\cdot(\vec{S}*B)=(\mathcal{U}*\check{B})\cdot\vec{S}\subset\mathcal{V}^0\cdot\vec{S},$$

from which it follows that $\mathcal{V}^0 \cdot \vec{S}$ is a compact disk of E since $\mathcal{V}^0$ is an equicontinuous subset of $\mathcal{S}$. Consequently the set $\{\vec{S}*\phi\}_{\phi \in B}$ is β_0-equibounded in $\mathcal{E}(E)$. The proof is completed.

Let $\mathcal{O}'_C$ be the space of rapidly decreasing distributions defined on R^N. It may be considered to be the topological linear subspace of $\mathfrak{L}_s(\mathcal{S}, \mathcal{S})$ restricted to the maps $\phi \rightarrow S*\phi$, $\phi \in \mathcal{S}$. Any $\phi \in \mathcal{S}$ can be decomposed into a convolution product of the form $\phi = \phi_1 * \phi_2$, $\phi_1, \phi_2 \in \mathcal{S}$ ([6], p. 530). Let $\mathcal{V}$ be any neighbourhood of 0 in $\mathcal{S}$. Since $\mathcal{S}$ is a space of type (**F**), we can find a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{S}$ such that $\mathcal{U}*\mathcal{U} \subset \mathcal{V}$. Then with the aid of these facts it is easy to check that the convolution map $(S, T) \rightarrow S*T$ of $\mathcal{O}'_C \times \mathcal{O}'_C$ into $\mathcal{O}'_C$ is continuous ([8], Chap. II, p. 104). Using this result we shall show

LEMMA 5. *Let $\vec{S} \in \mathcal{O}'_C(E)$ and let B be a bounded subset of $\mathcal{O}_C$, the strong dual of $\mathcal{O}'_C$. Then the set $\{\vec{S}*\phi\}_{\phi \in B}$ is β_0-equibounded in $\mathcal{O}_C(E)$.*

PROOF. Since the convolution map $\mathcal{O}'_C \times \mathcal{O}'_C \rightarrow \mathcal{O}'_C$ is continuous it follows that a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{O}'_C$ may be chosen so that $\check{\mathcal{U}}*\mathcal{U} \subset \check{B}^0$, whence $\mathcal{U}*\check{B} \subset \mathcal{U}^0$, which implies that

$$\mathcal{U} \cdot (\vec{S}*B) = (\mathcal{U}*\check{B}) \cdot \vec{S} \subset \mathcal{U}^0 \cdot \vec{S}.$$

Since $\mathcal{U}^0$ is an equicontinuous subset of $\mathcal{O}_C$, we see that $\mathcal{U}^0 \cdot \vec{S}$ is a compact disk of E. This shows that $\{\vec{S}*\phi\}_{\phi \in B}$ is β_0-equibounded in $\mathcal{O}_C(E)$. The proof is completed.

REMARK. Schwartz ([12], p. 149) has shown that if $\vec{S} \in \mathcal{D}'(E)$ and B is a bounded subset of $\mathcal{D}$, the set $\{\vec{S}*\phi\}_{\phi \in B}$ is locally β_0-equibounded in $\mathcal{E}(E)$. Another proof of this result may be carried out as in the proofs of our lemmas 4 and 5. In fact, we can find a sequence of positive numbers λ_p and a bounded subset B_1 of $\mathcal{D}$ so that we may have $D^p B \subset \lambda_p B_1$. Let Ω be a relatively compact open subset of R^N and let $\mathcal{V}$ be the neighbourhood of 0 in $\mathcal{E}$ defined by the condition: $\sup_{x \in \bar{\Omega}} |f(x)| \leqq 1$, $f \in \mathcal{E}$. On account of the hypocontinuity of the convolution map of $\mathcal{D}' \times \mathcal{D}$ into $\mathcal{E}$, we can find a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{D}'$ such that $\mathcal{U}*B_1 \subset \mathcal{V}$. Then we have for each p

$$\sup_{x \in \bar{\Omega}} |D^p \mathcal{U}*B| = \sup_{x \in \bar{\Omega}} |\mathcal{U}*D^p B| \leqq \lambda_p.$$

This shows that $\mathcal{U}*B$ is bounded in $\mathcal{E}_{\bar{\Omega}}$. Let α be any element of $\mathcal{D}_{\bar{\Omega}}$. If we put $B_2 = \alpha(\mathcal{U}*B)$, we see that $\mathcal{V} = B_2^0 \cap \mathcal{E}'$ is a neighbourhood of 0 in $\mathcal{E}'$. Now we take a compact disk K of E such that $< \vec{S}, K^0 > \subset \mathcal{U}$. Then we have

$$\begin{aligned} < \alpha(\vec{S}*B) \cdot \mathcal{V}, K^0 > &= | < \alpha(\vec{S}*B), K^0 > \cdot \mathcal{V} | \\ &= | \alpha(< \vec{S}, K^0 > *B) \cdot \mathcal{V} | \\ &= | \alpha(\mathcal{U}*B) \cdot \mathcal{V} | \\ &= | B_2 \cdot B_2^0 | \leqq 1. \end{aligned}$$

Consequently $\alpha(\vec{S}*\phi)$ maps $\mathcal{Q}$ into K whenever ϕ lies in B. Thus the set $\{\vec{S}*\phi\}_{\phi\in B}$ is locally β_0-equibounded.

Suppose that $\vec{S} \in \mathcal{S}'(E)$ and $\vec{T} \in \mathcal{S}'(F)$. Owing to Lemma 4 the multiplicative product $[(\check{\vec{S}}*\phi)\vec{T}]_\theta$ is well defined. We remark that the map $\phi \to [(\check{\vec{S}}*\phi)\vec{T}]_\theta$ is continuous. For when ϕ lies in a bounded subset of $\mathcal{S}$, Lemma 4 together with Lemma 1 shows that the set $[(\check{\vec{S}}*\phi)\vec{T}]_\theta$ is bounded. It follows that the map $\phi \to [(\check{\vec{S}}*\phi)\vec{T}]_\theta$ is continuous since $\mathcal{S}$ is bornological. The following proposition will be proved in an entirely similar way as in Proposition 5.

Proposition 7. *Each of the following conditions is equivalent to the condition* $(*)_{\mathcal{S}'}$:

(i) $[(\check{\vec{S}}*\phi)\vec{T}]_\theta \in \mathcal{D}'_{L^1}(G)$ *for every* $\phi \in \mathcal{S}$;

(ii) $[\vec{S}(\check{\vec{T}}*\phi)]_\theta \in \mathcal{D}'_{L^1}(G)$ *for every* $\phi \in \mathcal{S}$;

(iii) $\vec{S}(\hat{x}-\hat{y})\otimes_\theta \vec{T}(\hat{y}) \in \mathcal{S}'_x((\mathcal{D}'_{L^1})_y(G))$;

(iv) $\vec{S}(\hat{y})\otimes_\theta \vec{T}(\hat{x}-\hat{y}) \in \mathcal{S}'_x((\mathcal{D}'_{L^1})_y(G))$;

(v) $[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta \in L^1\varepsilon G$ *for every* $\phi \in \mathcal{S}$, $\psi \in \mathcal{D}$;

(vi) $[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta \in L^1\varepsilon G$ *for every* $\phi \in \mathcal{D}$, $\psi \in \mathcal{S}$;

(vii) $[(\check{\vec{S}}*\phi)(\vec{T}*\psi)]_\theta \in L^1\varepsilon G$ *for every* $\phi, \psi \in \mathcal{S}$.

Proof. (iii)$\to(*)_{\mathcal{S}'}$: It is known that for any $\phi \in \mathcal{S}$ the linear map $\vec{R} \to \phi\vec{R}$ of $\mathcal{S}'(E)$ into $\mathcal{O}'_C(E)$ is well defined and continuous since the map $R \to \phi R$ of $\mathcal{S}'$ into $\mathcal{O}'_C$ is continuous. Since $\mathcal{O}'_C$ is nuclear, we have

$$(\mathcal{O}'_C)_x \varepsilon (\mathcal{D}'_{L^1})_y = (\mathcal{O}'_C)_x \hat{\otimes}_\pi (\mathcal{D}'_{L^1})_y \subset (\mathcal{D}'_{L^1})_x \hat{\otimes}_\pi (\mathcal{D}'_{L^1})_y = (\mathcal{D}'_{L^1})_{x,y}.$$

Using these inequalities, we can infer that (iii) implies that $(\vec{S}(\hat{x}-\hat{y})\otimes_\theta \vec{T}(\hat{y}))\phi(\hat{x})$ is contained in $(\mathcal{D}'_{L^1})_{x,y}(G)$ for every $\phi \in \mathcal{S}$. Consequently, by change of variables, we obtain the condition $(*)_{\mathcal{S}'}$.

$(*)_{\mathcal{S}'}\to$(i)$\to$(iii), (i)$\to$(v)$\to$(i), $(*)_{\mathcal{S}'}\to$(ii)$\to$(iv)$\to(*)_{\mathcal{S}'}$ and (ii)$\to$(vi)$\to$(ii) are proved in an entirely same way as in the corresponding cases of Proposition 5. Thus the conditions $(*)_{\mathcal{S}'}$, (i), (ii), (iii), (iv), (v) and (vi) are equivalent.

(v)$\to$(vii): (v) implies $[((\vec{S}*\check{\phi})^\vee *\chi)(\vec{T}*\psi)]_\theta = [(\check{\vec{S}}*(\phi*\chi))(\vec{T}*\psi)]_\theta \in L^1\varepsilon G$ for every $\phi, \chi \in \mathcal{S}$ and $\psi \in \mathcal{D}$. It follows from the equivalence of $(*)_{\mathcal{S}'}$ and (v) already proved that $\vec{S}*\check{\phi}$ and $\vec{T}$ are $*_\theta$-$\mathcal{S}'$-composable. Since $(*)_{\mathcal{S}'}$ implies (vi), it follows that $[((\vec{S}*\check{\phi})^\vee *\alpha)(\vec{T}*\psi)]_\theta \in L^1\varepsilon G$ for every $\phi, \psi \in \mathcal{S}$ and $\alpha \in \mathcal{D}$. Now let $\{\rho_n\}$ be a sequence of regularizations. For the proof of (vii) it is sufficient to show that the sequence $\{[((\vec{S}*\check{\phi})^\vee *\rho_n)(\vec{T}*\psi)]_\theta\}_{1\leq n<\infty}$ forms a Cauchy sequence in $L^1\varepsilon G$. Using the closed graph theorem we can infer that the multilinear map $(\phi, \alpha, \psi, g') \to [((\vec{S}*\check{\phi})^\vee *\alpha)(\vec{T}*\psi)]_\theta, g'>$ of $\mathcal{S}_U \times \mathcal{D}_U \times \mathcal{S}_U \times G'_{A'}$ into L^1 is continuous, where U is the closed unit cube of R^N and A' is an equicontinuous disk of G'. Then we can make use of a parmetrix of an iterated Laplacian to conclude the statement.

The converse (vii)$\to$(v) is trivial.

Thus the proof is completed.

As in Proposition 6 we can show

PROPOSITION 8. *Suppose that* $\vec{S} \in \mathcal{S}'(E)$ *and* $\vec{T} \in \mathcal{S}'(F)$ *are* $*_\theta$-$\mathcal{S}'$-*composable. Then the following relations hold*:

$$\text{(i)} \quad \vec{S} *_{\theta, \mathcal{S}'} \vec{T} = \int \vec{S}(\hat{x}-y) \otimes_\theta \vec{T}(y) dy = \int \vec{S}(y) \otimes_\theta \vec{T}(\hat{x}-y) dy;$$

$$\text{(ii)} \quad \phi \cdot (\vec{S} *_{\theta, \mathcal{S}'} \vec{T}) = \int [(\check{\vec{S}} * \phi) \vec{T}]_\theta dx = \int [\vec{S}(\check{\vec{T}} * \phi)]_\theta dx \quad \textit{for every} \quad \phi \in \mathcal{S};$$

$$\text{(iii)} \quad (\phi * \check{\psi}) \cdot (\vec{S} *_{\theta, \mathcal{S}'} \vec{T}) = \int [(\check{\vec{S}} * \phi) (\vec{T} * \psi)]_\theta dx \quad \textit{for every} \quad \phi, \psi \in \mathcal{S}.$$

REMARK. O'_C is the set of distributions composable with every element of $\mathcal{S}'$ ([15], p. 22). However, in the case of the vector valued distributions, any two elements taken from $O'_C(E)$ and $\mathcal{S}'(F)$ respectively are not always $*_\theta$-composable if θ is assumed only to be separately continuous. While $\vec{S} \in O'_C(E)$ and $\vec{T} \in O'_C(F)$ are $*_\theta$-$\mathcal{S}'$-composable. These will be shown in the examples in Section 7.

§ 4. Strict convolution map

Let E, F and G be three LCSs, where G is assumed quasi-complete. Suppose θ is hypocontinuous with respect to the compact disks of E. We denote by θ' the bilinear map of $F \times G'_c$ into E'_c defined by $<\theta(e, f), g'> = <e, \theta'(f, g')>$. θ' is hypocontinuous with respect to the equicontinuous subsets of G'_c. We put $\tilde{\theta}'(g')$: $f \to \theta'(f, g')$, which is a continuous linear map of F into E'_c.

Let $\vec{S} \in \mathcal{D}'(E)$, $\vec{T} \in \mathcal{D}'(F)$, and $\phi \in \mathcal{D}$. $\check{\vec{T}} * \phi$ is locally β_0-bounded in $\mathcal{E}(F)$. If $F = E'_c$ and $\theta(e, e') = <e, e'>$, we write simply $\vec{S}(\check{\vec{T}} * \phi)$ instead of $[\vec{S}(\check{\vec{T}} * \phi)]_\theta$. It is to be noted that the bilinear form $<e, e'>$ on $E \times E'_c$ is hypocontinuous with respect to the compact disks of E and the equicontinuous subsets of E'. Now we can write for a given $\alpha \in \mathcal{D}$

$$\alpha(\check{\vec{T}} * \phi) = \sum_{\nu=1}^{\infty} \lambda_\nu (T_\nu \otimes f_\nu),$$

with T_ν in a compact disk of $\mathcal{D}$ and f_ν in a compact disk of F and $\sum_\nu |\lambda_\nu| < \infty$. The linear map $\{\lambda_\nu\} \to \sum_\nu \lambda_\nu (T_\nu \otimes f_\nu)$ of l^1 into $\mathcal{E}(F)$ transforms the unit ball of l^1 into a β_0-equibounded subset of $\mathcal{E}(F)$. Now using Lemma 1 we have the continuous linear map of l^1 into $\mathcal{D}'(G)$ defined by $\{\lambda_\nu\} \to [\vec{S}(\sum_\nu \lambda_\nu (T_\nu \otimes f_\nu)]_\theta$. Then

$$\alpha[\vec{S}(\check{\vec{T}} * \phi)]_\theta = \sum_\nu < [\vec{S}(\lambda_\nu (T_\nu \otimes f_\nu))]_\theta,$$

therefore

$$\begin{aligned}\alpha<[\vec{S}(\check{\vec{T}}*\phi)]_\theta, g'> &= \sum_\nu <[\vec{S}(\lambda_\nu T_\nu \otimes f_\nu)]_\theta, g'> \\ &= \sum_\nu \vec{S}(\lambda_\nu T_\nu \otimes \theta'(f_\nu, g')) \\ &= \vec{S}(\sum_\nu \lambda_\nu T_\nu \otimes \theta'(f_\nu, g')) \\ &= \alpha\vec{S}((I\otimes\tilde{\theta}'(g'))\check{\vec{T}}*\phi).\end{aligned}$$

Consequently

$$<[\vec{S}(\check{\vec{T}}*\phi)]_\theta, g'> = \vec{S}((I\otimes\tilde{\theta}'(g'))\check{\vec{T}}*\phi). \tag{1}$$

We first show

PROPOSITION 9. *Let $\mathcal{H}$ be a $\dot{\mathcal{B}}$-normal space of distributions. Let $\vec{T}$ be an F-valued distribution. Let u be a continuous linear map of $\mathcal{H}(E)$ into $\mathcal{D}'(G)$ such that the restriction of u to $\mathcal{D}\otimes E$ is of the form $u(\phi\otimes e)=(\phi\otimes e)*_\theta\vec{T}$. If $\mathcal{H}$ or E has the approximation property, then $\vec{T}$ is $*_\theta$-composable with every element $\vec{S}$ of $\mathcal{H}(E)$ and $u(\vec{S})=\vec{S}*_\theta\vec{T}$*

PROOF. Since $E'_c=(E_\gamma)'_c$, $\mathcal{H}(E_\gamma)$ coincides with $\mathcal{H}(E)$, but the former has a finer topology than the latter. If E has the approximation property, so does E_γ. Therefore we may assume that E has a γ-topology ([11], p. 17), then any compact disk of E'_c is an equicontinuous subset of E'.

Now let us denote by Γ the set of $\vec{S}\in\mathcal{H}(E)$ with the following properties:

$$[\vec{S}(\check{\vec{T}}*\phi)]_\theta \in \mathcal{D}'_{L^1}(G) \quad \text{for every} \quad \phi\in\mathcal{D}, \tag{2}$$

$$\phi\cdot u(\vec{S}) = \int [\vec{S}(\check{\vec{T}}*\phi)]_\theta dx \quad \text{for every} \quad \phi\in\mathcal{D}, \tag{3}$$

$$\text{if} \quad \vec{S}\in\Gamma, \quad \text{then} \quad \beta\vec{S}\in\Gamma \quad \text{for every} \quad \beta\in\dot{\mathcal{B}}. \tag{4}$$

Clearly Γ is linear and contains $\mathcal{D}\otimes E$ which is dense in $\mathcal{H}(E)$ since $\mathcal{H}$ or E has the approximation property. It is sufficient to show that Γ is closed in $\mathcal{H}(E)$. Suppose $\vec{S}$ tends to $\vec{S}_0$ in $\mathcal{H}(E)$, and $\vec{S}\in\Gamma$. The properties (2), (3), (4) together with the fact that $\mathcal{H}$ is $\dot{\mathcal{B}}$-normal imply that $\{[\vec{S}(\check{\vec{T}}*\phi)]_\theta\}$ is a Cauchy filter in $\mathcal{D}'_{L^1}(G)$, and therefore converges to an $\vec{X}$ in $\mathcal{D}'_{L^1}(\hat{G})$. On the other hand

$$<[\vec{S}(\check{\vec{T}}*\phi)]_\theta, g'> = \vec{S}((I\otimes\tilde{\theta}'(g'))\check{\vec{T}}*\phi). \tag{5}$$

$\check{\vec{T}}*\phi$ is locally γ-bounded in $\mathcal{E}(F)$. Since $\tilde{\theta}'(g')$ is a continuous map of F into E'_c, it is easy to see that $(I\otimes\tilde{\theta}'(g'))\check{\vec{T}}*\phi$ is locally γ-bounded in $\mathcal{E}(E'_c)$. Since the bilinear form $<e, e'>$ is hypocontinuous with respect to the equicontinuous subsets of E'_c, we can apply Corollary 1 to Proposition 32 of Schwartz [12] (p. 133) to conclude that $\vec{S}((I\otimes\tilde{\theta}'(g'))\check{\vec{T}}*\phi)\to\vec{S}_0((I\otimes\tilde{\theta}'(g'))\check{\vec{T}}*\phi)$ in $\mathcal{D}'$. Therefore (5) yields

$$<\vec{X}, g'> = <[\vec{S}_0(\check{\vec{T}}*\phi)]_\theta, g'>,$$

from which it follows that $\vec{X}=[\vec{S}_0(\check{\vec{T}}*\phi)]_\theta$ in $\mathcal{D}'(\hat{G})$. Let $\{\alpha_n\}$ be a sequence of multiplicators. Then for each α_n

$$\alpha_n[\vec{S}_0(\check{\vec{T}}*\phi)]_\theta \in \mathcal{D}'_{L^1}(G).$$

Passing to the limit as $n\to\infty$, we have $\vec{X}=[\vec{S}_0(\check{\vec{T}}*\phi)]_\theta \in \mathcal{D}'_{L^1}(G)$. Now since $[\vec{S}_0(\check{\vec{T}}*\phi)]_\theta$ is the limit of $\{[\vec{S}(\check{\vec{T}}*\phi)]_\theta\}$ in $\mathcal{D}'_{L^1}(G)$, the properties (2), (3), (4) required for $\vec{S}_0$ are immediate.

Thus the proof is completed.

Now we take F as $\mathfrak{L}_b(E; G)$ and θ as the bilinear map $E\times\mathfrak{L}_b(E; G)$ into G as usual, where b denotes the topology of bounded convergence. Then the map θ is hypocontinuous with respect to the bounded subset of E and the equicontinuous subsets of $\mathfrak{L}_b(E; G)$.

By making use of Proposition 9 we have

THEOREM 1. *Let u be a continuous linear map of $\mathcal{H}(E)$ into $\mathcal{D}'(G)$, where $\mathcal{H}$ is $\mathcal{k}$-normal and $\mathcal{H}$ or E is assumed to have the approximation property. Suppose $\delta \in \mathcal{H}$ or $\mathfrak{L}_b(E; G)$ is sequentially complete. If the restriction of u to $\mathcal{D}\otimes E$ is commutative with any translation, then there exists a unique $\vec{T} \in \mathcal{D}'(\mathfrak{L}_b(E; G))$ such that $\vec{T}$ is $*_\theta$-composable with any element $\vec{S}$ of $\mathcal{H}(E)$ and $u(\vec{S})=\vec{S}*_\theta\vec{T}$.*

PROOF. Let $S\otimes e \in \mathcal{H}\otimes E$ and $\phi \in \mathcal{D}$. Putting $L(S, \phi)e=\phi\cdot u(S\otimes e)$, since u is continuous, it follows that $L(S, \phi) \in \mathfrak{L}(E; G)$. Now if we put $\phi\cdot L(S)=L(S, \phi)$, then the map $L(S)$: $\phi\to L(S, \phi)$ of $\mathcal{D}$ into $\mathfrak{L}_b(E; G)$ is continuous. In fact, let B and C be any bounded subsets of $\mathcal{D}$ and E respectively. When ϕ and e run through B and C respectively, the set $\{(\phi\cdot L(S))e\} = \{\phi\cdot u(S\otimes e)\}$ is bounded in G. Since $\mathcal{D}$ is bornological, it follows that $L(S)$ is continuous, that is, $L(S) \in \mathcal{D}'(\mathfrak{L}_b(E; G))$.

Next we show that the map L: $S\to L(S)$ of $\mathcal{H}$ into $\mathcal{D}'(\mathfrak{L}_b(E; G))$ is continuous. This is easily seen from the equation

$$(\phi\cdot L(S))e = \phi\cdot u(S\otimes e)$$

and from the fact that u is continuous.

We have for any ϕ, $\psi \in \mathcal{D}$

$$\begin{aligned}(\phi\cdot L(\tau_h\psi))e &= \phi\cdot u(\tau_h\psi\otimes e) = \phi\cdot\tau_h u(\psi\otimes e)\\ &= \psi_{\tau_{-h}}\cdot u(\psi\otimes e) = (\phi_{\tau_{-h}}\cdot L(\psi))e\\ &= (\phi\cdot\tau_h L(\psi))e.\end{aligned}$$

Hence $L(\psi)$ is commutative with any translation. Consequently, owing to Proposition 4 there exists a unique distribution $\vec{T} \in \mathcal{D}'(\mathfrak{L}_b(E; G)^\frown)$ such that $L(S)=S*\vec{T}$ for every $S \in \mathcal{H}$. When $\delta \in \mathcal{H}$, then $\vec{T}=\delta*\vec{T}=L(\delta) \in \mathcal{D}'(\mathfrak{L}_b(E; G))$. Next suppose $\mathfrak{L}_b(E; G)$ is sequentially complete. Let $\{\rho_n\}$ be a sequence of regularizations. Then $\phi\cdot(\rho_n*\vec{T})=\check{\rho}_n\cdot(\check{\phi}*\vec{T}) \in \mathfrak{L}_b(E; G)$ for each n. Hence we have

$\phi\cdot\vec{T}=\lim_{n\to\infty}\check{\rho}_n\cdot(\check{\phi}*\vec{T})\in\mathfrak{L}_b(E;G)$. Therefore $\vec{T}\in\mathcal{D}'(\mathfrak{L}_b(E;G))$.

Next we have for any ϕ, $\psi\in\mathcal{D}$

$$\begin{aligned}\phi\cdot u(\psi\otimes e)&=(\phi\cdot L(\psi))e=\theta(e,\phi\cdot(\psi*\vec{T}))\\&=\phi\cdot(I\otimes\tilde{\theta}(e))(\psi*\vec{T})=\phi\cdot((\psi\otimes e)*_\theta\vec{T}).\end{aligned}$$

Consequently

$$u(\psi\otimes e)=(\psi\otimes e)*_\theta\vec{T}.$$

Hence we can apply the preceding proposition 9 to conclude our statements of the theorem. The proof is completed.

Here we note that when E and G are Banach spaces and $\mathcal{H}$ is any of $\mathcal{E}'$, $\mathcal{D}'$ and $\mathcal{D}'_{L^1}$, the theorem was proved by Lions ([5], p. 150).

If $\vec{T}\in\mathcal{D}'(F)$ is $*_\theta$-composable with any $\vec{S}\in\mathcal{H}(E)$ and moreover if the linear map u: $\vec{S}\to\vec{S}*_\theta\vec{T}$ is continuous from $\mathcal{H}(E)$ into a space of vector valued distributions $\mathcal{K}(G)$, we shall say that u is a *strict convolution map* of $\mathcal{H}(E)$ into $\mathcal{K}(G)$.

Lemma 6. *If $E\varepsilon F$ is barrelled, then E and F are also barrelled.*

Proof. Let A' be a weakly bounded subset of E'. Let f' be any fixed non-zero element of F' and choose $f\in F$ in such a way that $\langle f,f'\rangle=1$. For any $\xi\in E\varepsilon F$, ξ is considered to be a bilinear form on $E'_c\times F'_c$. Now $\xi(e',f')=\langle{}^t\xi(f'),e'\rangle$. This shows that the set $\xi(e',f')$ with e' in A' is bounded. Since, by assumption, $E\varepsilon F$ is barrelled, there exists a neighbourhood W of 0 in $E\varepsilon F$ such that

$$|\xi(e',f')|\leqq1\quad\text{for}\quad e'\in A',\ \xi\in W.$$

Let U be a neighbourhood of 0 in E such that $U\otimes f\subset W$. Then, for any $e\in U$ and $e'\in A'$, we have

$$|\langle e,e'\rangle|=|(e\otimes f)(e',f')|\leqq1.$$

This proves that E is barrelled. Similarly we see that F is barrelled.

We note that if E is a Banach space and if $\mathcal{H}$ is nuclear, then $\mathcal{H}(E)$ is barrelled when and only when $\mathcal{H}$ is barrelled.

Lemma 7. *Let $\vec{S}\in\mathcal{D}'(E)$, $\vec{T}\in\mathcal{D}'(F)$ and $\phi\in\mathcal{D}$. If E is barrelled, the map $\vec{S}\to[\vec{S}(\check{\vec{T}}*\phi)]_\theta$ of $\mathcal{D}'(E)$ into $\mathcal{D}'(G)$ is continuous.*

Proof. Since E is barrelled, the bilinear map θ is hypocontinuous with respect to the bounded subsets of F. Then Corollary 1 to Proposition 32 of Schwartz [12] (p. 133) shows that the map $\vec{S}\to[\vec{S}(\check{\vec{T}}*\phi)]_\theta$ of $\mathcal{D}'(E)$ into $\mathcal{D}'(G)$ is continuous.

By virtue of these lemmas we can show

PROPOSITION 10. *Let $\mathcal{H}$ be a space of distributions. Let $\mathcal{K}$ be a normal space of distributions with $\mathcal{D}$ strictly dense in $\mathcal{K}'_\sigma$. Suppose that $\vec{T} \in \mathcal{D}'(F)$ is $*_\theta$-composable with any $\vec{S} \in \mathcal{H}(E)$ and that $\vec{S} *_\theta \vec{T} \in \mathcal{K}(G)$. If $\mathcal{H}(E)$ is barrelled, the linear map $\vec{S} \to \vec{S} *_\theta \vec{T}$ of $\mathcal{H}(E)$ into $\mathcal{K}(G)$ is continuous.*

PROOF. Since E becomes barrelled by Lemma 6, the linear map $\vec{S} \to [\vec{S}(\check{\vec{T}} * \phi)]_\theta$ of $\mathcal{H}(E)$ into $\mathcal{D}'(G)$ is continuous by Lemma 7. But, by assumption, $[\vec{S}(\check{\vec{T}} * \phi)]_\theta \in \mathcal{D}'_{L^1}(G)$, whence, by Corollary to Proposition 2, the map $\vec{S} \to [\vec{S}(\check{\vec{T}} * \phi)]_\theta$ of $\mathcal{H}(E)$ into $\mathcal{D}'_{L^1}(G)$ is continuous. Therefore the map $\vec{S} \to \vec{S} *_\theta \vec{T}$ of $\mathcal{H}(E)$ into $\mathcal{D}'(G)$ is also continuous. Using again the corollary cited above one can conclude that the map $\vec{S} \to \vec{S} *_\theta \vec{T}$ of $\mathcal{H}(E)$ into $\mathcal{K}(G)$ is continuous, which completes the proof.

§ 5. Strict convolution map between two spaces of vector valued distributions (elementary case)

This section and the next are devoted to the investigations on the behavior of strict convolution maps between two spaces of vector valued distributions concerning the assigned continuity. Let us denote by $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ normal spaces of distributions and by E, F, G the locally convex spaces. We assume that G is quasi-complete. θ will stand for a separately continuous bilinear map of $E \times F$ into G. Recall that a convolution map $*$ of $\mathcal{H} \times \mathcal{K}$ into $\mathcal{L}$ is, by definition, a separately continuous bilinear map, coinciding on $\mathcal{D} \times \mathcal{D}$ with the usual convolution. Here, if we further assume that $\mathcal{H}$ or $\mathcal{K}$ is $\dot{\mathcal{B}}$-normal, the convolution map is strict in our sense (see Proposition 3).

In this section we shall deal with the case called elementary where $*$ or θ is continuous. For our later purpose we need the following lemmas.

LEMMA 8. *Let E, F, G be three LCSs. Let θ be a separately continuous bilinear map of $E \times F$ into G. If A is a bounded subset of E and B is a bounded completing subset of F, then $\theta(A, B)$ is bounded in G.*

PROOF. There exists an absolutely convex bounded subset $B_1 \supset B$ of F such that F_{B_1} is a Banach space with B_1 as the unit ball. The restriction of θ to $E \times F_{B_1}$ becomes hypocontinuous with respect to the bounded subsets of E. Hence it follows that $\theta(A, B)$ is bounded in G, which was to be proved.

LEMMA 9. *Let $\mathcal{H}$ be a $\dot{\mathcal{B}}$-normal space of distributions and E, F, G be LCSs, G being assumed to be quasi-complete. Let θ be a separately continuous bilinear map of $E \times F$ into G. Further we assume that $\mathcal{H} \otimes E$ is strictly dense in $\mathcal{H}(E)$. Let $\vec{T} \in \mathcal{H}'_c(F)$ be a nuclear linear map of $\mathcal{H}$ into F, that is, we can write*

$$\vec{T} = \sum_j \lambda_j h'_j \otimes f_j,$$

with h'_j in an equicontinuous $\mathcal{U}^0$ in $\mathcal{H}'_c$, $\mathcal{U}$ being a neighbourhood of 0 in $\mathcal{H}$ and f_j in a compact disk $\mathfrak{f}$ of F and $\sum_j |\lambda_j| < \infty$. Then for any $\vec{S} \in \mathcal{H}(E)$,

(i) $\Phi(\vec{S}, \vec{T})=\sum_j \lambda_j <h'_j\vec{S}, f_j>_\theta$ *converges in* $\mathcal{D}'_{L^1}(G)$, *where* $<h'\vec{S}, f>_\theta=(I\otimes\tilde{\theta}(f))\,(h'\vec{S})$. *And* $\Phi(\vec{S}, \vec{T})$ *is independent of the representations of* $\vec{T}$;

(ii) *Moreover if* $\vec{T}$ *is locally* β_0*-bounded in* $\mathcal{E}(F)$, *then* $\Phi(\vec{S}, \vec{T})=[\vec{S}\vec{T}]_\theta(=(I\otimes\tilde{\theta})[\vec{S}\vec{T}]_\iota)$, *where* $[\vec{S}\vec{T}]_\iota$ *denotes the multiplicative product of* $\vec{S}$ *and* $\vec{T}$ *in the sense of Schwartz* ([12], p. 134).

Proof. (i): Any element h' of $\mathcal{H}'$ is a multiplicator of $\mathcal{H}$ into $\mathcal{D}'_{L^1}$ since $\mathcal{H}$ is $\dot{\mathcal{B}}$-normal. It follows that $h'\vec{S}\in\mathcal{D}'_{L^1}(E)$ for $\vec{S}\in\mathcal{H}(E)$. And it is easy to see that if h' runs through an equicontinuous subset $\mathcal{U}^0$ of $\mathcal{H}'_c$ the set $\{h'\vec{S}\}$ is bounded in $\mathcal{D}'_{L^1}(E)$. Since $\mathfrak{f}$ is a compact disk of F, it follows from Lemma 8 that the set $\{<h'S, f>_\theta\}_{h'\in\mathcal{U}^0, f\in\mathfrak{f}}$ is bounded in $\mathcal{D}'_{L^1}(G)$. Therefore the series $\Phi(\vec{S}, \vec{T})$ converges in $\mathcal{D}'_{L^1}(G)$. Next we shall prove that $\Phi(\vec{S}, \vec{T})$ is independent of the representations of $\vec{T}$. To this end it is sufficient to prove the case $\vec{T}=0$. We note that the linear map $\vec{S}\to\Phi(\vec{S}, \vec{T})$ of $\mathcal{H}(E)$ into $\mathcal{D}'_{L^1}(G)$ is *quasi-continuous*, that is, the map is continuous on any bounded subset of $\mathcal{H}(E)$. Let $\vec{S}\to 0$ on a bounded subset of $\mathcal{H}(E)$. Then for any h' and f, $\{<h'\vec{S}, f>_\theta\}$ is bounded and converges to 0 in $\mathcal{D}'_{L^1}(G)$. In fact, let β run through a bounded subset of $\dot{\mathcal{B}}$. The equations $\beta\cdot<h'\vec{S}, f>_\theta=\theta(\beta\cdot h'\vec{S}, f)=\theta(h'\beta\cdot\vec{S}, f)$ show that since the set $\{h'\beta\}$ is equicontinuous in $\mathcal{H}'_c$, $\theta(h'\beta\cdot\vec{S}, f)\to 0$ in G as $\vec{S}\to 0$ and the set $\{h'\beta\cdot\vec{S}\}$ is relatively compact. Hence by Lemma 8 the set $\{\theta(h'\beta\cdot\vec{S}, f)\}$ is bounded in G. First suppose that $\vec{S}$ is decomposable: $\vec{S}=S\otimes e$, $S\in\mathcal{H}$, $e\in E$. The linear map $h'\to Sh'$ of $\mathcal{H}'_c$ into $\mathcal{D}'$ is continuous. Therefore the map $\vec{T}\to S\vec{T}$ of $\mathcal{H}'_c(F)$ into $\mathcal{D}'(F)$ is continuous. Hence we have

$$\sum_j \lambda_j Sh'_j\otimes f_j=S\sum_j \lambda_j h'_j\otimes f_j=S\vec{T}=0.$$

Consequently

$$\begin{aligned}\Phi(S\otimes e, \vec{T})&=\sum_j \lambda_j Sh'_j\otimes\theta(e, f_j)\\&=<e, \sum_j \lambda_j Sh'_j\otimes f_j>_\theta=0.\end{aligned}$$

Next consider the general case where $\vec{S}$ is any element of $\mathcal{H}(E)$. Since the map $\vec{S}\to\Phi(\vec{S}, \vec{T})$ is quasi-continuous and $\mathcal{H}\otimes E$ is, by assumption, strictly dense in $\mathcal{H}(E)$, we have $\Phi(\vec{S}, \vec{T})=0$, as desired.

(ii): Finally, let $\vec{T}$ be locally β_0-bounded in $\mathcal{E}(F)$. Then for any $\alpha\in\mathcal{D}$, $\alpha\vec{T}$ can be written as follows:

$$\alpha\vec{T}=\sum_j \mu_j\alpha_j\otimes\tilde{f}_j,$$

with α_j in an equicontinuous subset of $\mathcal{D}$ and $\tilde{f}_j$ in a compact disk of F and $\sum_j|\mu_j|<\infty$. On the other hand, $\alpha\vec{T}=\alpha\sum_j\lambda_j h'_j\otimes f_j=\sum_j\lambda_j\alpha h'_j\otimes f_j$. Because of the uniqueness of $\Phi(\vec{S}, \vec{T})$ proved above, we have

$$\alpha\Phi(\vec{S}, \vec{T})=\Phi(\vec{S}, \alpha\vec{T})=[\vec{S}(\alpha\vec{T})]_\theta=\alpha[\vec{S}\vec{T}]_\theta.$$

Consequently $\Phi(\vec{S}, \vec{T})=[\vec{S}\vec{T}]_\theta$. The proof is completed.

REMARK. In the preceding lemma we assumed that $\mathscr{H}\otimes E$ is strictly dense in $\mathscr{H}(E)$. If we assume instead that $\mathscr{H}\otimes E$ is dense in $\mathscr{H}(E)$ and that θ is hypocontinuous with respect to the compact disks of F, the conclusions of the preceding lemma remains valid. This is because that in virtue of the expression $\Phi(\vec{S}, \vec{T})$ the linear map $\vec{S}\to\Phi(\vec{S}, \vec{T})$ of $\mathscr{H}(E)$ into $\mathscr{D}'_{L^1}(G)$ will become continuous as seen from the proof of our lemma.

We shall now consider the case where the convolution map $*: \mathscr{H}\times\mathscr{K}\to\mathscr{L}$ is continuous.

THEOREM 2. *Let $\mathscr{H}$, $\mathscr{K}$, $\mathscr{L}$ be three normal spaces of distributions, $\mathscr{L}$ being assumed to be complete and E, F, G be three LCSs, G being assumed to be quasi-complete. We assume that $\mathscr{H}$ is nuclear, $\dot{\mathscr{B}}$-normal and $\mathscr{H}\otimes E$ is strictly dense in $\mathscr{H}(E)$. Suppose that the convolution map $*: \mathscr{H}\times\mathscr{K}\to\mathscr{L}$ is continuous and that the bilinear map $\theta: E\times F\to G$ is separately continuous. Then any $\vec{S}\in\mathscr{H}(E)$ and $\vec{T}\in\mathscr{K}(F)$ are $*_\theta$-composable and $\vec{S}*_\theta\vec{T}\in\mathscr{L}(G)$.*

(a) *The linear map $\vec{T}\to\vec{S}*_\theta\vec{T}$ of $\mathscr{K}(F)$ into $\mathscr{L}(G)$ is quasi-continuous. Moreover if $\mathscr{L}'_c$ is barrelled, the linear map $\vec{S}\to\vec{S}*_\theta\vec{T}$ of $\mathscr{H}(E)$ into $\mathscr{L}(G)$ is also quasi-continuous.*

(b) *If θ is hypocontinuous with respect to the compact disks of E, then the linear map $\vec{T}\to\vec{S}*_\theta\vec{T}$ is uniformly continuous with respect to the equicontinuous subsets of $\mathfrak{L}(E'_c; \mathscr{H})$.*

(c) *If θ is hypocontinuous with respect to the compact disks of F, then the linear map $\vec{S}\to\vec{S}*_\theta\vec{T}$ is uniformly quasi-continuous with respect to the equicontinuous subsets of $\mathfrak{L}(F'_c; \mathscr{K})$. Further, if F is quasi-complete, any compact subset of $\mathscr{K}(F)$ is an equicontinuous subset of $\mathfrak{L}(F'_c; \mathscr{K})$* ([11], p. 22), *therefore the linear map $\vec{S}\to\vec{S}*_\theta\vec{T}$ is uniformly quasi-continuous with respect to the compact subsets of $\mathscr{K}(F)$.*

(d) *If θ is hypocontinuous with respect to the bounded subsets of E and F, then so is $*_\theta$.*

Finally,

(e) *If θ is continuous, then so is $*_\theta$.*

PROOF. Let $\mathcal{W}$ be any disked neighbourhood of 0 in $\mathscr{L}$. Since the convolution map $*: \mathscr{H}\times\mathscr{K}\to\mathscr{L}$ is continuous, we can find two disked neighbourhoods $\mathscr{U}$ and $\mathcal{V}$ of 0 in $\mathscr{H}$ and $\mathscr{K}$ respectively such that $\mathscr{U}*\mathcal{V}\subset\mathcal{W}$. Now we choose a compact disk $\mathfrak{f}$ in F such that the image of $\mathfrak{f}^0$ by the map $\vec{T}$ is contained in $\mathcal{V}$. Since $\mathscr{H}$ is assumed to be nuclear, there exists a disked neighbourhood $\tilde{\mathscr{U}}$ of 0 in $\mathscr{H}$ such that the map $I: \hat{\mathscr{H}}_{\tilde{\mathscr{U}}}\to\hat{\mathscr{H}}_{\mathscr{U}}$ is nuclear. Here I is of the form:

$$\sum_{j=1}^{\infty}\lambda_j h'_j\otimes h_j,$$

where $h'_j\in\tilde{\mathscr{U}}^0$, $h_j\in\mathscr{U}$ and $\sum_j|\lambda_j|<\infty$. The bilinear map $(h, l')\to h*\check{l}'$ of $\mathscr{H}\times\mathscr{L}'_c$

into $\mathcal{K}'_c$, as the transposed map of the convolution map $\mathcal{H}\times\mathcal{K}\to\mathcal{L}$, is a strict convolution map since $\mathcal{H}$ is $\dot{\mathcal{B}}$-normal. Let $l'\in\mathcal{W}^0$ and $\vec{T}\in\mathcal{K}(F)$. Putting $<\check{\vec{T}}*_1 l', h> = <\check{\vec{T}}, h*\check{l}'>$, $\check{\vec{T}}*_1 l'\in\mathcal{H}'_c(F)$ is considered as a continuous map of $\mathcal{H}$ into F factorized as follows:

$$\mathcal{H}\xrightarrow{i_1}\hat{\mathcal{H}}_{\tilde{\mathcal{U}}}\xrightarrow{I}\hat{\mathcal{H}}_{\mathcal{U}}\xrightarrow{*\check{l}'}\mathcal{K}'_{\mathcal{V}^0}\xrightarrow{\check{\vec{T}}}F_{\mathfrak{f}}\xrightarrow{i_2}F,$$

where i_1, i_2 are the canonical maps and $*\check{l}'$ is the continuous extension of the convolution map $h\to h*\check{l}'$ of $\mathcal{H}_{\mathcal{U}}$ into $\mathcal{K}'_{\mathcal{V}^0}$. Therefore $\check{\vec{T}}*_1 l'$ can be written as

$$\check{\vec{T}}*l' = \sum_{j=1}^{\infty}\lambda_j h'_j\otimes f_{j,\vec{T},l'}, \tag{1}$$

where $f_{j,\vec{T},l'}=(h_j*\check{l}')\cdot\check{\vec{T}}\subset\mathcal{W}^0\cdot\vec{T}\subset\mathfrak{f}$, $h'_j\in\tilde{\mathcal{U}}^0$ and $\sum_j|\lambda_j|<\infty$. According to Lemma 9

$$\Phi(\vec{S},\check{\vec{T}}*_1 l')=\sum_j\lambda_j<h'_j\vec{S}, f_{j,\vec{T},l'}>_\theta\in\mathcal{D}'_{L^1}(G)$$

for every $\vec{S}\in\mathcal{H}(E)$. In particular, $\check{\vec{T}}*\phi$, $\phi\in\mathcal{D}$, is locally β_0-bounded in $\mathcal{E}(F)$, so that $\Phi(\vec{S},\check{\vec{T}}*\phi)=[\vec{S}(\check{\vec{T}}*\phi)]_\theta\in\mathcal{D}'_{L^1}(G)$. Therefore, by definition, $\vec{S}$ and $\vec{T}$ are $*_\theta$-composable.

We shall next prove that if $\mathcal{L}$ is complete, $\vec{S}*_\theta\vec{T}\in\mathcal{L}(G)$. To this end we put

$$\Psi(l')=\int\Phi(\vec{S},\check{\vec{T}}*_1 l')dx=\sum_j\lambda_j\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'}).$$

Then the map $l'\to\Psi(l')$ of $\mathcal{W}^0\subset\mathcal{L}'_c$ into G is continuous. In fact, $f_{j,\vec{T},l'}=(h_j*\check{l}')\cdot\check{\vec{T}}=l'\cdot(h_j*\vec{T})$ tends to 0 for each j when $l'\to 0$ in $\mathcal{W}^0$, and the set $\{\theta(h'_j\cdot\vec{S}, f_{j,T,l'})\}_{h'_j\in\tilde{\mathcal{U}}^0, l'\in\mathcal{W}^0}$ is bounded in G. According to Proposition 8 of Schwartz [11] (p. 41) asserting that a linear map of $\mathcal{L}'_c$ into an LCS is continuous, when $\mathcal{L}$ is complete and the map is continuous on any equicontinuous subset of $\mathcal{L}'_c$, the map $l'\to\Psi(l')$ is continuous from $\mathcal{L}'_c$ into G. We can conclude that there exists an element $\vec{\Psi}\in\mathcal{L}(G)$ such that $\Psi(l')=l'\cdot\vec{\Psi}$ for every $l'\in\mathcal{L}'_c$. Now l' may be chosen an arbitrary element of $\mathcal{D}$, so that we can write $l'\cdot\vec{\Psi}=l'\cdot(\vec{S}*_\theta\vec{T})$, and therefore $\vec{\Psi}=\vec{S}*_\theta\vec{T}\in\mathcal{L}(G)$, as desired.

From now on let $\mathcal{U}$, $\tilde{\mathcal{U}}$, $\mathcal{V}$, $\mathcal{W}$ be chosen as before. Here we note that $\mathcal{W}$ may be chosen an arbitrary fixed disked neighbourhood of 0 in $\mathcal{L}$.

(a): Suppose that $\vec{T}$ converges to 0 in $\mathcal{K}(F)$ lying in a bounded subset $\mathfrak{B}$ of $\mathcal{K}(F)$. We shall first show that $\vec{S}*_\theta\vec{T}\to 0$ in $\mathcal{L}(G)$ for every fixed $\vec{S}\in\mathcal{H}(E)$. It is known that any bounded subset of $\mathcal{K}(F)$ is an equicontinuous subset of $\mathfrak{L}(F'_b;\mathcal{K})$ ([11], p. 28). Therefore we can choose a bounded disk $\mathfrak{f}_1$ of F so that any $\vec{T}\in\mathfrak{B}$ maps $\mathfrak{f}_1^0$ into $\mathcal{W}$. As before, we write

$$\check{\vec{T}}*l' = \sum_{j=1}^{\infty}\lambda_j h'_j\otimes f_{j,\vec{T},l'},$$

where the set $\{f_{j,\vec{T},l'}\}$ is contained in $\mathfrak{f}_1$. For each j, $\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'})$ converges to 0 as $\vec{T}\to 0$ uniformly with respect to $l' \in \mathcal{W}^0$. And the set $\{\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'})\}_{h'_j\in\mathscr{U}^0,\vec{T}\in\mathfrak{B},l'\subset\mathscr{W}^0}$ becomes bounded by Lemma 8. Taking into account the equation $<\vec{S}*_\theta\vec{T}, l'> = \sum_j \lambda_j\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'})$, we see that $\vec{S}*_\theta\vec{T}\to 0$ when $\vec{T}\to 0$ in $\mathfrak{B}$.

Next suppose that $\mathscr{L}'_c$ is barrelled. Let $\vec{T}$ be fixed and let $\vec{S}$ converges to 0 in a bounded subset $\mathfrak{B}_1$ of $\mathscr{H}(E)$. We shall prove that $\vec{S}*_\theta\vec{T}\to 0$ in $\mathscr{L}(G)$. To this end, in view of the Banach-Steinhaus theorem it is sufficient to show that $<\vec{S}*_\theta\vec{T}, l'>$ converges boundedly to 0 for every fixed l', which may be assumed to be an element of $\mathcal{W}^0$ since we can take $\mathcal{W}^0$ such that $l' \in \mathcal{W}^0$ for a given l'. Also we may assume that $\vec{T}*_1 l'$ is of the form (1). Then we see that $\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'})$ converges boundedly to 0 for each j when $\vec{S}\to 0$ in $\mathfrak{B}_1$, and that the set $\{\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'})\}_{h'_j\in\mathscr{U}^0,\vec{S}\in\mathfrak{B}_1,f_{j,\vec{T},l'}\in\mathfrak{f}_1,}$ is bounded. Then, owing to the equation $<\vec{S}*_\theta\vec{T}, l'> = \sum_j \lambda_j\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'})$, we can conclude that $<\vec{S}*_\theta\vec{T}, l'>$ converges boundedly to 0 in G, as desired.

(b): Suppose that θ is hypocontinuous with respect to the compact disks of E. Let $\vec{S}$ lie in an equicontinuous subset $\mathfrak{A}$ of $\mathfrak{L}(E'_c; \mathscr{H})$. Then there exists a compact disk $\mathfrak{k}$ of E such that each $<\vec{S}, \mathfrak{k}^0>$ is contained in $\tilde{\mathscr{U}}$. By our assumption on θ we can find a neighbourhood V of 0 in such a way that $\theta(\mathfrak{k}, V) \subset W$ for a given neighbourhood W of 0 in G. Now consider the set $\boldsymbol{V}$ of the elements $\vec{T} \in \mathscr{K}(F)$ such that $\mathcal{W}^0\cdot\vec{T} \subset V$. $\boldsymbol{V}$ is, by definition, a neighbourhood of 0 in $\mathscr{K}(F)$. Then $h'_j\cdot\vec{S} \in \mathfrak{k}$ and $f_{j,\vec{T},l'} \in \mathcal{W}^0\cdot\vec{T} \subset V$ for $\vec{S} \in \mathfrak{A}$, $\vec{T} \in \boldsymbol{V}$ and $l' \in \mathcal{W}^0$. Therefore we have for every $\vec{S} \in \mathfrak{A}$, $\vec{T} \in \boldsymbol{V}$, $l' \in \mathcal{W}^0$

$$<\vec{S}*_\theta\vec{T}, l'> = \sum \lambda_j\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'}) \in \sum_j |\lambda_j| W,$$

which implies that the map $\vec{T}\to\vec{S}*_\theta\vec{T}$ is uniformly continuous with respect to the equicontinuous subsets of $\mathfrak{L}(E'_c; \mathscr{H})$.

(c): Suppose that θ is hypocontinuous with respect to the compact disks of F. Let $\vec{S}\to 0$ in a bounded subset $\mathfrak{B}$ of $\mathscr{H}(E)$ and let $\vec{T}$ lie in an equicontinuous subset $\mathfrak{A}$ of $\mathfrak{L}(F'_c; \mathscr{K})$. Then we can find a compact disk $\mathfrak{f}$ of F such that $<\vec{T}, \mathfrak{f}^0> \subset \mathcal{V}$ for every $\vec{T} \in \mathfrak{A}$. We may assume that $\vec{T}*_1 l'$ is of the form (1) with $f_{j,T,l'} \in \mathfrak{f}$. By our assumption on θ, $\theta(h'_j\cdot\vec{S}, f_{j,\vec{T},l'})\to 0$ in G for each j as $\vec{S}\to 0$ in $\mathfrak{B}$ and the set $\{\theta(h'_j\cdot\vec{S}, f_{j,\vec{T}l'})\}_{h'\in\mathscr{U}^0,\vec{S}\in\mathfrak{B},\vec{T}\in\mathfrak{A},l'\in\mathscr{W}^0}$ is bounded in G. Therefore $\vec{S}*_\theta\vec{T}\to 0$ uniformly in $\mathscr{L}(G)$ when $\vec{T}$ lies in $\mathfrak{A}$.

(d): Suppose that θ is hypocontinuous with respect to the bounded subsets of E and F. It is known that any bounded subset of $\mathscr{H}(E)$ (resp. $\mathscr{K}(F)$) is an equicontinuous subset of $\mathfrak{L}(E'_b; \mathscr{H})$ (resp. $\mathfrak{L}(F'_b; \mathscr{K})$) ([11], p. 28). From this fact together with our assumption on θ we can conclude just as in (b), (c) that the bilinear map $*_\theta$: $\mathscr{H}(E)\times\mathscr{K}(F)\to\mathscr{L}(G)$ becomes hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and $\mathscr{K}(F)$.

(e): Suppose that θ is continuous. Let W be a neighbourhood of 0 in G, then there exist two neighbourhoods U and V of 0 in E and F respectively such that $\theta(U, V)\subset W$. Let $\boldsymbol{U}$ (resp. $\boldsymbol{V}$) be a neighbourhood of 0 in $\mathscr{H}(E)$ (resp.

$\mathscr{K}(F)$) such that $\mathcal{U}^0 \cdot \boldsymbol{U} \subset U$ (resp. $\mathcal{V}^0 \cdot \boldsymbol{V} \subset V$). Then we have for $\vec{S} \in \boldsymbol{U}$, $\vec{T} \in \boldsymbol{V}$, $l' \in \mathcal{V}^0$

$$<\vec{S} *_\theta \vec{T}, l'> = \sum_j \lambda_j \theta(h'_j \cdot \vec{S}, f_{j, \vec{T}, l'}) \in \sum_j |\lambda_j| W,$$

which shows that the bilinear map $*_\theta$ is continuous.

Thus the proof is completed.

Remark. It is shown that $\mathscr{L}'_c$ is a space of type (β), *a fortiori* barrelled, if $\mathscr{L}$ is a complete Schwartz space ([11], p. 43, [7], p. 431).

Next we turn to the case where θ is continuous.

Theorem 3. *Let $\mathscr{H}$, $\mathscr{K}$, $\mathscr{L}$ be three normal spaces of distributions on R^N. Let E, F, G be three LCSs. We assume that $\mathscr{L}$, G are quasi-complete. Further we assume that $\mathscr{H}$ is nuclear and $\dot{\mathscr{B}}$-normal. Suppose the convolution map $(S, T) \to S*T$ of $\mathscr{H} \times \mathscr{K}$ into $\mathscr{L}$ is defined and γ-continuous, that is, hypocontinuous with respect to the compact disks of $\mathscr{H}$ and $\mathscr{K}$. Let θ be a continuous bilinear map of $E \times F$ into G.*

(a) *If $\mathscr{L}$ and G are complete, or if $\mathscr{H}$ or E has the strict approximation property, then any $\vec{S} \in \mathscr{H}(E)$ and $\vec{T} \in \mathscr{K}(F)$ are $*_\theta$-composable and $\vec{S} *_\theta \vec{T} \in \mathscr{L}(G)$, where the map $\vec{S} \to \vec{S} *_\theta \vec{T}$ of $\mathscr{H}(E)$ into $\mathscr{L}(G)$ is continuous. If we further assume that $\mathscr{H}'_c$ is nuclear, then the bilinear map $(\vec{S}, \vec{T}) \to \vec{S} *_\theta \vec{T}$ of $\mathscr{H}(E) \times \mathscr{K}(F)$ into $\mathscr{L}(G)$ is hypocontinuous with respect to the bounded subsets of $\mathscr{H}(E)$ and the compact subsets of $\mathscr{K}(F)$ whenever $\mathscr{H}$, $\mathscr{K}$ are quasi-complete.*

(b) *If the convolution map of $\mathscr{H} \times \mathscr{K}$ into $\mathscr{L}$ is β-continuous, that is, hypocontinuous with respect to the bounded subsets of $\mathscr{H}$ and $\mathscr{K}$, and if $\mathscr{L}$ and G are complete or if $\mathscr{H}$ or E has the strict approximation property, and if $\mathscr{H}$ is quasi-complete and $\mathscr{H}'_c$ is nuclear, then the bilinear map $(\vec{S}, \vec{T}) \to \vec{S} *_\theta \vec{T}$ of $\mathscr{H}(E) \times \mathscr{K}(F)$ into $\mathscr{L}(G)$ is β-continuous.*

Proof. Let $\vec{T} \in \mathscr{K}(F)$. First we define the map: $S \to S *_1 \vec{T}$ of $\mathscr{H}$ into $\mathscr{L}(F)$ as follows: $<S *_1 \vec{T}, f'> = S * <\vec{T}, f'>$ for any $f' \in F'$. Since the convolution map of $\mathscr{H} \times \mathscr{K}$ into $\mathscr{L}$ is γ-continuous, it is easy to verify that the linear map $S \to S *_1 \vec{T}$ of $\mathscr{H}$ into $\mathscr{L}(F)$ is continuous. By our assumption $\mathscr{H}$ is $\dot{\mathscr{B}}$-normal. According to Proposition 4 any $S \in \mathscr{H}$ is composable with $\vec{T}$ and $S *_1 \vec{T} = S * \vec{T}$. Then from Remark 3 in Section 2 it follows that $S \otimes e$ is $*_\theta$-composable with $\vec{T}$ and $(S \otimes e) *_\theta \vec{T} = (I \otimes \tilde{\theta}(e))(S * \vec{T})$. Thus we have a bilinear map $(S, e) \to (S \otimes e) *_\theta \vec{T}$ of $\mathscr{H} \times E$ into $\mathscr{L}(G)$. Now we show that the bilinear map thus obtained is continuous. To this end let A' be any equicontinuous subset of G'. For any $g' \in A'$ we have

$$(1) \qquad \begin{aligned} <(S \otimes e) *_\theta \vec{T}, g'> &= (I \otimes \tilde{\theta}'(g')) \circ (I \otimes \tilde{\theta}(e))\,(S * \vec{T}) \\ &= S * <\vec{T}, \theta''(e, g')>, \end{aligned}$$

where θ'' is defined by the equation $<\theta(e, f), g'> = <f, \theta''(e, g')>$. Since θ is continuous, we can find neighbourhoods U and V of 0 in E and F respectively so that for any $e \in U, f \in V, g' \in A'$ we have

$$|<\theta(e, f), g'>| \leqq 1.$$

Therefore the image $\theta''(U, A')(\subset V^0)$ is an equicontinuous subset of F'. Therefore since $S \to S*\vec{T}$ is continuous, if $\mathcal{W}$ is any neighbourhood of 0 in $\mathcal{L}$, we can find a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{H}$ in such a way that for any $S \in \mathcal{U}$ and $e \in U$ we have

$$(2) \qquad <S*\vec{T}, \theta''(e, g')> \in \mathcal{W}.$$

Consequently it follows from (1) that the map $(S, e) \to (S \otimes e)*_\theta \vec{T}$ under consideration is continuous.

$\mathcal{H}$ is nuclear by our assumption, and therefore $\mathcal{H} \otimes_\varepsilon E = \mathcal{H} \otimes_\pi E$ and $\mathcal{H}(E) \subset \mathcal{H} \hat{\otimes}_\pi E$. If $\mathcal{L}$ and G are complete, then we can extend the above map uniquely to the continuous linear map u of $\mathcal{H}(E)$ into $\mathcal{L}(G)$ which coincides on $\mathcal{D} \otimes E$ with the θ-convolution map by $\vec{T}$. Therefore we can apply Proposition 9 to infer that any $\vec{S} \in \mathcal{H}(E)$ is $*_\theta$-composable with $\vec{T}$ and that $\vec{S} \to \vec{S} *_\theta \vec{T}$ is a continuous linear map of $\mathcal{H}(E)$ into $\mathcal{L}(G)$. In case where $\mathcal{H}$ or E has the strict approximation property, we can reach the same conclusion by a similar way.

When $\mathcal{K}$ is quasi-complete (resp. when the convolution map $\mathcal{H} \times \mathcal{K} \to \mathcal{L}$ is β-continuous), the map $S \to S*<\vec{T}, \theta''(e, g')>$ is uniformly continuous when $\vec{T}$ lies in a compact subset (resp. a bounded subset) of $\mathcal{K}(F)$ and $e \in U, g' \in A'$. By the same way as above we can infer that the map $\vec{S} \to \vec{S} *_\theta \vec{T}$ is uniformly continuous with respect to the compact subsets (resp. bounded subsets) of $\mathcal{K}(F)$.

Now we shall assume that $\mathcal{H}'_c$ is nuclear and prove that the linear map $\vec{T} \to \vec{S} *_\theta \vec{T}$ of $\mathcal{K}(F)$ into $\mathcal{L}(G)$ is continuous.

First we define the convolution map $(\check{T}, R') \to \check{T} *_1 R'$ of $\check{\mathcal{K}} \times \mathcal{L}'_c$ into $\mathcal{H}'_c$ by the equation

$$<S*T, R'> = <S, \check{T} *_1 R'>.$$

It is easy to see thas $*_1$ is hypocontinuous with respect to the compact disks of $\check{\mathcal{K}}$ and the equicontinuous subsets of $\mathcal{L}'_c$. We then define $\check{\vec{T}} *_1 R'$ by the equation

$$<\check{\vec{T}} *_1 R', f'> = <\check{\vec{T}}, f'> *_1 R', f' \in F'.$$

Then the bilinear map $(\vec{T}, R') \to \check{\vec{T}} *_1 R'$ of $\mathcal{K}(F) \times \mathcal{L}'_c$ into $\mathcal{H}'_c(F)$ is hypocontinuous with respect to the equicontinuous subsets of $\mathcal{L}'_c$. Let A' be any equicontinuous subset of G' and choose neighbourhoods $U \subset E$, $V \subset F$ as before. Then $\theta''(U, A') \subset V^0$. Here we assume that U is a disked neighbourhood of E.

Since the map $\vec{S} \to \vec{S} *_\theta \vec{T}$ is continuous, $\vec{S} \to <R' \cdot (\vec{S} *_\theta \vec{T}), g'>$ is a continuous linear form on $\mathcal{H}(E)$, to which there corresponds an element $\vec{T}_{R', g'} \in \mathcal{H}'_c(E'; \varepsilon) = (\mathcal{H}(E))'$ ([12], p. 103). We show that $\vec{T}_{R', g'}$ transforms a neighbourhood of

0 in $\mathscr{H}$ into U^0 when $g' \in A'$, and that we can write $\vec{T}_{R',g'} = (I \otimes \tilde{\theta}'(g'))(\check{\vec{T}} *_1 R')$. Indeed, for any $\phi \in \mathscr{D}$ and $e \in E$ we have

$$\begin{aligned} < R' \cdot ((\phi \otimes e) *_\theta \vec{T}), g' > &= R' \cdot < \phi * \vec{T}, \theta''(e, g') > \\ &= \phi \cdot < \check{\vec{T}} *_1 R', \theta''(e, g') > \\ &= \phi \cdot < (I \otimes \tilde{\theta}'(g'))(\check{\vec{T}} *_1 R'), e > \\ &= < \phi \otimes e, (I \otimes \tilde{\theta}'(g'))(\check{\vec{T}} *_1 R') > \end{aligned}$$

Hence $\vec{T}_{R',g'} = (I \otimes \tilde{\theta}'(g'))(\check{\vec{T}} *_1 R')$. From the equations just obtained it follows that

$$S \cdot < \vec{T}_{R',g'}, e > = S \cdot < \check{\vec{T}} *_1 R', \theta''(e, g') >,$$

which implies that for $e \in U$, $g' \in A'$ there exists a neighbourhood $\mathscr{U}$ of 0 in $\mathscr{H}$ such that for every $S \in \mathscr{U}$

$$| S \cdot < \vec{T}_{R',g}, e > | \leqq 1.$$

Thus $\vec{T}_{R,g'}$ can be identified with an element of $(\mathscr{H}(\hat{E}_U))'$.

Since $\mathscr{H}'_c$ is nuclear, $\vec{S}$ is a subnuclear map of $\mathscr{H}'_c$ into E. Hence for $U \subset E$ we can find a compact disk C of $\mathscr{H}$ in such a way that the map $\vec{S}$ followed by the canonical map $E \to \hat{E}_U$ can be written as

$$\sum_{\nu=1}^{\infty} \lambda_\nu h_\nu \otimes \hat{e}_\nu,$$

h_ν lying in C and $\hat{e}_\nu$ lying in the unit cube of $\hat{E}_U$ and $\vec{T}_{R',g'}$ may be considered an element of $(\mathscr{H}(\hat{E}_U))'$.

Now we can write

$$< \vec{S}, \vec{T}_{R',g'} > = \sum_\nu \lambda_\nu < h_\nu \cdot \vec{T}_{R',g'}, \hat{e}_\nu >.$$

Let A' be any equicontinuous subset of $\mathscr{L}'$. Let V be the set of elements $\vec{T}$ of $\mathscr{K}(F)$ such that for $f' \in V^0$, $R' \in A'$, $h \in C$ we have

$$| h \cdot < \check{\vec{T}} *_1 R', f' > | \leqq 1.$$

Then V is, by definition, a neighbourhood of 0 in $\mathscr{K}(F)$. For every $\vec{T} \in V$ we have

$$| < h_\nu \cdot \vec{T}_{R',g'}, \hat{e}_\nu > | \leqq 1.$$

Therefore

$$| < \vec{S}, \vec{T}_{R',g'} > | \leqq \sum_\nu | \lambda_\nu |,$$

which proves that the map $\vec{T} \to \vec{S} *_\theta \vec{T}$ is continuous.

Finally we assume that $\mathscr{H}$ is quasi-complete. Let $\vec{S}$ run through a bounded subset of $\mathcal{H}(E)$, the maps $\vec{S}$ followed by the canonical map $E\to\hat{E}_U$ may be written as

$$\sum_\nu \lambda_\nu h_\nu \otimes \hat{e}_{\nu,\vec{S}}$$

with the same h_ν and λ_ν as before. Therefore we can infer in a similar way that $\vec{T}\to\vec{S}*_\theta\vec{T}$ is uniformly continuous with respect to the bounded subsets of $\mathcal{H}(E)$. Thus the proof is completed.

From the proof of the preceding theorem we can infer the following

COROLLARY. *Let $\mathcal{H}$, $\mathcal{K}$ be $\dot{\mathcal{B}}$-normal and nuclear. Let $\mathcal{L}$ be complete. Let E, F, G be three LCSs, G being assumed complete. Let θ be a continuous bilinear map of $E\times F$ into G. Suppose the convolution map of $\mathcal{H}\times\mathcal{K}$ into $\mathcal{L}$ is defined and β-continuous. Then any $\vec{S}\in\mathcal{H}(E)$ and $\vec{T}\in\mathcal{K}(F)$ are $*_\theta$-composable and the bilinear map $(\vec{S},\vec{T})\to\vec{S}*_\theta\vec{T}$ of $\mathcal{H}(E)\times\mathcal{K}(F)$ into $\mathcal{L}(G)$ is β-continuous.*

§ 6. Strict convolution map between two spaces of vector valued distributions (general case)

Let $\mathfrak{S}$ be a saturated family of bounded subsets of an LCS E. A family $\mathfrak{S}$ of subsets of E is called *saturated* if the following conditions are satisfied:

(i) if $A\in\mathfrak{S}$, then $\lambda A\in\mathfrak{S}$ for every $\lambda>0$;

(ii) if $A\in\mathfrak{S}$, then any subset of A belongs to $\mathfrak{S}$;

(iii) if $A\in\mathfrak{S}$, then the disked envelope of A belongs to $\mathfrak{S}$;

(iv) if $A, B\in\mathfrak{S}$, then $A\cup B\in\mathfrak{S}$;

(v) every one point subset of E belongs to $\mathfrak{S}$.

Let θ be a separately continuous bilinear map of $E\times F$ into G. Let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be a family of bounded subsets A (resp. of bounded completing subsets B) of E (resp. F) for which for any neighbourhood W of 0 in G there exists a neighbourhood V (resp. U) of 0 in F (resp. E) such that $\theta(A, V)\subset W$ (resp. $\theta(U, B)\subset W$). Then θ becomes $\mathfrak{S}$-$\mathfrak{T}$-hypocontinuous.

Now we shall show the following theorem which bears a very close analogy to Proposition 38 in Schwartz [12] (p. 159).

THEOREM 4. *Let $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ be normal spaces of distributions on R^N and E, F, G, be LCSs, where $\mathcal{L}'_c$ and G are assumed to be quasi-complete. Suppose that $\mathcal{K}\subset\mathcal{H}$ and the injection $\mathcal{K}\to\mathcal{H}$ is subnuclear, and that $\mathcal{K}$ is $\dot{\mathcal{B}}$-normal. Suppose the convolution map $(\check{S}, R)\to\check{S}*_1R$ of $\check{\mathcal{M}}\times\mathcal{L}$ into $\mathcal{K}$ is defined and separately continuous. Let θ be a separately continuous bilinear map of $E\times F$ into G. Then any $\vec{S}\in\mathcal{M}(E)$ and $\vec{T}\in\mathcal{H}'_c(F;\beta_0)$ are $*_\theta$-composable. Further, if the convolution map $*_1$ is hypocontinuous with respect to the compact disks of $\check{\mathcal{M}}$ or those of $\mathcal{L}$, then $\vec{S}*_\theta\vec{T}\in\mathcal{L}'_c(G)$.*

Suppose that the convolution map $*_1$ is hypocontinuous with respect to the *compact disks of $\mathcal{L}$. Let $\mathfrak{S}$ (resp. $\mathfrak{T}$) be a saturated family of bounded (resp. bounded completing) subsets of E (resp. F) defined above.*

(a) *Let $\vec{S}$ lie in a subset of type $\mathfrak{S}$ in $\mathcal{M}(E)$. If $\vec{T}$ tends to 0 in $\mathcal{H}'_c(F)$ while lying in a β_0-equibounded subset, then $\vec{S}*_\theta\vec{T}$ converges to 0 uniformly in $\mathcal{L}'_c(G)$ with respect to $\vec{S}$. In particular, this is also the case if the injection of $\mathcal{K}$ into $\mathcal{H}$ is nuclear, and if $\vec{T}$ tends to 0 in $\mathcal{H}'_c(F)$.*

(b) *If $\vec{S}$ tends to 0 in $\mathcal{M}(E)$ and $\vec{T}$ lies in a $\mathfrak{T}$-equibounded subset of $\mathcal{H}'_c(F)$, then $\vec{S}*_\theta\vec{T}$ converges to 0 uniformly in $\mathcal{L}'_c(G)$ with respect to $\vec{T}$.*

Proof. Let $\vec{S} \in \mathcal{M}(E)$ and $T\otimes f \in \mathcal{K}'_c(F)$. By assumption, $<\check{\vec{S}}, e'>*\phi \in \mathcal{K}$ for any $\phi \in \mathcal{D}$ and $e' \in E'$. The map $e' \to <\check{\vec{S}}, e'>*\phi$ of E'_c into $\mathcal{K}$ is continuous. Taking account of the fact that $\mathcal{K}$ is $\dot{\mathcal{B}}$-normal, it follows from the remark preceding Proposition 3 that $(\check{\vec{S}}*\phi)T \in \mathcal{D}'_{L^1}(E)$ holds, so that, by Remark 3 (§2), $\vec{S}$ and $T\otimes f$ become $*_\theta$-composable. Let $\vec{T}$ be any element of $\mathcal{H}'_c(F; \beta_0)$, that is, $\vec{T}$ maps a neighbourhood of 0 in $\mathcal{H}$ into a bounded completing absolutely convex subset B of F so that F_B is a Banach space. The injection $\mathcal{K}\to\mathcal{H}$ being subnuclear, a neighbourhood $\mathcal{U}$ of 0 in $\mathcal{K}$ may be found so that the map $\vec{T}$ can be written as

$$\sum_{j=1}^{\infty} \lambda_j k'_j \otimes f_j,$$

with k'_j in an equicontinuous subset $\mathcal{U}^0$, f_j in a compact disk C of F_B and $\|f_j\|_B \to 0$ as $j\to\infty$ and $\sum|\lambda_j|<\infty$. On the other hand $\check{\vec{S}}*\phi$ is locally β_0-bounded in $\mathcal{E}(E)$. Hence, considering $\vec{T}$ as an element of $\mathcal{D}'(E)$, we can define the multiplicative product $[(\check{\vec{S}}*\phi)\vec{T}]_\theta$, which is written as

$$[(\check{\vec{S}}*\phi)\vec{T}]_\theta = [(\check{\vec{S}}*\phi)\ (\sum_j \lambda_j k'_j \otimes f_j)]_\theta,$$

whence, by the remark preceding Proposition 9,

$$[(\check{\vec{S}}*\phi)\vec{T}]_\theta = \sum_{j=1}^{\infty} \lambda_j [(\check{\vec{S}}*\phi)\ (k'_j \otimes f_j)]_\theta \tag{1}$$

As k'_j lies in an equicontinuous subset $\mathcal{U}^0$, the set $\{(\check{\vec{S}}*\phi)k'_j\}_{j=1,2\ldots}$ is bounded in $\mathcal{D}'_{L^1}(E)$. According to Lemma 8 and the fact that f_j lies in a bounded completing subset of F, it follows that the set $\{[(\check{\vec{S}}*\phi)\ (k'_j\otimes f_j)]_\theta\}$ is bounded in $\mathcal{D}'_{L^1}(G)$, whence the right side of (1) converges in $\mathcal{D}'_{L^1}(G)$. Therefore $\vec{S}$ and $\vec{T}$ are $*_\theta$-composable, and by integrating both sides of (1) we obtain

$$\vec{S}*_\theta\vec{T} = \sum_{j=1}^{\infty} \lambda_j \vec{S}*_\theta(k'_j\otimes f_j) \in \mathcal{D}'(G). \tag{2}$$

Now let us define the convolution map $(\check{\vec{S}}, R)\to\check{\vec{S}}*_1R$ of $\check{\mathcal{M}}(E)\times\mathcal{L}$ into $\mathcal{K}(E)$ by the equation $<\check{\vec{S}}*_1R, e'>=<\check{\vec{S}}, e'>*_1R$, $e' \in E'$. We note that $\check{\vec{S}}*_1R$ forms a bounded subset of $\mathcal{K}(E)$ when R runs through a compact disk of $\mathcal{L}$. In fact, $<\check{\vec{S}}, e'>$ lies in a compact disk of $\check{\mathcal{M}}$ when e' runs through an equicontinuous disk of E'. Then by Lemma 8 the set $\{<\check{\vec{S}}, e'>*_1 R\}$ and therefore the set $\{\check{\vec{S}}*_1 R\}$ is bounded. Since $\mathcal{K}$ is $\mathcal{k}$-normal and $\check{\vec{S}}*_1 R$ belongs to $\mathcal{K}(E)$, it follows that $[(\check{\vec{S}}*_1 R)(k'_j \otimes f_j)]_\theta \in \mathcal{D}'_{L^1}(G)$. Accordingly

$$\int [(\check{\vec{S}}*_1 R)(k'_j \otimes f_j)]_\theta dx = \theta(k'_j \cdot (\check{\vec{S}}*_1 R), f_j) \in G. \tag{3}$$

First we suppose that the map $*_1$ is hypocontinuous with respect to the compact disks of $\check{\mathcal{M}}$. Then the map $R \to \theta(k'_j \cdot (\check{\vec{S}}*_1 R), f_j)$ of $\mathcal{L}$ into G is continuous. This is because $R \to 0$ implies $\check{\vec{S}}*_1 R \to 0$ in $\mathcal{K}(E)$ by the assumption just made, and in turn $k'_j \cdot (\check{\vec{S}}*_1 R) \to 0$ in E. For any $\phi \in \mathcal{D}$, as $\vec{S}$ and $k'_j \otimes f_j$ are $*_\theta$-composable,

$$\int [(\check{\vec{S}}*\phi)(k'_j \otimes f_j)]_\theta dx = \phi \cdot (\vec{S}*_\theta (k'_j \otimes f_j)). \tag{4}$$

This together with (3) implies that $\vec{S}*_\theta (k'_j \otimes f_j) \in \mathcal{L}'_c(G)$ and

$$\int [(\check{\vec{S}}*_1 R)(k'_j \otimes f_j)]_\theta dx = R \cdot (\vec{S}*_\theta (k'_j \otimes f_j)). \tag{5}$$

Using this we next show that $\sum_{j=1}^{\infty} \lambda_j \vec{S}*_\theta (k'_j \otimes f_j)$ converges in $\mathcal{L}'_c(G)$. Let R run through a compact disk C of $\mathcal{L}$, then the set $\{R \cdot (\vec{S}*_\theta (k'_j \otimes f_j))\}_{j=1,2,\dots,R \in C}$ is bounded in G. In fact, the set $\{k'_j \cdot (\check{\vec{S}}*_1 R)\}_{j=1,2,\dots}$ is bounded in E and f_j is an compact disk of F. Then, by Lemma 8, the set $\{\theta(k'_j \cdot (\check{\vec{S}}*_1 R), f_j)\}_{j=1,2,\dots,R \in C}$ is bounded in G. It follows from (3) and (5) that the set $\{\vec{S}*_\theta (k'_j \otimes f_j)\}_{j=1,2,\dots}$ is bounded in $\mathcal{L}'_c(G)$. $\sum_{j=1}^{\infty} \lambda_j$ being convergent, $\sum_{j=1}^{\infty} \lambda_j \vec{S}*_\theta (k'_j \otimes f_j)$ converges in $\mathcal{L}'_c(G)$. Now it follows from (2) that $\vec{S}*_\theta \vec{T} \in \mathcal{L}'_c(G)$.

Now we suppose that the map $*_1$ is hypocontinuous with respect to the compact disks of $\mathcal{L}$. Consider the convolution map $*'_1$ of $\mathcal{M} \times \mathcal{K}'_c$ into $\mathcal{L}'_c$ induced from the map $*_1$. The map $*'_1$ is hypocontinuous with respect to the equicontinuous subsets of $\mathcal{K}'_c$. We define $\vec{S}*'_1 k' \in \mathcal{L}'_c(E)$ for any $k' \in \mathcal{K}'_c$ by the equation $R \cdot (\vec{S}*'_1 k') = k' \cdot (\check{\vec{S}}*_1 R)$, $R \in \mathcal{L}$. Now the map $R \to \theta(R \cdot (\vec{S}*'_1 k'_j), f_j)$ of $\mathcal{L}_\gamma$ into G is continuous. For, if $R \to 0$ in $\mathcal{L}_\gamma$, then $R \cdot (\vec{S}*'_1 k'_j) \to 0$ in E. Noting that the right side of (3) is rewritten as $\theta(R \cdot (\vec{S}*'_1 k'_j), f_j)$, we can infer from (3) and (4) that $\vec{S}*_\theta (k'_j \otimes f_j) \in \mathcal{L}'_c(G)$. Proceeding just as before, we can conclude that $\sum \lambda_j \vec{S}*_\theta (k'_j \otimes f_j)$ converges in $\mathcal{L}'_c(G)$, so that $\vec{S}*_\theta \vec{T} \in \mathcal{L}'_c(G)$.

Now we shall turn to the proof of the cases (a) and (b) of the theorem. Suppose that the convolution map $*_1: \check{\mathcal{M}} \times \mathcal{L} \to \mathcal{K}$ is hypocontinuous with respect to the compact disks of $\mathcal{L}$. Then, as proved above, $\vec{S}*_\theta \vec{T} \in \mathcal{L}'_c(G)$ for every $\vec{S} \in \mathcal{M}(E)$ and $\vec{T} \in \mathcal{H}'_c(F; \beta_0)$. Let $\vec{S}$ lie in a subset $\mathfrak{A}$ *of type* $\mathfrak{S}$ in $\mathcal{M}(E)$, that is, for any equicontinuous subset $\mathfrak{B}'$ of $\mathcal{M}'$ the set $\bigcup_{\vec{S} \in \mathfrak{A}} \mathfrak{B}' \cdot \vec{S}$ is contained in an $A \in \mathfrak{S}$. Let $\vec{T}$ be in a β_0-equibounded subset of $\mathcal{H}'_c(F)$ and $\vec{T} \to 0$ in $\mathcal{H}'_c(F)$. $\vec{T}$ may be written as

$$\sum_{j=1}^{\infty} \lambda_j k'_j \otimes f_{j,\vec{T}}$$

with the same k'_j and λ_j, where k'_j lies in an equicontinuous subset of $\mathcal{K}'$, $\sum|\lambda_j|<\infty$ and $\{f_{j,\vec{T}}\}_{j=1,2,\dots,\vec{T}\in\mathfrak{A}}$ is contained in a bounded completing subset B of F. $\vec{T}\to 0$ implies that $f_{j,\vec{T}}\to 0$ for each j. Now consider the convolution map $*''_1$ of $\mathcal{L}\times\check{\mathcal{K}}'_c$ into $\mathcal{M}'_c$ induced from the map $*_1$. Then we have

$$R\cdot\left(\vec{S}*_\theta(k'_j\otimes f_{j,\vec{T}})\right)=\theta\left((R*''_1\check{k}'_j)\cdot\vec{S},\, f_{j,\vec{T}}\right). \tag{6}$$

Let R run through a compact disk C of $\mathcal{L}$. By our assumption on $*_1$, since $\{k'_j\}_{j=1,2,\dots}$ is contained in an equicontinuous subset of $\mathcal{K}'$ it follows that $\{R*''_1\check{k}'_j\}_{j=1,2,\dots,R\in C}$ is contained in an equicontinuous subset of $\mathcal{M}'_c$. As $\vec{S}$ lies in $\mathfrak{A}$, the set $\{(R*''_1\check{k}'_j)\cdot\vec{S}\}_{j=1,2,\dots,R\in C,\vec{S}\in\mathfrak{A}}$ is contained in an $A\in\mathfrak{S}$. The map $\theta\colon E\times F\to G$ is hypocontinuous with respect to $\mathfrak{S}$. Therefore by the equation (6), if $\vec{T}$ tends to 0 in $\mathcal{H}'_c(F)$, $R\cdot\left(\vec{S}*_\theta(k'_j\otimes f_{j,\vec{T}})\right)$ for each j converges uniformly in G to 0 when $R\in C$. On the other hand, the set $\{R\cdot\left(\vec{S}*_\theta(k_j\otimes f_{j,\vec{T}})\right)\}_{j=1,2,\dots,R\in C,\vec{S}\in\mathfrak{A}}$ is bounded in G. Consequently, it follows from (2) that $\vec{S}*_\theta\vec{T}$ converges to 0 uniformly in $\mathcal{L}'_c(G)$ as $\vec{T}\to 0$. In particular, if the injection $i\colon\mathcal{K}\to\mathcal{H}$ is nuclear, i can be written as $\sum_j\lambda_j(k'_j\otimes h_j)$ with k'_j in an equicontinuous subset of $\mathcal{K}'$ and any h_j in a compact disk of $\mathcal{H}$ and $\sum_j|\lambda_j|<\infty$. If we take for $\vec{T}$ to be an element of $\mathcal{K}'_c(F)$, we can write $\vec{T}$ in the form:

$$\sum_j \lambda_j k'_j\otimes f_{j,\vec{T}},$$

where $f_{j,\vec{T}}=h_j\cdot\vec{T}$. As in the preceding discussions with necessary modifications the equation (6) implies that $\vec{S}*_\theta\vec{T}\to 0$ uniformly in $\mathcal{L}'_c(G)$ when $\vec{S}$ lies in a subset of type $\mathfrak{S}$ in $\mathcal{M}(E)$ and $\vec{T}\to 0$ in $\mathcal{H}'_c(F)$.

Next, let $\vec{T}$ run through a $\mathcal{G}$-equibounded subset $\mathfrak{B}$ of $\mathcal{H}'_c(F)$, and suppose that $\vec{S}\to 0$ in $\mathcal{M}(E)$. We shall show that $\vec{S}*_\theta\vec{T}\to 0$ in $\mathcal{L}'_c(G)$. There exists a neighbourhood $\mathcal{V}$ of 0 in $\mathcal{H}$ such that

$$\bigcup_{\vec{T}\in\mathfrak{B}}\mathcal{V}\cdot\vec{T}\subset B_1,$$

where B_1 is an element of $\mathcal{G}$. Now we can write $\vec{T}=\sum_j\lambda_j k'_j\otimes f_{j,\vec{T}}$, where k'_j lie in an equicontinuous subset of $\mathcal{K}'$, $\sum_j|\lambda_j|<\infty$ and $f_{j,\vec{T}}$ lie in a $B_1\in\mathcal{G}$. Then we can infer in a similar way as above that, when $\vec{S}\to 0$ in $\mathcal{M}(E)$, $\vec{S}*_\theta\vec{T}$ converges to 0 in $\mathcal{L}'_c(G)$ with respect to $\vec{T}$.

Thus the proof is completed.

Remark. In the preceding theorem we have assumed that the convolution map $*_1\colon\check{\mathcal{M}}\times\mathcal{L}\to\mathcal{K}$ is separately continuous. We note that if $\mathcal{L}$ (resp. $\mathcal{M}$) is quasi-barrelled the map $*_1$ also becomes hypocontinuous with respect to the compact disks of $\check{\mathcal{M}}$ (resp. $\mathcal{L}$). In fact, let C be any compact disk of $\check{\mathcal{M}}$. If B

is a bounded subset of $\mathcal{L}$, then by Lemma 8 $C*B$ is a bounded subset of $\mathcal{K}$, and therefore absorbed in any given disked neighbourhood $\mathcal{W}$ of 0 in $\mathcal{K}$. Let $\mathcal{V}$ be the set of elements l such that $C*l \subset \mathcal{W}$. $\mathcal{V}$ will be a disk absorbing B. $\mathcal{L}$ being quasi-barrelled, $\mathcal{V}$ must be a neighbourhood of 0 in $\mathcal{L}$.

When θ is continuous, Theorem 4 yields a result concerning elementary convolution. However it is possible to obtain here a more general result by relaxing a little the conditions on $\vec{T}$. Given a disked neighbourhood W of 0 in G there exist two disked neighbourhoods U and V of 0 in E and F respectively such that $\theta(U, V) \subset W$, so that we shall have a continuous bilinear map $\hat{\theta}: \hat{E}_U \times \hat{F}_V \to \hat{G}_W$ deduced from θ in an obvious way. Generally $\vec{S}_U$, $\vec{S} \in \mathcal{H}(E)$, will stand for the continuous linear map of $\mathcal{H}'_c$ into $\hat{E}_U$ defined as the map $\vec{S}$; $\mathcal{H}'_c \to E$ followed by the canonical map $E \to \hat{E}_U$. Then by the definition of the multiplicative product ([12], p. 120) we have for any $\vec{S} \in \mathcal{D}'(E)$, $\vec{T} \in \mathcal{D}'(F)$, $\phi \in \mathcal{D}$

$$(7) \qquad [(\check{\vec{S}}*\phi)\vec{T}]_{\theta, W} = [(\check{\vec{S}}_U*\phi)\vec{T}_V]_{\hat{\theta}},$$

which will be made use in the proof of the following

COROLLARY 1. *Let E, F, G be LCSs, and $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$, $\mathcal{M}$ be normal spaces of distributions on R^N, where G and $\mathcal{L}'_c$ are assumed to be complete. Let θ be a continuous bilinear map of $E \times F$ into G and $*_1$ be a separately continuous convolution map of $\check{\mathcal{M}} \times \mathcal{L}$ into $\mathcal{K}$. If $\mathcal{K} \subset \mathcal{H}$ and the injection $\mathcal{K} \to \mathcal{H}$ is subnuclear, and if $\mathcal{K}$ is $\dot{\mathcal{B}}$-normal, then any $\vec{S} \in \mathcal{M}(E)$ and $\vec{T} \in \mathfrak{L}_c(\mathcal{H}; F)$ are $*_\theta$-composable and $\vec{S}*_\theta\vec{T} \in \mathcal{L}'_c(G)$.*

Further if the map $*_1$ is hypocontinuous with respect to the compact disks of $\mathcal{L}$, then we have

(a) *Let $\vec{S}$ lie in a bounded subset of $\mathcal{M}(E)$. If $\vec{T}$ tends to 0 lying in an equicontinuous subset of $\mathfrak{L}_c(\mathcal{H}; F)$, then $\vec{S}*_\theta\vec{T}$ converges to 0 uniformly in $\mathcal{L}'_c(G)$ with respect to $\vec{S}$. In particular, this is also the case if the injection of $\mathcal{K}$ into $\mathcal{H}$ is nuclear, and if $\vec{T}$ tends to 0 in $\mathfrak{L}_c(\mathcal{H}; F)$.*

(b) *If $\vec{S}$ tends to 0 in $\mathcal{M}(E)$ and $\vec{T}$ lies in an equicontinuous subset of $\mathfrak{L}_c(\mathcal{H}; F)$, then $\vec{S}*_\theta\vec{T}$ converges to 0 uniformly in $\mathcal{L}'_c(G)$ with respect to $\vec{T}$.*

PROOF. Let U, V, W be chosen as before, where W is an arbitrary given disked neighbourhood of 0 in G. Then $\vec{S}_U \in \mathcal{M}(\hat{E}_U)$, $\vec{T}_V \in \mathfrak{L}_c(\mathcal{H}, \hat{F}_V) = \mathcal{H}'_c(\hat{F}_V; \beta_0)$. By the preceding theorem and (7) we have for any $\phi \in \mathcal{D}$

$$[(\check{\vec{S}}*\phi)\vec{T}]_{\theta, W} = [(\check{\vec{S}}_U*\phi)\vec{T}_V]_{\hat{\theta}} \in \mathcal{D}'_{L^1}(\hat{G}_W).$$

It follows from this equation that $\{\alpha_n[(\check{\vec{S}}*\phi)\vec{T}]_\theta\}$ forms a Cauchy sequence in $\mathcal{D}'_{L^1}(G)$ for any sequence $\{\alpha_n\}$ of multiplicators. Since $\mathcal{D}'_{L^1}(G)$ is complete, $[(\check{\vec{S}}*\phi)\vec{T}]_\theta \in \mathcal{D}'_{L^1}(G)$, so that $\vec{S}$ and $\vec{T}$ are $*_\theta$-composable and

$$(\vec{S}*_\theta\vec{T})_W = \vec{S}_U *_{\hat{\theta}} \vec{T}_V.$$

The injection $\mathcal{K}\to\mathcal{H}$ being subnuclear, $\vec{T}_V$ is written as

$$\vec{T}_V=\sum_j \lambda_j(k_j'\otimes\hat{f}_j)$$

with k_j' in an equicontinuous subset of $\mathcal{K}_c'$ and $\hat{f}_j$ in a compact disk of $\hat{F}_V$ and $\sum|\lambda_j|<\infty$. Here we may take f_j in V so that $\hat{f}_j$ may be the canonical image of f_j. Then

$$(8)\qquad \vec{S}_U *_{\hat{\theta}} \vec{T}_V=\sum\lambda_j\big(\vec{S}_U *_{\hat{\theta}}(k_j'\otimes\hat{f}_j)\big).$$

We next show that $\vec{S}*_\theta\vec{T}\in\mathcal{L}_c'(G)$. For any $g'\in W^0$, we have

$$(9)\qquad \begin{aligned}<\vec{S}*_\theta\vec{T}, g'> &= <(\vec{S}*_\theta\vec{T})_W, g'>\\ &=\sum_{j=1}^{\infty}\lambda_j<\vec{S}_U *_1 k_j', \hat{\theta}'(\hat{f}_j, g')>\\ &=\sum_{j=1}^{\infty}\lambda_j<\vec{S}*_1 k_j', \theta'(f_j, g')>.\end{aligned}$$

Each $<\vec{S}_U *_1 k_j', \hat{\theta}(\hat{f}_j, g')>=<\vec{S}_U, \hat{\theta}'(\hat{f}_j, g')>*_1\check{k}_j'$ belongs to $\mathcal{L}_c'$ and for any $l\in\mathcal{L}$ we have $l\cdot<\vec{S}_U *_1 k_j', \hat{\theta}'(\hat{f}_j, g')>=k_j'\cdot(<\check{\vec{S}}_U, \hat{\theta}'(\hat{f}_j, g')>*_1 l)$. The set $\{<\check{\vec{S}}_U, \hat{\theta}'(\hat{f}_j, g')>\}$ is contained in a compact disk of $\mathcal{K}$ and $\{k_j'\}$ is equicontinuous in $\mathcal{K}_c'$. Hence the set $\{l\cdot<\vec{S}_U *_1 k_j', \hat{\theta}'(\hat{f}_j, g')>\}_{j=1,2,\dots,g\in W^0}$ is bounded, that is, the set $\{<\check{\vec{S}}_U *_1 k_j', \hat{\theta}'(\hat{f}_j, g')>\}_{j=1,2,\dots,g'\in W^0}$ is $\sigma(\mathcal{L}_c', \mathcal{L})$-bounded, and therefore is bounded in $\mathcal{L}_c'$. $\mathcal{L}_c'$ being complete, (9) implies $<\vec{S}*_\theta\vec{T}, g'>\in\mathcal{L}_c'$. If g' tends to 0 in G_c' lying in W^0, then each $\theta'(f_j, g')$ tends to 0 in E_c', so that $<\vec{S}*_1 k_j', \theta'(f_j, g')>$ converges to 0 in $\mathcal{L}_c'$. Therefore, it follows from (9) that $<\vec{S}*_\theta\vec{T}, g'>$ converges to 0 in $\mathcal{L}_c'$. According to Proposition 8 in Schwartz [11] (p. 41), the map $g'\to<\vec{S}*_\theta\vec{T}, g'>$ of G_c' into $\mathcal{L}_c'$ becomes continuous. Consequently $\vec{S}*_\theta\vec{T}\in\mathcal{L}_c'(G)$.

If $\mathfrak{S}$ and $\mathcal{T}$ are taken as the family of bounded subsets of $\hat{E}_U$ and $\hat{F}_V$ respectively, any bounded subset of $\mathcal{M}(\hat{E}_U)$ is a set of type $\mathfrak{S}$ in $\mathcal{M}(\hat{E}_U)$ and any equicontinuous subset of $\mathfrak{L}_c(\mathcal{H}; \hat{F}_V)$ is $\mathcal{T}$-equibounded. Then the cases (a) and (b) are immediate consequences of the preceding theorem, completing the proof.

As an immediate consequence of Corollary 1 we have

COROLLARY 2. *Let E, F, G be LCSs, and $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ be normal spaces of distributions, where G and $\mathcal{L}_c'$ are assumed to be complete. Suppose that $\mathcal{K}$ is nuclear, $\dot{\mathcal{B}}$-normal and has the γ-topology. Let θ be a continuous bilinear map of $E\times F$ into G and $*_1$ be a separately continuous convolution map of $\mathcal{H}\times\mathcal{K}_c'$ into $\mathcal{L}_c'$. Then any $\vec{S}\in\mathcal{H}(E)$ and $\vec{T}\in\mathcal{K}_c'(F)$ are $*_\theta$-composable and $\vec{S}*_\theta\vec{T}\in\mathcal{L}_c'(G)$.*

Further if the map $*_1$ is hypocontinuous with respect to the compact disks of $\mathcal{K}_c'$, then we have

(a) *If $\vec{S}$ lies in a bounded subset of $\mathcal{H}(E)$ and $\vec{T}$ tends to 0 lying in an equicontinuous subset of $\mathcal{K}_c'(F)$, then $\vec{S}*_\theta\vec{T}$ converges to 0 uniformly in $\mathcal{L}_c'(G)$*

with respect to $\vec{S}$.

(b) *If* $\vec{S}$ *tends to* 0 *in* $\mathcal{H}(E)$ *and* $\vec{T}$ *lies in an equicontinuous subset of* $\mathcal{K}'_c(F)$, *then* $\vec{S}*_\theta\vec{T}$ *converges to* 0 *uniformly in* $\mathcal{L}'_c(G)$ *with respect to* $\vec{T}$.

§ 7. Examples

Let E, F, G be three LCSs, G being assumed to be quasi-complete. Let θ be a separately continuous bilinear map of $E\times F$ into G.

EXAMPLE 1. O'_C, the space of convolution maps of $\mathcal{S}'$ into itself, is a nuclear barrelled space ([3], Chap. II, p. 131). It is easy to verify that O'_C is $\dot{\mathcal{B}}$-normal. As already indicated in Section 3 the convolution map $*: O'_C\times O'_C\to O'_C$ is continuous. Applying Theorem 2 to the case where $\mathcal{H}=\mathcal{K}=\mathcal{L}=O'_C$, we obtain:

The strict convolution map $*_\theta$: $O'_C(E)\times O'_C(F)\to O'_C(G)$ *is defined and is separately quasi-continuous. Further the map is hypocontinuous (resp. continuous) whenever* θ *is hypocontinuous (resp. continuous).*

We shall show that *for any* $\vec{S}\in O'_C(E)$ *and* $\vec{T}\in O'_C(F)$

$$\mathcal{F}(\vec{S}*_\theta\vec{T})=[\mathcal{F}(\vec{S})\mathcal{F}(\vec{T})]_\theta, \tag{1}$$

where $\mathcal{F}$ *stands for the Fourier transform.* We first note that the right side of (1) makes sense if we consider it as a multiplicative product $\in\mathcal{D}'(G)$ of elements of $\mathcal{D}'(E)$ and $\mathcal{E}(F)$ respectively. $\mathcal{F}(\vec{S})$ is locally β_0-bounded in $\mathcal{D}'(E)$. In fact, let $\alpha\in\mathcal{D}$ and put $\beta=\mathcal{F}^{-1}(\alpha)\in\mathcal{S}$. We have $\alpha\mathcal{F}(\vec{S})=\mathcal{F}(\beta*\vec{S})$. Owing to Lemma 5, $\beta*\vec{S}$ is β_0-bounded in $O_C(E)$, and therefore $\mathcal{F}(\beta*\vec{S})$ is β_0-bounded in $(O_M)'(E)$ and *a fortiori* in $\mathcal{D}'(E)$. Consequently, $\mathcal{F}(\vec{S})$ is locally β_0-bounded in $\mathcal{D}'(E)$. Therefore, considering $\mathcal{F}(\vec{T})\in O_C(F)$ to be an element of $\mathcal{E}(F)$, we can define the multiplicative product $[\mathcal{F}(\vec{S})\mathcal{F}(\vec{T})]_\theta\in\mathcal{D}'(G)$ ([12], p. 134). The bilinear maps $(\vec{S},\vec{T})\to\mathcal{F}(\vec{S}*_\theta\vec{T})\in\mathcal{D}'(G)$ and $(\vec{S},\vec{T})\to[\mathcal{F}(\vec{S})\mathcal{F}(\vec{T})]_\theta\in\mathcal{D}'(G)$ are separately quasi-continuous ([12], p. 134). If $\vec{S}$ and $\vec{T}$ are decomposable: $\vec{S}=S\otimes e$ and $\vec{T}=T\otimes f$, S, $T\in O'_C$, $e\in E$, $f\in F$, then we have

$$\begin{aligned}\mathcal{F}((S\otimes e)*_\theta(T\otimes f))&=\mathcal{F}(S*T)\otimes\theta(e,f)\\&=\mathcal{F}(S)\mathcal{F}(T)\otimes\theta(e,f)\\&=[\mathcal{F}(S\otimes e)\mathcal{F}(T\otimes f)]_\theta.\end{aligned}$$

It follows since $O'_C\otimes E$ and $O'_C\otimes F$ are strictly dense in $O'_C(E)$ and $O'_C(F)$ respectively that $\mathcal{F}(S*_\theta\vec{T})=[\mathcal{F}(\vec{S})\mathcal{F}(\vec{T})]_\theta$ for any $\vec{S}\in O'_C(E)$ and $\vec{T}\in O'_C(F)$.

Further we shall show that we have for any $x_0\in R^N$

$$\mathcal{F}(\vec{S}*_\theta\vec{T})\,(x_0)=\theta(\mathcal{F}(\vec{S})\,(x_0),\,\mathcal{F}(\vec{T})\,(x_0)). \tag{2}$$

Indeed, we consider the element $\delta_{x_0}\in(O_M)'$ and put $\phi=\mathcal{F}(\delta_{x_0})=e^{-2\pi i x_0\cdot\xi}\in O_C$, where $x_0\cdot\xi$ denotes the scalar product of $x_0\in R^N$ and $\xi\in\Xi^N$, the dual of R^N. Now $\check{\vec{S}}*\phi$ is β_0-bounded in $O_C(E)$ by Lemma 5. Using the notations in the proof of Theorem 2, $\Phi(\check{\vec{S}}*\phi, \vec{T}) \in \mathcal{D}'_{L^1}$ and $\int \Phi(\check{\vec{S}}*\phi, \vec{T})dx = <\vec{S}*_\theta \vec{T}, \phi>$. This yields

$$\begin{aligned}<\delta_{x_0}, \mathcal{F}(\vec{S}*_\theta\vec{T})> &= <\phi, \vec{S}*_\theta\vec{T}> \\ &= \int \Phi(\check{\vec{S}}*\phi, \vec{T})dx \\ &= \int \Phi(\phi\otimes\mathcal{F}(\vec{S})\,(x_0), \vec{T})dx \\ &= \int <\phi\vec{T}, \mathcal{F}(\vec{S})\,(x_0)>_\theta dx \\ &= \theta(\mathcal{F}(\vec{S})\,(x_0), \phi\cdot\vec{T}) \\ &= \theta(\mathcal{F}(\vec{S})\,(x_0), \mathcal{F}(\vec{T})\,(x_0))\end{aligned}$$

Consequently we have the equation (2). (1) and (2) show that $[\mathcal{F}(\vec{S})\mathcal{F}(\vec{T})]_\theta \in O_M(G)$ and $[\mathcal{F}(\vec{S})\mathcal{F}(\vec{T})]\,(x_0)=\theta(\mathcal{F}(\vec{S})\,(x_0), \mathcal{F}(\vec{T})\,(x_0))$.

Here we have:

The multiplicative product map $(\vec{S}, \vec{T})\to[\vec{S}\vec{T}]_\theta$ of $O_M(E)\times O_M(F)$ into $O_M(G)$ is well defined and is separately quasi-continuous and if $\vec{S}$ and $\vec{T}$ are decomposable, $[(S\otimes e)\,(T\otimes f)]_\theta = ST\otimes\theta(e, f)$ for $S, T\in O_M$, $e\in E$, $f\in F$. Further the map is hypocontinuous (resp. continuous) whenever θ is hypocontinuous (resp. continuous).

As a consequence of this assertion we can conclude the following assertion, the case in which θ is continuous is a simple consequence of the proposition in Schwartz ([12], p. 120).

The multiplicative product map $(\vec{S}, \vec{T})\to[\vec{S}\vec{T}]_\theta$ of $\mathcal{E}(E)\times\mathcal{E}(F)$ into $\mathcal{E}(G)$ is well defined and is separately quasi-continuous. Further the map is hypocontinuous (resp. continuous) whenever θ is hypocontinuous (resp. continuous).

In fact, let $\alpha, \gamma\in\mathcal{D}$ such that $\gamma=1$ on the support of α. Now $\alpha\vec{S}=\gamma\alpha\vec{S}\in O_M(E)$. Since there exists an element $\vec{S}_1\in O'_C(E)$ such that $\alpha\vec{S}=\mathcal{F}(\vec{S}_1)$. $\alpha\vec{S}=\gamma\mathcal{F}(\vec{S}_1)$ is β_0-bounded in $\mathcal{D}'(E)$ as indicated above. Therefore $[\vec{S}\vec{T}]_\theta$ makes sense if we consider it as a multiplicative product $\in\mathcal{D}'(G)$ of elements of $\mathcal{D}'(E)$ and $\mathcal{E}(F)$ respectively. Next we take $\alpha_1, \beta_1\in\mathcal{D}$ such that $\alpha_1(x)\beta_1(x)=1$ in a neighbourhood of $x_0\in R^N$. Since $\alpha_1\vec{S}\in O_M(E)$ and $\beta_1\vec{T}\in O_M(F)$, there exist $\vec{S}_1\in O'_C(E)$ and $\vec{T}_1\in O'_C(F)$ such that $\alpha_1\vec{S}=\mathcal{F}(\vec{S}_1)$ and $\beta_1\vec{T}=\mathcal{F}(\vec{T}_1)$ respectively. If we consider $[(\alpha_1\vec{S})\,(\beta_1\vec{T})]_\theta$ as a multiplicative product $\in O_M(G)$ of elements of $O_M(E)$ and $O_M(F)$ respectively, then from the preceding result we have

$$\begin{aligned}[(\alpha_1\vec{S})\,(\beta_1\vec{T})]_\theta(x) &= [\mathcal{F}(\vec{S}_1)\mathcal{F}(\vec{T}_1)]_\theta(x) \\ &= \theta(\mathcal{F}(\vec{S}_1)\,(x), \mathcal{F}(\vec{T}_1)\,(x)) \\ &= \theta(\alpha_1(x)\vec{S}(x), \beta_1(x)\vec{T}(x)) \\ &= \alpha_1(x)\beta_1(x)\theta(\vec{S}(x), \vec{T}(x)).\end{aligned}$$

On the other hand, if we consider $[(\alpha_1\vec{S})(\beta_1\vec{T})]_\theta$ as a multiplicative product $\in \mathcal{D}'(G)$, $[(\alpha_1\vec{S})(\beta_1\vec{T})]_\theta=\alpha_1\beta_1[\vec{S}\vec{T}]_\theta$ in $\mathcal{D}'(G)$, whence in a neighbourhood of x_0 $[(\alpha_1\vec{S})(\beta_1\vec{T})]_\theta(x)=\alpha_1(x)\beta_1(x)[\vec{S}\vec{T}]_\theta(x)$. Consequently, $[\vec{S}\vec{T}]_\theta(x_0)=\theta(\vec{S}(x_0), \vec{T}(x_0))$ for any $x_0 \in R^N$ and $[\vec{S}\vec{T}]_\theta \in \mathcal{E}(G)$.

In Theorems 2 and 3 we have assumed that the convolution map $*: \mathcal{H}\times \mathcal{K}\to\mathcal{L}$ or $\theta: E\times F\to G$ is continuous. Otherwise it is generally not possible to define the θ-convolution map between $\mathcal{H}(E)$ and $\mathcal{K}(F)$, or even when it is possible the θ-convolution need not belong to $\mathcal{L}(G)$ as the examples follow.

EXAMPLE 2. Let $E=(s)$, the space of the sequences $\vec{\xi}=(\xi_n)_{n=0,1,2,\dots}$ of complex numbers equipped with the usual topology. It is well known that (s) is a nuclear space of type (**F**). We take $F=(s)'$, the strong dual of (s), which is the space of sequences $\vec{\eta}=(\eta_n)_{n=0,1,2,\dots}$ of complex numbers with only a finite number of components not zero. We define $\theta(\vec{\xi}, \vec{\eta})=\sum_{n=0}^{\infty}\xi_n\eta_n$. Therefore G is assumed to be the space of complex numbers. Then θ is hypocontinuous, but not continuous. Let $\vec{e}_n$ be the unit vector of (s) such that the n-th component of $\vec{e}_n$ is 1, but the others are 0. $\{\vec{e}_n\}$ forms a Schauder basis of E. Let $\{\vec{e}'_n\}$ be its dual basis in E'. Then we can write $\vec{\xi}=\sum_{n=0}^{\infty}\xi_n\vec{e}_n$, $\vec{\eta}=\sum_{n=0}^{\infty}\eta_n\vec{e}'_n$. Now we consider the spaces of vector valued distributions $\mathcal{E}'(E)$ and $\mathcal{E}(F)$. It is to be noticed that the convolution map $*: \mathcal{E}'\times\mathcal{E}\to\mathcal{D}'$ is not continuous. We shall show that *there exist vector valued distributions* $\vec{S} \in \mathcal{E}'(E)$ *and* $\vec{T} \in \mathcal{E}(F)$ *which are not* $*_\theta$*-composable.* In fact, we put

$$\vec{S}=\sum_{n=0}^{\infty}\tau_n\delta\otimes\vec{e}_n,\quad \vec{T}=\sum_{n=0}^{\infty}\tau_{-n}\alpha\otimes\vec{e}'_n,$$

where α is a positive element of $\mathcal{D}$. It is easy to verify that $\vec{S} \in \mathcal{E}'(E)$ and $\vec{T} \in \mathcal{E}(F)$. Let ϕ be any positive element of $\mathcal{D}$. Then an easy calculation yields

$$[(\check{\vec{S}}*\phi)\vec{T}]_\theta=\sum_{n=0}^{\infty}\tau_{-n}(\phi\alpha) \in \mathcal{E}.$$

This is not an element of $\mathcal{D}'_{L^1}$. For, otherwise the sequence $\{\int[(\check{\vec{S}}*\phi)\vec{T}]_\theta\alpha_n(x)dx\}$ would be bounded for any sequence $\{\alpha_n\}$ of multiplicators, which is evidently a contradiction. Consequently $\vec{S}$ and $\vec{T}$ are not $*_\theta$-composable.

EXAMPLE 3. Taking E as the space of the rapidly decreasing suquences $\vec{\xi}=(\xi_n)_{n=0,1,2,\dots}$ of complex numbers with semi-norms $\|\xi\|_p=\sum_{n=0}^{\infty}n^p|\xi_n|$, $p=1, 2, \dots$, it is a nuclear space of type (**F**). Let F be the strong dual of E, which is the space of the slowly increasing sequences $\vec{\eta}=(\eta_n)_{n=0,1,2\dots}$ of complex numbers. We take $\theta(\vec{\xi}, \vec{\eta})=\sum_{n=0}^{\infty}\xi_n\eta_n$. Therefore G is assumed to be the space of complex numbers. Then θ is hypocontinuous, but not continuous. Consider the spaces of vector valued distributions $\mathcal{S}(E)$ and $\mathcal{S}'(F)$. Here the convolution map $*: \mathcal{S}\times\mathcal{S}'\to\mathcal{D}'$ is not continuous. Then we can show that *there exist vector valued distributions* $\vec{S}\in\mathcal{S}(E)$ *and* $\vec{T}\in\mathcal{S}'(F)$ *which are not* $*_\theta$*-composable.* Let $\vec{e}_n$, $\vec{e}'_n$ be chosen as in the preceding example, and put $\vec{S}=\sum_{n=0}^{\infty}\tau_n\alpha\otimes\vec{e}_n\in\mathcal{S}(E)$ and $\vec{T}=\sum_{n=0}^{\infty}\tau_{-n}\beta\otimes\vec{e}'_n\in\mathcal{S}'(F)$ for positive α, $\beta\in\mathcal{D}$. Then we have for any positive $\phi\in\mathcal{D}$

$$[\vec{S}(\check{\vec{T}}*\phi)]_\theta=\sum_{n=0}^{\infty}\tau_n\alpha(\tau_n\check{\beta}*\phi)\in\mathcal{E}.$$

If the θ-convolution $\vec{S}*_\theta\vec{T}$ is defined, we must have $[\vec{S}(\check{\vec{T}}*\phi)]_\theta\in L^1$. But it is a contradiction as follows:

$$\begin{aligned}\int[\vec{S}(\check{\vec{T}}*\phi)]_\theta dx&=\sum_{n=0}^{\infty}\int(\tau_n\alpha*\tau_{-n}\beta)\phi\,dx\\&=<\sum_{n=0}^{\infty}(\alpha*\beta),\phi>=\infty.\end{aligned}$$

It follows from this example that we cannot generally define the convolution map $*_\theta$ of $\mathcal{S}'(E)\times\mathcal{O}'_C(F)$ into $\mathcal{D}'(G)$.

Example 4. Let $E=\mathcal{E}$ and let $F=\mathcal{S}'$. Let θ be the multiplicative product map of $\mathcal{E}\times\mathcal{S}'$ into $\mathcal{D}'$. Then θ is hypocontinuous, but is not continuous. Now we shall consider two elements $\vec{S}=\delta(y-x)\in\mathcal{E}'(\mathcal{E})$ and $\vec{T}=\delta(x+y)\in\mathcal{S}(\mathcal{S}')$. Since $\mathcal{E}'$, $\mathcal{S}\subset\mathcal{O}'_C$, it follows from Example 1 that the θ-convolution $\vec{S}*_\theta\vec{T}$ can be defined and belongs to $\mathcal{O}'_C(\mathcal{D}')$. However, we can show that $\vec{S}*_\theta\vec{T}$ *does not belong to* $\mathcal{S}(\mathcal{D}')$. In fact, $\mathcal{F}(\vec{S})=e^{-2\pi i\xi\cdot y}$ and $\mathcal{F}(\vec{T})=e^{2\pi i\xi\cdot y}$. Therefore we have $\theta(\mathcal{F}(\vec{S})(\xi),\mathcal{F}(\vec{T})(\xi))=1$. On the other hand $\mathcal{F}(\delta\otimes 1)=1$, which leads us to a contradiction: $\vec{S}*_\theta\vec{T}=\delta\otimes 1\notin\mathcal{S}(\mathcal{D}')$.

The following example indicates some cases where the convolution map $*: \mathcal{H}\times\mathcal{K}\to\mathcal{L}$ is continuous and the spaces $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ satisfy the properties stipulated in Theorem 2, which we may apply to the existence and the assigned continuity of the convolution map $*_\theta$: $\mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(G)$.

Example 5. We shall consider the twelve cases given in the following table, where $\mathcal{H}$, $\mathcal{K}$, $\mathcal{L}$ are chosen from the classical spaces of distributions.

	$\mathcal{H}$	$\mathcal{K}$	$\mathcal{L}$
1	$\mathcal{D}$	$\mathcal{D}$	$\mathcal{D}$
2	$\mathcal{D}$	$\mathcal{E}$	$\mathcal{E}$
3*	$\mathcal{D}$	$\mathcal{E}'$	$\mathcal{E}'$
4	$\mathcal{D}$	$\mathcal{S}$	$\mathcal{S}$
5*	$\mathcal{D}$	$\mathcal{S}'$	$\mathcal{S}'$
6	$\mathcal{S}$	$\mathcal{S}$	$\mathcal{S}$

	$\mathcal{H}$	$\mathcal{K}$	$\mathcal{L}$
7*	$\mathcal{S}$	O'_c	O'_c
8	O'_c	O'_c	O'_c
9	$\mathcal{E}'$	$\mathcal{E}'$	$\mathcal{E}'$
10*	$\mathcal{E}'$	$\mathcal{S}$	O'_c
11	$\mathcal{E}'$	$\mathcal{S}'$	$\mathcal{S}'$
12	$\mathcal{E}'$	O'_c	O'_c

In each case the continuity of the convolution map $*: \mathcal{H}\times\mathcal{K}\to\mathcal{L}$ is easily proved. $\mathcal{H}$ is a nuclear $\dot{\mathcal{B}}$-normal space, possesses the strict approximation property and $\mathcal{L}'_c$ is barrelled. Therefore according to Theorem 2 we can infer that *the convolution map* $*_\theta$: $\mathcal{H}(E)\times\mathcal{K}(F)\to\mathcal{L}(G)$ *is defined and separately quasi-continuous* (*resp. hypocontinuous, resp. continuous*) *when* θ *is separately continuous* (*resp. hypocontinuous, resp. continuous*). It is to be noticed that in the cases 3*, 5*, 7*, 10* the spaces indicated in the last column cannot generally be replaced by the spaces $\mathcal{D}$, $\mathcal{S}$, $\mathcal{S}$ and $\mathcal{S}$ respectively. The proof may be carried out just as in Example 4 and shall be omitted.

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Maritime Safety Academy, Kure.