

Analysis of Partially Balanced Incomplete Block Designs

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1. Introduction and Summary.

In 1959, J. Ogawa [7] dealt with the analysis of association algebra, introduced by R. C. Bose [2], and the relationship algebra, introduced by A. T. James [5], of the partially balanced incomplete block designs (PBIBD). He obtained all of the irreducible constituents in the direct decomposition of the relationship algebra of the design. His results, however, were concerned with a restrictive case which might be considered to be regular PBIBD as was defined by Bose [1] in the case of group divisible incomplete block designs, because a restrictive condition " $BT_i^* \neq T_i^*B$ for all T_i^* " was imposed on the relationship matrices. Theorem I of this paper will throw light on the meaning of this restriction. Another restriction imposed implicitly on the relationship algebra of PBIBD, was that the ideal of the algebra related to the block relationship did not degenerate to zero.

We have succeeded in removing those restrictions and completing the analysis of partially balanced incomplete block designs. In PBIBD, NN' belongs to the association algebra \mathfrak{A} and it will be seen in Theorem I of this paper that the magnitude of each density ρ of its spectral expansion in \mathfrak{A} determines the property of the corresponding component of the treatment sum of squares (S. S.) to be either orthogonal to, or confounded with, or partially confounded with, the block space. The dimension of the relationship algebra \mathfrak{R} and the unique decomposition of its unit element into mutually orthogonal principal idempotents will be given in Theorem II. Examination of the ideal related to block relationship will give an inequality which appears in Theorem III. Although our inequality is essentially the same with the one due to W. S. Connor and W. H. Clatworthy [3] in a certain sense, the former is more substantial than the latter in that it will make clear the significance of the reduction in the lower bound of the number of blocks in a PBIBD. It includes, as its special cases, those inequalities given by Fisher in balanced incomplete block designs (BIBD) and by Bose [1] in group divisible incomplete block designs. Complete Table for the analysis of variance of PBIBD will also be given.

2. Association scheme and association algebra.

We say, following Bose [2], that an association scheme is defined in a set

of v elements or objects which are called treatments, if the elements of the set satisfy the following conditions (a), (b) and (c):

(a) Any two elements of the set are either 1st, or 2nd, ..., or m -th associates ($1 \leq m < v$), the relation of association being symmetrical. Each element α is said to be the 0-th associate of itself.

(b) Each element α of the set has n_i ($n_i \geq 1$) i -th associates, the number n_i being independent of the individual element α .

(c) If any two elements α and β of the set are i -th associates, then the number of elements γ which are j -th associates of α , and at the same time k -th associates of β , is p_{jk}^i , the number p_{jk}^i being independent of the pair of i -th associates α and β .

If we number those elements or treatments from 1 to v in some way but once for all, we can define the association matrices as a matrix representation of the association scheme as follows:

$$A_i = \|a_{\alpha\beta}^i\|, \quad \alpha, \beta = 1, 2, \dots, v; \quad i = 0, 1, \dots, m, \quad (1)$$

where

$$a_{\alpha\beta}^i = \begin{cases} 1, & \text{if } \alpha\text{-th and } \beta\text{-th treatments are } i\text{-th associates,} \\ 0, & \text{otherwise.} \end{cases}$$

The following are known [7] as the immediate consequences of the definition of association scheme:

(i) Each of the $v \times v$ matrices A_i is symmetric. In particular, $A_0 (= I_v)$ is the unit matrix.

$$(ii) \quad \sum_{i=0}^m A_i = G_v, \quad (2)$$

where G_v is the matrix whose elements are all unity. The relation shows that the association matrices A_0, \dots, A_m are linearly independent.

$$(iii) \quad A_i A_j = A_j A_i = \sum_{k=0}^m p_{ij}^k A_k. \quad (3)$$

Those statements show that if we consider the linear closure \mathfrak{A} of matrices A_0, A_1, \dots, A_m over the real field, \mathfrak{A} is $(m+1)$ -dimensional commutative algebra containing unit, which is called the association algebra of treatments [2] [7]. We shall denote the association algebra $\mathfrak{A} = [A_0, \dots, A_m]$ by indicating the basis of the algebra regarded as a vector space. Throughout this paper, we shall denote an algebra as $[C_1, C_2, \dots, C_s]$ by indicating the basis C_1, C_2, \dots, C_s of the algebra regarded as a vector space in the bracket [].

The association algebra \mathfrak{A} is completely reducible and each of its irreducible constituents is linear. In order to decompose the association algebra \mathfrak{A} into its irreducible constituents, it is sufficient to find all of the irreducible constituents of the regular representation of \mathfrak{A} [7].

Let

$$P_i = \|p_{ji}^k\|, \quad i = 0, 1, \dots, m, \tag{4}$$

then the relations $A_i (A_0, \dots, A_m) = (A_0, \dots, A_m) P_i' (i = 0, 1, \dots, m)$, show that the mapping of \mathfrak{A} onto itself defines the regular representation of the association algebra:

$$(\mathfrak{A}): \quad A_i \rightarrow P_i, \quad i = 0, 1, \dots, m. \tag{5}$$

The algebra \mathfrak{B} generated by P_0, P_1, \dots, P_m is isomorphic to \mathfrak{A} .

We may find a non-singular real matrix

$$C = \begin{bmatrix} c_{00} & \dots & c_{0m} \\ & \dots & \\ c_{m0} & \dots & c_{mm} \end{bmatrix}, \tag{6}$$

which makes all P_i diagonal simultaneously, such as

$$CP_iC^{-1} = \begin{bmatrix} z_{0i} & & \\ & z_{1i} & \\ & & \dots \\ & & & z_{mi} \end{bmatrix}, \quad i = 0, 1, \dots, m. \tag{7}$$

The matrix C is uniquely determined apart from the order of its rows and the proportionality factor of each row. We can choose $c_{0i} = 1$ and $z_{0i} = n_i$ for all i .

The mutually orthogonal idempotents of the algebra \mathfrak{B} will be obtained as

$$P_i^\# = \left(\sum_{u=0}^m c_{iu} z_{iu} \right)^{-1} \sum_{j=0}^m c_{ij} P_j, \quad i = 0, 1, \dots, m. \tag{8}$$

The mutually orthogonal idempotents of \mathfrak{A} corresponding to $P_i^\#$ are, therefore, given by

$$A_i^\# = \left(\sum_{u=0}^m c_{iu} z_{iu} \right)^{-1} \sum_{j=0}^m c_{ij} A_j, \quad i = 0, 1, \dots, m. \tag{9}$$

\mathfrak{A} may also be expressed by indicating its ideal basis as

$$\mathfrak{A} = [A_0^\#, \dots, A_m^\#]. \tag{10}$$

The rank or trace α_i of matrix $A_i^\#$, that is the multiplicity of the irreducible constituents in regular representation of \mathfrak{A} , is given by

$$\alpha_i = \text{tr}(A_i^\#) = c_{i0} v \left(\sum_{u=0}^m c_{iu} z_{iu} \right)^{-1}. \tag{11}$$

Clearly,

$$A_0^\# + A_1^\# + \dots + A_m^\# = I_v, \tag{12}$$

and

$$\alpha_0 = \text{tr}(A_0^\#) = 1, \quad \sum_{i=0}^m \alpha_i = v. \quad (13)$$

Each of the matrices $A_i^\#$ defines an invariant subspace under \mathfrak{A} in v -dimensional treatment space E_v generated by the coefficient vectors of all linear function of treatment parameters. These $m+1$ subspaces are mutually orthogonal to each other. Among them, the one-dimensional invariant subspace defined by $A_0^\# = v^{-1}G_v$ corresponds to the grand mean of the parameters, and the others defined by $A_i^\#$ are mutually orthogonal subspaces, each of which is generated by α_i ($=\text{tr}(A_i^\#)$) independent coefficient vectors of treatment contrasts. Hence, the set of matrices $A_i^\#$ ($i=1, \dots, m$) gives unique decomposition of $(v-1)$ -dimensional subspace of E_v , generated by the coefficient vectors of all treatment contrasts, into m mutually orthogonal subspaces. The decomposition is uniquely determined by the association scheme, and each $A_i^\#$ is the projection operator to each of those mutually orthogonal subspaces, respectively.

3. PBIBD and its relationship algebra.

Suppose an association scheme of m associate classes is defined among the v treatments and, consider the arrangements of these in r replications to the plots of b blocks each of which consists of k plots.

We say the design is PBIBD if the arrangements of those v treatments, satisfying (a), (b) and (c) in §1, satisfy the following conditions (d), (e) and (f):

- (d) Each of b (>1) blocks contains k (>1) different treatments.
- (e) Each of v (>1) treatments occurs in r (>1) blocks.
- (f) Any two treatments which are i -th associates occur together in λ_i (≥ 0) blocks.

$v, n_i, p_{jk}^i, b, r, k$ and λ_i are called the parameters of PBIBD.

We shall number those n ($=rv=bk$) plots from 1 to n in some way but once for all, and define the incidence matrices of a PBIBD such that, incidence matrices for

$$\begin{aligned} \text{treatments: } \Phi &= \|\varphi_{f\alpha}\|, & (n \times v), \\ \text{blocks} &: \Psi = \|\psi_{fa}\|, & (n \times b), \quad \text{and} \\ \text{design} &: N = \Phi'\Psi = \|n_{\alpha a}\|, & (v \times b), \end{aligned} \quad (14)$$

where

$$\varphi_{f\alpha} = \begin{cases} 1, & \text{if } \alpha\text{-th treatment occurs in } f\text{-th plot,} \\ 0, & \text{otherwise,} \end{cases}$$

$$v_{fa} = \begin{cases} 1, & \text{if } f\text{-th plot belongs to } a\text{-th block,} \\ 0, & \text{otherwise,} \end{cases}$$

$$n_{aa} = \begin{cases} 1, & \text{if } a\text{-th treatment occurs in } a\text{-th block,} \\ 0, & \text{otherwise.} \end{cases}$$

There are four groups of relationship matrices corresponding to four types of relationships induced among the plots [7]:

(1°) Identity relationship matrix: $I = \|\delta_{fg}\|$, *i.e.*, the unit matrix of degree n , where δ_{fg} is the Kronecker δ .

(2°) Universal relationship matrix: $G = \|g_{fg}\|$, $g_{fg} = 1$, *i.e.*, the matrix of degree n whose elements are all unity.

(3°) Block relationship matrix: $B = \Psi\Psi' = \|b_{fg}\|$,

where
$$b_{fg} = \begin{cases} 1, & \text{if } f\text{-th and } g\text{-th plots belong to the same block,} \\ 0, & \text{otherwise.} \end{cases}$$

(4°) Treatment relationship matrices: $T_i = \Phi A_i \Phi' = \|t_{fg}^i\|$, $i = 0, 1, \dots, m$,

where
$$t_{fg}^i = \begin{cases} 1, & \text{if } f\text{-th and } g\text{-th plots receive respectively } \alpha\text{-th} \\ & \text{and } \beta\text{-th treatments which are } i\text{-th associates,} \\ 0, & \text{otherwise.} \end{cases}$$

The following formulas are known as the immediate consequences of the definition of the relationship matrices [7]:

$$\begin{aligned} \text{(i)} \quad & \sum_{i=0}^m T_i = G, \\ \text{(ii)} \quad & G^2 = nG, \\ \text{(iii)} \quad & BG (= GB) = kG, \quad B^2 = kB, \\ \text{(iv)} \quad & T_i G (= GT_i) = rn_i G. \end{aligned} \tag{15}$$

The algebra

$$\mathfrak{R} = \{I, G, B, T_i; i = 1, 2, \dots, m\} \tag{16}$$

generated by the relationship matrices, $I, G, B, T_i (i = 1, 2, \dots, m)$, over the real field is called the relationship algebra of a PBIBD [7]. It is, however, convenient to use as the generators of the algebra the following (4*) in place of (4°), *i.e.*,

$$(4^*) \quad T_i^* = \Phi A_i^* \Phi' \left(= \left(\sum_{u=0}^m c_{iu} z_{iu} \right)^{-1} \sum_{j=0}^m c_{ij} T_j \right), \quad i = 0, 1, 2, \dots, m.$$

It is clear that the algebra generated by (1°), (2°), (3°) and (4*) is equivalent to \mathfrak{R} .

The following lemmas will be useful in the analysis of the relationship

algebra of PBIBD.

LEMMA 1. NN' belongs to the association algebra \mathfrak{A} and can be expressed as

$$NN' = \sum_{j=0}^m \lambda_j A_j = \sum_{i=0}^m \rho_i A_i^{\#}, \quad (17)$$

where the last member of the expression is the spectral expansion of NN' in \mathfrak{A} . The densities

$$\rho_i = \sum_{j=0}^m \lambda_j z_{ij}, \quad i = 0, 1, \dots, m, \quad (18)$$

are the latent roots of NN' and satisfy the inequality

$$0 \leq \rho_i \leq rk, \quad i = 0, 1, \dots, m. \quad (19)$$

The multiplicity of ρ_i is α_i , respectively. In particular,

$$\rho_0 = rk = \sum_{i=0}^m n_i \lambda_i. \quad (20)$$

PROOF. For any pair of α and β , the element of NN' in α -th row and β -th column can be expressed as

$$(NN')_{\alpha\beta} = \sum_{a=1}^b n_{\alpha a} n_{\beta a} = \lambda_0 a_{\alpha 0}^2 + \lambda_1 a_{\alpha 1}^2 + \dots + \lambda_m a_{\alpha m}^2. \quad (21)$$

Thus we have, $NN' = \sum_{j=0}^m \lambda_j A_j$. The formula shows that NN' belongs to \mathfrak{A} . We can, therefore, express NN' by using the ideal basis of \mathfrak{A} as the last member of (17). The expressions (17) and (9) lead to (18), in particular, to (20).

As $(NN' - \rho_i I)A_i^{\#} = 0$, ρ_i is the latent root of NN' and the column vectors of $A_i^{\#}$ are the latent vectors of NN' corresponding to the latent root ρ_i . The rank of $A_i^{\#}$, i.e., the multiplicity of ρ_i , is α_i .

Since NN' is a positive semi-definite matrix, ρ_i is non-negative. On the other hand, as the matrix $(rk)^{-1}NN'$ is doubly stochastic, the result due to Fréchet [4] on the bound of the latent roots shows that $|(rk)^{-1}\rho_i| \leq 1$. Thus we obtain (19).

- LEMMA 2. (i) *The $m+3$ matrices, $I, G, B, T_i^{\#}$ ($i=1, 2, \dots, m$), are linearly independent.*
- (ii) $T_i^{\#} T_j^{\#} = r \delta_{ij} T_i^{\#}, \quad i, j = 0, 1, \dots, m. \quad (22)$
- (iii) $T_0^{\#} = v^{-1}G.$
- (iv) $T_i^{\#} B T_j^{\#} = \delta_{ij} \rho_i T_i^{\#}, \quad i, j = 0, 1, \dots, m.$

PROOF. To prove (i), it is sufficient to prove that $I, B, T_0, T_1, \dots, T_m$ are linearly independent. It may easily be proved by the definition of B and T_i

with $r > 1$ and $k > 1$. As it is easy to prove (ii) and (iii), we shall omit to do so. The formula (iv) may be proved by using (17) as

$$\begin{aligned} T_i^\# BT_j^\# &= \mathcal{O} A_i^\# N N' A_j^\# \mathcal{O}' = \mathcal{O} \delta_{ij} \rho_i A_i^\# \mathcal{O}' \\ &= \delta_{ij} \rho_i T_i^\#. \end{aligned}$$

Each $T_i^\#$ ($i = 1, \dots, m$) is the relationship matrix corresponding respectively to $A_i^\#$. The column vectors of the former define an invariant subspace under $\mathcal{O}\mathcal{A}\mathcal{O}'$ in the n -dimensional observation space E_n generated by all of the coefficient vectors of the linear functions of observation vector. The projection operator to the subspace is $r^{-1}T_i^\#$. The subspaces defined by $T_1^\#, T_2^\#, \dots, T_m^\#$ are mutually orthogonal to each other and correspond to those defined by $A_1^\#, A_2^\#, \dots, A_m^\#$ in E_v , respectively. These subspaces are, of course, orthogonal to the subspace defined by $T_0^\#$ (or G) corresponding to $A_0^\#$.

4. Analysis of the relationship algebra.

The relationship algebra of a PBIBD is generated by symmetric matrices, and therefore it is completely reducible. All irreducible components of this algebra have been obtained by Ogawa [7] under certain restrictive conditions such that, $BT_i^\# \neq T_i^\#B$ for all $T_i^\#$ and, though implicitly stated, the ideal corresponding to B does not degenerate to zero. We shall present here the general results free from any of these restrictions. The following theorem will serve to complete the decomposition of \mathfrak{R} in relation to the treatment relationships.

THEOREM I. *Each of the components of the treatment sum of squares (S.S.) which corresponds respectively to each of the mutually orthogonal families of treatment contrasts induced by the association scheme, can be classified into one of the following three cases according to the magnitude of the corresponding density ρ in the spectral expansion of NN' :*

(A) *Orthogonal case. The conditions $\rho_i = 0$ and $BT_i^\# (= T_i^\#B) = 0$ are equivalent. In this case, $[T_i^\#]$ is the one-dimensional two-sided ideal of \mathfrak{R} , and the principal idempotent of the ideal is $r^{-1}T_i^\#$. The component S. S. of α_i degrees of freedom corresponding to $A_i^\#$ and being defined by $r^{-1}T_i^\#$ is orthogonal to the block space.*

(B) *Confounded case. The conditions $\rho_i = rk$ and $BT_i^\# (= T_i^\#B) = kT_i^\#$ ($i \neq 0$) are equivalent. In this case, $[T_i^\#]$ is the one-dimensional two-sided ideal of \mathfrak{R} , and the principal idempotent of the ideal is $r^{-1}T_i^\#$. The component S. S. of α_i degrees of freedom corresponding to $A_i^\#$ and being defined by $r^{-1}T_i^\#$ is completely confounded with the block space.*

(C) *Partially confounded case. Three conditions; $0 < \rho_i < rk$; $BT_i^\# \neq T_i^\#B$; and $T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B$ are linearly independent; are equivalent. In this case, $[T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B]$ is four-dimensional two-sided ideal of \mathfrak{R} and is isomorphic to the complete 2×2 matrix algebra. The principal idempotent of the ideal*

is

$$E_i^{\#(2)} = \frac{1}{rk - \rho_i} \left(kT_i^{\#} - BT_i^{\#} - T_i^{\#}B + \frac{r}{\rho_i} BT_i^{\#}B \right). \quad (23)$$

The component S. S. of $2\alpha_i$ degrees of freedom corresponding to $A_i^{\#}$ and being defined by $E_i^{\#(2)}$ is partially confounded with the block space, the confounding coefficient being ρ_i/rk . The non-principal idempotent $F_i^{\#(1)}$ of the ideal being orthogonal to the block space is

$$F_i^{\#(1)} = \frac{k}{rk - \rho_i} \left(I - \frac{1}{k}B \right) T_i^{\#} \left(I - \frac{1}{k}B \right). \quad (24)$$

The residual idempotent of the ideal being orthogonal to $F_i^{\#(1)}$ and confounded with the block space is

$$B_i^{\#(1)} = \frac{1}{k\rho_i} BT_i^{\#}B. \quad (25)$$

The degrees of freedom of these component idempotents are α_i .

As to the occurrence of these cases in a PBIBD, we can state the following:

- (i) When and only when a PBIBD is connected, there occurs no component of the treatment S. S. which is completely confounded with the block space.
- (ii) When and only when a PBIBD degenerates into the complete block design, all components of the treatment S. S. are orthogonal to the block space.
- (iii) In a PBIBD with three or more associate classes, all of the cases (A), (B) and (C) may occur simultaneously.

The following seven lemmas will be useful in the proof of this theorem.

LEMMA 3. The condition $\rho_i = 0$ is necessary and sufficient for the condition $BT_i^{\#} (= T_i^{\#}B) = 0$.

PROOF. Suppose $\rho_i = 0$, then Lemma 2 shows that $T_i^{\#}BT_i^{\#} = \rho_i T_i^{\#} = 0$. It follows that the symmetric matrix $BT_i^{\#}B$ is nilpotent of order 2. We have therefore $BT_i^{\#}B = 0$. Using the results we can prove by similar arguments as above that $BT_i^{\#} + T_i^{\#}B = 0$. Multiplying by B from the left, we have $BT_i^{\#} = 0$, and from the right, we have $T_i^{\#}B = 0$.

Conversely, suppose $BT_i^{\#} = 0$, then $0 = T_i^{\#}BT_i^{\#} = \rho_i T_i^{\#}$. As $T_i^{\#} \neq 0$, it follows $\rho_i = 0$.

LEMMA 4. The condition $\rho_i = rk$ is necessary and sufficient for the condition $BT_i^{\#} (= T_i^{\#}B) = kT_i^{\#}$.

PROOF. If we replace B in Lemma 3 by $kI - B$, the lemma will easily be proved.

LEMMA 5. Any one of the following four conditions is necessary and sufficient for the remaining three.

- (i) *The matrices $T_i^\#$ and $BT_i^\#$ are linearly dependent.*
- (ii) *The matrices $T_i^\#$ and B are commutative, i.e., $T_i^\#B = BT_i^\#$.*
- (iii) *$BT_i^\# = 0$ or $BT_i^\# = kT_i^\#$.*
- (iv) *$\rho_i = 0$ or $\rho_i = rk$.*

PROOF. We shall prove the lemma along the following steps, such as, (i)→(ii)→(iv)↔(iii)→(i). Suppose that the condition (i) holds, then there exist some constants α and β , not both zero, and $\alpha T_i^\# + \beta BT_i^\# = 0$ holds. As we can assume $\beta \neq 0$ without loss of generality, we have $BT_i^\# = aT_i^\#$ for some constant a . The expression shows that $T_i^\#B = BT_i^\#$, because the matrices $T_i^\#$ and B are both symmetric. Accordingly, the condition (i) implies the condition (ii).

Suppose that the condition (ii) holds, then multiplying both members of $T_i^\#B = BT_i^\#$ by $T_i^\#B$ from the left and by $T_i^\#$ from the right, we obtain the equation $(rk - \rho_i)\rho_i T_i^\# = 0$. As $T_i^\# \neq 0$, we obtain $\rho_i = 0$ or $\rho_i = rk$. The condition (ii), therefore, implies the condition (iv). Lemmas 3 and 4 show that the conditions (iii) and (iv) are equivalent. It is easy to see that the condition (iii) implies (i).

LEMMA 6. *The condition; $T_i^\#$ and $BT_i^\#$ are linearly independent; is necessary and sufficient for the condition; $T_i^\#, BT_i^\#, T_i^\#B$ and $BT_i^\#B$ are linearly independent.*

PROOF. Assume that, $T_i^\#$ and $BT_i^\#$ are linearly independent and for some constants a, b, c and d , we have

$$aT_i^\# + bBT_i^\# + cT_i^\#B + dBT_i^\#B = 0. \tag{26}$$

Transposing (26), we have $b = c$. Multiplying (26) by $T_i^\#$ from the right, we have $(ar + b\rho_i)T_i^\# + (br + d\rho_i)BT_i^\# = 0$. As $T_i^\#$ and $BT_i^\#$ are linearly independent, we have

$$ar + b\rho_i = 0, \quad br + d\rho_i = 0. \tag{27}$$

Multiplying (26) by $T_i^\#B$ from the left and by $BT_i^\#$ from the right, we have

$$(a + 2bk + dk^2)\rho_i^2 T_i^\# = 0. \tag{28}$$

As $T_i^\#$ is not equal to zero, we have

$$(a + 2bk + dk^2)\rho_i^2 = 0. \tag{29}$$

Lemma 5 shows that, under the assumption, ρ_i is equal neither to 0 nor to rk . Thus, using (27) and (29), we have $a = b (= c) = d = 0$. $T_i^\#, BT_i^\#, T_i^\#B$ and $BT_i^\#B$ are, therefore, linearly independent.

The converse is clear and the proof is complete.

LEMMA 7. *Any one of the following three conditions is necessary and sufficient for the remaining two.*

- (i) *A PBIBD is connected.*
- (ii) *There exists a positive integer p such that all elements of the matrix $(NN')^p$ are positive.*
- (iii) *Maximum latent root $\rho_0 = rk$ of the matrix NN' is simple.*

PROOF. Bose and Mesner [2] has pointed out that (iii) is necessary for (i). We shall, therefore, prove that (i) is necessary for (ii) and (ii) is necessary for (iii).

The condition (ii) states that, for any pair of α and β , we have,

$$((NN')^p)_{\alpha\beta} = \sum_{\alpha_1, \alpha_1 a_2, \dots, \alpha_{p-1} a_p} n_{\alpha a_1} n_{\alpha_1 a_2} n_{\alpha_2 a_3} \dots n_{\alpha_{p-1} a_p} n_{\beta a_p} > 0. \tag{30}$$

As every element of the matrix N is non-negative, the inequality (30) shows that there exists at least an alternately associated treatment-block chain:

$$\alpha, a_1, \alpha_1, a_2, \alpha_2, \dots, \alpha_{p-1}, a_p, \beta, \tag{31}$$

such that,

$$n_{\alpha a_1} n_{\alpha_1 a_2} n_{\alpha_2 a_3} \dots n_{\alpha_{p-1} a_p} n_{\beta a_p} > 0. \tag{32}$$

Hence the design is connected. Thus (i) is necessary for (ii).

From (17) of Lemma 1, we have

$$(NN')^p = (rk)^p \left[A_0^\# + \sum_{i=1}^m \left(\frac{\rho_i}{rk} \right)^p A_i^\# \right]. \tag{33}$$

Suppose that the condition (iii) holds. Lemma 1 shows that the inequalities $0 \leq \rho_i / rk < 1$ hold for all ρ_i except ρ_0 . Thus, for any positive number ϵ and for any pair α and β , we can choose an integer p such that the inequality

$$((NN')^p)_{\alpha\beta} \geq (rk)^p \left(\frac{1}{v} - \epsilon \right) \tag{34}$$

holds. The condition (ii) is, therefore, necessary for the condition (iii).

- LEMMA 8. (i) *BIBD is always connected.*
 (ii) *PBIBD is not always connected.*

PROOF. As $k > 1$, λ is necessarily positive in a BIBD, every element of NN' is positive. Hence, by Lemma 7, BIBD is always connected. In the case of a PBIBD, not all of the λ_j are necessarily positive. It is, therefore, easy to illustrate an unconnected PBIBD when $m \geq 2$.

LEMMA 9. *A PBIBD is reduced to a complete block design if and only if $\rho_0 = rk$, and $\rho_i = 0$ for all $i = 1, 2, \dots, m$.*

PROOF. If a design is complete block, then

$$v = k, \quad b = r \quad \text{and} \quad NN' = \begin{bmatrix} b & \dots & b \\ & \dots & \\ b & \dots & b \end{bmatrix}. \quad (35)$$

The latent roots of NN' are, therefore, $\rho_0 = rk$ and $\rho_i = 0$ for all $i = 1, 2, \dots, m$. Conversely, if the conditions $\rho_0 = rk$ and $\rho_i = 0$ for all $i = 1, 2, \dots, m$ hold, then we have

$$NN' = \rho_0 A_0^* = \frac{rk}{v} G_v. \quad (36)$$

The expression (36) implies that all of the elements of NN' are rk/v . Especially, for diagonal elements, we have $rk/v = r$. Thus we have $v = k$ and hence the design is complete block.

Proof of Theorem I.

Lemma 1 shows that it is possible to classify the spectral densities or latent roots ρ_i of NN' , corresponding to A_i^* ($i \neq 0$), into one of the three cases (A): $\rho_i = 0$, (B): $\rho_i = rk$, and (C): $0 < \rho_i < rk$.

(A) Lemma 3 shows that the condition $\rho_i = 0$ is necessary and sufficient for the condition $BT_i^* (= T_i^*B) = 0$. In this case, $IT_i^* = T_i^*I = T_i^*$, $BT_i^* = T_i^*B = 0$, $GT_i^* = T_i^*G = 0$ and $T_i^*T_j^* = r\delta_{ij}T_i^*$ show that $[T_i^*]$ is the one-dimensional two-sided ideal of \mathfrak{R} and, evidently, $r^{-1}T_i^*$ is the principal idempotent of the ideal. The relation $BT_i^* (= T_i^*B) = 0$ shows that the space generated by the row (or column) vectors of T_i^* is orthogonal to the block space. The component of S. S. with $\alpha_i (= \text{tr}(r^{-1}T_i^*) = \text{tr}(A_i^*))$ degrees of freedom corresponding to A_i^* and being defined by $r^{-1}T_i^*$ is orthogonal to the block space.

(B) Lemma 4 shows that the condition $\rho_i = rk$ is necessary and sufficient for the condition $BT_i^* (= T_i^*B) = kT_i^*$. In this case, $IT_i^* = T_i^*I = T_i^*$, $BT_i^* = T_i^*B = kT_i^*$, $GT_i^* = T_i^*G = 0$ and $T_i^*T_j^* = r\delta_{ij}T_i^*$ show that $[T_i^*]$ is the one-dimensional two-sided ideal of \mathfrak{R} and, evidently, $r^{-1}T_i^*$ is the principal idempotent of the ideal. As $k^{-1}BT_i^* = k^{-1}T_i^*B = T_i^*$, the component of S. S. with α_i degrees of freedom corresponding to A_i^* and being defined by $r^{-1}T_i^*$ is completely confounded with the block space.

(C) Lemma 5 shows that the condition $0 < \rho_i < rk$ is necessary and sufficient for the condition $BT_i^* \neq T_i^*B$ and, consequently, Lemma 6 shows that T_i^* , BT_i^* , T_i^*B , BT_i^*B are linearly independent. In this case, since the left representations of the generators of \mathfrak{R} :

$$I[T_i^*, BT_i^*, T_i^*B, BT_i^*B] = [T_i^*, BT_i^*, T_i^*B, BT_i^*B] \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$G[T_i^*, BT_i^*, T_i^*B, BT_i^*B] = [T_i^*, BT_i^*, T_i^*B, BT_i^*B] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad (37)$$

$$B[T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B] = [T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B] \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & k & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & k \end{bmatrix},$$

$${}^{(N_i)}T_j^\# [T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B] = [T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B] \delta_{ij} \begin{bmatrix} r & \rho_i & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & r & \rho_i \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

as well as the right representations of the generators of \mathfrak{R} in transposed form of (37) hold, the subalgebra $[T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B]$ is a four-dimensional two-sided ideal of \mathfrak{R} . It can be seen that the ideal is irreducible and is isomorphic to the complete 2×2 matrix algebra, because we can find a matrix basis $\{f_{ij}; i, j=1, 2\}$ of this ideal satisfying $f_{ij}f_{kl} = \delta_{jk}f_{il}$ for all $i, j, k, l=1, 2$ by using (37) as shown in the following:

$$\begin{aligned} f_{11} &= \frac{1}{rk - \rho_i} (kT_i^\# - T_i^\#B), \\ f_{12} &= \frac{1}{(rk - \rho_i)\rho_i} (rT_i^\#B - \rho_i T_i^\#), \\ f_{21} &= \frac{1}{rk - \rho_i} (kBT_i^\# - BT_i^\#B), \\ f_{22} &= \frac{1}{(rk - \rho_i)\rho_i} (rBT_i^\#B - \rho_i BT_i^\#). \end{aligned} \tag{38}$$

It can be seen that the principal idempotent $E_i^{\#(2)}$ of the ideal is (23), its trace being $2\alpha_i$. The relation $(r^{-1}T_i^\# - k^{-1}B)^2 T_i^\# = (1 - \rho_i/rk)T_i^\#$ shows that each vector of the subspace generated by the column vectors of $T_i^\#$ is partially confounded with the block space with constant confounding coefficient ρ_i/rk [6].

The decomposition of the idempotent $E_i^{\#(2)}$ into two, though non-principal, symmetric idempotents, one of which is orthogonal to the block space and the other is confounded with the block space, will give $F_i^{\#(1)}$ and $B_i^{\#(1)}$ as was described in (24) and (25). The degrees of freedom associated with both $F_i^{\#(1)}$ and $B_i^{\#(1)}$, i.e., the traces of those idempotents, are α_i .

Lemma 7 shows that when and only when the design is connected, the case (B) can never occur. Lemma 9 shows that when and only when the design is complete block, only the case of (A) can occur. A design composed of the direct arrangement of two singular group divisible designs [1] is an example of PBIBD with three associate classes in which all of the cases (A), (B) and (C) occur simultaneously.

Now, we shall give the direct decomposition of the relationship algebra \mathfrak{R} into its irreducible components. To this end, we shall rearrange ρ_i as well as corresponding $A_i^\#$ in (17) according to the magnitude of ρ_i as follows:

$$\begin{cases} \rho_i = 0, & i = 1, 2, \dots, s, \\ 0 < \rho_i < rk, & i = s + 1, \dots, c, \\ \rho_i = rk, & i = c + 1, \dots, m. \end{cases} \quad (0 \leq s \leq c \leq m) \quad (39)$$

In the following, we shall use the suffix u exclusively for the orthogonal parts, the suffix l exclusively for partially confounded parts and the suffix j exclusively for confounded parts.

The principal idempotent $E_G^{(1)}$ of the one-dimensional two-sided ideal $[G]$ of \mathfrak{R} is, of course,

$$E_G^{(1)} = \frac{1}{n} G. \quad (40)$$

The principal idempotents $E_u^{*(1)}$ ($u = 1, \dots, s$) and $E_j^{*(1)}$ ($j = c + 1, \dots, m$) corresponding respectively to the one-dimensional two-sided ideals $[T_u^\#]$ and $[T_j^\#]$, have already been mentioned in Theorem I as

$$E_u^{*(1)} = \frac{1}{r} T_u^\#, \quad E_j^{*(1)} = \frac{1}{r} T_j^\#. \quad (41)$$

The principal idempotent $E_i^{*(2)}$ corresponding to the four-dimensional two-sided ideal $[T_i^\#, BT_i^\#, T_i^\#B, BT_i^\#B]$ has also been given in (23) of Theorem I.

In order to obtain the remaining irreducible two-sided ideals of \mathfrak{R} and their principal idempotents, we shall consider the difference algebra of \mathfrak{R} modulo $(G, T_i^\#; i = 1, \dots, m)$, *i.e.*,

$$\mathfrak{R} - (G, T_1^\#, T_2^\#, \dots, T_m^\#) \quad (42)$$

where $(G, T_1^\#, T_2^\#, \dots, T_m^\#)$ is the ideal of \mathfrak{R} generated by G and $T_i^\#$ ($i = 1, \dots, m$) and the principal idempotent of the ideal is $E_G^{(1)} + \sum_{u=1}^s E_u^{*(1)} + \sum_{l=s+1}^c E_l^{*(2)} + \sum_{j=c+1}^m E_j^{*(1)}$.

In general, this difference algebra is isomorphic to the algebra $[I, B]$, *i.e.*, the algebra generated by I and B . The latter can be decomposed into the direct sum of two mutually orthogonal one-dimensional two-sided ideals $[B]$ and $[I - k^{-1}B]$, and their principal idempotents are $k^{-1}B$ and $(I - k^{-1}B)$, respectively. In some cases, however, it may happen that the ideal of the difference algebra corresponding to $[B]$ degenerates into the null algebra, though the ideal corresponding to $[I - k^{-1}B]$ cannot degenerate into null algebra because r and k are both greater than 1. In such a degenerate case, the algebra $\mathfrak{R} - (G, T_i^\#, i = 1, \dots, m)$ is one-dimensional, and is not isomorphic but homomorphic to the algebra $[I, B]$.

The principal idempotents $E_B^{(1)}$ and $E_e^{(1)}$ of the ideals of \mathfrak{R} corresponding to the principal idempotents $k^{-1}B$ and $I - k^{-1}B$ of the difference algebra may be obtained by dropping the modulo G and $T_i^\#$ ($i = 1, \dots, m$) of the following:

$$\begin{cases} \frac{1}{k}B = E_B^{(1)} \\ I - \frac{1}{k}B = E_e^{(1)} \end{cases} \pmod{G, T_i^\#; \quad i = 1, \dots, m.} \tag{43}$$

The results may be expressed as

$$\begin{aligned} E_B^{(1)} &= \frac{1}{k}B - F_2, \\ E_e^{(1)} &= I - \frac{1}{k}B - F_3, \end{aligned} \tag{44}$$

where

$$\begin{aligned} F_2 &= \frac{1}{n}G + \sum_{l=s+1}^c B_l^{\#(1)} + \sum_{j=c+1}^m E_j^{\#(1)}, \\ F_3 &= \sum_{u=1}^s E_u^{\#(1)} + \sum_{l=s+1}^c F_l^{\#(1)}, \end{aligned}$$

and

$$\begin{aligned} B_l^{\#(1)} &= \frac{1}{k\theta_l} B T_l^\# B, & l = s + 1, \dots, c, \\ E_j^{\#(1)} &= \frac{1}{r} T_j^\#, & j = c + 1, \dots, m, \\ E_u^{\#(1)} &= \frac{1}{r} T_u^\#, & u = 1, \dots, s, \\ F_l^{\#(1)} &= \frac{1}{kr - \theta_l} \left(I - \frac{1}{k}B \right) T_l^\# \left(I - \frac{1}{k}B \right) & l = s + 1, \dots, c, \end{aligned}$$

since $E_B^{(1)}, E_e^{(1)}, F_2$ and F_3 must satisfy the following equations:

$$\begin{bmatrix} E_B^{(1)} \\ E_e^{(1)} \\ F_2 \\ F_3 \end{bmatrix} \left[E_G^{(1)} + \sum_{u=1}^s E_u^{\#(1)} + \sum_{l=s+1}^c E_l^{\#(2)} + \sum_{j=c+1}^m E_j^{\#(1)} \right] = \begin{bmatrix} 0 \\ 0 \\ F_2 \\ F_3 \end{bmatrix}.$$

We may summarize the results obtained so far in the following theorem.

THEOREM II. *The dimension of the relationship algebra of a PBIBD with m -associate classes is $m + 3d + 2$ or $m + 3d + 3$ according as the idempotent $E_B^{(1)}$ is null or not, where $d (= c - s)$ is the number of treatment components which are partially confounded with the block space.*

The decomposition of the unit element of the relationship algebra \mathfrak{R} into mutually orthogonal idempotents

$$I = E_G^{(1)} + E_B^{(1)} + E_e^{(1)} + \sum_{u=1}^s E_u^{\#(1)} + \sum_{l=s+1}^c E_l^{\#(2)} + \sum_{j=c+1}^m E_j^{\#(1)} \tag{45}$$

is unique, where $E_B^{(1)}$ may or may not degenerate into null.

The decomposition of the idempotent $E_l^{\#(2)}$ ($l=s+1, \dots, c$) of four-dimensional ideal $[T_l^{\#}, BT_l^{\#}, T_l^{\#}B, BT_l^{\#}B]$ into two mutually orthogonal idempotents is not unique. It may, however, be relevant to decompose $E_l^{\#(2)}$ into two mutually orthogonal non-principal idempotents in relation to the block relationship as

$$E_l^{\#(2)} = \left(I - \frac{1}{k} B \right) E_l^{\#(2)} + \frac{1}{k} B E_l^{\#(2)}, \tag{46}$$

where $(I - k^{-1} B)E_l^{\#(2)} = F_l^{\#(1)}$ and $k^{-1} B E_l^{\#(2)} = B_l^{\#(1)}$ cited in (24) and (25). The former is the idempotent which defines the component of treatment S. S. corresponding to $A_l^{\#}$ and orthogonal to the block space, and the latter is the idempotent though corresponding to $A_l^{\#}$ but being confounded with the block space.

In general, the rank or trace of a non-null symmetric idempotent is a positive integer. The idempotent $E_B^{(1)}$ may, as was cited above, degenerate into null. We can, therefore, state that the rank or trace of $E_B^{(1)}$ is a non-negative integer. Enumerating the trace of $E_B^{(1)}$ we obtain the following theorem.

THEOREM III. *For PBIBD, the inequality*

$$b \geq v - \sum_{u=1}^s \alpha_u \tag{47}$$

holds, where $\sum_{u=1}^s \alpha_u$ is the sum of the degrees of freedom of the components of the treatment S. S. which are orthogonal to the block space.

The inequality (47) is more substantial than the one given by Conner and Clatworthy [3]. It is a generalization of Fisher's inequality for BIBD and, of course, includes some inequalities which appears in Bose and Connor [1].

5. Analysis of variance for PBIBD.

We are considering PBIBD in which the observation vector $\mathbf{x}' = (x_1, x_2, \dots, x_n)$ satisfies the linear model

$$\mathbf{x} = \mathbf{j}_n \gamma + \Phi \tau + \Psi \boldsymbol{\beta} + \mathbf{e}, \tag{48}$$

where γ is the general mean, $\boldsymbol{\tau}' = (\tau_1, \dots, \tau_v)$ is the treatment parameter vector and $\boldsymbol{\beta}' = (\beta_1, \dots, \beta_b)$ is the block parameter vector being subjected to the restrictions

Table I. Analysis of variance for PBIBD with m -associate classes

Source of variation		Families of treatment contrasts		Idempotents	d. f.	S. S.	Expectation of mean squares
Treatments, eliminating blocks	Orthogonal	1st~ s -th	$E_u^{(1)}$	α_u	$\mathbf{x}'E_u^{(1)}\mathbf{x}$	$\frac{r}{\alpha_u}\tau'A_u^\# + \sigma^2, u=1, \dots, s$	
	Partially confounded	s -th~ c -th	$F_l^{(1)}$	α_l	$\mathbf{x}'F_l^{(1)}\mathbf{x}$	$\frac{r}{\alpha_l}\left(1-\frac{\rho_l}{rk}\right)\tau'A_l^\# + \sigma^2, l=s+1, \dots, c$	
	Confounded	$(c+1)$ -th~ m -th	/	/	/	/	
	Total						
Blocks, ignoring treatments	Treatment- components	Orthogonal	1st~ s -th	/	/	/	/
		Partially confounded	$(s+1)$ -th~ c -th	$B_l^{(1)}$	α_l	$\mathbf{x}'B_l^{(1)}\mathbf{x}$	$\frac{1}{\alpha_l\rho_l k}(\rho_l^2\tau'A_l^\# + 2k\rho_l\tau'A_l^\#N\beta + k^2\beta'N'A_l^\#N\beta) + \sigma^2, l=s+1, \dots, c$
		Confounded	$(c+1)$ -th~ m -th	$E_j^{(1)}$	α_j	$\mathbf{x}'E_j^{(1)}\mathbf{x}$	$\frac{1}{\alpha_j r}(r^2\tau'A_j^\# + 2r\beta'N'A_j^\#\tau + \beta'N'A_j^\#N\beta) + \sigma^2, j=c+1, \dots, m$
	Remainder		$E_B^{(1)}$	α_B^*	by subtract.	$\frac{k}{\alpha_B}\beta'(I_B - \sum_l \frac{1}{\rho_l}N'A_l^\#N - \sum_j \frac{1}{rk}N'A_j^\#N)\beta + \sigma^2$	
	Total		$\frac{1}{k}B - \frac{1}{n}G$	$b-1$	$\mathbf{x}'\left(\frac{1}{k}B - \frac{1}{n}G\right)\mathbf{x}$	$\frac{1}{b-1}\left(\frac{1}{k}\tau'NN'\tau + 2\beta'N'\tau + k\beta'\beta\right) + \sigma^2$	
Intra-block error			$E_e^{(1)}$	α_E^*	by subtract.	σ^2	
Total			$I - \frac{1}{n}G$	$n-1$	$\mathbf{x}'\left(I - \frac{1}{n}G\right)\mathbf{x}$	$\frac{1}{n-1}(rr'\tau + 2\tau'N\beta + k\beta'\beta) + \sigma^2$	

* $\alpha_B = b - v + \sum_{u=1}^s \alpha_u \geq 0$

** $\alpha_E = n - b - v + 1 + \sum_{j=c+1}^m \alpha_j \geq 1$

$$\sum_{\alpha=1}^v \tau_{\alpha} = 0 \quad \text{and} \quad \sum_{a=1}^b \beta_a = 0, \quad (49)$$

respectively, and $\mathbf{e}' = (e_1, \dots, e_n)$ is the error vector being normally distributed with mean vector zero and covariance matrix $\sigma^2 I_n$. Φ and Ψ are the incidence matrices defined in (14) and $\mathbf{j}'_n = (1, 1, \dots, 1)$.

The complete table of the analysis of variance for this design will be given in Table I.

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