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Parallel Mappings and Comparability Theorem in Affine Matroid Lattices

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1. Introduction.

In the theory of continuous geometry, the following theorems are significant.

Theorem (1.1). (Perspective mappings). *In a modular lattice L, when $a\sim_x b$, put*

$$Ta_1=(a_1\cup x)\cap b \quad for \quad a_1\in L(0,a),$$
$$Sb_1=(b_1\cup x)\cap a \quad for \quad b_1\in L(0,b).$$

Then T and S are mutually inverse, isomorphic mappings between $L(0,a)$ and $L(0,b)$. In order that a_1, b_1 correspond by these mappings, it is necessary and sufficient that $a_1\cup x=b_1\cup x$ holds. And in this case $a_1\sim_x b_1$.

Here $a\sim_x b$ means $a\cup x=b\cup x$ and $a\cap x=b\cap x=0$. (Cf. [6] p. 18 and [4] p. 59).

Theorem (1.2). (Comparability theorem). *Let a, b be any elements in an upper continuous complemented modular lattice L. Then there exist a', a'', b', b'' such that*

(1°) $$a=a'\cup a'', \quad a'\cap a''=0,$$
$$b=b'\cup b'', \quad b'\cap b''=0.$$

(2°) $$a'\sim b' \quad and \quad e(a'')\cap e(b'')=0.$$

In this case $e(a')=e(b')=e(a)\cap e(b)$.

Here $e(a)$ means the smallest element z such that $a\leqq z$, $z\in Z$, where Z is the center of L. (Cf. [6] p. 265 and [4] p. 87.)

Theorem (1.3). (Distributivity and perspectivity). *Let a, b be elements in a complete complemented modular lattice L. Then the following three propositions are equivalent.*

(α) $a\nabla b$.

(β) *There do not exist nonzero elements a_1, b_1, with $a_1\sim b_1$, $a_1\leqq a$, $b_1\leqq b$.*

(γ) $e(a)\cap e(b)=0$.

Here $a\nabla b$ means $a\cap b=0$ and $(a,b)D$ (i.e. $(c\cup a)\cap b=(c\cap b)\cup(a\cap b)$ for every $c \in L$). (Cf. [6] pp. 243–244; [4] p. 70 and Remark (8.1) below).

The object of this paper is to obtain formally analogous theorems with respect to the parallelism instead of the perspectivity.

As Wilcox [7] considered, the basic lattices, in which the parallelism is investigated, may be weakly modular symmetric lattices. Corresponding to Theorem (1.1) we have the following theorem.

THEOREM (3.1). (Parallel mappings). *In a weakly modular symmetric lattice L, Let* $a \| b$ *and* p, q *be points with* $p \leqq a$, $q \leqq b$. *Put*

$$Ta_1 = (a_1 \cup q) \cap b \quad \textit{for} \quad a_1 \in L(p, a),$$
$$Sb_1 = (b_1 \cup p) \cap a \quad \textit{for} \quad b_1 \in L(q, b).$$

Then T and S are mutually inverse, isomorphic mappings between $L(p, a)$ *and* $L(q, b)$. *In order that* a_1, b_1 *correspond by these mappings, it is necessary and sufficient that* $a_1 \cup q = b_1 \cup p$ *holds. And in this case* $a_1 \| b_1$.

Let r be a fixed point in an affine matroid lattice L, then for any incomplete element a in L, there exists one and only one element $r(a)$, such that $r \leqq r(a)$ and either $r(a) \| a$ or $r(a) = a$. When a is a point p, put $r(p) = r$. Then $r(a)$ is an element of $R = L(r, I(r))$, and $a \| b$ or $a = b$ if and only if $r(a) = r(b)$. Since R is a modular sublattice of L, we may call R a *modular contraction* of L. Now $r(a)$ is the smallest element such that $a \leqq | \omega$, $\omega \in R$. Using this R, we can prove easily the following theorem.

THEOREM (5.1). (Comparability theorem). *Let* a, b *be incomplete elements in an affine matroid lattice L, and* p, q *be points with* $p \leqq a$ *and* $q \leqq b$. *Then there exist* a', a'', b', b'', *such that*

(1°) $$a = a' \cup a'', \quad a' \cap a'' = p,$$
$$b = b' \cup b'', \quad b' \cap b'' = q.$$

(2°) $$a' \| b' \quad \textit{or} \quad a' = b' \quad \textit{and} \quad r(a'') \cap r(b'') = r.$$

In this case $r(a') = r(b') = r(a) \cap r(b)$.

Lastly, corresponding to Theorem (1.3), I have the following theorem.

THEOREM (7.3). (Modularity and parallelism). *Let* a, b *be incomplete elements in an affine matroid lattice L and* $a \cap b = 0$. *Then the following three propositions are equivalent.*

(α) $a \perp b$.

(β) *There do not exist incomplete elements* a_1, b_1 *with* $a_1 \| b_1$, $a_1 \leqq a$, $b_1 \leqq b$.

(γ) $r(a) \cap r(b) = r$.

Here $a \perp b$ means $a \cap b = 0$ and $(a, b)M$ (i.e. $(c \cup a) \cap b = c \cup (a \cap b)$ for every $c \leqq b$). The equivalence of (β) and (γ) follows directly from the comparability theorem (5.1). In order to prove the equivalence of (α) and (β), I use the property of the following Wilcox lattice. Let Λ be a complemented modular lattice and $S \subset \Lambda$ be an ideal with 0 deleted. Then $L \equiv \Lambda - S$ is a weakly modular symmetric lattice, where the equivalence of (α) and (β) holds. Since an affine matroid lattice is a Wilcox lattice, Theorem (7.3) is proved.

Thus we obtain the theorems which are formally analogus to Theorems (1.1), (1.2) and (1.3). To the center Z corresponds the modular contraction $R = (r, I(r))$, whereas, in the preceding paper [5], the modular center M corresponds to Z. Although the modular center M and the modular contraction R of an affine matroid lattice L are the different modular sublattices of L, they are both projective geometries.

2. Preliminary.

Definition (2.1). In a lattice L with 0, $a \perp b$ means $a \cap b = 0$, $(a, b)M$; and $a \perp\!\!\!\perp b$ means $a \cap b = 0$, $(a, b)\bar{M}$ ($\bar{M}$ being the negation of the relation M). If $a \perp b$ and $a_1 \leqq a$, $b_1 \leqq b$, then $a_1 \perp b_1$ (Cf. [7] p. 492). When b covers a, we write $a \lessdot b$.

If $a \perp b$ implies $b \perp a$, L is called a *symmetric lattice* (Cf. [7] p. 495). And if $a \cap b \neq 0$ implies $(a, b)M$, L is called a *weakly modular lattice* (Cf. [5] (1.1)). A matroid lattice is a relatively atomic, upper continuous, symmetric lattice (cf. [5] (1.2), (1.8) and (1.9)). The converse statement follows from (2.3) below.

Remark (2.2). In a symmetric lattice L, if p is a point and $a \cap p = 0$, then $(p, a)M$. For, since $a \cap p = 0$ and $(a, p)M$, we have $a \perp p$. Hence $p \perp a$ and $(p, a)M$.

Lemma (2.3). *In a symmetric lattice L, if p is a point and $a \cap p = 0$, then $a \lessdot a \cup p$.*

Proof. Take c such that $a \leqq c \leqq a \cup p$. When $p \leqq c$, since $a \cup p \leqq c$, we have $c = a \cup p$. When $p \nleqq c$, then $c \cap p = 0$, hence by (2.2) we have $(p, c)M$. Therefore $c = (a \cup p) \cap c = a$. Consequently $a \lessdot a \cup p$.

Definition (2.4). In [1] p. 272 and [5] (2.1), the *parallelism* in a lattice with 0 is defined as follows. Let a, b be nonzero elements of L, if (1°) $a \cap b = 0$ and (2°) $b \lessdot a \cup b$, then we write $a <| b$. And if $a <| b$ and $b <| a$, then we write $a \| b$.

Remark (2.5). When $a <| b$ in a lattice L with 0, then $a_1 \cup b = a \cup b$ for every a_1 such that $0 < a_1 \leqq a$ (cf. [5] (2.3)). Hence when $a <| b$ and $0 < a_1 \leqq a$, we have $a_1 <| b$. For, $a_1 \cap b \leqq a \cap b = 0$ and $b \lessdot a \cup b = a_1 \cup b$.

Theorem (2.6). *In a weakly modular symmetric lattice L, if $a <| b$ and p is a point with $p \leqq b$, then $a \| (a \cup p) \cap b$.*

Proof. Using (2.3), we can prove as the proof (i) of [5] (2.8).

Definition (2.7). An *affine matroid lattice* L is a weakly modular matroid lattice of length ≥ 4, which satisfies the weak Euclid's parallel axiom (cf. [5] (3.3)). A line l in L is called *incomplete*, when for any point $p \not\leq l$, there exists a line k such that $l \| k$ and $p \leq k$, an element a of length ≥ 2 is called *incomplete*, when any line contained in a is incomplete (cf. [5] (3.4)). When L is not modular, for any point p in L, there exists a maximal incomplete element $I(p)$ which contains p. If $I(p)=1$, then L satisfies the strong Euclid's parallel axiom. If $I(p) \neq 1$, then $I(p)=I(q)$ or $I(p) \| I(q)$ for any points p, q in L. When L is modular, put $I(p)=p$. (Cf. [5] (4.1) and (4.2).)

In what follows, the assertion is trivial when the affine matroid lattice is modular. Hence we omit the explanations for the modular case.

THEOREM (2.8). *Let a be an incomplete element of an affine matroid lattice L, and r be a point such that $r \not\leq a$. Then there exists one and only one element b such that $a \| b$ and $r \leq b$.*

Proof. Cf. [1] p. 307.

LEMMA (2.9). *In an affine matroid lattice L, if $a <| b$ and a is not a point, then a is an incomplete element.*

Proof. Let l be any line such that $l \leq a$. Then by (2.5) we have $l <| b$. Since L is relatively atomic, b contains a point. Hence by (2.6) there exists an element k such that $l \| k$. Therefore l is incomplete, and a is an incomplete element.

LEMMA (2.10). *Let r be a point in an affine matroid lattice L. Then $L(r, I(r))$ is an irreducible modular matroid sublattice of L.*

Proof. When $r=I(r)$, it is trivial. Hence assume that $r<I(r)$, and put $R=L(r, I(r))$. Then a point in R means a line $l=r \cup p$ in L, where p is a point contained in $I(r)$ and $r \neq p$. Hence by [5] (1.4) and (1.5) we can easily prove that R is a relatively atomic, upper continuous sublattice of L. For $a, b \in R$, since $a \cap b \geq r>0$ and L is weakly modular, we have $(a, b)M$. Therefore R is modular. And R is a modular matriod sublattice of L. (Cf. also [1] p. 270.) To prove the irreducibility of R, let $l_1=r \cup p_1$ and $l_2=r \cup p_2$ be two different points in R, then $r \not\leq p_1 \cup p_2$. Since $p_1 \cup p_2$ is a line of L contained in $I(r)$, by (2.8) there exists a line $l_3=r \cup p_3$ such that $r \cup p_3 \| p_1 \cup p_2$. Then

$$l_3=r \cup p_3 \leq r \cup p_1 \cup p_2=l_1 \cup l_2.$$

Hence the line $l_1 \cup l_2$ in R contains a third point l_3 in R. Therefore, by [4] p. 80 Satz 2.4, R is irreducible.

3. Parallel mappings in weakly modular symmetric lattices.

THEOREM (3.1). *In a weakly modular symmetric lattice L, let $a \| b$ and p, q be points such that $p \leq a$, $q \leq b$. Put*

$$Ta_1 = (a_1 \cup q) \cap b \quad \textit{for} \quad a_1 \in L(p, a),$$
$$Sb_1 = (b_1 \cup p) \cap a \quad \textit{for} \quad b_1 \in L(q, b).$$

Then T and S are mutually inverse, isomorphic mappings between $L(p, a)$ and $L(q, b)$.

In order that a_1, b_1 correspond by these mappings, it is necessary and sufficient that

$$a_1 \cup q = b_1 \cup p \tag{1}$$

holds. And in this case $a_1 \| b_1$.

Proof. (i) It is evident that $Ta_1 \in L(q, b)$ and $Sb_1 \in L(p, a)$. Since by (2.5) $a_1 <| b$ and $q \leqq b$, by (2.6) we have $a_1 \| Ta_1$. Similarly we have $b_1 \| Sb_1$.

Since $a_1 \| Ta_1$ and $p \leqq a_1$, $q \leqq Ta_1$, we have by (2.5) $p \cup Ta_1 = a_1 \cup Ta_1 = a_1 \cup q$. Similarly we have $q \cup Sb_1 = b_1 \cup p$. Thus (1) holds.

(ii) Conversely assume that (1) holds. Since $p \cap b \leqq a \cap b_1 = 0$, by (2.2) we have $(p, b)M$. Hence

$$Ta_1 = (a_1 \cup q) \cap b = (b_1 \cup p) \cap b = b_1.$$

Similarly $Sb_1 = a_1$. Thus a_1 and b_1 correspond by T and S.

(iii) Next we shall prove that T and S are mutually inverse, isomorphic mappings. Put $b_1 = Ta_1$. Then by (i), (1) holds. Hence by (ii) $STa_1 = Sb_1 = a_1$. Similarly $TSb_1 = b_1$. Therefore by T and S, there exists a one-one correspondence between $L(p, a)$ and $L(q, b)$ preserving the order. Hence $L(p, a)$ and $L(q, b)$ are isomorphic.

Definition (3.2). We call T and S in (3.1) *parallel mappings* between $L(p, a)$ and $L(q, b)$.

4. Modular contractions of affine matroid lattices.

Definition (4.1). In an affine matroid lattice L, for any incomplete element a and any point r with $r \not\leqq a$, by (2.8), there exists one and only one element b such that $a \| b$ and $r \leqq b$. In this case we write $r(a) = b$. When $r \leqq a$, we write $r(a) = a$. And, since either $p \| r$ or $p = r$ for any point p, we write $r(p) = r$. We call $r(a)$ a *$\|$-image of a at r.*

Remark (4.2). In an affine matroid lattice L, the parallel mappings in (3.1) may be written as: $Ta_1 = q(a_1)$ and $Sb_1 = p(b_1)$. Hence, since $I(p) \| I(q)$, for any $a \in L(p, I(p))$, $b \in L(q, I(q))$, we have

$$p(q(a)) = a \quad \text{and} \quad q(p(b)) = b.$$

And for any $a, b \in L(p, I(p))$, we have

$$r(a \cup b) = r(a) \cup r(b) \quad \text{and} \quad r(a \cap b) = r(a) \cap r(b).$$

DEFINITION (4.3). In an affine matroid lattice L, when $a \parallel b$ or $a=b$, we write $a \| b$, and when $a < | b$ or $a \leqq b$, we write $a \leqq | b$.

Remark (4.4). In an affine matroid lattice L, $a \| b$ is an equivalence relation, and (1°) $a \leqq | a$, (2°) $a \leqq | b$, $b \leqq | a$ imply $a \| b$, and (3°) $a \leqq | b$, $b \leqq | c$ imply $a \leqq | c$. (cf. [1] pp. 310–311.)

Remark (4.5). In an affine matroid lattice L, let a, b be incomplete elements or points. Then

(1°) $\quad a \| b$ if and only if $r(a)=r(b)$,

(2°) $\quad a \leqq | b$ if and only if $r(a) \leqq r(b)$.

Proof. Since $a \| r(a)$ and $b \| r(b)$, by (4.4), $a \leqq | b$ if and only if $r(a) \leqq | r(b)$. But $r(a) \frown r(b) \geqq r$, hence $r(a) \leqq | r(b)$ means $r(a) \leqq r(b)$. Thus we have (2°). Similarly we have (1°).

Since $r(a)$ and $r(b)$ are elements in $R=L(r, I(r))$, by $a \to r(a)$, all incomplete elements and points in L are transposed into R preserving the order in the sense of (1°) and (2°). By (2.10) R is an irreducible modular matroid lattice. Hence we may call $R=L(r, I(r))$ a *modular contraction* of L. Since $I(p) \| I(q)$ for any points p, q in L, by (3.1) $L(p, I(p))$ and $L(q, I(q))$ are isomorphic. Hence the modular contraction of L is uniquely determined up to isomorphism. (This is an extension of Ex. 3 in [1] p. 317.) By (1°) and (2°), we may say that $r(a)$ is the smallest element ω such that $a \leqq | \omega$, $\omega \in R$.

Remark (4.6). In an affine matroid lattice L, let b be an incomplete element and $a < | b$. Then for any point $p \leqq a$, by (2.8), there exists one and only one element a_2 such that $a_2 \parallel b$ and $p \leqq a_2$. In this case $a \leqq a_2$. For, by (2°) in (4.5), we have $a=p(a) \leqq p(b)=a_2$. Therefore a_2 is uniquely determined irrespective of $p \leqq a$.

5. Comparability theorem in affine matroid lattices.

THEOREM (5.1). *Let a, b be incomplete elements in an affine matroid lattice L, and p, q be points such that $p \leqq a$ and $q \leqq b$. Then there exist a', a'', b', b'' such that*

(1°) $$a=a' \cup a'', \quad a' \frown a''=p,$$
$$b=b' \cup b'', \quad b' \frown b''=q.$$

(2°) $$a' \| b' \quad \text{and} \quad r(a'') \frown r(b'')=r.$$

In this case $r(a')=r(b')=r(a) \frown r(b)$.

Proof. Put $\omega=r(a) \frown r(b)$. Since $R=L(r, I(r))$ is a complemented modular lattice, if we take u and v such that

(1) $$r(a)=\omega \cup u, \quad \omega \frown u=r,$$

(2) $$r(b)=\omega\cup v,\quad \omega\cap v=r,$$

then we have

(3) $$u\cap v=r.$$

(Cf. [4] p. 14 Hilfssatz 1. 12.) Put $a'=p(\omega)$, $a''=p(u)$, $b'=q(\omega)$, $b''=q(v)$. Then by (4.2) and (1) we have

$$a=p(r(a))=p(\omega)\cup p(u)=a'\cup a'',$$
$$a'\cap a''=p(\omega)\cap p(u)=p(\omega\cap u)=p(r)=p.$$

Similarly from (2) we have

$$b=b'\cup b'',\quad b'\cap b''=q.$$

Since $a'\|\|\omega$ and $b'\|\|\omega$, we have $a'\|\|b'$, and from (3) we have

$$r(a'')\cap r(b'')=r(p(u))\cap r(q(v))=u\cap v=r,$$

and $$r(a')=r(p(\omega))=\omega=r(a)\cap r(b),$$

similarly $$r(b')=r(a)\cap r(b).$$

Theorem (5.2). *Let a, b be incomplete elements in an affine matroid lattice L. Then the following two propositions (α) and (β) are equivalent.*

(α) *There exist no incomplete elements a_1, b_1 such that $a_1\|\|b_1$, $a_1\leqq a$, $b_1\leqq b$.*

(β) $r(a)\cap r(b)=r.$

Proof. $(\alpha)\rightarrow(\beta)$. When $r(a)\cap r(b)>r$, from (5.1), there exist a', b' such that $a'\leqq a$, $b'\leqq b$, and $r(a')=r(b')=r(a)\cap r(b)>r$. Then a', b' are incomplete, in contradiction to (α).

$(\beta)\rightarrow(\alpha)$. If there exist incomplete elements a_1, b_1 such that $a_1\|\|b_1$, $a_1\leqq a$, $b_1\leqq b$, then by (4.5) $r(a_1)=r(b_1)$. Hence $r(a)\cap r(b)\geqq r(a_1)\cap r(b_1)=r(a_1)$. Since a_1 is incomplete, we have $r(a_1)>r$, which contradicts (β).

6. Parallelism in Wilcox lattices.

Definition (6.1). Let S be a subset of a lattice L. If $a, b\in S$ implies $a\cup b\in S$, and $a\in S$, $b\leqq a$ imply $b\in S$, then S is called an *ideal* of L.

Theorem (6.2). *Let A be a given complemented modular lattice partially ordered by a relation $a\leqq b$, and having the operations $a\vee b$, $a\wedge b$. Let $S\subset A$ be a fixed ideal of A with 0 deleted. Define $L\equiv A-S$. Then L is a weakly modular symmetric lattice partially ordered by the relation $a\leqq b$, with the operations $a\cup b$, $a\cap b$ which satisfy the following conditions:*

(6.2.1) $$a\cup b=a\vee b,$$

(6.2.2) $$a \cap b = \begin{cases} a \wedge b & \text{if } a \wedge b \in L, \\ 0 & \text{if } a \wedge b \in S. \end{cases}$$

And for $a, b \in L$,

(6.2.3) $a \perp b$ *in L* *if and only if* $a \wedge b = 0$,

(6.2.4) $a \perp\!\!\!\perp b$ *in L* *if and only if* $a \wedge b \in S$.

Proof. Cf. [7] pp. 497–498.

DEFINITION (6.3). When a weakly modular symmetric lattice L arises from a complemented modular lattice Λ in the manner described in (6.2) we call L a *Wilcox lattice*, and Λ the *modular extension* of L.

The characterization of Wilcox lattices is as yet unsolved (cf. [7] p. 505). In Wilcox lattices, we can define the parallelism by (2.4).

THEOREM (6.4). *In a Wilcox lattice L, let a be an element which is neither zero nor a point. Then the following three propositions are equivalent.*

(α) $a <| b$.

(β) $a \cap b = 0$ *and* $a_1 \cup b = a \cup b$ *for every* a_1 *such that* $0 < a_1 \leqq a$.

(γ) $a \wedge b \in S$ *and* $a \wedge b \lessdot a$ *in* Λ.

Proof. (α)→(β) follows from [5] (2.3).

(β)→(γ). Since a is not a point, there exists a_1 such that $0 < a_1 < a$. Hence by (β)

$$(a_1 \cup b) \cap a = (a \cup b) \cap a = a > a_1 = a_1 \cup (b \cap a).$$

Thus $(b, a)\bar{M}$. Therefore by (6.2.4) $a \wedge b \in S$. In Λ, take an element c such that

(1) $$a = (a \wedge b) \vee c, \quad (a \wedge b) \wedge c = 0.$$

Since $a \in L$ and $a \wedge b \in S$, we have $c \in L$. Lex x be any element of L such that $0 < x \leqq c$. Since $0 < x \leqq c \leqq a$, by (β), we have $x \cup b = c \cup b = a \cup b$, therefore by (6.2.1) we have

$$x \vee b = c \vee b = a \vee b,$$

and from (1), $x \wedge b \leqq c \wedge b = 0$. That is, x and c are relative complements of b in $a \vee b$, such that $x \leqq c$. By the modularity of Λ, we have $x = c$ (cf. [4] p. 6 Satz 1. 4). Consequently c is a point, hence from (1) we have $a \wedge b \lessdot a$ in Λ.

(γ)→(α). Since $a \wedge b \lessdot a$, there exists a point p in Λ such that

$$a = (a \wedge b) \vee p, \quad (a \wedge b) \wedge p = 0.$$

Since $a \in L$ and $a \wedge b \in S$, we have $p \in L$. Then

$$a \vee b = (a \wedge b) \vee p \vee b = p \vee b \quad \text{and} \quad b \wedge p = 0.$$

Hence $b \lessdot a \vee b$ in A. Therefore by (6.2.1) we have $b \lessdot a \cup b$ in L. Since $a \wedge b \in S$, by (6.2.2), we have $a \cap b = 0$. Consequently $a < | b$.

Reference. Hsu [2] defined (*)-parallelism using (β), and I have proved in [5] (2.3) that (**)-parallelism is equivalent to that defined by (α). (2.4) shows that in a Wilcox lattice, these two parallelisms coincide (when a is a point p, by (2.3) $p < | b$ for any element $b > 0$ such that $p \cap b = 0$, and (β) also holds), and from above proof, when $a \| b$, a contains at least one point. In [5] (2.4) Reference, I noted that the same statement holds in a left complemented lattice.

Definition (6.5). Let a be an element in a Wilcox lattice L. If there exist a point $p \in L$ and $u \in S$ such that $a = p \vee u$ in A, then we call a a *singular element* of L.

Remark (6.6). In a Wilcox lattice L, when a is not a point and $a < | b$, by (6.4), we have $a \wedge b \in S$ and $a \wedge b \lessdot a$ in A. Hence there exists a point $p \in A$ such that $a = p \vee (a \wedge b)$. If $p \in S$ then $a \in S$, which contradicts $a \in L$. Hence p is a point in L, and a is a singular element of L. Especially when $a \| b$ and a, b are not points, there exist points p, $q \in L$ such that

$$a = p \vee (a \wedge b), \quad b = q \vee (a \wedge b) \quad \text{and} \quad a \wedge b \in S.$$

Theorem (6.7). *Let a be a singular element in a Wilcox lattice L. Then for any point $q \in L$ with $q \nleqq a$, there exists a singular element $b \in L$ such that $a \| b$ and $q < b$.*

Proof. By (6.5) there exist a point $p \in L$ and $u \in S$ such that $a = p \vee u$. Since $a \cap q = 0$ and $(a, q)M$, we have $a \perp q$. Hence by (6.2.3) we have $a \wedge q = 0$. Therefore, if we put $b = q \vee u$, then $a \wedge b = a \wedge (q \vee u) = u \in S$, and $a \wedge b = u \lessdot b$. Hence by (6.4), we have $b < | a$. Similarly, from $a = p \vee u$ we have $a \wedge b = u \lessdot a$, that is $a < | b$. Consequently $a \| b$.

Lemma (6.8). *In a Wilcox lattice L, if $a \perp\!\!\!\perp b$ and p is a point with $p < a$, then $a_1 = p \vee (a \wedge b)$ is a singular element of L such that*

$$a_1 < | b \quad \textit{and} \quad p < a_1 \leqq a.$$

In this case $a_1 \wedge b = a \wedge b \in S$.

Proof. From (6.2.4), we have $a \wedge b \in S$. Hence $a_1 = p \vee (a \wedge b)$ is a singular element and $p < a_1 \leqq a$ in L. If $p \wedge b = p$, then by (6.2.2), we have $p \wedge b = p \cap b \leqq a \cap b = 0$, which is absured. Hence $p \wedge b = 0$, and we have $a_1 \wedge b = \{p \vee (a \wedge b)\} \wedge b = a \wedge b$. Therefore, since $a_1 = p \vee (a_1 \wedge b)$, we have $a_1 \wedge b \lessdot a_1$. Consequently from (6.4), $a_1 < | b$ holds.

Theorem (6.9). *In a Wilcox lattice L, if $a \perp\!\!\!\perp b$ and p, q are points with $p < a$ and $q < b$, then $a_1 = p \vee (a \wedge b)$ and $b_1 = q \vee (a \wedge b)$ are singular elements of L such* *that*

$$a_1 \| b_1 \quad and \quad p < a_1 \leqq a, \quad q < b_1 \leqq b.$$

In this case $a_1 \wedge b_1 = a \wedge b \in S$.

Proof. Since $a \perp\!\!\!\perp b$ and $p<a$, from (6.8), $a_1 = p \vee (a \wedge b)$ is a singular element such that

$$a_1 < | b, \quad p < a_1 \leqq a \quad \text{and} \quad a_1 \wedge b = a \wedge b \in S.$$

Hence by [5] (2.5) $a_1 \perp\!\!\!\perp b$ and $q<b$. Applying (6.8) again, $b_1 = p \wedge (a_1 \wedge b) = p \wedge (a \wedge b)$ is a singular element such that

$$b_1 < | a_1, \quad q < b_1 \leqq b \quad \text{and} \quad a_1 \wedge b_1 = a_1 \wedge b \in S.$$

Since $a_1 < | b$, by (6.4), we have $a_1 \wedge b \lessdot a_1$. Then $a_1 \wedge b_1 \lessdot a_1$ and we have $a_1 < | b_1$. Consequently $a_1 \| b_1$.

THEOREM (6.10). *Let a and b be elements in a Wilcox lattice L, each of which contains at least one point, and $a \cap b = 0$. Then the following two propositions are equivalent.*

(α) $a \perp b$.

(β) *There do not exist singular elements a_1, b_1 such that*

$$a_1 \| b_1, \quad a_1 \leqq a, \quad b_1 \leqq b.$$

Proof. (α)$\to$(β). If there exist singular elements a_1, b_1 such that $a_1 \| b_1$, $a_1 \leqq a$, $b_1 \leqq b$, then from [5] (2.5) we have $(a_1, b_1)\bar{M}$. But from $a \perp b$, we have $a_1 \perp b_1$, which is absured.

(β)$\to$(α) follows from (6.9).

Remark (6.11). In (6.10), we can not delete the condition "$a \cap b = 0$", even if we write $a_1 \| \! | b_1$ instead of $a_1 \| b_1$ in (β). For example, in an affine matroid lattice L, let a, b be two different lines which intersect at a point. Then (α) does not hold, although (β) holds.

7. Modularity and parallelism in affine matroid lattices.

DEFINITION (7.1). Let L be an affine matroid lattice with the operations $a \cup b$, $a \cap b$. Since by (4.4) $a \| \! | b$ is an equivalence relation, we put $[a] = \{b;\ b \| \! | a\}$, and denote by S the set of all $[a]$, where a is any incomplete element of L. Define $\Lambda \equiv L \cup S$.

In Λ, we can define a partial order $\alpha \leqq \beta$ by the following convention:

1° When $a, b \in L$, $a \leqq b$ in Λ means $a \leqq b$ in L.

2° When $[a] \in S$, $b \in L$, $[a] < b$ in Λ means $a \leqq | b$ in L.

3° When $[a]$, $[b] \in S$, $[a] \leqq [b]$ in Λ means $a \leqq | b$ in L.

4° For $[a] \in S$, there exists no nonzero element $b \in L$ such that $b < [a]$ in Λ.

5° For every element $[a] \in S$, $0 < [a]$ in Λ.

Then, in [1] pp. 311–314, it is proved that Λ is a modular matroid lattice with the operations $\alpha \vee \beta$, $\alpha \wedge \beta$, which satisfy the following conditions: For $a, b \in L$,

$$(7.1.1) \qquad a \vee b = a \cup b,$$

$$(7.1.2) \qquad a \wedge b \begin{cases} = a \cap b & \text{if } a \cap b \neq 0, \\ \in S \text{ or } = 0 & \text{if } a \cap b = 0. \end{cases}$$

And $S = \{\alpha \in \Lambda;\ 0 < \alpha \leqq [I(r)]\}$, where r is a point in L (cf. (2.7)). S is isomorphic to the modular contraction $R = L(r, I(r))$ with r deleted.

Theorem (7.2). *An affine matroid lattice L is a Wilcox lattice. And $a \in L$ is singular if and only if a is incomplete.*

Proof. In (7.1), $L = \Lambda - S$, and (7.1.2) is equivalent to (6.2.2), from (6.2) L is a Wilcox lattice. When $a \in L$ is singular, by (6.7), there exists an element $b \in L$ such that $a \| b$. Hence by (2.9), a is incomplete. Similarly, when $a \in L$ is incomplete, by (2.8), there exists an element $b \in L$ such that $a \| b$. Hence by (6.6), a is singular.

Theorem (7.3). *Let a, b be incomplete elements in an affine matroid lattice L, and $a \cap b = 0$. Then the following three propositions are equivalent.*

(α) $a \perp b$.

(β) *There do not exist incomplete elements a_1, b_1 such that*

$$a_1 \| b_1, \quad a_1 \leqq a, \quad b_1 \leqq b.$$

(γ) $r(a) \cap r(b) = r$.

Proof. $(\alpha) \rightleftarrows (\beta)$ from (6.10), and $(\beta) \rightleftarrows (\alpha)$ from (5.2).

8. Appendix.

Remark (8.1). In [4] p. 70, Theorem (1.3) is proved when L is an upper continuous complemented modular lattice. But as Kaplansky [3, p. 537] suggested, this theorem can be proved without the use of the upper continuity.

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