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Modular Centers of Affine Matroid Lattices

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It is already known that a relatively atomic, upper continuous, modular lattice L_p (which we may call a modular matroid lattice) characterizes a generalized projective geometry which satisfies

(PG 1) Two distinct points are in one and only one line,
(PG 2) If a line intersects two sides of a triangle at different points, it also intersects the third side.

And L_p is a direct sum of $L(0, e_\alpha)$ $(\alpha \in I)$, where $L(0, e_\alpha)$ satisfies not only (PG 1) and (PG 2), but also

(PG 3) Every line contains at least three points.

When p, q are points such that $p \leqq e_\alpha$, $q \leqq e_\beta (\alpha \neq \beta)$, then the line $p \cup q$ consists of only two points p and q.

The sublattice Z generated by $e_\alpha (\alpha \in I)$ is the center of L_p, which is isomorphic to the lattice of all subsets of the set $\{e_\alpha; \alpha \in I\}$. Thus the center Z is an atomic, complete Boolean algebra. Therefore it is distributive.

In this paper, applying the above consideration to the generalized affine geometry, I shall show an example of a lattice decomposition which is induced by a modular center.

In [1] p. 304, a generalized affine geometry (which we may call an affine matroid lattice) is defined as a weakly modular matroid lattice L_a of length $\geqq 4$, which satisfies the following weak Euclid's parallel axiom:

Let l be a line in a matroid lattice L. If p is a point such that $p \not\leqq l$, then there exists at most one line k such that $l \| k$ and $p \leqq k$.

A line l is called incomplete, when for any point $p \not\leqq l$ there exists a line k such that $l \| k$ and $p \leqq k$. And an element a is called incomplete, when any line contained in a is incomplete. Denote by $I(p)$ the greatest incomplete element which contains p. In [1] p. 309, it is proved that $I(p) = I(q)$ or $I(p) \| I(q)$ for any points p, q in L_a, and $L(0, I(p))$ satisfies the following strong Euclid's parallel axiom:

If p is a point such that $p \not\leqq l$, then there exists one and only one line k such that $l \| k$ and $p \leqq k$.

Therefore when p, q are points such that $I(p) \neq I(q)$, then the line $p \cup q$ has no parallel line.

In this paper, I shall show that the set Ω_0 of all $I(p)$, p being any point in L_a, is an irreducible projective space and the sublattice M generated by Ω_0 is an irreducible modular matroid lattice which is isomorphic to the lattice of all linear sets of Ω_0. Thus we may call M the modular center of L_a.

1. Matroid lattices.

In this section we deal with a given lattice L with 0.

DEFINITION (1.1). Let $a, b \in L$.

$(a, b)M$ means $(c \cup a) \cap b = c \cup (a \cap b)$ for every $c \leqq b$.

In a lattice L, if $a \cap b \neq 0$ implies $(a, b)M$ then we call L a *weakly modular* lattice. (Cf. [1] p. 68.) If $(a, b)M$ implies $(b, a)M$ then L is called a *M-symmetric* lattice. (Cf. [12] p. 453.)

L is called *left complemented* if $a, b \in L$ implies the existence of b_1 such that

$$a \cup b = a \cup b_1, \quad a \cap b_1 = 0, \quad (b_1, a)M, \quad b_1 \leqq b.$$

(Cf. [12] p. 453.)

LEMMA (1.2). *A left complemented lattice is M-symmetric.*

Proof. Cf. [12] p. 454.

DEFINITION (1.3). In a lattice L, we say that b *covers* a and write $a \lessdot b$ if $a < b$ and $a \leqq c \leqq b$ implies $c = a$ or $c = b$. A *point* is an element $p \in L$ such that $0 \lessdot p$.

If $a < b$ implies $a < a \cup p \leqq b$ for some point p, then L is called *relatively atomic.*

A lattice L is called *atomistic* if every element a of L is the join of points contained in a.

The set of all points in an atomistic lattice L is called the *point space* of L and we denote it by $\Omega(L)$.

Let $\{a_\delta;\ \delta \in D\}$ be a directed set of a complete lattice L. When $a_\delta \uparrow a$ implies $a_\delta \cap b \uparrow a \cap b$, we say that L is an *upper continuous* lattice.

LEMMA (1.4). *A lattice L is relatively atomic if and only if L is atomistic.*

Proof. Cf. [6] p. 70.

LEMMA (1.5). *In a relatively atomic, complete lattice L, the following two propositions* (α) *and* (β) *are equivalent:*

(α) *L is upper continuous.*

(β) *Let p be a point and S be a set of points in L. Then $p \leqq \bigcup_{q \in S} q$ implies $p \leqq q_1 \cup \cdots \cup q_n$ for some $q_i \in S$ $(i = 1, \cdots, n)$.*

Proof. Cf. [6] p. 71.

LEMMA (1.6). *Let p be a point and a_α ($\alpha \in I$) be elements in a relatively atomic, upper continuous lattice L, if $p \leqq {}_{\alpha\in I}\bigvee a_\alpha$, then there exists a finite subset J of I such that $p \leqq {}_{\alpha\in J}\bigvee a_\alpha$.*

Proof. By (1.4) each a_α is a join of points. Hence ${}_{\alpha\in I}\bigvee a_\alpha$ is a join of a set S of points. Therefore by (1.5) $p \leqq q_1 \cup \cdots \cup q_n$ for some $q_i \in S$ ($i=1, \cdots, n$). When $q_i \leqq a_{\alpha_i}$ ($i=1, \cdots, n$), then $p \leqq a_{\alpha_1} \cup \cdots \cup a_{\alpha_n}$.

THEOREM (1.7). *In a relatively atomic, upper continuous lattice L, the following condition (η'), (η'') and (ξ') are equivalent:*

(η') *If p, q are points such that $q \not\leqq a$ and $q \leqq a \cup p$, then $p \leqq a \cup q$.*
(η'') *If p is a point, then either $p \leqq a$ or $a \lessdot a \cup p$.*
(ξ') *If $c \lessdot a$, $c \lessdot b$ and $a \neq b$, then $a \lessdot a \cup b$ and $b \lessdot a \cup b$.*

Proof. Cf. [5] p. 180.

DEFINITION (1.8). Let L be a relatively atomic, upper continuous lattice. If L satisfies (η'), L is called an *exchange lattice* in [4] p. 456, if L satisfies (η''), L is called a *geometric lattice* in [1] p. 264, and if L satisfies (ξ'), L is called a *matroid lattice* in [5] p. 181.

By (1.7) these three lattices are identical.

The matroid lattice characterizes the lattice of all subspaces of a geometry which has the exchange property. (Cf. [3] p. 191)

In a matroid lattice, the terms "line" and "plane" are used in the ordinal geometical sense, defining by the dimension.

LEMMA (1.9). *A matroid lattice L is left complemented.*

Proof. Cf. [7] p. 331.

LEMMA (1.10). *A matroid lattice L is irreducible, if and only if any two points in L are perspective.*

Proof. Cf. [11] p. 188.

LEMMA (1.11). *A matroid lattice L is weakly modular, if and only if L satisfies the following condition:*

(SP) *If $p \leqq q \cup a$, $r \leqq a$, where p, q, r are points, then there exists a point s such that $p \leqq q \cup r \cup s$ and $s \leqq a$.*

Proof. Cf. [8] p. 232 and [9] p. 414.

The weakly modular matroid lattice characterizes the lattice of all subspaces of a strongly planer geometry. (Cf. [10] p. 422 and [3] p. 193.)

Remark (1.12). A *projective space Ω* is a system of points and lines satisfying the following two conditions:

(PG 1) Two distinct points are in one and only one line.
(PG 2) If a line intersects two sides of a triangle at different points, it also intersects the third side.

If a projective space Ω satisfies the following condition:

(PG 3) Every line contains at least three points,

then Ω is called *irreducible.*

A *linear set* is a set of points which contains the line through p and q if it contains p and q.

The set $L(\Omega)$ of all linear sets of a projective space Ω forms a relatively atomic, upper continuous, modular lattice, which we may call a *modular matroid lattice.* $L(\Omega)$ is irreducible if and only if Ω is irreducible. (Cf. [6] p. 83.) We call $L(\Omega)$ a *generalized projective geometry* on Ω; when Ω is irreducible, we call $L(\Omega)$ a *projective geometry* on Ω.

Conversely, when L is a modular matroid lattice, the point space $\Omega(L)$ of L is a projective space, and the generalized projective geometry $L(\Omega(L))$ is isomorphic to L. (Cf. [6] p. 84.)

Especially when L is a distributive matroid lattice, it is an atomistic, complete Boolean algebra, and $\Omega(L)$ is merely a set and L is isomorphic to the lattice of all subsets of $\Omega(L)$.

2. Parallelism in lattices.

In this section we deal with a given lattice L with 0.

DEFINITION (2.1). Let a, b be non-zero elements in a lattice L. If

(1°) $$a \frown b = 0,$$

(2°) $$b \lessdot a \smile b,$$

then we write $a < | b$.

If $a < | b$ and $b < | a$, then we say that a and b are *parallel* and write $a \| b$.

Reference. This is the definition of parallelism used in [1] p. 272, it is written $b \| a$ instead of $a < | b$.

Remark (2.2). Let l, k be two lines in a matroid lattice L. Then $l \| k$ if and only if l and k are contained in a plane and do not intersect. And $p \| q$ for any different points p, q in L.

THEOREM (2.3). *Let a, b be non-zero elements of a lattice L. Then $a < | b$ if and only if*

(α) $a \frown b = 0$,

(β) $a_1 \smile b = a \smile b$ *for every a_1 such that* $0 < a_1 \leqq a$, *and*

(γ) *there exists no b_2 such that* $a \smile b_2 = a \smile b$, $b < b_2$, $a \frown b_2 = 0$.

Proof. (i) Necessity. For any a_1 such that $0 < a_1 \leqq a$, we have $b \leqq a_1 \smile b \leqq a \smile b$. If $b = a_1 \smile b$ then $a_1 = a_1 \frown b \leqq a \frown b = 0$, which contradicts $a_1 > 0$. Therefore $b < a_1 \smile b$. Since $b \lessdot a \smile b$, it must be that $a_1 \smile b = a \smile b$, and ($\beta$) holds.

When $a \smile b_2 = a \smile b$ and $b < b_2$, we have $b < b_2 \leqq a \smile b$. Since $b \lessdot a \smile b$ we have $b_2 = a \smile b$. Hence $a \frown b_2 = a \frown (a \smile b) = a > 0$. Thus ($\gamma$) holds.

(ii) Sufficiency. In order to prove $b \lessdot a \cup b$, let $b \leqq x \leqq a \cup b$. When $a \cap x \neq 0$, since $0 < a \cap x \leqq a$, from (β) we have $a \cup b = (a \cap x) \cup b \leqq x$, and $x = a \cup b$. When $a \cap x = 0$, since $a \cup x = a \cup b$, from (γ) it must be that $b = x$. Therefore $b \lessdot a \cup b$, and $a <| b$.

THEOREM (2.4). *Let a, b be non-zero elements in a left complemented lattice L. Then $a <| b$ if and only if (α) and (β) in (2.3) hold.*

Proof. From (2.3) necessity is evident. We shall prove sufficiency. Since L is left complemented, there exists a_1 such that

$$(1) \qquad a \cup b = a_1 \cup b, \quad a_1 \cap b = 0, \quad (a_1, b)M, \quad a_1 \leqq a.$$

Let $0 < c \leqq a_1$, then from (β) $c \cup b = a \cup b$. Since by (1.2) $(b, a_1)M$, we have $a_1 = (a \cup b) \cap a_1 = (c \cup b) \cap a_1 = c$. Therefore a_1 is a point, say p. Then from (1), we have $a \cup b = p \cup b$.

Let $b \leqq x \leqq b \cup p$. When $p \leqq x$, then $b \cup p \leqq x$ and we have $x = b \cup p$. When $p \not\leqq x$, then $p \cap x = 0$. Since $(x, p)M$, we have $(p, x)M$. Hence $x = (b \cup p) \cap x = b$. Therefore $b \lessdot b \cup p = a \cup b$, and we have $a <| b$.

Reference. Hsu [2] defined $(*)$-parallelism using (α) and (β), and $(**)$-parallelism using (α), (β) and (γ). From (2.4), in a left complemented lattice these two parallelisms coincide, and from the above proof, when $a \| b$, a contains at least one point.

LEMMA (2.5). *In a lattice L, if $a <| b$ and a is not a point, then $(b, a)M$ does not hold.*

Proof. Since a is not a point, there exists a_1 such that $0 < a_1 < a$. By (2.3) we have $a_1 \cup b = a \cup b$. Hence

$$(a_1 \cup b) \cap a = (a \cup b) \cap a = a > a_1 = a_1 \cup (b \cap a).$$

Therefore $(b, a)M$ does not hold.

LEMMA (2.6). *Let L be a weakly modular matroid lattice. If every line in L has no parallel line, then L is modular.*

Proof. (i) I shall first prove that the point space $\Omega(L)$ is a projective space when a line l is defined by the set $\{r \in \Omega(L); r \leqq p \cup q\}$ for different points p and q.

When $r \leqq p \cup q$ and $r \neq p$, then by (η') in (1.7), we have $p \cup q = p \cup r$. Hence (PG 1) in (1.12) holds.

Let p, q and r be three points which form a triangle. Take two different points s and t, such that $s \leqq p \cup q$ and $t \leqq q \cup r$. We may assume that s and t are different from p, q and r. Then $s \cup t$ and $p \cup r$ are two lines contained in the plane $p \cup q \cup r$, but they are not parallel. Hence by (2.2) they intersect. Therefore (PG 2) holds.

Consequently by (1.12) $\Omega(L)$ is a projective space and $L(\Omega(L))$ is a modular matroid lattice.

(ii) Let S be a linear set in $\Omega(L)$. I shall prove by induction that $r \leqq p_1 \cup \cdots \cup p_n (p_i \in S)$ implies $r \in S$. When $n=2$, this is evident. Next assume that the assertion holds when $n=i-1$. It is evident when $r=p_i$ or $p_i \leqq p_1 \cup \cdots \cup p_{i-1}$. Hence let $r \neq p_i$ and $p_i \nleqq p_1 \cup \cdots \cup p_{i-1}$. Since $r \leqq p_i \cup (p_1 \cup \cdots \cup p_{i-1})$, by (1.11) there exists a point s such that

$$(1) \qquad r \leqq p_i \cup p_1 \cup s \quad \text{and} \quad s \leqq p_1 \cup \cdots \cup p_{i-1}.$$

When $s \leqq p_i \cup p_1$, then since $r \leqq p_i \cup p_1$, we have $r \in S$. When $s \nleqq p_i \cup p_1$, then $r \cup p_i$ and $p_1 \cup s$ are lines contained in the plane $p_i \cup p_1 \cup s$. Hence by (2.2) there exists a point q such that

$$(2) \qquad q \leqq (r \cup p_i) \cap (p_1 \cup s).$$

By (1) and (2), $q \leqq p_1 \cup s \leqq p_1 \cup \cdots \cup p_{i-1}$. Hence $q \in S$ and $q \neq p_i$. But by (2) $q \leqq r \cup p_i$, hence by (η') in (1.7) we have $r \leqq q \cup p_i$. Therefore $r \in S$. Thus the assertion holds for $n=i$.

(iii) For $a \in L$ and $S \in L(\Omega(L))$, define

$$S(a) = \{p \in \Omega(L); p \leqq a\} \quad \text{and} \quad a(S) = \bigvee (p; p \in S).$$

Since L is atomistic, we have $a = \bigvee (p; p \leqq a)$. Hence

$$a(S(a)) = \bigvee (p; p \in S(a)) = \bigvee (p; p \leqq a) = a.$$

Let q be a point such that $q \leqq a(S)$. Then by (1.5), there exists a finite subset $\{p_1, \cdots, p_n\}$ of S such that $q \leqq p_1 \cup \cdots \cup p_n$. Hence by (ii) $q \in S$. Therefore $S(a(S)) \leqq S$. On the other hand, when $p \in S$ then $p \leqq a(S)$, and we have $p \in S(a(S))$. Therefore $S(a(S)) = S$.

(iv) By (iii), $a \to S(a)$ and $S \to a(S)$ define a one-to-one correspondence between L and $L(\Omega(L))$ preserving the order. Consequently L is isomorphic to $L(\Omega(L))$, and L is modular.

Reference. In [1] p. 307, (2.6) is proved using the hyperplane.

THEOREM (2.7). *A weakly modular matroid lattice L is modular, if and only if every line in L has no parallel line.*

Proof. By (2.5) and (2.6).

THEOREM (2.8). *A non-modular matroid lattice L is weakly modular, if and only if L satisfies the following condition:*

(PE) *For a point p, if $a <| b$ and $p \leqq b$, then there exists an element b_1 such that $a \| b_1$ and $p \leqq b_1 \leqq b$.*

Proof. (i) When L is weakly modular and $a <| b, p \leqq b$, let $b_1 = b \cap (a \cup p)$. Since $a \cup b_1 \leqq a \cup p$ and $p \leqq b_1$, we have $a \cup b_1 = a \cup p$. Being $a \cap p \leqq a \cap b = 0$, we have $a \lessdot a \cup p = a \cup b_1$. But $a \cap b_1 \leqq a \cap b = 0$, hence $b_1 <| a$.

Next we shall prove that $a<|b_1$, that is, $b\frown(a\smile p)=b_1\lessdot a\smile b_1=a\smile p$. Let $b\frown(a\smile p)\leqq x\leqq a\smile p$. When $x\leqq b$, being $x\leqq a\smile p$, we have $b\frown(a\smile p)=x$. When $x\nleqq b$, since $b<x\smile b\leqq a\smile p\smile b=a\smile b$ and $b\lessdot a\smile b$, we have $x\smile b=a\smile b$. Being $b\frown(a\smile p)\geqq p>0$, we have $(b, a\smile p)M$. Therefore $(x\smile b)\frown(a\smile p)=x\smile\{b\frown(a\smile p)\}=x$. Thus $a\smile p=x$. Consequently $b\frown(a\smile p)\lessdot a\smile p$ and $a<|b_1$.

Then $a\|b_1$ and $p\leqq b_1\leqq b$. Thus (PE) is proved.

(ii) Conversely assume that L satisfies (PE). In order to prove that L is weakly modular, by (1.11), it is sufficient to prove that the following condition (SP) holds.

(SP) If $p\leqq q\smile a$, $r\leqq a$, where p, q, r are points, then there exists a point s such that $p\leqq q\smile r\smile s$ and $s\leqq a$.

To prove (SP) we may assume that $p\neq q$ and $q\nleqq a$, for otherwise (SP) is evident. Let $l=p\smile q$. When $l\frown a\neq 0$, there exists a point s such that $s\leqq l\frown a$. Since $s\leqq p\smile q$ and $s\frown q\leqq a\frown q=0$, we have $p\leqq q\smile s$, and (SP) holds. When $l\frown a=0$, since $a\lessdot a\smile q=a\smile p\smile q=a\smile l$, we have $l<|a$. Hence by (PE), there exists an element k such that $l\|k$ and $r\leqq k\leqq a$. Since k is a line, there exists a point s such that $k=r\smile s$ and $s\leqq a$. From $p\smile q\|r\smile s$, we have $p\leqq q\smile r\smile s$ and (SP) holds.

Remark (2.9). Instead of (PE) in Theorem (2.8), we may put the following condition (PL):

(PL) For a point p and a line l, if $l<|a$ and $p\leqq a$, then there exists a line k such that $l\|k$ and $p<k\leqq a$.

This is evident from the proof (ii) in (2.8).

Reference. The part (i) in the proof of (2.8) is already given in [1] p. 305, where the hyperplane is used.

3. Affine matroid lattices.

STRONG EUCLID'S PALALLEL AXIOM (3.1). Let l be a line in a matroid lattice L. If p is a point such that $p\nleqq l$, then there exists one and only one line k such that $l\|k$ and $p\leqq k$.

WEAK EUCLID'S PALALLEL AXIOM (3.2). Let l be a line in a matroid lattice L. If p is a point such that $p\nleqq l$, then there exists at most one line k such that $l\|k$ and $p\leqq k$.

DEFINITION (3.3). Let L be a weakly modular matroid lattice L of length $\geqq 4$. When L satisfies the weak Euclid's parallel axiom, we call L an *affine matroid lattice.*

Reference. In [1] p. 304, the above-defined affine matroid lattice is called a generalized affine lattice or a generalized affine geometry.

DEFINITION (3.4). In an affine matroid lattice L, a line l is called *incomplete*, when for any point $p \not\leq l$, there exists a line k such that $l \| k$ and $p \leq k$. And a line l is called *complete*, when there exists no line parallel to l. An element a of length ≥ 2 is called *incomplete*, when any line contained in a is incomplete.

When a line l is not complete, then l is incomplete. (Cf. [1] p. 306.)

REMARK (3.5). An affine matroid lattice L satisfies the strong Euclid's parallel axiom if and only if all lines in L are incomplete, and L is modular if and only if all lines in L are complete. (Cf. (2.7).)

LEMMA (3.6). *When an affine matroid lattice L is not modular, every complete line l in L contains at least there points.*

Proof. Cf. [1] p. 314.

THEOREM (3.7). *When an affine matroid lattice L is not modular, then L is irreducible.*

Proof. Let p, q be any two different points in L. When the line $p \cup q$ is complete, by (3.6) $p \cup q$ contains a third point r. Then $p \cup r = q \cup r$, $p \cap r = 0$, $q \cap r = 0$. Hence $p \sim q$. When $p \cup q$ is incomplete, there exists a line l such that $p \cup q \| l$. Then by (2.3) $p \cup l = (p \cup q) \cup l = q \cup l$. And $p \cap l \leq (p \cup q) \cap l = 0$, similarly $q \cap l = 0$. Therefore $p \sim q$. Hence in any case $p \sim q$. Consequently by (1.10) L is irreducible.

Reference. In [1] p. 315, (3.7) is proved using the hyperplanes in L.

4. Decomposition spaces of affine matroid lattices.

THEOREM (4.1). *Let L be an affine, non-modular, matroid lattice, then for any point p in L, there exists a maximal incomplete element $I(p)$ which contains p. If $I(p) = 1$, then L satisfies the strong Euclid's parallel axiom. If $I(p) \neq 1$, then $I(p) = I(q)$ or $I(p) \| I(q)$ for any points p, q in L.*

Proof. Cf. [1] p. 309.

Remark (4.2). When an affine matroid lattice L is modular, there exists no incomplete element containing a point p. Hence we put $I(p) = p$.

DEFINITION (4.3). Let L be an affine matroid lattice. From (4.1) and (4.2) there exists a set $\Omega_0 = \{I(t_\alpha); \alpha \in I\}$ such that for any point p in L, there exists $\alpha \in I$ with $p \leq I(t_\alpha)$, and $I(t_\alpha) \| I(t_\beta)$ when $\alpha \neq \beta$. We call Ω_0 a *decomposition space* of L.

When L is modular, then $\Omega_0 = \Omega$ and Ω_0 is a projective space. Hence we may expect that this fact holds in the non-modular case. This is Theorem (4.8) below.

In what follows, the assertion is evident when the affine matroid lattice is modular. Hence we omit the explanations for the modular case.

In the following (4.4) — (4.7), L is an affine matroid lattice.

Remark (4.4). Let p, q be any two points in L, then $p\cup q$ is a complete line if and only if p and q are contained in different $I(t_\alpha)$ and $I(t_\beta)$ in Ω_0.

Proof. Let $p\leqq I(t_\alpha)$ and $q\leqq I(t_\beta)$. Then $p\cup q\leqq I(t_\alpha)\cup I(t_\beta)$. Since $I(t_\alpha)$ and $I(t_\beta)$ are incomplete elements and $I(t_\alpha)=I(t_\beta)$ or $I(t_\alpha)\frown I(t_\beta)=0$, $p\cup q$ is a complete line if and only if $I(t_\alpha)\frown I(t_\beta)=0$, that is $I(t_\alpha)\neq I(t_\beta)$.

Lemma (4.5). *Let r be a point in L if $r\leqq I(p)\cup I(q)$ and $r\nleqq I(p)$, then there exists a point q' such that $r\leqq p\cup q'$ and $q'\leqq I(q)$.*

Proof. Since $r\nleqq I(p)$, we have $I(p)\neq I(q)$. Hence $I(p)\| I(q)$. Therefore by (2.3) we have

$$r\leqq I(p)\cup I(q)=p\cup I(q).$$

Since L is weakly modular, from (1.11), there exists a point t such that

$$r\leqq p\cup q\cup t \quad \text{and} \quad t\leqq I(q).$$

When $t=q$, then q is the required q'. When $t\neq q$, since $p\frown(t\cup q)\leqq p\frown I(q)=0$, $p\cup t\cup q$ is a plane. Then $r\cup p$ and $q\cup t$ are lines in the plane $p\cup q\cup t$. But since $r\nleqq I(p)$, by (4.4) $r\cup p$ is a complete line. Therefore, from (2.2), there exists a point q' such that $q'\leqq(r\cup p)\frown(q\cup t)$. Then $q'\leqq q\cup t\leqq I(q)$, and therefore $q'\nleqq I(p)$, that is, $q'\frown p=0$. Since $q'\leqq r\cup p$, by (η') in (1.7), we have $r\leqq p\cup q'$.

Lemma (4.6). *Let r be a point in L, $r\leqq I(p)\cup I(q)$ implies $I(r)\leqq I(p)\cup I(q)$.*

Proof. When $r\leqq I(p)$, this is evident. Hence we assume that $r\nleqq I(p)$ and hence $I(p)\neq I(q)$. From (4.5) there exists a point q' such that

$$r\leqq p\cup q' \quad \text{and} \quad q'\leqq I(q).$$

Take any point r' such that $r'\leqq I(r)$ and $r'\neq r$. If $r'\leqq p\cup q'$, then $r\cup r'=p\cup q'$ which is absurd, since $r\cup r'$ is incomplete and $p\cup q'$ is complete. Hence $r'\nleqq p\cup q'$ and $p\cup q'\cup r'$ is a plane. Since $r\cup r'$ is an incomplete line and $r\nleqq I(p)$, there exists a point p' such that $r\cup r'\| p\cup p'$. Hence $p'\leqq p\cup r\cup r'\leqq p\cup q'\cup r'$.

If $p'\leqq p\cup q'$, then $p\cup p'=p\cup q'$, which is absurd, since $p\cup q'$ is a complete line. Hence $p'\nleqq p\cup q'$. Therefore by (η') in (1.7) we have $r'\leqq p\cup q'\cup p'$. Since $p\cup p'$ is an incomplete line, we have $p\cup p'\leqq I(p)$. Therefore $r'\leqq I(p)\cup q'\leqq I(p)\cup I(q)$. Consequently $I(r)\leqq I(p)\cup I(q)$.

Lemma (4.7). *Let r be a point in L, $r\leqq I(t_1)\cup\cdots\cup I(t_n)$ implies $I(r)\leqq I(t_1)\cup\cdots\cup I(t_n)$.*

Proof. We shall prove this lemma by induction. When $n=2$, the assertion follows from (4.6). Next assume that the assertion holds when $n=i-1$.

It is evident when $r \leqq I(t_i)$ or $I(t_i) \leqq I(t_1) \cup \cdots \cup I(t_{i-1})$. Hence let $r \not\leqq I(t_i)$ and $I(t_i) \not\leqq I(t_1) \cup \cdots \cup I(t_{i-1})$. Since $I(t_1) \| I(t_i)$, we have $I(t_1) \cup I(t_i) = I(t_1) \cup t_i$. Consequently $r \leqq I(t_1) \cup \cdots \cup I(t_i)$ implies $r \leqq t_i \cup \{I(t_1) \cup \cdots \cup I(t_{i-1})\}$. Since L is weakly modular, by (1.11), there exists a point s such that

$$(1) \qquad r \leqq t_i \cup t_1 \cup s \quad \text{and} \quad s \leqq I(t_1) \cup \cdots \cup I(t_{i-1}).$$

When $s \leqq t_i \cup t_1$, then from (1) we have $r \leqq t_i \cup t_1 \leqq I(t_i) \cup I(t_1)$. Then from (4.6) we have $I(r) \leqq I(t_i) \cup I(t_1)$, and the assertion holds. When $s \not\leqq t_i \cup t_1$, then $t_i \cup t_1 \cup s$ is a plane, and since $r \cup t_i \leqq t_i \cup t_1 \cup s$ from (1), $r \cup t_i$ and $t_1 \cup s$ are two lines on the plane $t_i \cup t_1 \cup s$. Since $r \not\leqq I(t_i)$, $r \cup t_i$ is a complete line. Hence, from (2.2), there exists a point p such that

$$(2) \qquad p \leqq (r \cup t_i) \cap (t_1 \cup s).$$

Since from (1)

$$(3) \qquad p \leqq t_1 \cup s \leqq I(t_1) \cup \cdots \cup I(t_{i-1}),$$

we have $p \not\leqq I(t_i)$, because if $p \leqq I(t_i)$ then $I(t_i) = I(p) \leqq I(t_1) \cup \cdots \cup I(t_{i-1})$ which contradicts the assumption. Therefore $p \cap t_i = 0$. Since from (2) $p \leqq r \cup t_i$, by (η') in (1.7) we have $r \leqq p \cup t_i \leqq I(p) \cup I(t_i)$. Therefore by (4.6) we have

$$(4) \qquad I(r) \leqq I(p) \cup I(t_i).$$

Since from (3) $I(p) \leqq I(t_1) \cup \cdots \cup I(t_{i-1})$, we have $I(r) \leqq I(t_1) \cup \cdots \cup I(t_i)$ and the assertion holds for $n = i$. The lemma is completely proved.

THEOREM (4.8). *The decomposition space $\Omega_0 = \{I(t_\alpha); \alpha \in I\}$ of an affine, non-modular, matroid lattice L is an irreducible projective space, where the line determined by different points $I(t_\alpha)$ and $I(t_\beta)$ is a set of elements of Ω_0 contained in $I(t_\alpha) \cup I(t_\beta)$.*

Proof. (i) Let $I(t_\alpha)$, $I(t_\beta)$ and $I(t_\gamma)$ be three different elements in Ω_0 such that $I(t_\gamma) \leqq I(t_\alpha) \cup I(t_\beta)$. Then from (4.5) there exists a point p such that $t_\gamma \leqq t_\alpha \cup p$ and $p \leqq I(t_\beta)$. Since $t_\gamma \cap p = 0$, from (η') in (1.7), we have $t_\alpha \leqq t_\gamma \cup p \leqq I(t_\gamma) \cup I(t_\beta)$. Therefore, since $I(t_\alpha) \cup I(t_\beta) \leqq I(t_\gamma) \cup I(t_\beta) \leqq I(t_\alpha) \cup I(t_\beta)$, we have $I(t_\alpha) \cup I(t_\beta) = I(t_\gamma) \cup I(t_\beta)$. Consequently the line is determined by the two different elements contained in it, and (PG 1) in (1.12) holds.

(ii) Let $I(t_\alpha)$, $I(t_\beta)$ and $I(t_\gamma)$ are elements in Ω_0 which form a triangle. Take $I(p)$ and $I(q)$ such that $I(p) \leqq I(t_\alpha) \cup I(t_\beta)$, $I(q) \leqq I(t_\alpha) \cup I(t_\gamma)$ where $I(p)$ and $I(q)$ are different from any $I(t_\alpha)$, $I(t_\beta)$ and $I(t_\gamma)$. By (i), we have $I(t_\beta) \leqq I(t_\alpha) \cup I(p)$, hence from (4.5) we may take p such that $t_\beta \leqq t_\alpha \cup p$. Therefore $p \leqq t_\alpha \cup t_\beta$. Similarly we may take q such that $q \leqq t_\alpha \cup t_\gamma$. Then $p \cup q$ is a line on the plane $t_\alpha \cup t_\beta \cup t_\gamma$. By (4.4) $p \cup q$ is a complete line. Hence by (2.2) there exists a point r such that $r \leqq p \cup q$ and $r \leqq t_\beta \cup t_\gamma$. Therefore by (4.6), we have $I(r) \leqq I(p) \cup I(q)$ and $I(r) \leqq I(t_\beta) \cup I(t_\gamma)$. Consequently (PG 2) holds in Ω_0.

(iii) Let $I(t_\alpha)\cup I(t_\beta)$ be any line in Ω_0. Since $t_\alpha\cup t_\beta$ is a complete line, by (3.6), $t_\alpha\cup t_\beta$ contains a third point p. Hence by (4.6) $I(p)\leqq I(t_\alpha)\cup I(t_\beta)$. Since $t_\alpha\cup t_\beta=t_\alpha\cup p$ is a complete line, it must be that $I(p)\neq I(t_\alpha)$. Similarly $I(p)\neq I(t_\beta)$. Consequently $I(t_\alpha)\cup I(t_\beta)$ contains a third point $I(p)$, that is, (PG 3) holds in Ω_0.

Thus Ω_0 is an irreducible projective space.

LEMMA (4.9). *Let S be a linear set of $\Omega_0=\{I(t_\alpha);\alpha\in I\}$ of an affine matroid lattice L, and $I(t_1),\cdots,I(t_n)\in S$. Then $r\leqq I(t_1)\cup\cdots\cup I(t_n)$ implies $I(r)\in S$.*

Proof. We shall prove the lemma by induction. When $n=2$, the assertion follows from (4.6). Next assume that the assertion holds when $n=i-1$. As (1) in the proof of (4.7)

$$r\leqq t_i\cup t_1\cup s\quad\text{and}\quad s\leqq I(t_1)\cup\cdots\cup I(t_{i-1}).$$

When $s\leqq t_i\cup t_1$, then $r\leqq t_i\cup t_1\leqq I(t_i)\cup I(t_1)$. Hence $I(r)\in S$. When $s\not\leqq t_i\cup t_1$, as (3) and (4) in the proof of (4.7)

$$I(r)\leqq I(p)\cup I(t_i)\quad\text{and}\quad p\leqq I(t_1)\cup\cdots\cup I(t_{i-1}).$$

Since $I(p)\in S$, we have $I(r)\in S$. Consequently the assertion holds when $n=i$. And the lemma is completely proved.

DEFINITION (4.10). Let a be an element of an affine matroid lattice L. If $r\leqq a$ implies $I(r)\leqq a$, then a is called a *$\|$-closed element* of L. We shall say that 0 is a $\|$-closed element.

THEOREM (4.11). *The set M of all $\|$-closed elements of an affine, non-modular, matroid lattice L is an irreducible modular matroid sublattice of L, and is isomorphic to the projective geometry $L(\Omega_0)$ on the decomposition space $\Omega_0=\{I(t_\alpha);\alpha\in I\}$ of L.*

Proof. (i) For $m\in M$ and $S\in L(\Omega_0)$, define

$$(1)\qquad S(m)=\{I(r);r\leqq m\}\quad\text{and}\quad a(S)=\bigvee\big(I(t);I(t)\in S\big).$$

When $r\leqq a(S)$, from (1.6) we have $r\leqq I(t_1)\cup\cdots\cup I(t_n)$ where $I(t_i)\in S$ $(i=1,\cdots,n)$. Hence from (4.7) $I(r)\leqq I(t_1)\cup\cdots\cup I(t_n)\leqq a(S)$. Therefore $a(S)\in M$.

Next we shall show that $S(m)\in L(\Omega_0)$. When $I(r)\leqq I(p)\cup I(q)$ for some p, $q\leqq m$ and $I(r)\neq I(p)$, then from (4.5), there exists a point q' such that $r\leqq p\cup q'$ and $q'\leqq I(q)$. Since m is $\|$-closed, $r\leqq I(r)\leqq I(p)\cup I(q')=I(p)\cup I(q)\leqq m$. Hence $I(r)\in S(m)$. Consequently $S(m)\in L(\Omega_0)$.

(ii) From (1), we have

$$a\big(S(m)\big)=\bigvee\big(I(t);I(t)\in S(m)\big)=\bigvee\big(I(t);t\leqq m\big).$$

Since $I(t)\leqq m$ for any $t\leqq m$, we have $a\big(S(m)\big)\leqq m$. On the other hand, $t\leqq m$ implies $I(t)\in S(m)$, and therefore $t\leqq I(t)\leqq a\big(S(m)\big)$. Hence $m\leqq a\big(S(m)\big)$. Consequently $a(S(m))=m$.

(iii) When $r \leqq a(S)$, by (1.6), $r \leqq I(t_1) \cup \cdots \cup I(t_n)$ for some $I(t_i) \in S$ $(i=1, \cdots, n)$. Then by (4.9) $I(r) \in S$. Hence $S(a(S)) \leqq S$. On the other hand, when $I(r) \in S$, we have $r \leqq I(r) \leqq a(S)$. Hence $I(r) \in S(a(S))$. Therefore $S(a(S))=S$.

(iv) Let $a={}_{\alpha \in J}\bigvee m_\alpha$ where $m_\alpha \in M$ $(\alpha \in J)$. By (ii) $m_\alpha = a(S(m_\alpha)) = \bigvee (I(t); t \leqq m_\alpha)$. Hence by (1.6) $r \leqq a$ implies $r \leqq I(t_1) \cup \cdots \cup I(t_n)$ for $t_i \leqq m_{\alpha_i}$ $(i=1, \cdots, n)$. Then by (4.7) $I(r) \leqq I(t_1) \cup \cdots \cup I(t_n) \leqq a$. Therefore $a \in M$.

Next let $b={}_{\alpha \in J}\bigwedge m_\alpha$ where $m_\alpha \in M$ $(\alpha \in J)$. Then $r \leqq b$ implies $r \leqq m_\alpha$, that is, $I(r) \leqq m_\alpha$ for all $\alpha \in J$. Hence $I(r) \leqq b$, and $b \in M$.

Consequently, M is a complete sublattice of L.

(v) By (ii) and (iii), $m \rightarrow S(m)$ and $S \rightarrow a(S)$ define a one-to-one correspondence between M and $L(\Omega_0)$, preserving the order. Hence M is isomorphic to $L(\Omega_0)$. Consequently M is an irreducible modular matroid sublattice of L.

DEFINITION (4.12). We call the set M of all $\|$-closed elements of an affine matroid lattice L the *modular center* of L. And when M is composed of only two elements 0 and 1, we say that L is *modularly irreducible.*

COROLLARY (4.13). *An affine, non-modular, matroid lattice L is modularly irreducible if and only if L satisfies the strong Euclid's parallel axiom.*

Proof. This follows from (4.1) and (4.4).

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