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Equilibrium in a Stochastic n-Person Game

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Heuristically, a stochastic game is described by a sequence of states which are determined stochastically. The stochastic element arises from a set of transition probability measures. The determination of the particular transition probability measure to be used at a move of the game is controlled in part by each of the n players and it is this determination scheme which gives rise to the strategies.

One might consider the following economics problem. We have n firms competing for a market. Each will make a strategy decision periodically. During each period the n firms play an n-person game. Across the infinite horizon then we have a sequence of games. The economic situation behaves in such a way that the game played in a given period depends on the game played in the previous period and the strategies used in this period. The dependence is not deterministic but is stochastic. A player's strategy will reflect a concern both for the game now being played and for the situation that will be probably confronted in the next period. The relative strength of these two concerns will be in a geometric ratio, the so-called discounting over the infinite horizon. We shall call the outcome of each game a cost to each player. Negative cost is thus a gain.

More precisely we are considering a game which is described by a finite set of states I. A play is a sequence of states $\{i_n\}_{n=0}^{\infty}$, $i_n \in I$. If the game is in the state i, each player h may choose an alternative $j_h \in J^h(i)$. This choice is made with knowledge of the state i. Once each player has made his choice, the game proceeds to the state k with probability $P_{ij_1\cdots j_nk}$. Here $P_{ij_1\cdots j_nk} \geq 0$ and $\sum_k P_{ij_1\cdots j_nk} = 1$. The cost to player h of being in state i and having the vector $\bar{j} = (j_1, \cdots, j_n)$ chosen is $C_{hi\bar{j}}$. Each player furthermore chooses to discount his projected cost by a factor α_h, where $0 \leq \alpha_h < 1$. Thus if a sequence of states $\{i_n\}$ and alternative choices $\{\bar{j}_n\}$ have been made, the cost to player h is given by

$$g(h) = \sum_{n=0}^{\infty} \alpha_h^n C_{hi_n\bar{j}_n}. \tag{1}$$

The analysis of this game is simplified by a slight change in the outlook and notation. Let $g(h, i)$ be the cost to player h given that the game started at the state i. Then equation (1) can be rewritten

$$g(h, i_0) = C_{hi_0\bar{j}_0} + \alpha_h g(h, i_1). \tag{2}$$

Since i_1 is not strictly determined, we are interested in the expected value of $g(h, i)$. If the selection of the vector $\bar{j}_0$ is a function of the state i_0, i.e. a pure strategy, one would get for every such strategy a relation

$$\begin{aligned} E[g(h, i_0)] &= C_{hi_0\bar{j}_0} + \alpha_h E[\sum_k P_{i_0\bar{j}_0k} g(h, k)] \\ &= C_{hi_0\bar{j}_0} + \alpha_h \sum_k P_{i_0\bar{j}_0k} E[g(h, k)]. \end{aligned} \tag{3}$$

Of course, we may expect that the players will not use pure strategies. Mixed strategies thus must be introduced. For every state i of the game, player h will give a probability distribution $x^h(i)$ on the alternative set $J^h(i)$, i.e. $x^h(i) = (x_1^h(i), \cdots, x_m^h(i))$ where $x_j^h(i) \geq 0$, $\sum_j x_j^h(i) = 1$ and m is the cardinality of $J^h(i)$. If such a set of probability distributions x is given for every alternative set, then letting $e_{hi}(x) = E[g(h, i)]$ in (3) we have

$$e_{hi}(x) = \sum_{\bar{j}} \prod_m x_{j_m}^m(i) [C_{hi\bar{j}} + \alpha_h \sum_k P_{i\bar{j}k} e_{hk}(x)], \qquad \begin{array}{l} i = 1, \cdots, \sigma. \\ h = 1, \cdots, n. \end{array} \tag{4}$$

LEMMA 1. *For every* $x = \{x^m(i) \mid x^m(i)$ *is a probability vector*$\}$, *there exist numbers* $e_{hi}(x)$ *satisfying relation* (4). *This set furthermore is unique.*

Proof: The relation (4) for fixed h is a linear system whose coefficient matrix A has the following properties.

$$a_{ii} = 1 - \alpha_h (\sum_{\bar{j}} \prod_m x_{j_m}^m (i) P_{i\bar{j}i}),$$

$$a_{ik} = -\alpha_h \sum_{\bar{j}} \prod_m x_{j_m}^m (i) P_{i\bar{j}k}, \quad (i \neq k).$$

By Hadamard's theorem all eigenvalues of A have absolute value at least $a_{ii} - \sum_{k \neq i} |a_{ik}| = 1 - \alpha_h > 0$. Thus zero is not an eigenvalue of A and the linear system has a unique solution.

Now each player h will use a strategy that tends to minimize his expected cost. That is, if $v_{hi} = \min_{x^h(i)} e_{hi}(x)$, then from (4) we get

$$v_{hi} = \min_{x^h(i)} \sum_{\bar{j}} \prod_m x_{j_m}^m(i) [C_{hi\bar{j}} + \alpha_h \sum_k P_{i\bar{j}k} v_{hk}], \quad \begin{array}{l} i = 1, \cdots, \sigma. \\ h = 1, \cdots, n. \end{array} \tag{5}$$

We will say that the vector x is an equilibrium point if and only if no player can improve his cost by changing his strategy, i.e., if relation (5) is considered for every player h and every state i, the minimization of the right hand side leads back to the same vector x. We will show that such equilibrium points do in fact exist.

Let $X \equiv \{(x^1(1), \cdots, x^1(\sigma), x^2(1), \cdots, x^2(\sigma), \cdots, x^n(1), \cdots, x^n(\sigma));\ x^k(i)$ is a probability distribution on $J^k(i)\}$, with euclidean metric; and

$R \equiv \{(v_{11}, \cdots, v_{n\sigma});\ v_{ij}$ real$\}$, $d(u, v) = \max\limits_{i,j} |u_{ij} - v_{ij}|$. X is compact and R is complete. In the proofs that follow one notes that one can confine the discussion to any closed convex subset of X and results are not altered. Thus we simultaneously establish the existence of equilibrium points in constrained games.

To ease the notational bulk we introduce the vector function f. Define

$$f(x, y, v)_{hi} = \sum_{\bar{j}} \prod_{m \neq h} x^m_{j_m}(i) y^h_{j_h}(i) [C_{hi\bar{j}} + \alpha_h \sum_k P_{i\bar{j}k} v_{nk}] \tag{6}$$

where $x \in X$ and $y^h(i)$ is a probability vector. Thus (5) can be written

$$v_{hi} = \min_{y^h(i)} f(x, y, v)_{hi}. \tag{7}$$

We note that $f(x, y, v)$ has the following properties:

(a) $f(x, y, v)$ is continuous;

(b) $f(x, y, v)_{hj} - f(x, y, u)_{hj} \leq \alpha \max\limits_{k,l} |v_{kl} - u_{kl}|$, $\alpha = \max\limits_h \alpha_h$;

(c) $f(x, y, v)$ is linear in y.

Let $x \in X$ and define a mapping T_x of R into itself by the equation

$$(T_x v)_{hj} = \min_{y^h(j)} f(x, y, v)_{hj} \tag{8}$$

THEOREM 1: *For every $x \in X$, T_x is a contraction mapping of R.*

Proof: Let u and v be two elements of R and let x be a fixed element of X. Let $T_x u = f(x, y, u)$ and $T_x v = f(x, z, v)$, then $T_x u \leq f(x, z, u)$ and $T_x v \leq f(x, y, v)$. Thus

$(T_x u)_{hj} - (T_x v)_{hj} \leq f(x, z, u)_{hj} - f(x, z, v)_{hj} \leq \alpha d(v, u)$ by (b) and

$(T_x v)_{hj} - (T_x u)_{hj} \leq f(x, y, v)_{hj} - f(x, y, u)_{hj} \leq \alpha d(v, u)$.

Hence

$$d(T_x u, T_x v) = \max_{h,j} |(T_x v)_{hj} - (T_x u)_{hj}| \leq \alpha d(v, u).$$

Corollary 1. *For every $x \in X$, T_x has a unique fixed point v.*

Corollary 2. *The set $\{T_x | x \in X\}$ is equicontinuous.*

Let $x \in X$ and define mappings ϕ and β by

$$\beta(x) = \{v \mid v = \min_y f(x, y, v)\} \tag{9}$$

$$\phi(x) = \{y \mid \beta(x) = f(x, y, \beta(x))\}. \tag{10}$$

We note that $\beta(x)$ is a single-valued function by Corollary 1. By (c) it is clear that $\phi(x)$ is convex and closed for every $x \in X$.

LEMMA 2. *The range of $\beta(x)$ is bounded.*

Proof. By Theorem 1 the sequence $v_0=0$, $v_{n+1}=T_x v_n$ converges to $\beta(x)$. Furthermore, we have $d(v_m, v_{m-1}) \leq \alpha d(v_{m-1}, v_{m-2}) \leq \cdots \leq \alpha^{m-1} d(v_1, v_0)$ so that $d(v_n, v_0) \leq d(v_n, v_{n-1}) + d(v_{n-1}, v_{n-2}) + \cdots + d(v_1, v_0) \leq [\alpha^{n-1} + \alpha^{n-2} + \cdots + 1] d(v_1, v_0) \leq \frac{1}{1-\alpha} d(v_1, v_0)$. Thus $d(\beta(x), 0) \leq \frac{1}{1-\alpha} d(T_x 0, 0) = \frac{1}{1-\alpha} \max_{h,i} |(T_x 0)_{hi}| \leq \frac{1}{1-\alpha} \max_{hij} |C_{hij}|$. Hence $\beta(x)$ is bounded as x takes on all values in X.

Define $S_v(x) = T_x v$. S_v is a mapping of X into R.

LEMMA 3. *S_v is continuous on X. Furthermore, $\{S_v \mid v$ is bounded$\}$ is equicontinuous.*

Proof: Let $S_v(x) = T_x v = f(x, y, v) \leq f(x, z, v)$ and $S_v(x') = T_{x'} v = f(x', z, v) \leq f(x', y, v)$; then $S_v(x') - S_v(x) \leq f(x', y, v) - f(x, y, v)$ and $S_v(x) - S_v(x') \leq f(x, z, v) - f(x', z, v)$.

If v is restrained to be in a bounded region, then the right hand sides can be made uniformly small because of the uniform continuity of f on compact sets.

LEMMA 4. *If $x_n \to x$ and $\beta(x_n) \to v_0$, then $\beta(x) = v_0$.*

Proof: $d(v_0, T_x v_0) \leq d(v_0, \beta(x_n)) + d(\beta(x_n), T_x \beta(x_n)) + d(T_x \beta(x_n), T_x v_0) = d(v_0, \beta(x_n)) + d(S_{\beta(x_n)}(x_n), S_{\beta(x_n)}(x)) + d(T_x \beta(x_n), T_x v_0) \to 0$ as $n \to \infty$ because $\beta(x_n) \to v_0$ and $\{\beta(x_n)\}$ is bounded by Lemma 2 so that Lemma 3 applies to the second term.

LEMMA 5. *If $x_n \to x$, $y_n \to y$ and $y_n \in \phi(x_n)$, then $y \in \phi(x)$.*

Proof: By taking subsequences we can consider $\beta(x_n) \to v_0$. By Lemma 4 $\beta(x) = v_0$. Now $d(f(x, y, v_0), v_0) \leq d(f(x, y, v_0), f(x_n, y_n, \beta(x_n))) + d(f(x_n, y_n, \beta(x_n)), v_0) = d(f(x, y, v_0), f(x_n, y_n, \beta(x_n))) + d(\beta(x_n), v_0) \to 0$ as $n \to \infty$. Thus $v_0 = f(x, y, v_0)$, and by Lemma 4, v_0 is the fixed point so that $f(x, y, v_0) = v_0 = \min_z f(x, z, v_0)$, thus $y \in \phi(x)$.

THEOREM 2. *There exist $x \in X$, $v \in R$ such that $v = f(x, x, v) = \min_y f(x, y, v)$.*

i.e. $x \in \phi(x)$.

Proof: This is a consequence of the Kakutani fixed point theorem, i.e. the mapping ϕ takes points into convex closed sets, by Lemma 5 the set $\bigcup_x \phi(x)$ is closed, hence sequentially compact and by Lemma 5 it is an upper semi-continuous set function.

It is also interesting to note that $\beta(x)$ is continuous (Lemmas 2 and 4) and thus its range is compact and connected. In case the set of states I is denumerable, one replaces R by the space l^∞ of bounded sequences and requires that $|C_{hij}| \leq M$ for some M, and the results are still valid for X the appropriate cartesian product of probability spaces. One also notes that Theorem 1 and its corollaries are valid if the cardinality of $J^h(i)$ is arbitrary and min is replaced by inf. In this case the techniques of this paper yield ε-effective strategies.

If $n=2$ the results of this paper are those given by Shapely [2] and for $n=1$ the problem is a well-known dynamic program problem. See Takahashi [3] for a different generalization of Shapely's results.

References

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