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Convolution Maps and Semi-group Distributions

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The main purpose of this paper is to extend the theory of semi-group distributions developed by J. L. Lions [5] to a more general case where the underlying vector space E is a locally convex space, while in his theory the space E is confined to a Banach space.

With this end in view, we shall first discuss the continuity behaviours of the θ-convolution map between the spaces of vector valued distributions $\bar{\mathscr{D}}'_+(E)$ and $\bar{\mathscr{D}}'_+(F)$ with separately continuous bilinear map $\theta: E\times F\to G$, where E, F, G are locally convex spaces, G being assumed to be quasi-complete. In Section 1 we shall show that if L is a continuous linear map of $\bar{\mathscr{D}}'_+(E)$ into $\bar{\mathscr{D}}'_+(G)$ such that the restriction of L to $\mathscr{D}\otimes E$ is commutative with every translation τ_h, $-\infty<h<\infty$, then L is the convolution map $L(\vec{S})=\vec{S}*_\theta\vec{T}$, where $\vec{T}\in\bar{\mathscr{D}}'_+(\mathscr{L}_b(E; G))$ is uniquely determined by L and θ denotes the bilinear map $E\times\mathscr{L}_b(E; G)\to G$ defined in an obvious way. The result will be used in Section 2 to make a characterization of a semi-group distribution. Concerning this, we follow in most parts the way of the proof carried out by J. L. Lions [5] and show that, roughly speaking, under certain conditions any semi-group distribution under consideration is no more than the Green operator of a differential equation of the form:

$$-A\vec{u}+\frac{d}{dt}\vec{u}=\vec{T},\quad \vec{T}\in\bar{\mathscr{D}}'_+(E),$$

where A is the infinitesimal generator of the semi-group distribution. Finally we shall make a remark about the relation between his results and ours.

§ 1. θ-convolution map of $\bar{\mathscr{D}}'_+(E)\times\bar{\mathscr{D}}'_+(F)$ into $\bar{\mathscr{D}}'_+(G)$

Let us denote by $\mathscr{D}$ (resp. $\mathscr{D}_+$, resp. $\mathscr{D}_-$) the space of all C^∞-functions on R^1, 1-dimensional Euclidean space, with compact supports (resp. with supports bounded on the left, resp. with supports bounded on the right). These spaces are provided with usual topologies of L. Schwartz ([6]). By $\mathscr{D}'$ (resp. $\mathscr{D}'_+$) we shall mean the strong dual of $\mathscr{D}$ (resp. $\mathscr{D}_-$). Let $\mathscr{H}$ be a normal space of distributions, that is, a linear subspace $\subset\mathscr{D}'$ with continuous injections $\mathscr{D}\to\mathscr{H}$, $\mathscr{H}\to\mathscr{D}'$ such that $\mathscr{D}$ is dense in $\mathscr{H}$. Let F be a locally convex Hausdorff topological vector space. For the sake of brevity we shall refer to such a space as LCS. The continuous linear maps of $\mathcal{H}'_c$ into F form a linear space $\mathcal{H}(F)$, called a space of F-valued distributions, on which we take the topology of uniform convergence with respect to the equicontinuous subsets of $\mathcal{H}'$. It is also considered as the space of the continuous linear maps of F'_c into $\mathcal{H}$. However without specific mention about $\mathcal{H}(F)$, we understand it the space of the continuous linear maps in the first sense. For any $\vec{T} \in \mathcal{H}(F)$ and $\phi \in \mathcal{H}'$, the image of ϕ by $\vec{T}$ is denoted by $\phi \cdot \vec{T}$. A subset $\mathfrak{A}$ of $\mathcal{H}(F)$ is called γ-equibounded when there is a disked neighbourhood $\mathcal{U}$ of 0 in $\mathcal{H}'_c$ such that $\mathcal{U} \cdot \mathfrak{A}$ is contained in a compact disk of F ([8], p. 54).

PROPOSITION 1. *Let $\alpha, \beta \in \mathcal{D}_+$ and B be a bounded disk of $\mathcal{D}$. Let F be an LCS and $\mathfrak{B}$ be an equicontinuous subset of $\mathcal{L}(F'_c; \mathcal{D}')$. If we put $M_{\phi,\vec{T}} = \alpha((\beta\vec{T})^{\vee} * \phi) \in \mathcal{D}(F)$ for every $\vec{T} \in \mathcal{D}'(F)$ and $\phi \in \mathcal{D}$, then the set $\{M_{\phi,\vec{T}}\}_{\phi \in B, \vec{T} \in \mathfrak{B}}$ is γ-equibounded in $\mathcal{D}(F)$, and there exists a disked neighbourhood $\mathcal{U}$ of 0 in $\mathcal{D}'$ and a compact disk K of F such that each $M_{\phi,\vec{T}}$ can be written as*

$$\sum_i \lambda_i h_{i,\phi} \otimes f_{i,\vec{T}}$$

with $h_{i,\phi} \in \mathcal{U}^\circ \subset \mathcal{D}$, $f_{i,\vec{T}} \in K$ and $\sum |\lambda_i| < \infty$, that is, for any $S \in \mathcal{D}'$

$$S \cdot M_{\phi,\vec{T}} = \sum \lambda_i \langle S, h_{i,\phi} \rangle f_{i,\vec{T}}.$$

The proposition will be obtained from the next two lemmas, in which α, β denote the elements of $\mathcal{D}_+$ as in the proposition 1.

LEMMA 1. *$(\alpha S) * (\beta T)$ exists for any $S, T \in \mathcal{D}'$ and the map $(S, T) \to (\alpha S) * (\beta T)$* is a continuous bilinear map of *$\mathcal{D}' \times \mathcal{D}'$ into $\mathcal{D}'_+$.*

PROOF. $(\alpha S)((\beta T)^{\vee} * \phi) \in \mathcal{E}' \subset \mathcal{D}'_{L^1}$ for every $\phi \in \mathcal{D}$, where the symbol $\vee$ means the symmetrization. Therefore the convolution $(\alpha S) * (\beta T)$ is well defined and belongs to $\mathcal{D}'_+$ ([6], II, p. 12, [9], p. 23). Next we shall show that the bilinear map $(S, T) \to (\alpha S) * (\beta T)$ of $\mathcal{D}' \times \mathcal{D}'$ into $\mathcal{D}'_+$ is continuous. Let B be any bounded disk of $\mathcal{D}$. It is well known that the supports of elements of B are contained in a finite interval I of R^1. We can choose a function γ of $\mathcal{D}$ equal to 1 on a finite interval, depending only on I, α, β, such that for every $\phi \in B$

$$\begin{aligned} \langle (\alpha S) * (\beta T), \phi \rangle &= \langle (\alpha S)_x \otimes (\beta T)_y, \phi(x+y) \rangle \\ &= \langle (\alpha S)_x \otimes (\beta T)_y, \gamma(x)\phi(x+y) \rangle. \end{aligned}$$

Now the set $\mathfrak{B} = \{\gamma(x)\phi(x+y)\}_{\phi \in B}$ is bounded in $\mathcal{D}_{x,y}$, the space of all C^∞-functions with compact supports on $R^1 \times R^1$. Any compact subset of the complete projective tensor product $E_1 \hat{\otimes}_\pi E_2$ of the spaces E_1 and E_2 of type **(F)** is contained in the absolutely convex closure of $A_1 \otimes A_2$, where A_i is a compact subset of E_i, $i=1, 2$ ([4], Chap. I, p. 52). Therefore there exists a bounded subset B_1 of $\mathcal{D}$ such that $\mathfrak{B}$ is contained in the absolutely convex closure of $B_1 \otimes B_1$. Consider two disked neighbourhoods $\mathcal{U}=\{S;\ \alpha S \in B_1^\circ \text{ and } S \in \mathcal{D}'\}$ and $\mathcal{V}=\{T;\ \beta T \in B_1^\circ \text{ and } T \in \mathcal{D}'\}$ of 0 in $\mathcal{D}'$. Then we have

$$
\begin{aligned}
|<(\alpha\mathcal{U})*(\beta\mathcal{V}), B>| &\leqq |<(\alpha\mathcal{U})\otimes(\beta\mathcal{V}), B_1\otimes B_1>| \\
&= |<\alpha\mathcal{U}, B_1>|\ |<\beta\mathcal{V}, B_1>| \\
&\leqq 1,
\end{aligned}
$$

which implies that the map $(S, T)\to(\alpha S)*(\beta T)$ is continuous. The proof is completed.

Lemma 2. *Let B be a bounded disk of $\mathcal{D}$. If we put $L_\phi(T)=\alpha((\beta T)^\vee * \phi)$ for every $T \in \mathcal{D}'$ and $\phi \in \mathcal{D}$, then the set $\{L_\phi\}_{\phi \in B}$ is γ-equibounded in $\mathcal{D}(\mathcal{D})$ and there exist two disked neighbourhoods $\mathcal{U}$ and $\tilde{\mathcal{U}}(\subset\mathcal{U})$ of 0 in $\mathcal{D}'$ such that L_ϕ can be written as*

$$\sum_i \lambda_i d_i \otimes h_{i,\phi}$$

with $d_i \in \tilde{\mathcal{U}}^\circ$, $h_{i,\phi} \in \mathcal{U}^\circ$ and $\sum|\lambda_i|<\infty$, that is, for every $T \in \mathcal{D}'$

$$L_\phi(T)=\sum_i \lambda_i <T, d_i> h_{i,\phi}.$$

Proof. Owing to Lemma 1, we can find for the given B a disked neighbourhood $\mathcal{U}$ of 0 in $\mathcal{D}'$ such that

$$|<(\alpha\mathcal{U})*(\beta\mathcal{U}), B>|=|<\mathcal{U}, \alpha((\beta\mathcal{U})^\vee * B)>|\leqq 1.$$

This implies that $\{L_\phi(\mathcal{U})\}_{\phi\in B}=\alpha((\beta\mathcal{U})^\vee * B)\subset\mathcal{U}^\circ$. Consequently $\{L_\phi\}_{\phi\in B}$ is γ-equibounded in $\mathcal{D}(\mathcal{D})$. Since $\mathcal{D}'$ is a nuclear space, there exists a disked neighbourhood $\tilde{\mathcal{U}}$ of 0 in $\mathcal{D}'$ such that the natural map J: $\hat{\mathcal{D}}'_{\tilde{\mathcal{U}}}\to\hat{\mathcal{D}}'_{\mathcal{U}}$ is nuclear, that is, $J=\sum\lambda_i d_i\otimes d'_i$ with $d_i \in \tilde{\mathcal{U}}^\circ$, $d'_i \in \mathcal{U}$ and $\sum|\lambda_i|<\infty$ ([4], Chap. I, p. 80). Now the map L_ϕ is factorized as follows:

$$\mathcal{D}' \xrightarrow{i_1} \hat{\mathcal{D}}'_{\tilde{\mathcal{U}}} \xrightarrow{J} \hat{\mathcal{D}}'_{\mathcal{U}} \xrightarrow{\bar{L}_\phi} \mathcal{D}_{\mathcal{U}^\circ} \xrightarrow{i_2} \mathcal{D},$$

where i_1, i_2 are the canonical maps and $\bar{L}_\phi$ is the induced map derived from L_ϕ. Therefore, for any $T \in \mathcal{D}'$

$$L_\phi(T) = i_2 \circ \bar{L}_\phi(\sum \lambda_i < T, d_i > d_i')$$
$$= \sum \lambda_i < T, d_i > h_{i,\phi},$$

where $h_{i,\phi} = \alpha((\beta d_i')\check{} * \phi) \in \mathcal{U}^\circ$, $d_i \in \tilde{\mathcal{U}}^\circ$ and $\sum |\lambda_i| < \infty$, which completes the proof.

Proof of Proposition 1. Let $\mathcal{U}$, $\tilde{\mathcal{U}}$ be disked neighbourhoods of 0 in $\mathcal{D}'$ as chosen in the proof of Lemma 2. By assumption, $\mathfrak{B}$ is an equicontinuous subset of $\mathcal{L}(F_c'; \mathcal{D}')$, so that we can choose a compact disk K of F such that $<\vec{T}, K^\circ> \subset \tilde{\mathcal{U}}$ for every $\vec{T} \in \mathfrak{B}$. Then we have for $\phi \in B$, $\vec{T} \in \mathfrak{B}$

$$|<\mathcal{U} \cdot M_{\phi,\vec{T}}, K^\circ>| = |<\mathcal{U} \cdot \alpha((\beta \vec{T})\check{} * \phi), K^\circ>|$$
$$= |\phi \cdot (\alpha \mathcal{U} * \beta <\vec{T}, K^\circ>)|$$
$$\leqq |\phi \cdot (\alpha \mathcal{U} * \beta \tilde{\mathcal{U}})|$$
$$\leqq 1.$$

This means that $\{M_{\phi,\vec{T}}\}_{\phi \in B, \vec{T} \in \mathfrak{B}}$ is γ-equibounded in $\mathcal{D}(F)$.

In virtue of Lemma 2, we have for any $f' \in F'$

$$<M_{\phi,\vec{T}}, f'> = \alpha((\beta <\vec{T}, f'>) * \phi)$$
$$= \sum_i \lambda_i (d_i \cdot <\vec{T}, f'>) h_{i,\phi}$$
$$= <\sum_i \lambda_i h_{i,\phi} \otimes (d_i \cdot \vec{T}), f'>.$$

Therefore if we put $f_{i,\vec{T}} = d_i \cdot \vec{T} \in \tilde{\mathcal{U}}^\circ \cdot \vec{T} \subset K$, then we can write

$$M_{\phi,\vec{T}} = \sum \lambda_i h_{i,\phi} \otimes f_{i,\vec{T}},$$

where $h_{i,\phi} \in \mathcal{U}^\circ$, $f_{i,\vec{T}} \in K$ and $\sum |\lambda_i| < \infty$, which completes the proof.

Let E be an LCS. We denote by $\mathcal{D}'_{[a,\infty)}(E)$ the space of the E-valued distributions on R^1 with supports contained in the half-line $[a, \infty)$, where a denotes any real number. On the space $\mathcal{D}'_{[a,\infty)}(E)$ we take the topology induced by that of the space of E-valued distributions $\mathcal{D}'(E)$. Further by $\bar{\mathcal{D}}'_+(E)$ we denote the space $\bigcup_a \mathcal{D}'_{[a,\infty)}(E)$ equipped with the topology of the inductive limit of $\{\mathcal{D}'_{[a,\infty)}(E)\}_{-\infty < a < \infty}$. $\bar{\mathcal{D}}'_+(E)$ is a subspace of $\mathcal{D}'(E)$ but not topologically in general. If E is a space of type (**DF**), it is not difficult to see that $\bar{\mathcal{D}}'_+(E) = \mathcal{D}'_+(E)$ algebraically as well as topologically ([4], Chap. I, p. 47). It is to be noticed that if E is normable, $\mathcal{D}'_+(E)$ is bornological and moreover, if E is a Banach space, $\mathcal{D}'_+(E)$ is barrelled. This follows from a more general situation as follows.

PROPOSITION 2. *Let E, F be two LCSs such that E, E'_c are nuclear and F is normable. Then the ε-product $E\varepsilon F$* ([7], p. 18) *is bornological whenever E is bornological.*

PROOF. Let G be any complete LCS. It suffices to show that any linear map u which transforms any bounded subset of $E\varepsilon F$ into a bounded subset of G is continuous. Let W be any disked neighbourhood of 0 in G. The set $\mathfrak{B}=u^{-1}(W)$ is absolutely convex and absorbs every bounded subset of $E\varepsilon F$. Let B be any bounded subset of E and V be the unit ball of F which we may consider to be a normed linear space. Clearly $B\otimes V$ is bounded in $E\varepsilon F$, so that it is absorbed by $\mathfrak{B}$. If we put $U=\{e;\, e\otimes V\subset\mathfrak{B} \text{ and } e\in E\}$, it is an absolutely convex subset of E which absorbs every bounded subset of E. Since E is bornological, it follows that U is a neighbourhood of 0 in E. This means that the restriction of u to $E\otimes_\pi F$ $(=E\otimes_\varepsilon F$ since E is nuclear) is continuous. Therefore it may be extended uniquely to a continuous linear map v of $E\varepsilon F$ into G. For any $\xi\in E\varepsilon F$, it is considered to be an element of $\mathscr{L}_\varepsilon(E'_c;\, F)$, so that there exists a compact disk K of E such that the image $\xi(K^\circ)$ is contained in V. Now since the space E'_c is nuclear, there exists a compact disk $K_1(\supset K)$ of E such that the natural map $\hat{E}'_{K_1^\circ}\to\hat{E}'_{K^\circ}$ is nuclear, from which we can infer that ξ may be written in the form:

$$\xi=\sum_{i=1}^{\infty}\lambda_i e_i\otimes f_i,$$

where $e_i\in K_1$, $f_i\in V$ and $\sum|\lambda_i|<\infty$. If we put $\rho_n=\sum_{i=n+1}^{\infty}|\lambda_i|$, the set $\left\{\frac{1}{\rho_n}\sum_{i=n+1}^{\infty}\lambda_i e_i\otimes f_i\right\}$ is bounded in $E\varepsilon F$, whence the set $\left\{\frac{1}{\rho_n}u(\sum_{i=n+1}^{\infty}\lambda_i e_i\otimes f_i)\right\}$ is bounded by the assumption imposed on u, and therefore $u(\sum_{i=n+1}^{\infty}\lambda_i e_i\otimes f_i)\to 0$ as $n\to\infty$. Consequently,

$$\begin{aligned}u(\xi)&=u(\sum\lambda_i e_i\otimes f_i)\\&=\sum_{i=1}^{n}\lambda_i u(e_i\otimes f_i)+u(\sum_{i=n+1}^{\infty}\lambda_i e_i\otimes f_i)\\&=\sum_{i=1}^{n}\lambda_i v(e_i\otimes f_i)+u(\sum_{i=n+1}^{\infty}\lambda_i e_i\otimes f_i).\end{aligned}$$

Passing to the limit as $n\to\infty$, and taking into account the fact that v is continuous, we can see that $u(\xi)=v(\xi)$. As ξ is any element of $E\varepsilon F$, u coincides with v, that is, u is continuous, which completes the proof.

REMARK. Suppose E satisfies the strict Mackey condition for convergence ([3], p. 105), that is, for any bounded subset $A\subset E$, there exists a bounded disk B ($\supset A$) such that the topology of A induced by E coincides with that induced by E_B. Let F be a Banach space. Here we assume that $F\neq(0)$. If $E\varepsilon F$ is bornological, then E is also bornological. In fact, for any $\xi\in E\varepsilon F$ which we may consider to be an element of $\mathcal{L}_\varepsilon(F'_c; E)$, there exists a bounded disk A of E such that ξ may be an element of $\mathcal{L}_\varepsilon(F'_c; E_A)$, that is, $\xi\in E_A\varepsilon F$. Indeed, let V be the unit ball of F. $\xi(V^\circ)=K$ is a compact disk of E, so that, by assumption on E, there exists a bounded disk $A\supset K$ such that the topology of A induced by E coincides with that induced by E_A. Then the map ξ restricted to V° is continuous of V° into E_A. Since F is complete, owing to a proposition of L. Schwartz ([7], p. 41), the map $\xi: F'_c\to E_A$ is continuous, that is, $\xi\subset E_A\varepsilon F$. Let u be any linear map of E into a complete LCS G such that it transforms any bounded subset of E into a bounded subset of G. Let us denote by u_A the restriction of u to E_A, which is a continuous linear map of E_A into G since E_A is a normed linear space. Therefore $u_A\otimes I$, I being the identical map of F into itself, is a continuous linear map of $E_A\varepsilon F$ into $G\varepsilon F$. Let us define the linear map v of $E\varepsilon F$ into $G\varepsilon F$ by the relation $v(\xi)=(u_A\otimes I)(\xi)$, where A is chosen as indicated above. That the choice of A has no effect on the definition of v is easily seen. If ξ runs through a bounded subset of $E\varepsilon F$, we can take A as the same bounded disk for these ξ, so that the map v becomes continuous. Let $f_0\in F$, $f'_0\in F'$ be chosen so that $<f_0, f'_0>=1$. Clearly the map $\theta: e\to e\otimes f_0$ of E into $E\varepsilon F$ and the map $I\otimes f'_0$ of $G\varepsilon F$ into G are continuous. Let us consider the map $w=(I\otimes f'_0)\circ v\circ\theta$ which is a continuous linear map of E into G. Now it is easy to see that $w(e)=u(e)$ for every $e\in E$, which implies that u is continuous.

Let $\mathfrak{S}$ be a *saturated family* of bounded subsets of an LCS F ([8], p. 198), that is, (i) if $A\in\mathfrak{S}$, then $\lambda A\in\mathfrak{S}$ for every $\lambda>0$; (ii) if $A\in\mathfrak{S}$, then any subset of A belongs to $\mathfrak{S}$; (iii) if $A\in\mathfrak{S}$, then the disked envelope of A belongs to $\mathfrak{S}$; (iv) if $A, B\in\mathfrak{S}$, then $A\cup B\in\mathfrak{S}$; (v) every one point subset of F belongs to $\mathfrak{S}$. We shall say that a subset $\mathfrak{A}$ of $\mathcal{D}'_{[b,\infty)}(F)$ is *of type* $\mathfrak{S}$ in $\mathcal{D}'_{[b,\infty)}(F)$, if $\mathfrak{A}$, considered as a subset of $\mathcal{D}'(F)$, is of type $\mathfrak{S}$ in $\mathcal{D}'(F)$, that is, for any bounded subset B of $\mathcal{D}$ the set $\bigcup_{\vec{T}\in\mathfrak{A}} B\cdot\vec{T}$ is contained in an $A\in\mathfrak{S}$.

First we prove

PROPOSITION 3. *Let E, F, G be three LCSs, where G is assumed to be quasi-complete. Let θ be a separately continuous bilinear map of $E\times F$ into G. Then any $\vec{S}\in\mathcal{D}'_{[a,\infty)}(E)$ and $\vec{T}\in\mathcal{D}'_{[b,\infty)}(F)$ are $*_\theta$-composable and $\vec{S}*_\theta\vec{T}\in\mathcal{D}'_{[a+b,\infty)}(G)$.*

(a) *The bilinear map $(\vec{S}, \vec{T})\to\vec{S}*_\theta\vec{T}$ of $\mathcal{D}'_{[a,\infty)}(E)\times\mathcal{D}'_{[b,\infty)}(F)$ into $\mathcal{D}'_{[a+b,\infty)}(G)$ is separately quasi-continuous.*

(b) *If θ is hypocontinuous with respect to the compact disks of F, then the linear map $\vec{S}\to\vec{S}*_\theta\vec{T}$ is uniformly continuous with respect to the equicontinuous subsets of $\mathcal{L}(F'_c; \mathcal{D}'_{[b,\infty)})$.*

(c) *If θ is hypocontinuous with respect to the bounded subsets of E and F, then so is $*_\theta$.*

(d) *If θ is continuous, then so is $*_\theta$.*

Finally, let $\mathfrak{G}$ be a saturated family of bounded subsets of F.

(e) *If θ is hypocontinuous with respect to the sets of $\mathfrak{G}$, then the linear map $\vec{S}\to\vec{S}*_\theta\vec{T}$ is uniformly continuous with respect to the subsets of type $\mathfrak{G}$ in $\mathcal{D}'_{[b,\infty)}(F)$.*

PROOF. For any $\phi\in\mathcal{D}$, $\check{\vec{T}}*\phi\in\mathcal{E}(F)$ is locally γ-bounded and $\vec{S}\in\mathcal{D}'(E)$, the multiplicative product $[\vec{S}(\check{\vec{T}}*\phi)]_\theta$ is well defined as an element of $\mathcal{D}'(G)$ ([8], p. 133). Clearly

$$[\vec{S}(\check{\vec{T}}*\phi)]_\theta\in\mathcal{E}'(G)\subset\mathcal{D}'_{L^1}(G).$$

Therefore, by definition ([10], p. 182), $\vec{S}$ and $\vec{T}$ are $*_\theta$-composable, and it is known that $\vec{S}*_\theta\vec{T}\in\mathcal{D}'_{[a+b,\infty)}(G)$ ([8], p. 167).

Choose two elements $\alpha,\beta\in\mathcal{D}_+$ such that α and β equal 1 on $[a,\infty)$ and $[b,\infty)$ respectively. Let B be any bounded disk of $\mathcal{D}$. Then, owing to Proposition 1, we can choose $\mathcal{U}$, $\tilde{\mathcal{U}}$ (resp. K), depending only on B, α, β (resp. $\tilde{\mathcal{U}}$, $\vec{T}$), as indicated in the same proposition and we can write $\alpha((\beta\vec{T})\check{}*\phi)$ in the form:

$$\alpha((\beta\vec{T})\check{}*\phi)=\sum_i\lambda_i h_{i,\phi}\otimes f_{i,\vec{T}}$$

with $h_{i,\phi}\in\mathcal{U}^\circ$, $f_{i,\vec{T}}\in\tilde{\mathcal{U}}^\circ\cdot\vec{T}\subset K$ and $\sum|\lambda_i|<\infty$. Taking into account a proposition of L. Schwartz ([8], p. 70), we have for any $\phi\in B$

$$\begin{aligned}\phi\cdot(\vec{S}*_\theta\vec{T})&=\phi\cdot(\alpha\vec{S}*_\theta\beta\vec{T})\\&=\vec{S}\cdot_\theta(\alpha((\beta\vec{T})\check{}*\phi))\\&=\vec{S}\cdot_\theta(\sum\lambda_i h_{i,\phi}\otimes f_{i,\vec{T}})\\&=\sum\lambda_i\theta(h_{i,\phi}\cdot\vec{S},f_{i,\vec{T}}).\end{aligned}\tag{1}$$

(a): Suppose $\vec{S}$ converges in $\mathcal{D}'_{[a,\infty)}(E)$ to 0, running through a bounded subset $\mathfrak{B}$ of $\mathcal{D}'_{[a,\infty)}(E)$. We shall show that $\vec{S}*_\theta\vec{T}$ converges to 0 in $\mathcal{D}'_{[a+b,\infty)}(G)$ for every $\vec{T}\in\mathcal{D}'_{[b,\infty)}(F)$. Since $\{h_{i,\phi}\cdot\vec{S}\}_{\phi\in B,\vec{S}\in\mathfrak{B},i=1,2,\dots}$ is bounded in E and $\{f_{i,\vec{T}}\}_{i=1,2,\dots}$ is contained in a compact disk K of F, it follows that the set $\{\theta(h_{i,\phi}\cdot\vec{S},f_{i,\vec{T}})\}_{\phi\in B,\vec{S}\in\mathfrak{B},i=1,2,\dots}$ is bounded in G ([10], p. 194). And for each i, $\theta(h_{i,\phi}\cdot\vec{S},f_{i,\vec{T}})$ converges to 0 as $\vec{S}\to 0$ in $\mathfrak{B}$. Therefore from (1) it follows that $\phi\cdot(\vec{S}*_\theta\vec{T})$ converges to 0 in $\mathcal{D}'_{[a+b,\infty)}(G)$ uniformly with respect to ϕ of B as $\vec{S}\to 0$ in $\mathfrak{B}$. Therefore, by symmetry, the bilinear map $*_\theta$ is separately quasi-continuous.

(b): Suppose θ is hypocontinuous with respect to the compact disks of F. Let $\vec{T}$ lie in an equicontinuous subset $\mathfrak{A}$ of $\mathcal{L}(F'_c; \mathcal{D}'_{[b,\infty)})$. Then there exists a compact disk K of F such that $<\vec{T}, K^\circ>$ is contained in $\tilde{\mathcal{U}}$. By our assumption on θ, we can find a neighbourhood U of 0 in E in such a way that $\theta(U, K) \subset W$ for a given neighbourhood W of 0 in G. Now consider the set $\boldsymbol{U}$ of the elements $\vec{S} \in \mathcal{D}'_{[a,\infty)}(E)$ such that $\mathcal{U}^\circ \cdot \vec{S} \subset U$. $\boldsymbol{U}$ is, by definition, a neighbourhood of 0 in $\mathcal{D}'_{[a,\infty)}(E)$. Then $h_{i,\phi} \cdot \vec{S} \in \mathcal{U}^\circ \cdot \vec{S} \subset U$ and $f_{i,\vec{T}} \in \tilde{\mathcal{U}}^\circ \cdot \vec{T} \subset K$ for $\vec{S} \in \boldsymbol{U}$, $\vec{T} \in \mathfrak{A}$, $\phi \in B$. Therefore it follows from (1) that for every $\vec{S} \in \boldsymbol{U}$, $\vec{T} \in \mathfrak{A}$, $\phi \in B$

$$\phi \cdot (\vec{S} *_\theta \vec{T}) = \sum \lambda_i \theta(h_{i,\phi} \cdot \vec{S}, f_{i,\vec{T}}) \in \sum |\lambda_i| W,$$

which implies that the map: $\vec{S} \to \vec{S} *_\theta \vec{T}$ is uniformly continuous with respect to the equicontinuous subsets of $\mathcal{L}(F'_c; \mathcal{D}'_{[b,\infty)})$.

(c): Suppose θ is hypocontinuous with respect to the bounded subsets of E and F. It is known that any bounded subset of $\mathcal{D}'_{[a,\infty)}(E)$ (resp. $\mathcal{D}'_{[b,\infty)}(F)$) is an equicontinuous subset of $\mathcal{L}(E'_b; \mathcal{D}'_{[a,\infty)})$ (resp. $\mathcal{L}(F'_b; \mathcal{D}'_{[b,\infty)})$) ([7], p. 28). From this fact together with the assumption on θ, we can conclude just as in (b) that the bilinear map $*_\theta$ becomes hypocontinuous with respect to the bounded subsets of $\mathcal{D}'_{[a,\infty)}(E)$ and $\mathcal{D}'_{[b,\infty)}(F)$.

(d): We can infer in a similar way as above that if θ is continuous, then $*_\theta$ is also continuous.

(e): Finally we assume that θ is hypocontinuous with respect to the subsets of $\mathfrak{S}$. Let $\mathfrak{B}$ be a subset of type $\mathfrak{S}$ in $\mathcal{D}'_{[b,\infty)}(F)$. Then $\mathcal{U}^\circ \cdot \mathfrak{B}$ is contained in an element K of $\mathfrak{S}$. By our assumption on θ, we find a neighbourhood U of 0 in E such that $\theta(U, K) \subset W$ for any given neighbourhood W of 0 in G. Therefore we can infer in a similar way as in the proof of (b) that $\vec{S} \to \vec{S} *_\theta \vec{T}$ is uniformly continuous with respect to the subsets of type $\mathfrak{S}$ in $\mathcal{D}'_{[b,\infty)}(F)$.

Thus the proof is completed.

$\bar{\mathcal{D}}'_+(E)$ is the strict inductive limit of $\{\mathcal{D}'_{[a,\infty)}(E)\}_{-\infty<a<\infty}$. It is known that if an LCS G is a strict inductive limit of closed linear subspaces G_n, then a subset of G is bounded if and only if it is contained in a G_n and is bounded there ([1], p. 8). Therefore $\mathfrak{B}$ is bounded in $\bar{\mathcal{D}}'_+(E)$ if and only if $\mathfrak{B}$ is contained in a $\mathcal{D}'_{[a,\infty)}(E)$ and is bounded there. We shall say that a subset $\mathfrak{A}$ of $\bar{\mathcal{D}}'_+(F)$ is *of type* $\mathfrak{S}$ in $\bar{\mathcal{D}}'_+(F)$, if $\mathfrak{A}$ is contained in a $\mathcal{D}'_{[b,\infty)}(F)$ and is of type $\mathfrak{S}$ there.

As an immediate consequence of the preceding proposition we have

COROLLARY. *Let E, F, G be three LCSs, G being assumed quasi-complete. Let θ be a separately continuous bilinear map of $E \times F$ into G. Then any $\vec{S} \in \bar{\mathcal{D}}'_+(E)$ and $\vec{T} \in \bar{\mathcal{D}}'_+(F)$ are $*_\theta$-composable and the bilinear map $(\vec{S}, \vec{T}) \to \vec{S} *_\theta \vec{T}$ of*

$\bar{\mathcal{D}}'_+(E)\times\bar{\mathcal{D}}'_+(F)$ into $\bar{\mathcal{D}}'_+(G)$ is separately quasi-continuous. Let $\mathfrak{S}$ be a saturated family of bounded subsets of F and θ be hypocontinuous with respect to the subsets of $\mathfrak{S}$. Then the map $\vec{S}\to\vec{S}*_\theta\vec{T}$ of $\bar{\mathcal{D}}'_+(E)$ into $\bar{\mathcal{D}}'_+(G)$ is uniformly continuous with respect to the subsets of type $\mathfrak{S}$ in $\bar{\mathcal{D}}'_+(F)$. In particular, if E is normable and θ is a separately continuous bilinear map of $E\times F$ into G, then the map $\vec{S}\to\vec{S}*_\theta\vec{T}$ of $\bar{\mathcal{D}}'_+(E)$ into $\bar{\mathcal{D}}'_+(G)$ is continuous.

We note that the last statement follows from the fact that if E is normable, then $\bar{\mathcal{D}}'_+(E)$ is bornological.

Next we shall consider a convolution map of $\bar{\mathcal{D}}'_+(E)$ into $\bar{\mathcal{D}}'_+(F)$. By $\mathcal{L}_b(E;F)$ we denote the space of all continuous linear maps of E into F, where b denotes the topology of bounded convergence. We take θ as the bilinear map of $E\times\mathcal{L}_b(E;F)$ into F defined by the relation: $\theta(e, u)=u(e)$, $e\in E$, $u\in\mathcal{L}_b(E;F)$, then the map θ is hypocontinuous with respect to the bounded subsets of E and the equicontinuous subsets of $\mathcal{L}_b(E;F)$, which is a saturated family of bounded subsets of $\mathcal{L}_b(E;F)$.

Next we prove the following proposition which will play a fundamental rôle in the next section.

Proposition 4. *Let E, F be two LCSs, where F is assumed to be quasi-complete. Let L be a continuous linear map of $\bar{\mathcal{D}}'_+(E)$ into $\bar{\mathcal{D}}'_+(F)$. If the restriction of L to $\mathcal{D}\otimes E$ is commutative with any translation τ_h, $-\infty<h<\infty$, then there exists a unique $\vec{T}\in\bar{\mathcal{D}}'_+(\mathcal{L}_b(E;F))$ such that $\vec{T}$ is $*_\theta$-composable with any element $\vec{S}$ of $\bar{\mathcal{D}}'_+(E)$ and $L(\vec{S})=\vec{S}*_\theta\vec{T}$, where θ is the bilinear map of $E\times\mathcal{L}_b(E;F)$ into F defined above. Conversely, for any $\vec{T}\in\bar{\mathcal{D}}'_+(\mathcal{L}_b(E;F))$ which maps any bounded subset of $\mathcal{D}$ into an equicontinuous subset of $\mathcal{L}_b(E;F)$ the map $\vec{S}\to\vec{S}*_\theta\vec{T}$ of $\bar{\mathcal{D}}'_+(E)$ into $\bar{\mathcal{D}}'_+(F)$ is continuous and its restriction to $\mathcal{D}\otimes E$ is commutative with any τ_h.*

Proof. Let $S\in\mathcal{D}'_+$, $e\in E$, $\psi\in\mathcal{D}_-$. Putting $M(S,\psi)e=\psi\cdot L(S\otimes e)$, since L is continuous, it follows that $M(S,\psi)\in\mathcal{L}(E;F)$. Further if we put $\psi\cdot M(S)=M(S,\psi)$, then the map $M(S)$: $\psi\to M(S,\psi)$ of $\mathcal{D}_-$ into $\mathcal{L}_b(E;F)$ is continuous. In fact, when ψ and e run through any bounded subsets of $\mathcal{D}_-$ and E respectively, the set $\{(\psi\cdot M(S))e\}=\{\psi\cdot L(S\otimes e)\}$ is bounded in F. Since $\mathcal{D}_-$ is bornological, it follows that $M(S)$ is continuous, that is, $M(S)\in\mathcal{D}'_+(\mathcal{L}_b(E;F))$. We note that the map M: $S\to M(S)$ of $\mathcal{D}'_+$ into $\mathcal{D}'_+(\mathcal{L}_b(E;F))$ is continuous. Further, let $\phi\in\mathcal{D}$. Then we have for any translation τ_h

$$\begin{aligned}(\psi\cdot M(\tau_h\phi))e &= \psi\cdot L(\tau_h\phi\otimes e)\\ &= \psi\cdot\tau_h L(\phi\otimes e)\\ &= (\psi\cdot\tau_h M(\phi))e.\end{aligned}$$

Hence the restriction M to $\mathcal{D}$ is commutative with any translation τ_h. Consequently, owing to Proposition 4 in Shiraishi ([10], p. 179), there exists a unique distribution $\vec{T} \in \mathcal{D}'(\mathcal{L}_b(E; F))$ such that $M(S)=S*\vec{T}$ for every $S \in \mathcal{D}'_+$. And $\vec{T}$ is $*_\theta$-composable for any $S\otimes e$, $S \in \mathcal{D}'_+$ and $e \in E$ (see Remark 3 of [10], p. 186).

Next we shall prove that there exists a real number a such that $\vec{T} \in \mathcal{D}'_{[a,\infty)}(\mathcal{L}_b(E; F))$. To do so, it is sufficient to prove that there exists a number a such that $L(\delta\otimes e)=\vec{T}e \in \mathcal{D}'_{[a,\infty)}(F)$ for all $e \in E$. Contrary assumed, there exists a sequence $\{e_n\}$, $e_n \in E$, such that $L(\delta\otimes e_n) \notin \mathcal{D}'_{[-2c_n,\infty)}(F)$, where $c_n \to \infty$ as $n\to\infty$ and c_n's are positive numbers. Now, as L is continuous and $\{\delta_{c_n}\otimes e_n\}$ is bounded in $\mathcal{D}'_{[0,\infty)}(E)$ since $c_n\to\infty$, it follows that $\{L(\delta_{c_n}\otimes e_n)\}$ is bounded in $\mathcal{D}'_+(F)$, and therefore it is bounded in a $\mathcal{D}'_{[a,\infty)}(F)$, while, on the other hand, $L(\delta_{c_n}\otimes e_n) \notin \mathcal{D}'_{[-c_n,\infty)}(F)$, $n=1, 2, \cdots$, which is a contradiction.

Finally, let us denote by Γ the set of $\vec{S} \in \bar{\mathcal{D}}'_+(E)$ such that $L(\vec{S})=\vec{S}*_\theta\vec{T}$. Clearly Γ is linear and contains $\mathcal{D}\otimes E$ which is strictly dense in $\bar{\mathcal{D}}'_+(E)$ ([7], p. 46). Since the map $\vec{S}\to\vec{S}*_\theta\vec{T}$ is quasi-continuous and L is continuous, it follows that $\bar{\mathcal{D}}'_+(E)=\Gamma$.

Let $\mathfrak{G}$ be the family of the equicontinuous subsets of $\mathcal{L}_b(E; F)$. Then θ is hypocontinuous with respect to the subsets of $\mathfrak{G}$ and $\vec{T} \in \bar{\mathcal{D}}'_+(\mathcal{L}_b(E; F))$ is of type $\mathfrak{G}$ in $\bar{\mathcal{D}}'_+(\mathcal{L}_b(E; F))$. Therefore we can apply Corollary to Proposition 3 to conclude the last statement of the proposition.

Thus the proof is completed.

When E and F are Banach spaces, the proposition was proved by J. L. Lions ([5], p. 150).

We note that if E happens to be a barrelled space, $\mathcal{D}'_+(\mathcal{L}_s(E; F))=\bar{\mathcal{D}}'_+(\mathcal{L}_b(E; F))$ algebraically, because any bounded subset of $\mathcal{L}_s(E; F)$ becomes equicontinuous since E is barrelled ([1], p. 27).

§ 2. On characterization of semi-group distribution

Let E be a quasi-complete LCS. We shall consider a vector valued distribution $\mathfrak{G} \in \mathcal{D}'_{[0,\infty)}(\mathcal{L}_b(E; E))$. For a given $x \in E$, $\mathfrak{G}x \in \mathcal{D}'_{[0,\infty)}(E)$ is defined by

$$\phi\cdot\mathfrak{G}x=(\phi\cdot\mathfrak{G})x \qquad \text{for any} \quad \phi \in \mathcal{D}.$$

In the sequel we shall use the notation $\mathfrak{G}(\phi)$ instead of $\phi\cdot\mathfrak{G}$. Following J. L. Lions ([5], p. 142) $\mathfrak{G}$ is referred to as a *semi-group distribution in* E if the following conditions are satisfied:

(i) $\mathfrak{G}(\phi*\psi)=\mathfrak{G}(\phi)\mathfrak{G}(\psi)$ for any $\phi, \psi \in \mathcal{D}_{[0,\infty)}$;

(ii) for any $y=\mathfrak{G}(\phi)x$, $\phi \in \mathcal{D}_{[0,\infty)}$, $x \in E$, the distribution $\mathfrak{G}y \in \bar{\mathcal{D}}'_+(E)$ is a function $u(t)$ such that $u(t)=0$ for $t<0$;

(iii) the set $\{\mathfrak{G}(\phi)x; \phi \in \mathcal{D}_{[0,\infty)}, x \in E\}$ is total;

(iv) if, for a given $x \in E$, $\mathfrak{S}(\phi)x=0$ for any $\phi \in \mathcal{D}_{[0,\infty)}$, then $x=0$.

Remark. From (i) and (ii) it is easy to see that we can take $u(t)=\mathfrak{S}(\tau_t\phi)x$ for $t\geqq 0$, because of the equation $\mathfrak{S}(\psi)\mathfrak{S}(\phi)x=\int_0^\infty \psi(t)\mathfrak{S}(\tau_t\phi)x dt$. J. L. Lions [5] has treated the case where E is a Banach space, where, as remarked in the preceding section, $\mathcal{D}'_{[0,\infty)}(\mathcal{L}_b(E; E))=\mathcal{D}'_{[0,\infty)}(\mathcal{L}_s(E; E))$ algebraically.

Let $\mathfrak{S}$ be a semi-group distribution in E. For any $T \in \mathcal{E}'_{[0,\infty)}$ we define an operator $\tilde{\mathfrak{S}}(T)$ as follows: $x \in \mathfrak{D}_{\tilde{\mathfrak{S}}(T)}$ (domain of $\tilde{\mathfrak{S}}(T)$) if and only if there exists an element y such that

$$(1) \qquad \mathfrak{S}(T*\phi)x = \mathfrak{S}(\phi)y \qquad \text{for every} \quad \phi \in \mathcal{D}_{[0,\infty)}.$$

The element y is, if it exists, uniquely determined because of (iv). And we put $\tilde{\mathfrak{S}}(T)x=y$. Now it is easy to see that any $\mathfrak{S}(\phi)x$ belongs to $\mathfrak{D}_{\tilde{\mathfrak{S}}(T)}$ and therefore the domain $\mathfrak{D}_{\tilde{\mathfrak{S}}(T)}$ is a dense linear subspace of E and that it is also a closed linear operator. Then for any $x \in \mathfrak{D}_{\tilde{\mathfrak{S}}(T)}$ we have

$$(2) \qquad \mathfrak{S}(T*\phi)x = \mathfrak{S}(\phi)\tilde{\mathfrak{S}}(T)x = \tilde{\mathfrak{S}}(T)\mathfrak{S}(\phi)x.$$

For example $\tilde{\mathfrak{S}}(\delta)=I_E$ (the idential map of E into itself). Especially $A=\tilde{\mathfrak{S}}(-\delta')$ is called *infinitesimal generator* of the semi-group distribution under consideration. If, for any $\psi \in \mathcal{D}$, we denote by ψ_+ the function equal to $\psi(t)$ for $t\geqq 0$ and 0 for $t<0$, then $\psi_+ \in \mathcal{E}'_{[0,\infty)}$ and we can show that $\tilde{\mathfrak{S}}(\psi_+)=\mathfrak{S}(\psi)$, and $\mathfrak{S}(\psi)\tilde{\mathfrak{S}}(T)z=\tilde{\mathfrak{S}}(\psi_+*T)z$ for any $z \in \mathfrak{D}_{\tilde{\mathfrak{S}}(T)}$. Indeed, for $y=\mathfrak{S}(\phi)x$, $\phi \in \mathcal{D}_{[0,\infty)}$, we have

$$\begin{aligned}\mathfrak{S}(\psi)y = \int_0^\infty u(t)\psi(t)\,dt &= \int_0^\infty \psi(t)\mathfrak{S}(\tau_t\phi)x\,dt\\ &= \mathfrak{S}(\psi_+ *\phi)x\\ &= \tilde{\mathfrak{S}}(\psi_+)y.\end{aligned}$$

Consequently, this together with (iii) implies that $\tilde{\mathfrak{S}}(\psi_+)=\mathfrak{S}(\psi)$. The second part follows from the equalities:

$$\mathfrak{S}(\phi)\mathfrak{S}(\psi)\tilde{\mathfrak{S}}(T)z = \mathfrak{S}(\phi*\psi_+)\tilde{\mathfrak{S}}(T)z = \mathfrak{S}(\phi*\psi_+*T)z.$$

Similarly if $z \in \mathfrak{D}_{\tilde{\mathfrak{S}}(T*\psi_+)}$, then $\mathfrak{S}(\psi)z$ belongs to $\mathfrak{D}_{\tilde{\mathfrak{S}}(T)}$ and

$$\tilde{\mathfrak{S}}(T)\tilde{\mathfrak{S}}(\psi_+)z = \tilde{\mathfrak{S}}(T*\psi_+)z.$$

For example, for any $x \in E$ and for any $\psi \in \mathcal{D}$, $\mathfrak{G}(\psi)x$ belongs to $\mathfrak{D}_A$ and

$$(3) \qquad A\mathfrak{G}(\psi)x = \tilde{\mathfrak{G}}(-\delta' * \psi_+)x$$
$$= -\mathfrak{G}(\psi')x - \psi(0)x.$$

Now we take on $\mathfrak{D}_A$ the weakest topology which makes the maps $x \to x$, $x \to Ax$ of $\mathfrak{D}_A$ into E continuous. Such a topology we shall refer to as the graph topology. It follows from (3) that $\mathfrak{G}$ may be considered to be a continuous linear map of $\mathcal{D}$ into $\mathcal{L}_b(E; \mathfrak{D}_A)$, or more precisely we can write $\mathfrak{G}$ in the form:

$$\mathfrak{G} = (\delta \otimes j) * \mathfrak{H},$$

where $\mathfrak{H} \in \mathcal{D}'_{[0,\infty)}(\mathcal{L}_b(E; \mathfrak{D}_A))$ and j is the continuous injection $\mathfrak{D}_A \to E$ and $((\delta \otimes j) * \mathfrak{H})(\phi)$ means $j(\mathfrak{H}(\phi))$. Then (3) is rewritten in the form

$$(4) \qquad (-\delta \otimes A + \delta' \otimes j) * \mathfrak{H} = \delta \otimes I_E,$$

where convolutions, say, $(\delta' \otimes j) * \mathfrak{H}$ means that $((\delta' \otimes j) * \mathfrak{H})(\phi) = j\left(\frac{d}{dt}\mathfrak{H}(\phi)\right)$ for any $\phi \in \mathcal{D}$. Similarly, we have

$$(5) \qquad \mathfrak{G} * (-\delta \otimes A + \delta' \otimes j) = \delta \otimes I_{\mathfrak{D}_A}.$$

By making use of (4), (5) and Proposition 4, we can conclude that the differential equation

$$-A\vec{u} + j\frac{d}{dt}\vec{u} = \vec{T}, \quad \vec{T} \in \bar{\mathcal{D}}'_+(E),$$

admits a unique solution $\vec{u} \in \bar{\mathcal{D}}'_+(\mathfrak{D}_A)$ such that $\vec{u} \in \mathcal{D}'_{[0,\infty)}(\mathfrak{D}_A)$ if $\vec{T} \in \mathcal{D}'_{[0,\infty)}(E)$, besides, if $\mathfrak{H}$ maps any bounded subset of $\mathcal{D}$ into an equicontinuous subset of $\mathcal{L}(E; \mathfrak{D}_A)$, then the map $\vec{T} \to \vec{u}$ is continuous, which is the case when E is barrelled.

Our main purpose of this section is to show the converse of the preceding statement. Hereafter we shall assume E satisfies the following conditions:

(*) *if, for a given sequence* $\{x_n\}$, $x_n \in E$, *and for any* $x' \in E'$, *the sequence* $\{\langle x_n, x' \rangle\}$ *is ultimately equal to zero, then* $\{x_n\}$ *is also ultimately equal to zero*;

(*)$'$ *if, for a given sequence* $\{x'_n\}$, $x'_n \in E'$, *and for any* $x \in E$, *the sequence* $\{\langle x, x'_n \rangle\}$ *is ultimately equal to zero, then* $\{x'_n\}$ *is also ultimately equal to zero.*

The space of type (**F**) considered by Gelfand and Shilov satisfies these conditions ([2], p. 37). More generally if E is a space of type (**F**) with a continuous norm p, then E satisfies the conditions (*) and (*)′. Indeed, let $U=\{x; p(x)\leq 1\}$, then E'_{U^0} is a Banach space. Putting $A'_k=\{x'; <x_k, x'>=<x_{k+1}, x'>=\cdots=0$ and $x'\in E'_{U^0}\}$ which is a closed linear subspace of E'_{U^0}, we have $E'_{U^0}=\cup A'_k$, whence, by a theorem of Baire $E'_{U^0}=A'_k$ for some k, which implies that $p(x_k)=p(x_{k+1})=\cdots=0$, and therefore $x_k=x_{k+1}=\cdots=0$. Thus the condition (*) is verified. The condition (*)′ holds for any space of type (**F**). This may be shown in a similar way. Of course there are spaces which do not satisfy these conditions: (s), $(\mathfrak{S})$, $(\mathfrak{S}')$, $(\mathcal{D})$, $(\mathcal{D}')$. Most of the classical spaces of distributions considered by L. Schwartz [6] satisfy these conditions. For example, consider the space $(\mathcal{O}'_C)$. It is known that $\mathcal{S}\subset\mathcal{O}'_C\subset\mathcal{B}$, where injections are continuous. Since the conditions (*) and (*)′ are valid for $\mathcal{S}$ and $\mathcal{B}$, it is easy to see that these conditions are also valid for $(\mathcal{O}'_C)$.

Now we show

THEOREM. *Let E, F be two quasi-complete LCSs such that E satisfies the conditions (*) and (*)′. We assume that there is a continuous injection j_0: $F\to E$ such that $j_0(F)$ is dense in E. Let A_0 be a continuous linear map of F into E. If the equation*

$$(G) \qquad -A_0\vec{u}+j_0\frac{d}{dt}\vec{u}=\vec{T}, \quad \vec{T}\in\bar{\mathcal{D}}'_+(E),$$

admits a unique solution $\vec{u}\in\bar{\mathcal{D}}'_+(F)$ for every $\vec{T}\in\bar{\mathcal{D}}'_+(E)$ and the map $\vec{T}\to\vec{u}$ is continuous and $\vec{u}\in\mathcal{D}'_{[0,\infty)}(F)$ whenever $\vec{T}\in\mathcal{D}'_{[0,\infty)}(E)$, then there exists a unique semi-group distribution $\mathfrak{G}\in\mathcal{D}'_{[0,\infty)}(\mathcal{L}_b(E; E))$ with the following properties: Let A be the infinitesimal generator of $\mathfrak{G}$ in E with domain $\mathfrak{D}_A$ equipped with the graph topology. Let j be the natural injection of $\mathfrak{D}_A$ into E. There exists then an isomorphism j_1 of $\mathfrak{D}_A$ onto F such that $j=j_0\circ j_1$ and $A=A_0\circ j_1$.

Moreover we can write $j_0\vec{u}=\mathfrak{G}*_\theta\vec{T}$.

PROOF. According to proposition 4, there exists a unique vector valued distribution $\mathfrak{H}_0\in\mathcal{D}'_{[a,\infty)}(\mathcal{L}_b(E; F))$ such that $\vec{u}=\mathfrak{H}_0*_\theta\vec{T}$, where we may assume that $a=0$, since, by assumption, $\vec{u}\in\mathcal{D}'_{[0,\infty)}(F)$ for every $\vec{T}\in\mathcal{D}'_{[0,\infty)}(E)$. Putting $\mathfrak{G}=j_0\mathfrak{H}_0$, we shall first show that $\mathfrak{G}$ is a semi-group distribution $\in\mathcal{D}'_{[0,\infty)}(\mathcal{L}_b(E; E))$ with requisite properties.

Conditions (i) and (ii) are valid for $\mathfrak{G}$. For we may carry out the proof following the way of the corresponding proof due to J. L. Lions ([5], pp. 150–152). Hence the proof thereof is omitted.

Condition (iv): Let x be an element of E such that $\mathfrak{G}(\phi)x=0$ for every $\phi\in\mathcal{D}_{[0,\infty)}$. This means that $\mathfrak{G}x$ is a vector valued distribution $\in\mathcal{D}'_{[0,\infty)}(E)$ with support in 0. Therefore, for any $x'\in E'$ we can write $<\mathfrak{G}x, x'>$ in the following from:

$$<\mathfrak{G}x,\ x'> = \sum_k a_k(x')\delta^{(k)}$$

where $\{a_k(x')\}$ is a sequence of complex numbers which is ultimately equal to zero. Now $a_k(x')$ is a continuous linear form on E'_c, whence we can write $a_k(x') = <x_k,\ x'>$ for some $x_k \in E$. Therefore we can write

(1) $$\mathfrak{G}x = \sum_k \delta^{(k)} \otimes x_k, \quad x_k \in E.$$

Since $\mathfrak{G}x = j_0\mathfrak{H}_0x$ and j_0 is a continuous injection, the support of $\mathfrak{H}_0x$ is also in 0 and we can write similarly as above

(2) $$\mathfrak{H}_0x = \sum_k \delta^{(k)} \otimes y_k, \quad y_k \in F,$$

where $x_k = j_0y_k$. Putting $\vec{u} = \mathfrak{H}_0x$ into the equation (G) we have

$$-A_0(\sum_k \delta^{(k)} \otimes y_k) + \sum_k \delta^{(k+1)} \otimes j_0y_k = \delta \otimes x,$$

which yields the equations

(3) $$x = -A_0y_0, \quad j_0y_0 = A_0y_1, \ \cdots, \quad j_0y_k = A_0y_{k+1}, \ \cdots.$$

By hypothesis, E satisfies the condition $(*)$. Hence $\{x_k\}$ is ultimately equal to zero, so that $\{y_k\}$ is also ultimately equal to zero and in turn it follows from (3) that $x=0$.

Condition (iii): Let x' be any element of E' such that $<\mathfrak{G}(\phi)x,\ x'> = 0$ for every $\phi \in \mathcal{D}_{[0,\infty)}$ and for every $x \in E$. We define ${}^t\mathfrak{H}_0 \in \mathcal{D}'_{[0,\infty)}(\mathcal{L}_b(F'_c;\ E'_c))$ by the relation: ${}^t\mathfrak{H}_0(\psi) = {}^t(\mathfrak{H}_0(\psi))$ for every $\psi \in \mathcal{D}$. Then the relations $<\mathfrak{G}(\phi)x,\ x'> = <j_0\mathfrak{H}_0(\phi)x,\ x'> = <x,\ {}^t\mathfrak{H}_0(\phi){}^tj_0x'>$ yield ${}^t\mathfrak{H}_0(\phi){}^tj_0x' = 0$ for every $\phi \in \mathcal{D}_{[0,\infty)}$, therefore ${}^t\mathfrak{H}_0{}^tj_0x'$ is a vector valued distribution $\in \mathcal{D}'_{[0,\infty)}(E'_c)$ with support in 0. As in the preceding proof we can write

(4) $${}^t\mathfrak{H}_0{}^tj_0x' = \sum_k \delta^{(k)} \otimes x'_k,$$

where $\{x'_k\}$ is a sequence of elements of E' which becomes ultimately equal to zero.

Now we show

(5) $$-\mathfrak{H}_0(\psi)A_0 + \frac{d}{dt}\mathfrak{H}_0(\psi)j_0 = \psi(0)I_F, \quad \psi \in \mathcal{D},$$

where I_F is the identical map of F into itself. Indeed, if we put $\vec{u} = \check{\psi} \otimes y$ into the equation (G), we obtain

$$-A_0\vec{u}+j_0\frac{d}{du}\vec{u}=-\check{\phi}\otimes A_0y+(\check{\phi})'\otimes j_0y,$$

whence

$$\check{\phi}\otimes y=-(\mathfrak{H}_0*\check{\phi})A_0y+(\mathfrak{H}_0'*\check{\phi})j_0y,$$

consequently for $t=0$ we obtain the equation (5).

Putting $\vec{v}={}^t\mathfrak{H}_0y'$, where $y'={}^tj_0x'$, we can verify that $\vec{v}$ satisfies the equation

$$(6)\qquad -{}^tA_0\vec{v}+{}^tj_0\frac{d}{dt}\vec{v}=\delta\otimes y'.$$

In fact, by making use of the equation (5), we have for any $y\in F$

$$\begin{aligned}&<-{}^tA_0{}^t\mathfrak{H}_0y'+{}^tj_0\frac{d}{dt}{}^t\mathfrak{H}_0y',\ y>\\ &\qquad=<y',\ -\mathfrak{H}_0A_0y+\mathfrak{H}_0'j_0y>\\ &\qquad=<y',\ \delta\otimes y>\\ &\qquad=<\delta\otimes y',\ y>,\end{aligned}$$

which yields the equation (6).

Now from the condition $(*)'$ together with the equations (4) and (6) we can conclude as before that $x'=0$, that is, the set $\{\mathfrak{G}(\phi)x;\ \phi\in\mathcal{D}_{[0,\infty)}$ and $x\in E\}$ is total in E.

Thus we have shown that $\mathfrak{G}$ is a semi-group distribution in E. Let A be its infinitesimal generator with domain $\mathfrak{D}_A$ equipped with the graph topology.

The solution $\vec{u}$ of the equation:

$$(7)\qquad -A_0\vec{u}+j_0\frac{d}{dt}\vec{u}=-\delta\otimes Ax+\delta'\otimes jx,\quad x\in\mathfrak{D}_A,$$

is given by $\vec{u}=-\mathfrak{H}_0Ax+\mathfrak{H}_0'jx$. On the other hand, $\delta\otimes x$ is the solution of the equation:

$$(8)\qquad -A\vec{v}+j\frac{d}{dt}\vec{v}=-\delta\otimes Ax+\delta'\otimes jx,$$

therefore

$$\delta \otimes x = -\mathfrak{G}Ax + \mathfrak{G}' jx = j_0 \vec{u},$$

consequently we can write $\vec{u}$ in the form $\vec{u}=\delta\otimes y$. Now we put $y=j_1x$. j_1 is a continuous injection of $\mathfrak{D}_A$ into F. For it is clear that j_1 is an injection. The linear map $x\to-\delta\otimes Ax+\delta'\otimes jx$ of $\mathfrak{D}_A$ into $\mathcal{D}'_{[0,\infty)}(E)$ is continuous. Hence the equation (7) shows that the map $x\to\delta\otimes y$ of $\mathfrak{D}_A$ into $\mathcal{D}'_{[0,\infty)}(F)$ is continuous, so that we can conclude that j_1 is continuous. From the equation

$$-A_0(\delta\otimes y)+j_0\frac{d}{dt}(\delta\otimes y)=-\delta\otimes A_0 y+\delta'\otimes j_0 y, \quad y\in F,$$

and the equation (7) we have for $y=j_1x$

$$\delta\otimes Ax+\delta'\otimes jx=-\delta\otimes A_0y+\delta'\otimes j_0y,$$

therefore

$$Ax=A_0j_1x \qquad \text{and} \qquad jx=j_0j_1x,$$

that is,

$$A=A_0\circ j_1 \qquad \text{and} \qquad j=j_0\circ j_1.$$

Next consider any element $y\in F$. The solution $\vec{v}$ of the equation

$$-A\vec{v}+j\frac{d}{dt}\vec{v}=-\delta\otimes A_0y+\delta'\otimes j_0y,$$

is given by $\vec{v}=-\mathfrak{G}A_0y+\mathfrak{G}'j_0y$. On the other hand,

$$-A_0(\delta\otimes y)+j_0\frac{d}{dt}(\delta\otimes y)$$
$$=-\delta\otimes A_0y+\delta'\otimes j_0y.$$

Therefore we can write $\delta\otimes y=-\mathfrak{H}_0A_0y+\mathfrak{H}'_0j_0y$. By making use of the relation $\mathfrak{G}=j_0\mathfrak{H}_0$ we see that

$$\vec{v}=\delta\otimes j_0y$$

which implies that $j_0y\in\mathfrak{D}_A$. $j_0y=j(j_0y)=j_0j_1j_0y$ imply that $y=j_1j_0y$, therefore j_1 in an onto map. On the other hand, $Aj_0y=A_0j_1j_0y=A_0y$. Therefore when $y\to0$ in F, then $j_0y\to0$, $Aj_0y\to0$ in E, that is, when $y\to0$ in F, then $j_0y\to0$ in $\mathfrak{D}_A$. This implies that $y\to j_0y$ of F into $\mathfrak{D}_A$ is continuous. Thus we have shown that j_1 is an isomorphism of $\mathfrak{D}_A$ onto F.

Let $\mathfrak{G}^*$ be another semi-group distribution in E with infinitesimal generator A^* whose domain $\mathfrak{D}_{A^*}$ is equipped with the graph topology. Suppose there exists an isomorphism j_1^* of $\mathfrak{D}_{A^*}$ onto F such that $A^*=A_0\circ j_1^*$ and $j^*=j_0\circ j_1^*$, where j^* is the natural injection of $\mathfrak{D}_{A^*}$ into E. Then it is not difficult to see that $j_1=j_1^*$ and therefore $A^*=A$, and in turn $\mathfrak{G}=\mathfrak{G}^*$. In fact, $\mathfrak{D}_{A^*}=\mathfrak{R}_{j^*}=\mathfrak{R}_{j_0}=\mathfrak{R}_j=\mathfrak{D}_A$, where $\mathfrak{R}$ denotes the range of the map indicated in the suffix. Then for any $x \in \mathfrak{D}_A(=\mathfrak{D}_{A^*})$ we have $j_0(j_1^*x)=j_0(j_1x)$, so that $j_1^*x=j_1x$. This means that $j_1^*=j_1$.

Thus the proof is completed.

From the preceding theorem we have as an immediate consequence the following

Corollary. *Let E be a quasi-complete LCS with the properties* $(*)$ *and* $(*)'$. *Let A be a closed linear operator in E with domain $\mathfrak{D}_A$ dense in E, where we take on $\mathfrak{D}_A$ the graph topology and let j be the natural injection of $\mathfrak{D}_A$ into E. Suppose the equation*

$$-A\vec{u}+j\frac{d}{dt}\vec{u}=\vec{T},\quad \vec{T}\in\bar{\mathscr{D}}'_+(E),$$

admits a unique solution $\vec{u}\in\bar{\mathscr{D}}'_+(\mathfrak{D}_A)$ such that the map $\vec{T}\to\vec{u}$ is continuous and $\vec{u}\in\mathfrak{D}'_{[0,\infty)}(\mathfrak{D}_A)$ whenever $\vec{T}\in\mathfrak{D}'_{[0,\infty)}(E)$. Then A is an infinitesimal generator of a semi-group distribution $\mathfrak{G}$ which is uniquely determined by A. Moreover we can write $j\vec{u}=\mathfrak{G}*_\theta\vec{T}$.

Remark. In this corollary, if there exists another closed linear operator B with the same properties as A such that $B\subset A$, that is, $\mathfrak{D}_B\subset\mathfrak{D}_A$ and $Bx=Ax$ for any $x\in\mathfrak{D}_B$, then we can conclude that $B=A$. In fact, take any element $x\in\mathfrak{D}_A$ and define $\vec{T}$ by the equation

$$\vec{T}=-\delta\otimes Ax+\delta'\otimes jx,$$

then the corresponding solution $\vec{u}$ of the equation

$$-B\vec{u}+j\frac{d}{dt}\vec{u}=\vec{T}$$

is an element of $\bar{\mathscr{D}}'_+(\mathfrak{D}_B)\subset\bar{\mathscr{D}}'_+(\mathfrak{D}_A)$ and we see that $\vec{u}=\delta\otimes x$, which implies that $x\in\mathfrak{D}_B$, that is $A=B$.

Finally, let us assume that E is a Banach space. Let $\mathfrak{G}$ be a semi-group distribution in E. The infinitesimal generator A of $\mathfrak{G}$ was introduced by J. L. Lions as the closure of the operator $\mathfrak{G}(-\delta')$ which is defined as follows: $x\in\mathfrak{D}_{\mathfrak{G}(-\delta')}$ if and only if there exists a sequence of regularization $\{\rho_n\}\subset\mathscr{D}_{[0,\infty)}$, which may depend on x, such that (1) $\mathfrak{G}(\rho_n)x\to x$ and (2) $\mathfrak{G}(-\delta'*\rho_n)$ tends to some element $y \in E$ as $n \to \infty$ and he put $\mathfrak{G}(-\delta')x=y$.

For this infinitesimal generator A, the assumptions made in the above corollary are valid as seen from his result (Theorem 5.1, [5], p. 149) and the fact that the convolution map $\vec{T} \to \mathfrak{G}*_\theta \vec{T}$ of $\mathcal{D}'_+(E)$ into $\mathcal{D}'_+(\mathfrak{D}_A)$ is, by Proposition 4, continuous. Therefore from the above remark the infinitesimal generator A of J. L. Lions coincides with that given in our discussions.

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