J. Sci. Hiroshima Univ. Ser. A-I
28 (1964), 53-58

On Weak and Unstable Components

Makoto Ohtsuka
(Received March 18, 1964)

Introduction

Let D be a bounded plane domain and C be a component of the boundary of D consisting of a single point. It is called by Sario [7] *weak* if its image under any conformal mapping of D consists of a single point, and called *unstable* if it is not weak. Jurchescu [3] proved that it is weak if and only if the extremal length of the family of all closed curves in D, which separate D from the outer boundary of D, vanishes.

Adopting the latter point of view, we shall define more generally the weakness and the instability as follows. Let E be any bounded set and C be a component of E consisting of a single point. Let D be a disc containing E and denote by Γ the family of all rectifiable closed curves in $D-E$ which separate C from the boundary ∂D. We shall say that C is a weak (unstable resp.) component of E if the extremal length $\lambda(\Gamma)=0$ (>0 resp.). Likewise we can define the weakness and the instability in the higher dimensional case but we shall limit ourselves to the plane case in this paper.

Let X be any subset of $0\leqq x<1$ such that $z=0$ is a non-isolated component of X, and $f(x)$ be a non-negative bounded function defined on X with $f(0)=0$. We are interested in the weakness and instability of $z=0$ which is a component of the set

$$E(f;X)=\{(x,y);\, x\in X,\ -f(x)\leqq y\leqq f(x)\}.$$

We shall generalize some results obtained in Oikawa [6] and Akaza and Oikawa [1] which are concerned with the weakness and instability in case $E(f;X)$ is a compact set. A part of the results in the present paper is found in [4].

§ 1. Weakness

We begin with quoting Theorem 3 of the preceding paper [5] as

Lemma 1. *Let $\{c_x\}$ be a family of vertical segments such that each vertical line contains at most one element. Denote by $l(c_x)$ the length of the member c_x contained in the vertical line with coordinate x. Then it holds that*

$$\overline{\int}_A \frac{dx}{l(c_x)} \leqq M\{c_x\} = \frac{1}{\lambda\{c_x\}},$$

where $A=\{x;\ c_x$ *exists*$\}$.

Likewise we can obtain

LEMMA 2. *Let* $\{c_r\}$ *be a family of concentric arcs such that each circle around* $z=0$ *contains at most one arc, and denote by* $l(c_r)$ *the length of the member* c_r *contained in* $\{|z|=r\}$. *Then it holds that*

$$\overline{\int}_A \frac{dr}{l(c_r)} \leqq M\{c_r\} = \frac{1}{\lambda\{c_r\}},$$

where $A=\{r;\ c_r$ *exists*$\}$.

We shall prove

THEOREM 1. *Define* $g(x)$ *by* $\sup\{f(\xi);\ 0<\xi\leqq x,\ \xi\in X\}$. *If*

$$(1)\quad \overline{\int}_{[0,1)-X} \frac{dx}{x+g(x)} = \infty \quad \textit{or equivalently} \quad \overline{\int}_{[0,1)-X} \frac{dx}{\max(x,\ g(x))} = \infty,$$

then $z=0$ *is a weak component of* $E(f;\ X)$.

PROOF. For $x\in[0,1)-X$ we define a segment c_x' by $\{z;\ \mathrm{Re}\,z=x,\ |\mathrm{Im}\,z| \leqq x+g(x)\}$ and an arc c_x'' by $\{z;\ |z|^2=(x+g(x))^2+x^2,\ \arctan(1+g(x)/x)\leqq |\arg z|\leqq\pi\}$. Set $c_x=c_x'+c_x''$. For a large disc D this is a Jordan curve in D surrounding $z=0$. Since $g(x)$ is increasing on $[0,1)-X$, c_x belongs to Γ and $c_{x'}$ lies inside c_x if $x'<x$. It suffices to show $\lambda\{c_x;\ x\in[0,1)-X\}=0$.

Suppose that ρ is admissible in association with $\{c_x;\ x\in[0,1)-X\}$ and that $\iint\rho^2\,dx\,dy<\infty$. We set

$$X_1=\left\{x\in[0,1)-X;\ \int_{c_x'}\rho ds\geqq\frac{1}{2}\right\} \text{ and } X_2=\left\{x\in[0,1)-X;\ \int_{c_x''}\rho ds\geqq\frac{1}{2}\right\}.$$

By our assumption (1) at least one of

$$\overline{\int}_{X_1}\frac{dx}{x+g(x)}=\infty \qquad\text{or}\qquad \overline{\int}_{X_2}\frac{dx}{x+g(x)}=\infty$$

is true. Suppose that the former is true. By Lemma 1 we have

$$M\{c_x';\ x\in X_1\}\geqq\frac{1}{2}\overline{\int}_{X_1}\frac{dx}{x+g(x)}=\infty.$$

This is impossible because 2ρ is admissible in association with $\{c_x';\ x\in X_1\}$ and

$\iint\rho^2\,dx\,dy<\infty$. Next suppose that the latter is true. We denote by $h(x)$ the function $\sqrt{(x+g(x))^2+x^2}$ and set $S=\{r=h(x);\,x\in X_2\}$. For the family $\Gamma'=\{\{|z|=r\};\,r\in S\}$ of circles, we have by Lemma 2

$$M\{c''_x;\,x\in X_2\}\geqq M(\Gamma')\geqq\overline{\int_S}\frac{dr}{r}.$$

Take any measurable set $S'\supset S$ and set $X'_2=\{x;\,h(x)\in S'\}$. It holds that

$$\int_{S'}\frac{dr}{r}\geqq\int_{X'_2}\frac{(x+g(x))\,(1+g'(x))+x}{(x+g(x))^2+x^2}\,dx$$

$$\geqq\overline{\int_{X_2}}\frac{2x+g(x)}{(x+g(x))^2+x^2}\,dx\geqq\overline{\int_{X_2}}\frac{1}{x+g(x)}\,dx=\infty.$$

Consequently $\overline{\int_S}dr/r=\infty$ and hence $M\{c''_x;\,x\in X_2\}=\infty$. This is, however, again impossible because 2ρ is admissible in association with $\{c''_x;\,x\in X_2\}$ and $\iint\rho^2\,dx\,dy<\infty$. Our theorem is now proved.

We shall show that this theorem is a generalization of Theorem 1 of Akaza and Oikawa [1]. It reads as follows:

Let X be a closed subset of $0\leqq x<1$ such that $x=0$ is a non-isolated component of X. Let $f(x)$ be a finite valued non-negative function defined on $[0,1)$ which is upper semicontinuous as a function on E and satisfies $f(0)=0$. If $\sqrt{x^2+f^2(x)}$ is a non-decreasing function on $0\leqq x<1$ with the derivative (existing almost everywhere) bounded away from zero and if

$$\int_{[0,1)-X}\frac{dx}{\sqrt{x^2+f^2(x)}}=\infty, \tag{2}$$

then $z=0$ is a weak component of $E(f;X)$.

Actually, for any ξ, $0<\xi<x$, we have $\sqrt{\xi^2+f^2(\xi)}\leqq\sqrt{x^2+f^2(x)}$ and hence max $(x,g(x))\leqq\sqrt{x^2+f^2(x)}$. Thus (2) implies (1) and Theorem 1 is applied.

In case X is a compact subset containing $x=0$ of $0\leqq x<1$, we decompose $[0,1)-X$ into mutually disjoint intervals (x_1,x'_1), (x_2,x'_2), $\cdots$ Condition (1) is now

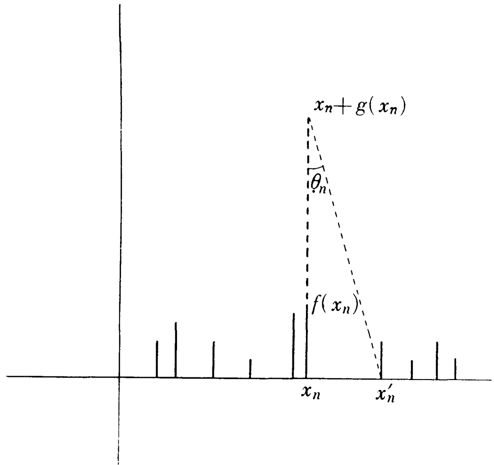
$$\sum_{n=1}^{\infty}\int_{x_n}^{x'_n}\frac{dx}{x+g(x_n)}=\sum_{n=1}^{\infty}\log\frac{x'_n+g(x_n)}{x_n+g(x_n)}=\sum_{n=1}^{\infty}\log\Big(1+\frac{x'_n-x_n}{x_n+g(x_n)}\Big)=\infty.$$

We see easily that this is equivalent to

$$\sum_{n=1}^{\infty}\frac{x_n'-x_n}{x_n+g(x_n)}=\infty.$$

If we use the angle θ_n defined as in the figure, the condition is expressed as

$$\sum_{n=1}^{\infty}\tan\theta_n=\infty.$$



§ 2. Instability

We prove

THEOREM 2. *Suppose that we can cover* $[0, 1)-X$ *by intervals* (x_1, x_1'), (x_2, x_2'), $\cdots$ *(these intervals may overlap) such that all* x_n *and* x_n' *are in* X *and*

$$\sum_n \min\left(\frac{x_n'-x_n}{a_n}, \frac{1}{\log^+\frac{x_n}{x_n'-x_n}}\right)<\infty, \tag{3}$$

where $a_n=\min(f(x_n), f(x_n'))$. *Then* $z=0$ *is an unstable component of* $E(f; X)$.

PROOF. First we infer from (3) that $x_n>0$ for each n. Let D be a large disc containing E. Let Γ_n be the family of rectifiable closed curves in $D-E$ each of which contains a curve connecting the segment $x_n<x<x_n'$ on the real axis with the negative real axis. If $a_n>0$, we consider $\rho=1/a_n$ in the rectangle $\{x_n<x<x_n', -a_n<y<a_n\}$ and $=0$ elsewhere. Each curve of Γ_n meets the interval $x_n<x<x_n'$ on the real axis and at least one of the upper and lower sides of the rectangle. Hence we obtain

$$\frac{1}{\lambda(\Gamma_n)}\leqq 2\frac{x_n'-x_n}{a_n};$$

this is true even if $a_n=0$.

Denote by Γ_n' the family of rectifiable curves in the plane each of which connects the segment $x_n<x<x_n'$ on the real axis with the negative real axis. It holds that $\lambda(\Gamma_n)\geqq\lambda(\Gamma_n')>0$ for each $n>0$. Each member of the family $\Gamma_0=\Gamma-\bigcup_{n=1}^{\infty}\Gamma_n$ meets both an interval of the form $0<\alpha<x<\beta<\infty$ on the real axis and the negative real axis, so that $\lambda(\Gamma_0)>0$.

From (3) we infer that

$$\lim_{n\to\infty}\frac{\max(a_n, x_n)}{x_n'-x_n}=\infty.$$

We choose n_0 such that $\max(a_n, x_n)\,(x_n'-x_n)^{-1}>1$ for every $n\geqq n_0$. If $n\geqq n_0$ and $x_n\leqq a_n$,

$$\frac{x_n'-x_n}{a_n}\leqq\frac{1}{\log\dfrac{a_n}{x_n'-x_n}}\leqq\frac{1}{\log^+\dfrac{x_n}{x_n'-x_n}}.$$

Next let us be concerned with n for which $x_n\geqq a_n$. It is known that

$$\lambda(\Gamma_n')=\frac{1}{\pi}\log\frac{x_n}{x_n'-x_n}+\frac{1}{\pi}\log 16+o(1)$$

if $x_n(x_n'-x_n)^{-1}$ is large; cf. Chap. II of [2], for instance. We choose $n_0'\geqq n_0$ such that, whenever $n\geqq n_0'$ and $x_n\geqq a_n$,

$$\lambda(\Gamma_n)\geqq\lambda(\Gamma_n')\geqq\frac{1}{\pi}\log\frac{x_n}{x_n'-x_n}.$$

Accordingly in any case

$$\frac{1}{\lambda(\Gamma_n)}\leqq\pi\min\left(\frac{x_n'-x_n}{a_n},\ \frac{1}{\log^+\dfrac{x_n}{x_n'-x_n}}\right)\quad\text{if}\quad n\geqq n_0',$$

and

$$\frac{1}{\lambda(\Gamma)}\leqq\sum_{n=0}^{\infty}\frac{1}{\lambda(\Gamma_n)}\leqq\sum_{n=0}^{n_0'-1}\frac{1}{\lambda(\Gamma_n)}+\pi\sum_{n=n_0'}^{\infty}\min\left(\frac{x_n'-x_n}{a_n},\ \frac{1}{\log^+\dfrac{x_n}{x_n'-x_n}}\right)<\infty.$$

We shall show that this theorem includes the sufficient condition for the instability given in Theorem 8 of Oikawa [6] and includes Theorem 2 of Akaza and Oikawa [1]. Oikawa's theorem in [6] is stated as follows:

Consider $X=\{0\}\cup\bigcup_{n=1}^{\infty}[x_{n+1}', x_n]$, *where* $0<x_{n+1}'<x_n<x_n'<1$ *and* $x_n\to 0$ *as* $n\to\infty$. *Then, under the assumption that* $\lim_{n\to\infty}x_n'/x_n=1$ *and* $x_n'/x_{n+1}'\geqq 1+\delta$ *for some* $\delta>0$, $z=0$ *is a weak component if and only if*

$$\sum_{n=1}^{\infty}\frac{1}{\log\dfrac{x_n}{x_n'-x_n}}=\infty.$$

It is obvious that Theorem 2 is a generalization of this theorem so far as the only-if part is concerned.

Next we quote Theorem 2 of Akaza and Oikawa [1]:

Let $f(x)$ be an increasing function on $0 \leqq x < 1$, X be a closed subset of $0 \leqq x < 1$. Assume that there is a constant $K > 0$ such that, for any $x \in X$, we can find $x' \in X$ satisfying $x < x'$ and $f(x') \leqq Kf(x)$; we shall refer to this assumption as (*) below. If $\int_{[0,1)-X} dx/f(x) < \infty$, $z=0$ is unstable.

Let us see that our Theorem 2 extends this result too. In fact if $\int_{[0,1)-X} dx/f(x) < \infty$,

$$\sum_n \frac{x_n' - x_n}{f(x_n)} \leqq K \sum_n \frac{x_n' - x_n}{f(x_n')} \leqq K \int_{[0,1)-X} \frac{dx}{f(x)} < \infty$$

and thus (3) is satisfied.

Akaza and Oikawa asked whether or not $z=0$ is unstable whenever $\int_0^1 dx/f(x) = \infty$ and $\int_{[0,1)-X} dx/f(x) < \infty$ without imposing the above condition (*); see lines 3–4 at p. 167 of [1]. Here we give a negative answer with the following example: $x_n = e^{-n^2}/n$, $x_n' = x_n + e^{-(n-1)^2}/n^2$ and $f(x) = e^{-n^2}$ on $x_{n+1} < x \leqq x_n$. In fact, $\int_0^1 dx/f(x) = \infty$, $\int_{[0,1)-X} dx/f(x) < \infty$ and

$$\sum_n \int_{x_n}^{x_n'} \{\max(x, f(x_n))\}^{-1} dx = \sum_n \frac{x_n' - x_n}{f(x_n)} = \infty.$$

On account of our Theorem 1 the last relation asserts that $z=0$ is weak. In this example $f(x)$ is not continuous but easily we can make it smooth and increasing while keeping all the above relations.

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Department of Mathematics,
Faculty of Science,
Hiroshima University.