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On Affinely Connected Spaces without Conjugate Points

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Introduction. Manifolds without conjugate points have been treated in Riemannian case by M. Morse, G. A. Hedlund, E. Hopf, L. W. Green and others, and discussed in more general metric spaces by H. Busemann and E. M. Zaustinsky.

In the present paper, we shall at first extend the notion of conjugate points to differentiable manifolds with affine connection (Section 1), and investigate a two-dimensional affinely connected manifold without conjugate points. In Section 2 we shall study a condition of the non-existence of conjugate points in two-dimensional affinely connected manifold (Theorem 1). In Section 3 we shall show by an example that the result of flatness of a two-dimensional Riemannian torus without conjugate points, given by Morse, Hedlund [9] and Hopf [6] no longer holds for the affinely connected torus in general (Theorem 2).

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1. Let M be a differentiable (C^∞) manifold with an affine connection ∇. Let $\gamma(t)$ $(t \in I)$ be a geodesic in M, where I is an open interval and t is an affine parameter.

Definition. A family of curves $t \to \Gamma(t, u)$ $(t \in I, |u| < \delta)$ in M will be called a *geodesic deviation* of $\gamma(t)$, if $\Gamma(t, u)$ is differentiable with respect to (t, u), $\Gamma(t, 0) = \gamma(t)$ and for sufficiently small $|u|$ the curve $t \to \Gamma(t, u)$ $(t \in I)$ represents a geodesic by neglecting the order of u^2.

Let J be a relatively compact open subinterval of I. The restriction of $\gamma(t)$ on J is also written $\gamma(t)$ $(t \in J)$.

Proposition. *A family of curves $\Gamma(t, u)$ $(t \in J, |u| < \delta)$ containing a geodesic $\gamma(t)$ $(t \in J)$ as $\Gamma(t, 0) = \gamma(t)$ is a geodesic deviation of $\gamma(t)$, if and only if a family of vectors $Y(t) = d\Gamma\left(\frac{\partial}{\partial u}\right)_{(t,0)}$ tangent to M at $\gamma(t)$ $(t \in J)$ satisfies the following equation,*

$$(1.1) \qquad (\nabla_X \nabla_X Y - R(X, Y)X - \nabla_X(T(X, Y)))_{\gamma(t)} = 0, \qquad \textit{for } t \in J,$$

where X, Y are vector fields on M satisfying $X_{\gamma(t)} = \dot{\gamma}(t)$ and $Y_{\gamma(t)} = Y(t)$ $(t \in J)$, *and T and R are respectively the torsion and the curvature tensor fields on M.*

Proof. From the above definition, $\Gamma(t, u)$ $(t \in J,\ |u| < \delta)$ is a geodesic deviation of $\gamma(t)$, if and only if there exist vector fields X, Y on M such that $X_{\gamma(t)} = \dot{\gamma}(t)$ and $Y_{\gamma(t)} = Y(t)$, and

$$(1.2) \qquad (Y \nabla_X X)_{\gamma(t)} = 0, \qquad \text{for} \quad t \in J.$$

Since $(\nabla_X X)_{\gamma(t)} = 0$, we get

$$(1.3) \qquad (\nabla_Y \nabla_X X)_{\gamma(t)} = 0.$$

But two equations (1.1) and (1.3) are equivalent. In fact, using the relations

$$T(X, Y) = \nabla_X Y - \nabla_Y X - [X, Y],$$

$$R(X, Y) = \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$$

we have

$$\nabla_Y \nabla_X X = \nabla_X \nabla_X Y - R(X, Y)X - \nabla_X \big(T(X, Y)\big) - \nabla_X [X, Y] - \nabla_{[X,Y]} X.$$

We can choose new vector fields $\tilde{X}$, $\tilde{Y}$ such that $\tilde{X}_{\gamma(t)} = X_{\gamma(t)}$, $\tilde{Y}_{\gamma(t)} = Y_{\gamma(t)}$ and $[\tilde{X}, \tilde{Y}] = 0$ in an open neighbourhood containing the geodesic $\gamma(t)$. Hence there exist vector fields X, Y on M satisfying (1.3) with $X_{\gamma(t)} = \dot{\gamma}(t)$, if and only if there exist $\tilde{X}$, $\tilde{Y}$ satisfying (1.1) and $\tilde{X}_{\gamma(t)} = \dot{\gamma}(t)$. Our proposition is thereby proved.

DEFINITION. In a manifold M with an affine connection ∇, a differentiable family of vectors $Y(t)$ along a geodesic $\gamma(t)$ $(t \in I)$ is called a *Jacobi field* along $\gamma(t)$ if it satisfies (1.1) for the arbitrary relatively compact open subinterval J of I. Two points $p = \gamma(t_0)$ and $q = \gamma(t_1)$ on a geodesic $\gamma(t)$ $(t \in J)$ are said to be *mutually conjugate* if there exists a non-trivial Jacobi field $Y(t)$ along $\gamma(t)$ $(t \in J)$ such that $Y(t_0) = Y(t_1) = 0$.

In particular, if $\Gamma(t, u)$ $(t \in I,\ |u| < \delta)$ is a one-parameter family of geodesics around a geodesic $\gamma(t)$, i.e., for each u $(|u| < \delta)$ the curve $t \to \Gamma(t, u)$ $(t \in I)$ is a geodesic with affine parameter $t \in I$ and $\Gamma(t, 0) = \gamma(t)$, then from the above proposition it is seen that $Y(t) = d\Gamma\left(\frac{\partial}{\partial u}\right)_{(t,0)}$ is a non-trivial Jacobi field along $\gamma(t)$. In this case, the conjugate points p, q on $\gamma(t)$ are the points of intersection of geodesics contained in the family $\Gamma(t, u)$.

2. In the following, M is assumed to be two-dimensional. Let $\gamma(t)$ $(t \in I)$ be a geodesic, $\gamma(t_0) = p$ be any point on $\gamma(t)$ which is not a double point of $\gamma(t)$

and let τ_t denote the parallel translation of tangent vectors from p to a point $\gamma(t)$ along the geodesic. Let X be a vector tangent to M at p such that X and $\dot{\gamma}(t_0)$ are linearly independent in the tangent vector space M_p. We now consider two families $X_1(t)$ and $X_2(t)$ of vectors along $\gamma(t)$ defined by $X_1(t)=\tau_t(\dot{\gamma}(t_0))$ and $X_2(t)=\tau_t(X)$. Then at each point $\gamma(t)$, the vectors $X_1(t)$ and $X_2(t)$ are linearly independent and differentiable with respect to t. Moreover, there exists a coordinate neighbourhood U of p with the coordinates (x_1, x_2) on U such that for some subinterval J of I containing t_0, U contains a geodesic arc $\gamma(t)$ $(t \in J)$ without double points in it and

$$\left(\frac{\partial}{\partial x_i}\right)_{\gamma(t)} = X_i(t), \qquad (i=1, 2).$$

Any geodesic $\gamma(t)$ $(t \in I)$ is covered by such coordinate neighbourhoods, and any finite geodesic arc can be considered as contained in such a coordinate neighbourhood. These local coordinates are called *parallel coordinates* along $\gamma(t)$. On a neighbourhood U of the above parallel coordinates we denote the vector fields $X_i=\dfrac{\partial}{\partial x_i}$ $(i=1, 2)$ and put $T(X_i, X_j)=T^k_{ij}X_k$, $R(X_i, X_j)X_k=R^l_{kij}X_l$ $(i, j, k, l=1, 2)$.

The equation (1.1) along a geodesic $\gamma(t)$ is expressed in U as follows:

$$\begin{aligned}(2.1)\qquad &\frac{d^2f^i(t)}{dt^2}-\frac{df^j(t)}{dt}T^i_{1j}(\gamma(t))\\ &\qquad -f^j(t)\left(R^i_{11j}(\gamma(t))+\frac{dT^i_{1j}(\gamma(t))}{dt}\right)=0, \quad t \in J, \qquad (i, j=1, 2)\end{aligned}$$

where $Y=f^iX_i$, $f^i(t)=f^i(\gamma(t))$ and J is a relatively compact subinterval of I.

In fact, since $X=X_1$ in (1.1) and $(\nabla_{X_1}X_i)_{\gamma(t)}=0$, we have

$$\begin{aligned}&(X_1X_1f^i)X_i-(X_1f^i)T(X_1, X_i)\\ &\qquad -f^i\big(R(X_1, X_i)X_1+\nabla_{X_1}(T(X_1, X_i))\big)=0 \quad \text{along} \quad \gamma(t).\end{aligned}$$

THEOREM 1. *Let M be a two-dimensional differentiable manifold which has an affine connection with zero torsion, and $\gamma(t)$ $(t \in I)$ be a geodesic in M. If the Ricci curvature Q of M is positive semi-definite along $\gamma(t)$, there is no pair of conjugate points on it.*

Proof. Since the torsion of M is zero, the system of equations (2.1) of Jacobi field $Y(t)=f^i(t)X_i$ along a geodesic arc $\gamma(t)$ $(t \in J)$ can be written in parallel coordinates (x_1, x_2) as follows:

$$\begin{cases} \dfrac{d^2 f^1(t)}{dt^2} - R^1_{112}(\gamma(t)) f^2(t) = 0 & (2.2) \\ \dfrac{d^2 f^2(t)}{dt^2} - R^2_{112}(\gamma(t)) f^2(t) = 0. & (2.3) \end{cases}$$

If

$$R^2_{112}(\gamma(t)) \geqq 0, \qquad \text{for} \quad t \in J, \tag{2.4}$$

then the comparison theorem of Sturm implies that in the values of t there is at most one value of t for which non-trivial solution $f^2(t)$ of the equation (2.3) vanishes. Hence the geodesic arc $\gamma(t)$ $(t \in J)$ has no pair of conjugate points. But changing (x_1, x_2) to any local coordinates (y_1, y_2), the condition (2.4) is expressed in the following

$$\sum_{i,j} (\tilde{R}^1_{ij1} + \tilde{R}^2_{ij2}) \frac{\partial y_i}{\partial x_1} \frac{\partial y_j}{\partial x_1} \geqq 0, \tag{2.5}$$

where $\tilde{R}^i_{jkl}$ are the coefficients of the curvature tensor in the local coordinates (y_1, y_2). Since $\tilde{Q}_{ij} = \tilde{R}^k_{ijk}$ is positive semi-difinite on $\gamma(t)$, the relation (2.5) is valid at any point on the geodesic. Therefore no pair of conjugate points exists on the whole $\gamma(t)$ $(t \in I)$.

REMARK. In particular, if M is a two-dimensional Riemannian manifold and ∇ a corresponding Riemannian connection, the condition of positive definiteness of the Ricci curvature Q is equivalent to the condition $\kappa(t) < 0$ along $\gamma(t)$, where $\kappa(t)$ is the Gaussian curvature at $\gamma(t)$.

3. On a two-dimensional manifold T_2 homeomorphic to the torus, it is known that the Riemannian structure on T_2 without conjugate points is only possible when the structure is flat [6]. On the other hand, we have

THEOREM 2. *There is an affine connection on T_2 without conjugate points and not flat.*

Proof. An affine connection on T_2 is given by a family of periodic functions $\Gamma^i_{jk}(x_1, x_2)$ $(i, j, k = 1, 2)$ on the two-dimensional Euclidean space E_2 with coordinates (x_1, x_2) which is considered as the covering space of T_2. Owing to THEOREM 1, our problem is reduced to finding a family of periodic functions $\Gamma^i_{jk}(x_1, x_2)$ $(i, j, k = 1, 2)$ not identically zero, and in such a way as the torsion is zero and the Ricci curvature is positive definite, considering $\Gamma^i_{jk}(x_1, x_2)$ as coefficients of an affine connection.

Such a family of functions can be constructed. At first, we consider the following system of equations:

$$(3.1) \qquad \begin{cases} \dfrac{\partial F^1(x_1, x_2)}{\partial x_1} - \dfrac{\partial F^2(x_1, x_2)}{\partial x_2} = 0 \\[2ex] \dfrac{\partial F^1(x_1, x_2)}{\partial x_2} - \dfrac{\partial F^2(x_1, x_2)}{\partial x_1} = 0 \\[2ex] (F^1(x_1, x_2))^2 - (F^2(x_1, x_2))^2 > 0. \end{cases}$$

The functions $F^1(x_1, x_2)$, $F^2(x_1, x_2)$ *periodic* in x_1, x_2 and satisfying the above equations can be found actually. For example, we are only to set

$$\begin{cases} F^1(x_1, x_2) = \sin(x_1 + x_2) + c \\ F^2(x_1, x_2) = \sin(x_1 + x_2) \end{cases}$$

where c is a constant greater than 2. By using these functions we define $\Gamma^i_{jk}(x_1, x_2)$ as follows:

$$\Gamma^1_{11} = 0, \qquad \Gamma^1_{22} = \Gamma^2_{12} = \Gamma^2_{21} = F^1(x_1, x_2)$$

$$\Gamma^2_{22} = 2F^2(x_1, x_2), \qquad \Gamma^1_{21} = \Gamma^1_{12} = \Gamma^2_{11} = F^2(x_1, x_2).$$

Then such functions define an affine connection with desired properties on T_2.

In fact, since $\Gamma^i_{jk} = \Gamma^i_{kj}$ this connection has zero torsion. The coefficients Q_{ij} of the Ricci curvature in the coordinates (x_1, x_2) are the following

$$Q_{11} = R^2_{112} = \frac{\partial F^1}{\partial x_1} - \frac{\partial F^2}{\partial x_2} + (F^1)^2 - (F^2)^2$$

$$Q_{12} = R^1_{121} = -\frac{\partial F^2}{\partial x_1}$$

$$Q_{21} = R^2_{212} = 2\frac{\partial F^2}{\partial x_1} - \frac{\partial F^1}{\partial x_2}$$

$$Q_{22} = R^1_{221} = \frac{\partial F^2}{\partial x_1} - \frac{\partial F^1}{\partial x_2} + (F^1)^2 - (F^2)^2.$$

By means of (3.1)

$$Q_{11} = Q_{22} = (F^1)^2 - (F^2)^2 > 0,$$

$$Q_{12} + Q_{21} = \frac{\partial F^2}{\partial x_1} - \frac{\partial F^1}{\partial x_2} = 0.$$

Hence the Ricci curvature Q is positive definite.

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