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Point-free Parallelism in Wilcox Lattices

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1. Introduction.

In the previous papers [4] and [5], I have investigated the properties of affine matroid lattices, using the parallelism given in [1], and I have seen that the points have significant roles. Hence this parallelism can not be applied to the non atomic lattices. Hsu [2] gave an apparently point-free parallelism, but in [4] Theorem (2.3), I have shown that this parallelism is coincident with that of [1].

In the present paper, I give a point-free parallelism using the modular elements instead of points, and applying to the Wilcox lattices, I obtain the same theorems as in [4] and [5].

In appendix, I investigate the modular centers of affine matroid lattices from the standpoint of the Wilcox lattice, and I obtain the same results as in the preceding paper [4].

2. Point-free parallelism in weakly modular symmetric lattices.

Definition (2.1). In a lattice L, we write $(a, b)M$ if $(c\cup a)\cap b=c\cup(a\cap b)$ for every $c\leqq b$. When b covers a, we write $a\lessdot b$.

In a lattice L with 0, $a\perp b$ means $a\cap b=0$, $(a, b)M$; and $a\parallel b$ means $a\cap b=0$, $(a, b)\bar{M}$ ($\bar{M}$ being the negation of the relation M). If $a\perp b$ implies $b\perp a$, then L is called a *symmetric lattice* (cf. [8] p. 495); and if $a\cap b\neq 0$ implies $(a, b)M$, then L is called a *weakly modular lattice* (cf. [4] (1.1)).

A relatively atomic, upper continuous, symmetric lattice is called a *matroid lattice* (cf. [5] (2.1)).

In this paper, we deal with a given lattice L with 0.

Definition (2.2). In a lattice L, a is called a *modular element* of L, if $(b, a)M$ for every $b\in M$ (cf. [6] p. 326). A point p, if it exists, is a modular element.

Remark (2.3). Especially when L is a weakly modular symmetric lattice, since $a\cap b\neq 0$ implies $(a, b)M$, a is a modular element if and only if $a\cap b=0$ implies $a\perp b$ for every $b\in L$.

Reference (2.4). In [1] p. 273, the parallelism in a matroid lattice L is defined as follows: Let a, b be nonzero elements of L, if

(2.4.1) $$a\cap b=0,$$

(2.4.2) $$a\lessdot a\cup b \quad \text{and} \quad b\lessdot a\cup b,$$

then we write $a\|b$, and say that a and b are *parallel*. But (2.4.2) is equivalent to the following condition:

(2.4.3) there exist points p and q such that

$$a\cup q=b\cup p, \qquad p\not\leqq a, \quad q\not\leqq b.$$

(Since $a, b\leqq a\cup q=b\cup p\leqq a\cup b$, we have $a\cup q=b\cup p=a\cup b$.)

In [1], [4] and [5], using the above definition of parallelism, the properties of weakly modular matroid lattices are obtained. In these investigations, the points p, q in (2.4.3) have significant roles. Hence we shall say that the above parallelism is a *point-set parallelism*. Since this parallelism can not be applied to non atomic lattices, we introduce a new parallelism.

DEFINITION (2.5). Let a, b be nonzero elements of a lattice L. When

(2.5.1) $$a\cap b=0,$$

(2.5.2) there exists a modular element m such that

$$m\cup b=a\cup b, \qquad m\leqq a,$$

then we write $a\underset{(m)}{<|}b$. Of course $m<0$.

If $a\underset{(m)}{<|}b$ and $b\underset{(n)}{<|}a$ both hold, then we write $a\underset{(m,n)}{\|}b$ and we say that a, b are *parallel with axes* m, n.

THEOREM (2.6). *Let a, b be nonzero elements in a lattice L. In order that $a\underset{(m,n)}{\|}b$, it is necessary and sufficient that the following conditions both hold.*

(2.6.1) $$a\cap b=0,$$

(2.6.2) *there exists modular elements m, n such that*

$$a\cup n=b\cup m, \qquad m\leqq a, \quad n\leqq b.$$

Proof. Necessity is evident from (2.5).

Sufficiency. From (2.6.2), we have

$$a\leqq a\cup n=b\cup m\leqq a\cup b \quad \text{and} \quad b\leqq b\cup m\leqq a\cup b,$$

hence $a\cup n=b\cup m=a\cup b$. Therefore $a\underset{(m)}{<|}b$ and $b\underset{(n)}{<|}a$. Consequently $a\underset{(m,n)}{\|}b$.

REMARK (2.7). If we require the equi-dimensionality of a, b in case $a\underset{(m,n)}{\|}b$, we must set a condition

$$m\sim n, \tag{2.7.1}$$

where "$\sim$" means some equi-dimensional relation in L. In this case we write $a\underset{(m\sim n)}{\|}b$. Cf. (7.3) below.

LEMMA (2.8). *In a lattice L, if $a\underset{(m)}{<|}b$ and $m<a$, then $b\perp\!\!\!\perp a$.*

Proof. By (2.5), $a\cap b=0$ and $m\cup b=a\cup b$. Hence

$$(m\cup b)\cap a=(a\cup b)\cap a=a>m=m\cup(b\cap a).$$

Consequently $(b, a)\bar{M}$, and since $b\cap a=0$ we have $b\perp\!\!\!\perp a$.

LEMMA (2.9). *In a lattice L, if $a\underset{(m)}{<|}b$ and $m\leqq a_1\leqq a$, then $a_1\underset{(m)}{<|}b$.*

Proof. $a_1\cap b\leqq a\cap b=0$. And $m\cup b\leqq a_1\cup b\leqq a\cup b$. Hence by (2.5.2) we have $m\cup b=a_1\cup b$. Consequently $a_1\underset{(m)}{<|}b$.

LEMMA (2.10). *In a lattice L, let $a\underset{(m)}{<|}b$ and $b<b_2$. If $a\cap b_2=0$ then $a\underset{(m)}{<|}b_2$, and if $m<b_2$ then $a<b_2$.*

Proof. By (2.5.2), we have

$$m\cup b_2=m\cup b\cup b_2=a\cup b\cup b_2=a\cup b_2.$$

Hence if $a\cap b_2=0$ then we have $a\underset{(m)}{<|}b_2$. And if $m<b_2$ the we have $b_2=a\cup b_2$, that is, $a<b_2$, since $a=b_2$ contradicts $a\cap b=0$.

LEMMA (2.11). *In a weakly modular lattice L, let $a\underset{(m)}{<|}b$ and n be a modular element with $0<n\leqq b$. Set $b_1=(a\cup n)\cap b$, then $a\underset{(m,n)}{\|}b_1$.*

Proof. From (2.5.2), we have $m\cup b=a\cup b$, and since $(a\cup n)\cap b\geqq n>0$, we have $(b, a\cup n)M$. Being $m\leqq a\cup n$, we have

$$b_1\cup m=\{(a\cup n)\cap b\}\cup m=(a\cup n)\cap(b\cup m)=(a\cup n)\cap(a\cup b)=a\cup n.$$

Since $a\cap b_1\leqq a\cap b=0$, by (2.6) $a\underset{(m,n)}{\|}b_1$ holds.

THEOREM (2.12). (Parallel mappings). *In a weakly modular summetric lattice L, let* $a \underset{(m,n)}{\|} b$. *Put*

$$Ta_1 = (a_1 \cup n) \cap b \quad for \quad a_1 \in L(m, a),$$

$$Sb_1 = (b_1 \cup m) \cap a \quad for \quad b_1 \in L(n, b).$$

Then T and S are mutually inverse, isomorphic mappings between $L(m, a)$ *and* $L(n, b)$.

In order that a_1, b_1 *correspond by these mappings, it is necessary and sufficient that*

$$(1) \qquad a_1 \cup n = b_1 \cup m$$

holds. And in this case $a_1 \underset{(m,n)}{\|} b_1$.

Proof. (i) It is evident that $Ta_1 \in L(n, b)$ and $Sb_1 \in L(m, a)$. By (2.9) we have $a_1 \underset{(m)}{<|} b$. Hence by (2.11) we have $a_1 \underset{(m,n)}{\|} Ta_1$. Similarly we have $Sb_1 \underset{(m,n)}{\|} b_1$. Thus by (2.6), (1) holds and we have $a_1 \underset{(m,n)}{\|} b_1$.

(ii) Conversely assume that (1) holds. Since L is symmetric and $m \cap b = 0$, we have $(m, b)M$. Hence

$$Ta_1 = (a_1 \cup n) \cap b = (b_1 \cup m) \cap b = b_1,$$

similarly $Sb_1 = a_1$. Thus a_1 and b_1 correspond by T and S.

(iii) Next we shall prove that T and S are mutually inverse, isomorphic mappings. Put $b_1 = Ta_1$. Then by (i), (1) holds. Hence by (ii), we have $STa_1 = Sb_1 = a_1$. Similarly we have $TSb_1 = b_1$. Therefore by T and S, there exists a one to one correspondence between $L(m, a)$ and $L(n, b)$ preserving the order. Hence $L(m, a)$ and $L(n, b)$ are isomorphic.

3. Point-free parallelism in Wilcox lattices.

DEFINITION (3.1). A Wilcox lattice L is constructed in the following manner. Let Λ be a given complemented modular lattice partially ordered by a relation $a \leqq b$, and having the operations $a \vee b$, $a \wedge b$. Let $S \subset \Lambda$ be a fixed set with the following properties:

(3.1.1) $\quad 0 \notin S$; and $a \in S, 0 < b \leqq a$ implies $b \in S$.

(3.1.2) $\quad a, b \in S$ implies $a \vee b \in S$.

Define $L \equiv \Lambda - S$. Then L is a weakly modular, symmetric lattice partially ordered by the relation $a \leqq b$, with the operations $a \cup b$, $a \cap b$ which satisfy the following conditions:

(3.1.3) $a \cup b = a \vee b,$

(3.1.4) $a \cap b = \begin{cases} a \wedge b & \text{if } a \wedge b \in L, \\ 0 & \text{if } a \wedge b \in S. \end{cases}$

And for $a, b \in L$,

(3.1.5) $a \perp b$ in L if and only if $a \wedge b = 0$,

(3.1.6) $a \perp\!\!\!\perp b$ in L if and only if $a \wedge b \in S$.

(Cf. [8] pp. 497–498.) We call L a *Wilcox lattice* and Λ the *modular extension* of L.

Remark (3.2). In the above construction of the Wilcox lattice, instead of (3.1.2), we may use the following condition:

(3.2.1) $a \in L$, $b \leqq a$ implies the existence of $c \in L$ with $a = b \vee c$, $b \wedge c = 0$.

In this case, L is a weakly modular, left complemented lattice, and is a special case of the Wilcox lattice given in [9] pp. 456–457. (Cf. [8] p. 499). Some investigations in what follows hold also in this kind of Wilcox lattices. But we use (3.1.2) in (3.5) below.

Definition (3.3). In a Wilcox lattice L, an element u in S is called an *imaginary element* of L, and a nonzero element a of L is called a *regular element* when $a \wedge u = 0$ for every $u \in S$. The set of all regular elements in L is denoted by R. If $a \in R$ and $0 < a_1 \leqq a$, then $a_1 \in R$.

When $a \in L$ is expressed as

$$a = m \vee u, \qquad m \in R, \quad u \in S,$$

then a is called an *irregular element* of L. And we write $u = \iota(a)$. When a is a regular element we put $\iota(a) = 0$.

Lemma (3.4). *In a Wilcox lattice L, a regular element a is a modular element.*

Proof. Let b be an element of L such that $a \cap b = 0$. Assume that $a \wedge b = u \in S$. Then $a \wedge u = u \in S$, which contradicts the regularity of a. Therefore by (3.1.4) $a \wedge b = 0$, and by (3.1.5) we have $a \perp b$. Hence by (2.3) a is a modular element.

LEMMA (3.5). *In a Wilcox lattice L, if*

$$a = m \vee u = n \vee v, \qquad m, n \in R, \quad u, v \in S,$$

then $u=v$. *Therefore* $\iota(a)$ *is uniquely determined with respect to* a.

Proof. By the assumption, we have

$$a = m \vee u = m \vee (u \vee v).$$

Since by (3.1.2) $u \vee v \in S$, we have $m \wedge (u \vee v) = 0$. Hence u and $u \vee v$ are relative complements of m in a, and $u \leqq u \vee v$. Therefore, by the modularity of Λ, we have $u = u \vee v$, that is $v \leqq u$. Similarly $u \leqq v$, and we have $u = v$.

LEMMA (3.6). *Let* a, b *be irregular elements in a Wilcox lattice L. Then* $a \leqq b$ *implies* $\iota(a) \leqq \iota(b)$.

Proof. Let

$$a = m \vee u, \quad b = n \vee v, \qquad m, n \in R, \quad u, v \in S.$$

Since $u < a \leqq b$, we have $b = n \vee (u \vee v)$. Hence by (3.5) we have $v = u \vee v$. Therefore $u \leqq v$, that is, $\iota(a) \leqq \iota(b)$.

REMARK (3.7). In a Wilcox lattice L, by (3.4), regular elements are modular elements. Hence for the parallelism $a \underset{(m,n)}{\|} b$, we use regular elements m, n.

THEOREM (3.8). *In a Wilcox lattice L, when* $m < a$, $m \in R$, *the following two propositions are equivalent.*

(α) $$a \underset{(m)}{<|} b.$$

(β) $$a \wedge b \in S \quad \text{and} \quad a = m \vee (a \wedge b).$$

Proof. $(\alpha) \rightarrow (\beta)$. Since from (2.8) $a \perp\!\!\perp b$, by (3.1.6) we have $a \wedge b \in S$. Since $m \cup b = a \cup b$ by (2.5.2), we have, by (3.1.3) and the modularity of Λ,

$$m \vee (a \wedge b) = a \wedge (m \vee b) = a \wedge (a \vee b) = a.$$

$(\beta) \rightarrow (\alpha)$. From $a \wedge b \in S$, by (3.1.4) we have $a \cap b = 0$. Since

$$a \vee b = m \vee (a \wedge b) \vee b = m \vee b,$$

we have $a \cup b = m \cup b$. Therefore $a \underset{(m)}{<|} b$.

LEMMA (3.9). *In a Wilcox lattice L, when* $m<a$, $n<b$ *and* $m, n \in R$, $a \underset{(m,n)}{\|} b$ *if and only if*

$$a = m \vee (a \wedge b), \quad b = n \vee (a \wedge b) \qquad \textit{and} \qquad a \wedge b \in S.$$

In this case $\iota(a) = \iota(b) = a \wedge b$.

Proof. This is evident from (3.8) and (3.5).

LEMMA (3.10). *In a Wilcox lattice L, for* $m, n \in R$ *and* $u \in S$, *if*

$$a = m \vee u, \quad b = n \vee u \qquad \textit{and} \qquad a \cap n = 0,$$

then $a \underset{(m,n)}{\|} b$.

Proof. By (3.4), n is a modular element, hence by (2.3) $a \cap n = 0$ implies $a \perp n$. Therefore by (3.1.5) we have $a \wedge n = 0$. Since

$$a \wedge b = a \wedge (n \vee u) = (a \wedge n) \vee u = u \in S,$$

by (3.9) we have $a \underset{(m,n)}{\|} b$.

THEOREM (3.11). *Let a be an irregular element in a Wilcox lattice L, such that*

$$a = m \vee u, \qquad m \in R, \quad u \in S.$$

Then for any regular element n with $a \cap n = 0$, *there exists one and only one irregular element b such that* $a \underset{(m,n)}{\|} b$. *In this case* $b = n \vee u$.

Proof. Put $b = n \vee u$, then by (3.10) we have $a \underset{(m,n)}{\|} b$. If there exists b' such that $a \underset{(m,n)}{\|} b'$, then by (3.9) we have

$$a = m \vee (a \wedge b'), \quad b' = n \vee (a \wedge b') \qquad \text{and} \qquad a \wedge b' \in S.$$

Since by (3.5), $u = a \wedge b'$, we have $b = b'$.

REMARK (3.12). (3.11) is a form of Euclid's parallel axiom, this is due to (3.1.2).

THEOREM (3.13). *In a Wilcox lattice L, let* $a \underset{(m)}{<|} b$ *and* $m<a$, $n<b$, $m, n \in R$. *Then there exists one and only one element* b_1 *such that* $a \underset{(m,n)}{\|} b_1$. *And in this case* $b_1 \leqq b$.

Proof. By (3.8) we have $a = m \vee (a \wedge b)$ and $a \wedge b \in S$. Since $a \cap n \leqq a \cap b = 0$, by (3.11) there exists one and only one element b_1 such that $a \underset{(m,n)}{\|} b_1$, and $b_1 = n \vee (a \wedge b) \leqq b$. Cf. (2.11).

LEMMA (3.14). *In a Wilcox lattice L, let a, b be irregular elements such that* $m < a$, $n < b$, $m, n \in R$. *Then* $a \underset{(m)}{<|} b$ *implies* $\iota(a) \leqq \iota(b)$.

Proof. By (3.13), there exists b_1 such that $a \underset{(m,n)}{\|} b_1$ and $b_1 \leqq b$. Then by (3.6) and (3.9), we have $\iota(a) = \iota(b_1) \leqq \iota(b)$.

THEOREM (3.15). *In a Wilcox lattice L, let a, b be irregular elements such that* $a \underset{(m)}{<|} b$ *and* $m < a$, $n < b$, $m, n \in R$. *Then there exists one and only one* a_2 *such that* $a_2 \underset{(m,n)}{\|} b$ *and* $a \leqq a_2$.

Proof. Put $a_2 = m \vee \iota(b)$, then by (3.10) we have $a_2 \underset{(m,n)}{\|} b$. Since $a \underset{(m)}{<|} b$ by (3.14) we have $\iota(a) \leqq \iota(b)$. Hence

$$a = m \vee \iota(a) \leqq m \vee \iota(b) = a_2.$$

Since $b \frown m = 0$, the uniqueniss follows from (3.11).

Reference (3.16). (3.15) is a form of parallel axiom used in [2] p. 4.

4. Comparability theorem in Wilcox lattices.

THEOREM (4.1). (Comparability theorem). *Let a, b be irregular elements in a Wilcox lattice L, and* $a \frown b = 0$. *Then there exist* $a', a'', b', b'' \in L$ *and* $m, n \in R$ *such that*

(1°) $$a = a' \cup a'', \qquad a' \frown a'' = m,$$

$$b = b' \cup b'', \qquad b' \frown b'' = n,$$

(2°) $$a' \underset{(m,n)}{\|} b' \qquad \text{and} \qquad \iota(a'') \wedge \iota(b'') = 0.$$

In this case $\iota(a') = \iota(b') = \iota(a) \wedge \iota(b)$.

Proof. Since a, b are irregular elements, there exist $m, n \in R$ such that

$$a = m \vee \iota(a) \qquad \text{and} \qquad b = n \vee \iota(b).$$

Denote by W the set S with 0 adjoined. Then W is a relatively complemented modular lattice. Since $\iota(a), \iota(b) \in W$, if we put $w = \iota(a) \wedge \iota(b)$, then there exist $u, v \in W$ such that

(1) $\iota(a) = w \vee u, \qquad w \wedge u = 0,$

(2) $\iota(b) = w \vee v, \qquad w \wedge v = 0,$

(3) $u \wedge v = 0.$

(Cf. [3] p. 14 Hilfssatz 1. 12).

Put $a' = m \vee w$, $a'' = m \vee u$, $b' = n \vee w$, $b'' = n \vee v$. Then $a', a'', b', b'' \in L$ and by (1) we have

$$a = m \vee \iota(a) = (m \vee w) \vee (m \vee u) = a' \vee a'' = a' \cup a''.$$

Since $m \wedge (w \vee u) = m \wedge \iota(a) = 0$ and $w \wedge u = 0$, we have $(m, w, u)\perp$. Therefore $w \wedge a'' = w \wedge (m \vee u) = 0$ (cf. [3] p. 13 Satz 1.8). Since $m \leqq a''$, by the modularity of Λ, we have $a' \wedge a'' = (m \vee w) \wedge a'' = m$. Hence by (3.1.4), we have $a' \cap a'' = m$. Similarly by (2), we have $b = b' \cup b''$ and $b' \cap b'' = n$.

From (3.10) we have $a' \underset{(m,n)}{\|} b'$, and from (3) we have $\iota(a'') \wedge \iota(b'') = u \wedge v = 0$. Now $\iota(a') = \iota(b') = w = \iota(a) \wedge \iota(b)$.

THEOREM (4.2). *Let a, b be irregular elements in a Wilcox lattice L. If $a \not\perp b$, then there exist irregular elements a_1, b_1 such that*

$$a_1 \underset{(m,n)}{\|} b_1, \quad a_1 \leqq a, \quad b_1 \leqq b \qquad \textit{and} \qquad a_1 \wedge b_1 = a \wedge b.$$

Proof. By (3.3), there exist $m, n \in R$ such that $m < a$ and $n < b$. Put

$$a_1 = m \vee (a \wedge b) \qquad \text{and} \qquad b_1 = n \vee (a \wedge b).$$

Then $a_1 \leqq a$ and $b_1 \leqq b$. Since, by (3.1.6) $a \wedge b \in S$ and $a_1 \cap n \leqq a \cap b = 0$, by (3.10) we have $a_1 \underset{(m,n)}{\|} b_1$. Since $a_1 \cap n = 0$ and n is a modular element, by (2.3) and (3.1.5) we have $a_1 \wedge n = 0$. Hence

$$a_1 \wedge b_1 = a_1 \wedge \{n \vee (a \wedge b)\} = a \wedge b.$$

THEOREM (4.3). (Modularity and parallelism). *Let a, b be irregular elements in a Wilcox lattice L, and $a \cap b = 0$. Then the following three propositions are equivalent.*

(α) $a \perp b$.

(β) *There do not exist irregular elements a_1, b_1 such that*

$$a_1 \underset{(m,n)}{\|} b_1, \qquad a_1 \leqq a, \quad b_1 \leqq b.$$

(γ) $\iota(a) \wedge \iota(b) = 0.$

Proof. $(\alpha)\to(\beta)$. If there exist irregular dements a_1, b_1 such that $a_1 \underset{(m,n)}{\|} b_1$, $a_1 \leqq a$, $b_1 \leqq b$, then from (2.8) we have $a_1 \perp\!\!\!\perp b_1$. On the other hand, $a \perp b$ implies $a_1 \perp b_1$ (cf. [8] p. 492), contrary to $a_1 \perp\!\!\!\perp b_1$.

$(\beta)\to(\alpha)$ follows from (4.2).

$(\beta)\to(\gamma)$. When $\iota(a)\wedge\iota(b)>0$, from (4.1), there exist a', b' such that $a'\leqq a$, $b'\leqq b$ and $\iota(a')=\iota(b')=\iota(a)\wedge\iota(b)>0$. Then a', b' are irregular elements in contradiction to (β).

$(\gamma)\to(\beta)$. If there exist irregular elements a_1, b_1 such that

$$a_1 \underset{(m,n)}{\|} b_1, \qquad a_1 \leqq a, \quad b_1 \leqq b,$$

then by (3.9) we have $\iota(a_1)=\iota(b_1)$, and by (3.6) we have $\iota(a_1)\leqq\iota(a)$, $\iota(b_1)\leqq\iota(b)$. Hence

$$\iota(a)\wedge\iota(b)\geqq\iota(a_1)\wedge\iota(b_1)=\iota(a_1)>0,$$

which contradicts (γ).

Reference (4.4). Theorems (2.12), (4.1) and (4.3) correspond to the Theorems (3.1), (5.1) and (7.3) in [5].

5. Modular centers of Wilcox lattices.

DEFINITION (5.1). Let L be a Wilcox lattice such that $L\equiv\Lambda-S$. For an element a of L, if $u<a$ for every $u\in S$, then we write $S\subset a$, and we call a a $\|$*-closed element* of L.

We sall say that 0 is a $\|$-closed element. Denote by M the set of all $\|$-closed elements of L.

REMARK (5.2). When a is a $\|$-closed element of a Wilcox lattice L, let m be any regular element with $m<a$. Then for any irregular element b such that $m<b$, we have $b\leqq a$.

THEOREM (5.3). *In a Wilcox lattice L, the set M is a modular sublattice of L.*

Proof. (i) We shall first show that M is a sublattice of L. Let $a, b\in M$. If one of a and b is zero, it is evident that $a\cup b\in M$ and $a\cap b\in M$. Hence assume that $a, b\neq 0$, then we have $S\subset a$ and $S\subset b$. Since $S\subset a\leqq a\vee b$, we have $a\cup b\in M$. When $a\wedge b\in L$, by (3.1.4) we have $S\subset a\wedge b=a\cap b$; and when $a\wedge b\in S$ by (3.1.4) we have $a\cap b=0$. Hence in both cases, we have $a\cap b\in M$.

(ii) Next we shall show that M is modular. Let $a, b, c\in M$ and $c\leqq b$. If one of a, b, c is zero, then

(1) $$(c \cup a) \cap b = c \cup (a \cap b)$$

is evident. Hence assume that a, b, c are all nonzero. Since Λ is a modular lattice, we have

(2) $$(c \vee a) \wedge b = c \vee (a \wedge b).$$

If $a \wedge b \in L$, then by (3.1.1) we have $(c \vee a) \wedge b \in L$. Hence by (3.1.3) and (3.1.4) we have (1).

If $a \wedge b \in S$, then by (5.1) $a \wedge b \leqq c$. Hence we have

(3) $$(c \vee a) \wedge b = c \vee (a \wedge b) = c \in L.$$

Therefore by (3.1.3) and (3.1.4), we have

$$(c \vee a) \wedge b = (c \cup a) \cap b \qquad \text{and} \qquad a \cap b = 0.$$

Hence from (3) we have

$$(c \cup a) \cap b = c = c \cup (a \cap b).$$

Thus $(a, b)M$ holds for all cases.

Definition (5.4). In a Wilcox lattice L, the set M of all $\|$-closed elements is called the *modular center* of L. And when M is composed of only two elements 0 and 1, we say that L is *modularly irreducible*. (Cf. [4] (4.12).)

6. Modular centers of Wilcox lattices with imaginary units.

Definition (6.1). In a Wilcox lattice $L \equiv \Lambda - S$, if S has the greatest element i, then we call i the *imaginary unit* of L, and we say that L is a *Wilcox lattice with i*.

In this case, $S \equiv \{x \in \Lambda;\ 0 < x \leqq i\}$.

Remark (6.2). In a Wilcox lattice L with i, a nonzero element a of L is a regular element if and only if $a \wedge i = 0$.

This is evident from Definition (3.3).

Definition (6.3). In a Wilcox lattice L with i for a regular element m of L, set $I(m) = m \vee i$, and $I(0) = 0$.

Theorem (6.4). *Let a be a nonzero element of a Wilcox lattice L with i. Then the following three propositions are equivalent.*

(α) *a is a $\|$-closed element.*
(β) $i < a$.
(γ) *There exists a regular element m such that* $a = I(m)$.

Proof. (α)$\rightleftarrows$(β) is evident from Definition (5.1).
(β)$\to$(γ). Since Λ is a complemented modular lattice, from $i<a$, there exists an element m such that

$$a = m \vee i, \qquad m \wedge i = 0.$$

Then from (6.2), m is a regular element and $a = I(m)$. Of course, m is not necessarily unique.
(γ)$\to$(β). If $a=I(m)=m\vee i$, then $i<a$.

THEOREM (6.5). *Let a, b be nonzero elements in the modular center M of a Wilcox lattice L with i. If $a\cap b=0$, then $a \underset{(m,n)}{\|} b$.*

Proof. By (6.4), there exist regular elements m, n such that

$$a = m \vee i \qquad \text{and} \qquad b = n \vee i.$$

Since $a\cap n \leqq a\cap b=0$, by (3.10) we have $a \underset{(m,n)}{\|} b$.

THEOREM (6.6). *The modular center M of a Wilcox lattice L with i is a complemented modular sublattice of L, and M is isomorphic to $\Lambda(i, 1) \equiv \{a \in \Lambda;\ i \leqq a\}$.*

Proof. By (6.4) M is the set $M_0 = \{a \in \Lambda;\ i<a\}$ with 0 adjoined. Hence there exists a one to one correspondence between $\Lambda(i, 1)$ and M such that if $i<a$ $a\to a$, and $i\to 0$. And by (3.1.3) and (3.1.4), we have, if $i<a, b$ $a\vee b\to a\cup b$, if $i<a$ $a\vee i=a\to a=a\cup 0$, if $i<a\wedge b$ $a\wedge b\to a\cap b$, and if $i<a$ $a\wedge i=i\to 0=a\cap 0$. Hence the above correspondence preserves the lattice operations. Therefore M is isomorphic to $\Lambda(i, 1)$, which is a complemented modular sublattice of Λ.

REMARK (6.7). A Wilcox lattice L with i is modularly irreducible if and only if i is the hyperplane of Λ.

Proof. Since i is the hyperplane of Λ if and only if $\Lambda(i, 1)$ consists of only i and 1, this remark is evident from Definition (5.4).

7. Appendix. Modular centers of affine matroid lattices.

PRELIMINARIES (7.1). As in (2.4) refered, in a matroid lattice L, we define the point-set parallelism. A weakly modular matroid lattice L of length $\geqq 4$ is called an *affine matroid lattice* (cf. [4] (3.3)), when L satisfies the follownig weak Euclid's parallel axiom:

Let l be a line in L. If p is a point such that $p \not\leqq l$, then there exists at most one line k such that $l \| k$ and $p \leqq k$.

In this section, we treat only the affine matroid lattice which are not modular.

In an affine matroid lattice L, a line l is called *incomplete*, when for any point $p \not\leqq l$, there exists a line k such that $l \| k$ and $p \leqq k$. And a line l is called *complete*, when there exists no line parallel to l. An element a of length $\geqq 2$ is called *incomplete*, when any line contained in a is incomplete (cf. [4] (3.4)). For any point p in L, there exists a maximal incomplete element $I(p)$ which contains p. If $I(p) \neq 1$, then either $I(p)=I(q)$ or $I(p) \| I(q)$ for any points p, q in L (cf. [4] (4.1)). If $I(p)=1$, then L satisfies the following strong Euclid's parallel axiom:

Let l be a line in L. If p is a point such that $p \not\leqq l$, then there exists one and only one line k such that $l \| k$ and $p \leqq k$.

An affine matroid lattice L is a Wilcox lattice with the modular extension $\Lambda \equiv L \cup S$ and with the imaginary unit $i \equiv [I(r)]$, r being any point in L (cf. [5] (7.1)). Let $\Omega_0 \equiv \{I(t_\alpha);\ \alpha \in I\}$ be the decomposition space of L (cf. [4] (4.3)). When p, q be any different points in L, the line $p \cup q$ is a complete line if and only if p and q are contained in different $I(t_\alpha)$ and $I(t_\beta)$ in Ω_0 (cf. [4] (4.4)).

Now we have the following lemma.

Lemma (7.2). *In an affine matroid lattice L, an element a of length $\geqq 2$ is a regular element if and only if any line l contained in a is complete, that is, for any different points p, $q \leqq a$, we have $I(p) \neq I(q)$.*

Proof. (i) Necessity. If there exist p, $q \leqq a$ $(p \neq q)$ such that $I(p)=I(q)$, then the line $l = p \cup q$ is an incomplete line, and $[l] < l \leqq a$. Since $[l] \in S$, this contradicts the regularity of a. (Cf. for detail [5] (7.1).)

(ii) Sufficiency. If a is not a regular element, then there exists $u \in S$ with $a \wedge u \neq 0$. Set $u_1 = a \wedge u$, then by (3.1.1) $u_1 \in S$. Since Λ is atomistic, there exists an incomplete line l in L, such that $[l] \leqq u_1 < a$, $[l]$ being a point in Λ. Since $l \leqq | a$ by [4] (2.8), there exists a line l' such that $l' = l$ or $l' \| l$ and $l' \leqq a$. Let p, q be points such that $l' = p \cup q$, then p, $q \leqq a$ $(p \neq q)$ and $I(p) = I(q)$, which contradicts the assumption.

Remark (7.3). When an affine matroid lattice L satisfies the strong Euclid's parallel axiom, any line in L is incomplete, hence by (7.2), only the points are regular elements. Therefore all point-free parallelisms are in the from $a \underset{(p,q)}{\|} b$ which is nothing but the point-set parallelism $a \| b$ (cf. (2.4)). Since L is an irreducible matroid lattice (cf. [4] (3.7)), the perspectivity of points $p \sim q$ is transitive (cf. [7] p. 186), hence we may also write $a \underset{(p \sim q)}{\|} b$.

REMARK (7.4). In an affine matroid lattice L, by [5] (7.2) and [5] (6.5), an incomplete element a is written in the from $a=p\vee u$, where p is a point in L and $u\in S$. Hence the maximal incomplete element $I(p)$ containg p, is expressed as $p\vee i$. Therefore $I(p)$ is coincident with $I(m)$ in (6.3) when the regular element m is the point p. And $p\leqq a$ implies $I(p)=p\vee i\leqq a$, if and only if $i<a$. Hence by (6.4) the $\|$-closed element of L defined in (5.1) is coincident with that of [4] (4.10). And the modular centers of L defined in (5.4) and [4] (4.12) coincide.

By (6.6), M is isomorphic to $\Lambda(i, 1)$ and $I(p)=p\vee i$ is a point of $\Lambda(i, 1)$. Consequently Theorem (6.6) is an alternative proof of [4] Theorem (4.11), except the irreducibility of M.

To prove the irreducibility of M, we shall prove the irreducibility of $\Lambda(i, 1)$. Let $p\vee i$, $q\vee i$ be two different points in $\Lambda(i, 1)$. Since $I(p)\neq I(q)$, by [4] (4.4), $p\cup q$ is a complete line of L. Hence by [4] (3.6), $p\cup q$ contains a third point r. Since $p\cup r=q\cup r=p\cup q$ is a complete line, $I(r)\neq I(p)$ and $I(r)\neq I(q)$. Hence $r\vee i$ is a third point contained in the line $(p\vee i)\vee(q\vee i)$ in $\Lambda(i, 1)$. Consequently by [4] (1.12), $\Lambda(i, 1)$ is irreducible.

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