J. Sci. Hiroshima Univ. Ser. A-I
28 (1964), 1-17

Weak Domination Principle

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(Received November 27, 1963)

Introduction and Preliminaries

In a paper [1] on "Modèles finis" of the potential theory, Choquet and Deny obtained the following interesting result. If a positive kernel on a space of a finite number of points is non-degenerate and satisfies the weak balayage (or equivalently domination) principle, then it satisfies the ordinary balayage (domination) principle or the inverse balayage (domination) principle. In this paper we shall extend this result to a positive continuous (in the extended sense) kernel on a locally compact Hausdorff space. Similar extension was tried by Ninomiya [4] for positive continuous symmetric kernels. His result states that if a positive symmetric kernel is of positive [negative resp.] type and satisfies the weak balayage principle, then it satisfies the ordinary [inverse resp.] balayage principle.

Let Ω be a locally compact Hausdorff space and G be a positive continuous (in the extended sense) kernel on Ω such that $G(x, y)$ is finite at any point $x \neq y$. Throughout this paper we assume that every compact subset of Ω is separable[1], and we shall use the same notations as in the author's paper [2].

First we define domination principles which we shall consider in this paper.

(I) *Weak domination principle.* If $G\mu \leq G\nu$ on $S\mu \cup S\nu$ for $\mu \in \mathfrak{E}_0$ and $\nu \in \mathfrak{M}_0$, then the same inequality holds in Ω[2].

(II) *Ordinary domination principle* (or simply, *domination principle*). If the above inequality holds on $S\mu$, so it does in Ω.

(III) *Inverse domination principle.* If the above inequality holds on $S\nu$, so it does in Ω.

(IV) *Elementary domination principle.* If $aG(x_1, x_1) \leq G(x_1, x_2)$ with $a > 0$, then $aG(z, x_1) \leq G(z, x_2)$ for any point z in Ω.

(V) *Elementary inverse domination principle.* If $aG(x_2, x_1) \leq G(x_2, x_2)$ with $a > 0$, then $aG(z, x_1) \leq G(z, x_2)$ for any point z in Ω.

(VI) *Strong elementary domination principle.* If $G\mu \leq G\varepsilon_{x_0}$ on $S\mu$ for $\mu \in \mathfrak{E}_0$ and $x_0 \notin S\mu$, then $G\mu \leq G\varepsilon_{x_0}$ in Ω.

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1) All the results in this paper hold with a slight modification for positive continuous kernels on Ω compact subsets of which are not necessarily separable (cf. Nakai [3]).

2) $\mathfrak{M}_0$ is the totality of positive measures with compact support and $\mathfrak{E}_0$ is the totality of positive measures in $\mathfrak{M}_0$ with finite energy.

(VII) *Strong elementary inverse domination principle.* If $G\mu \geq G\varepsilon_{x_0}$ on $S\mu$ for $\mu \in \mathfrak{M}_0$ and $x_0 \notin S\mu$, then $G\mu \geq G\varepsilon_{x_0}$ in Ω.

The following implications are immediate consequences of the above definitions:

Ordinary domination principle $\Rightarrow$ weak domination principle.
Inverse domination principle $\Rightarrow$ weak domination principle.
Ordinary domination principle $\Rightarrow$ strong elementary domination principle $\Rightarrow$ elementary domination principle.
Inverse domination principle $\Rightarrow$ strong elementary inverse domination principle $\Rightarrow$ elementary inverse domination principle.[3)]

In the preceding paper [2] we obtained the following results.

THEOREM 1. *Assume that G and its adjoint kernel $\check{G}$ satisfy the continuity principle. Then G satisfies the ordinary domination principle if and only if it satisfies the* ordinary balayage principle, *that is, for any $\mu \in \mathfrak{M}_0$ and any compact set K, there exists a positive measure μ', supported by K, such that*

$$G\mu' = G\mu \qquad G\text{-}p.p.p. \quad on \quad K,$$

$$G\mu' \leq G\mu \qquad in \quad \Omega.$$

THEOREM 2. *Under the same assumptions as above, G satisfies the ordinary domination principle if and only if its adjoint kernel $\check{G}$ satisfies it.*

THEOREM 3. *Under the same assumptions as above, G satisfies the ordinary domination principle if and only if it satisfies the strong elementary domination principle.*

These theorems were obtained by using the following fundamental existence theorem.

THEOREM 4. *If $\check{G}$ satisfies the continuity principle and $u(x)$ is a positive finite upper semi-continuous function on a compact set K, then there exists a positive measure λ, supported by K, such that*

$$G\lambda \geq u \qquad G\text{-}p.p.p. \quad on \quad K,$$

$$G\lambda \leq u \qquad on \quad S\lambda.$$

§ 1. Elementary domination principle

LEMMA 1.[4)] *Let $\check{G}$ satisfy the continuity principle and G satisfy the*

3) The last two implications hold under the assumption that $G(x, x) < +\infty$ for any x.

4) Cf. Choquet-Deny [1], Lemme 3. The measure λ which they constructed does not necessarily satisfy the condition: $G\lambda \leq G\varepsilon_{x_0}$ on $S\lambda$.

elementary domination principle. Then for any given compact set K and any given point $x_0 \notin K$, *there exists a point* x_1 *in* K *such that*

$$G\varepsilon_{x_1}/G\varepsilon_{x_0} \equiv a \text{ constant in } \Omega,$$

or there exists a positive measure λ, *supported by* K, *such that*

$$G\lambda \geq G\varepsilon_{x_0} \quad G\text{-p.p.p.} \quad on \quad K,$$

$$G\lambda \leq G\varepsilon_{x_0} \quad on \quad S\lambda,$$

$$G\lambda(x_0) < G\varepsilon_{x_0}(x_0).$$

Proof. Let us assume that $G\varepsilon_x/G\varepsilon_{x_0} \not\equiv$ any constant for any $x \in K$. In order to prove our lemma, it is sufficient to consider the case that $G(x_0, x_0)$ is finite. In fact, by the existence theorem (Theorem 4), there exists a positive measure λ, supported by K, such that

$$G\lambda \geq G\varepsilon_{x_0} \quad G\text{-p.p.p.} \quad \text{on} \quad K,$$

$$G\lambda \leq G\varepsilon_{x_0} \quad \text{on} \quad S\lambda.$$

Since $G(x_0, y)$ is finite and continuous at any point $y \neq x_0$, $G\lambda(x_0)$ is finite and hence this λ is what we want if $G(x_0, x_0) = +\infty$.

Now we put, for any point $x \in K$,

$$m_x = \frac{G(x_0, x_0)}{G(x_0, x)}.$$

Then by the elementary domination principle,

(1) $$G\varepsilon_{x_0}(z) \leq m_x G\varepsilon_x(z) \qquad \text{for any } z \text{ in } \Omega.$$

Suppose that there is a point $x_1 \in K$ such that

$$G\varepsilon_{x_0}(x_1) = m_{x_1} G\varepsilon_{x_1}(x_1).$$

Then again by the elementary domination principle,

$$G\varepsilon_{x_0}(z) \geq m_{x_1} G\varepsilon_{x_1}(z) \qquad \text{for any } z \text{ in } \Omega.$$

Therefore by (1), $G\varepsilon_{x_0} \equiv m_{x_1} G\varepsilon_{x_1}$. This is excluded. Hence we have

(2) $$G\varepsilon_{x_0}(x) < m_x G\varepsilon_x(x) \qquad \text{for any} \quad x \in K.$$

Now by (1)

$$G\lambda(x_0)=\int G(x_0, x)d\lambda(x)$$
$$\leq\int\frac{G(x_0, x_0)}{G(z, x_0)}G(z, x)d\lambda(x)=\frac{G(x_0, x_0)}{G(z, x_0)}G\lambda(z),$$

where λ is the measure mentioned above. Taking z on $S\lambda$, we have

$$G\lambda(x_0)\leq\frac{G(x_0, x_0)}{G(z, x_0)}G\varepsilon_{x_0}(z)=G(x_0, x_0).$$

Here the equality does not hold, otherwise it holds that

$$G\lambda(z)=G\varepsilon_{x_0}(z)\qquad\text{for any}\quad z\in S\lambda,$$

and

$$G(x_0, x)=\frac{G(x_0, x_0)}{G(z, x_0)}G(z, x)\qquad\lambda\text{-a.e. in } x \text{ for any } z\in S\lambda. \tag{3}$$

Take an arbitrarily fixed point z on $S\lambda$ and a neighborhood ω of z. Then by (3) there exists a point x in ω for which the equality (3) holds. Hence making ω tend to z, we obtain that x converges to z and

$$G(x_0, z)=\frac{G(x_0, x_0)}{G(z, x_0)}G(z, z),$$

since G is continuous. This shows that $G\varepsilon_{x_0}(z)=m_zG\varepsilon_z(z)$, which contradicts (2). This completes the proof.

REMARK. Lemma 1 holds for a positive lower semi-continuous kernel G if $G(x, y)$ is locally bounded at any point $(x, y)\in\Omega\times\Omega$ with $x\neq y$ and $\check{G}$ satisfies the continuity principle.

COROLLARY. *Let $\check{G}$ satisfy the continuity principle and G satisfy the elementary domination principle. Then for any given compact set K and any given point $x_0\notin K$, there exists a positive measure λ, supported by K, such that*

$$G\lambda(x)\geq G\varepsilon_{x_0}(x)\qquad G\text{-p.p.p.}\quad \text{on}\quad K,$$
$$G\lambda(x)\leq G\varepsilon_{x_0}(x)\qquad \text{on } S\lambda \text{ and at } x_0.$$

LEMMA 2[5]. *If G satisfies the weak domination principle, it satisfies the*

5) This is essentially due to Ohtsuka (cf. Lemma 2.3 in [5]).

continuity principle.

Proof. Let $G\mu$ be finite continuous as a function on the compact set $S\mu$ and x_0 be a point of $S\mu$. We shall show that $G\mu$ is continuous at x_0 as a function in Ω. We may assume that $G(x_0, x_0)=+\infty$. Let K be a compact neighborhood of x_0. By Dini's theorem, there exists, for any $\varepsilon>0$, a compact neighborhood ω_ε of x_0 such that $\omega_\varepsilon\subset K$ and $G\mu_\varepsilon\leq\varepsilon$ on $S\mu$, where μ_ε is the restriction of μ to ω_ε. We put

$$a_0=\inf\{G(x, y);\ x, y\in K\}.$$

Then for a fixed point x_1 in $K-S\mu$ and for any $\varepsilon>0$

$$G\mu_\varepsilon\leq\varepsilon a_0^{-1}G\varepsilon_{x_1}\qquad\text{on}\quad\omega_\varepsilon\cap S\mu.$$

If $G(x_1, x_1)=+\infty$, this inequality holds at x_1. When $G(x_1, x_1)$ is finite, we put

$$M(\varepsilon)=\max\left\{\frac{G\mu_\varepsilon(x_1)}{G(x_1, x_1)},\ \varepsilon a_0^{-1}\right\}.$$

Then $M(\varepsilon)\to 0$, as $\varepsilon\to 0$, since $G(x_0, x_0)=+\infty$ and $\mu(\{x_0\})=0$. Evidently we have

$$G\mu_\varepsilon\leq M(\varepsilon)G\varepsilon_{x_1}\qquad\text{on } \omega_\varepsilon\cap S\mu \text{ and at } x_1.$$

Consequently in any case, the above inequality holds on $S\mu_\varepsilon\cup\{x_1\}$ with $M(\varepsilon)\to 0$ as $\varepsilon\to 0$, and hence the inequality holds everywhere in Ω by the weak domination principle. From this follows immediately the continuity at x_0 of $G\mu$.

THEOREM 5. *Let $\check{G}$ satisfy the continuity principle. If G satisfies the elementary domination principle and the weak domination principle, then G satisfies the ordinary domination principle.*

Proof. First we shall show that $\check{G}$ satisfies the ordinary domination principle. Let $\check{G}\mu\leq\check{G}\nu$ on $S\mu$ for $\mu\in\mathfrak{E}_0$ and $\nu\in\mathfrak{M}_0$. We take a point x_0 in $\Omega-S\mu$. Then by the corollary to Lemma 1, there exists a positive measure λ, supported by $S\mu$, such that

$$G\lambda\geq G\varepsilon_{x_0}\qquad G\text{-p.p.p.}\quad\text{on}\quad S\mu,$$

$$G\lambda\leq G\varepsilon_{x_0}\qquad\text{on } S\lambda \text{ and at } x_0.$$

Then by the weak domination principle, $G\lambda\leq G\varepsilon_{x_0}$ in Ω. Hence

$$\check{G}\mu(x_0) = \int G\varepsilon_{x_0} d\mu \leq \int G\lambda d\mu = \int \check{G}\mu d\lambda$$

$$\leq \int \check{G}\nu d\lambda = \int G\lambda d\nu \leq \int G\varepsilon_{x_0} d\nu = \check{G}\nu(x_0).$$

This proves that $\check{G}$ satisfies the domination principle. Consequently G satisfies the domination principle by Lemma 2 and Theorem 2.

§ 2. Inverse existence theorem

In this section we shall give an inverse existence theorem.

THEOREM 4′. *Let G be a positive finite continuous kernel on Ω, and K be a compact subset of Ω and $u(x)$ be a positive finite continuous function on K. Then there exists a positive measure λ, supported by K, such that*

$$G\lambda(x) \leq u(x) \quad \text{on} \quad K,$$

$$G\lambda(x) = u(x) \quad \text{on} \quad S\lambda.$$

Proof. Without loss of generality we may suppose that $u(x) \equiv 1$ on K. Let M be a positive number such that

$$M > \max_{K \times K} G(x, y).$$

We put $G'(x, y) = M - G(x, y)$ on $K \times K$. Then G' is a positive finite continuous kernel on K. Hence by Theorem 4, there exists a positive measure λ', supported by K, such that

$$G'\lambda'(x) \geq 1 \quad \text{on} \quad K,$$

$$G'\lambda'(x) = 1 \quad \text{on} \quad S\lambda'.$$

Consequently

$$M\int d\lambda' - 1 \geq G\lambda' \quad \text{on} \quad K,$$

$$M\int d\lambda' - 1 = G\lambda' \quad \text{on} \quad S\lambda'.$$

Therefore $c = M\int d\lambda' - 1$ is positive and $\lambda = c^{-1}\lambda'$ is a positive measure what we want.

COROLLARY. *Assume the same for G and K as above. If $u(x)$ is a positive finite lower semi-continuous function on K, then we have the same conclusion as above.*

REMARK. For a positive lower semi-continuous kernel G we can obtain the following inverse existence theorem: if $\check{G}$ satisfies the continuity principle and $u(x)$ is a positive finite lower semi-continuous function on a compact set K, then there exists a positive measure λ, supported by K, such that

$$G\lambda \leq u \quad \text{on} \quad K,$$

$$G\lambda \geq u \quad G\text{-p.p.p.} \quad \text{on} \quad S\lambda.$$

§ 3. Inverse domination principle

In this section we consider only positive finite continuous kernels on Ω. By the inverse existence theorem we have

THEOREM 1′. *A positive finite continuous kernel G satisfies the inverse domination principle if and only if it satisfies the inverse balayage principle, that is, for any $\mu \in \mathfrak{M}_0$ and any compact set K, there exists μ', supported by K, such that*

$$G\mu' = G\mu \quad on \quad K,$$

$$G\mu' \geq G\mu \quad in \quad \Omega.$$

THEOREM 2′. *A positive finite continuous kernel G satisfies the inverse domination principle if and only if $\check{G}$ satisfies it.*

THEOREM 3′. *A positive finite continuous kernel G satisfies the inverse domination principle if and only if it satisfies the strong elementary inverse domination principle.*

We omit the proof of these theorems (cf. [2], Chapter II).

LEMMA 1′. *Let a positive finite continuous kernel G satisfy the elementary inverse domination principle. Then for any compact set K and any point $x_0 \notin K$, there exists a point x_1 in K such that*

$$G\varepsilon_{x_1}/G\varepsilon_{x_0} \equiv a \ constant \ in \ \Omega,$$

or there exists a positive measure λ, supported by K, such that

$$G\lambda \leq G\varepsilon_{x_0} \quad on \quad K,$$

$$G\lambda = G\varepsilon_{x_0} \quad on \;\; S\lambda,$$

$$G\lambda(x_0) > G\varepsilon_{x_0}(x_0).$$

This can be verified in the similar way by using the inverse existence theorem.

COROLLARY. *Let G satisfy the elementary inverse domination principle. Then for any compact set K and any point* $x_0 \notin K$, *there exists a positive measure* λ, *supported by K, such that*

$$G\lambda(x) \leq G\varepsilon_{x_0}(x) \quad on \;\; K,$$

$$G\lambda(x) \geq G\varepsilon_{x_0}(x) \quad on \; S\lambda \; and \; at \; x_0.$$

From this corollary follows

THEOREM 5′. *If a positive finite continuous kernel G satisfies the elementary inverse domination principle and the weak domination principle, then it satisfies the inverse domination principle.*

§ 4. First main theorem

Let G be a positive continuous (in the extended sense) kernel on Ω such that $G(x, y)$ is finite at any point $x \neq y$.

DEFINITION. If there exist different points x_1 and x_2 such that

$$G\varepsilon_{x_1}/G\varepsilon_{x_2} \equiv \text{a constant in } \Omega,$$

G is called *degenerate.* If G is not degenerate, it is called *non-degenerate.*

In this section we shall prove the following first main theorem.

THEOREM 6. *Let* $\check{G}$ *satisfy the continuity principle. If G is non-degenerate and satisfies the weak domination principle, then it satisfies the ordinary domination principle or the inverse domination principle.*

To prove this we require several lemmas.

LEMMA 3.[6] *Let G satisfy the weak domination principle and* $\check{G}$ *satisfy the continuity principle. Then G satisfies the ordinary domination principle if for any different points x and y,*

6) This corresponds to the result of Ninomiya for symmetric kernels [4].

$$\Gamma(x, y)=G(x, x)G(y, y)-G(x, y)G(y, x)\geq 0.$$

Proof. By Theorem 5, it is sufficient to show that G satisfies the elementary domination principle. Let $G\varepsilon_{x_1}(x_1)\leq aG\varepsilon_{x_2}(x_1)$ with $a>0$. Then, being $\Gamma(x_1, x_2)\geq 0$,

$$G\varepsilon_{x_1}(x_2)\leq\frac{G(x_1, x_1)}{G(x_1, x_2)}G\varepsilon_{x_2}(x_2)\leq aG\varepsilon_{x_2}(x_2).$$

Hence by the weak domination principle, $G\varepsilon_{x_1}(z)\leq aG\varepsilon_{x_2}(z)$ for any $z\in\Omega$. Thus G satisfies the elementary domination principle and hence the ordinary domination principle.

Similarly we have

LEMMA 3′. *Let G be a positive finite continuous kernel and satisfy the weak domination principle. If for any different points x and y, $\Gamma(x, y)\leq 0$, G satisfies the inverse domination principle.*

LEMMA 4[7]. *Let G be a positive (finite or infinite-valued) kernel on a space $\Omega=\{x_1, x_2, x_3\}$. If G is non-degenerate and satisfies the weak domination principle, then it satisfies the ordinary domination principle or the inverse domination principle.*

Proof. We put $g_{ij}=G(x_i, x_j)$ and $\gamma_{ij}=\Gamma(x_i, x_j)$. Since G is non-degenerate and satisfies the weak domination principle, $\gamma_{ij}\neq 0$ for any $i\neq j$. The proof will be accomplished by virtue of Lemmas 3 and 3′ when it will be shown that there occur only two cases: γ_{12}, γ_{23} and γ_{31} are simultaneously positive or negative.

First we consider the case that g_{ii}'s are all finite, and we show that

(4) if $\gamma_{ij}>0$ ($\gamma_{ij}<0$ resp.),

$$\frac{g_{ii}}{g_{ij}}\geq\frac{g_{ki}}{g_{kj}}\geq\frac{g_{ji}}{g_{jj}}\quad\left(\frac{g_{ii}}{g_{ij}}\leq\frac{g_{ki}}{g_{kj}}\leq\frac{g_{ji}}{g_{jj}}\text{ resp.}\right).$$

Suppose that $\gamma_{ij}>0$. Then

$$G\varepsilon_i(x_i)=\frac{g_{ii}}{g_{ij}}G\varepsilon_j(x_i)$$

$$G\varepsilon_i(x_j)<\frac{g_{ii}}{g_{ij}}G\varepsilon_j(x_j),$$

7) This is, in a certain sense, a special case of Choquet-Deny's theorem for a positive kernel on a space consisting of a finite number of points (cf. Theorem 4′ in [1]).

where ε_i is the unit measure at x_i. By these inequalities and the weak domination principle we get

$$G\varepsilon_i(x_k) \leq \frac{g_{ii}}{g_{ij}} G\varepsilon_j(x_k),$$

namely

$$\frac{g_{ii}}{g_{ij}} \geq \frac{g_{ki}}{g_{kj}}.$$

Interchanging i and j, we obtain

$$\frac{g_{ki}}{g_{kj}} \geq \frac{g_{ji}}{g_{jj}}.$$

Similarly we obtain that if $\gamma_{ij}<0$,

$$\frac{g_{ii}}{g_{ij}} \leq \frac{g_{ki}}{g_{kj}} \leq \frac{g_{ji}}{g_{jj}}\text{ [8]}.$$

Now we show that if $\gamma_{12}>0$, then $\gamma_{23}>0$ and $\gamma_{31}>0$. In fact, if $\gamma_{23}<0$ and $\gamma_{31}<0$, then by (4)

$$\frac{g_{11}}{g_{21}} g_{23} \leq g_{13} \leq \frac{g_{12}}{g_{22}} g_{23}$$

and

$$0 \leq \left(\frac{g_{12}}{g_{22}} - \frac{g_{11}}{g_{21}} \right) g_{23},$$

which is a contradiction, since $\gamma_{12}>0$ and the right-hand term is negative. If $\gamma_{23}>0$ and $\gamma_{31}<0$, then

$$\frac{g_{11}}{g_{31}} g_{33} < g_{13} \leq \frac{g_{12}}{g_{32}} g_{33}$$

and

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8) We can also derive (4) as follows: for any finite real numbers t_i and t_j,

$$\left.\begin{array}{l} g_{ii}t_i + g_{ij}t_j \leq 0 \\ g_{ji}t_i + g_{jj}t_j \leq 0 \end{array}\right\} \Rightarrow g_{ki}t_i + g_{kj}t_j \leq 0,$$

since G satisfies the weak domination principle. From this implication follows immediately (4).

$$0<\left(\frac{g_{12}}{g_{32}}-\frac{g_{11}}{g_{31}}\right)g_{33},$$

which is a contradiction, since $\gamma_{12}>0$ and the right-hand term is non-positive by (4). Similarly $\gamma_{23}<0$ and $\gamma_{31}>0$ lead to a contradiction. Consequently if $\gamma_{12}>0$, then γ_{23} and γ_{31} are positive and hence γ_{12}, γ_{23} and γ_{31} are simultaneously positive or negative.

Next we consider the case that some g_{ii}'s are infinite. In this case it is sufficient to consider the case that only one of them, say g_{11}, is infinite, since otherwise $\gamma_{ij}>0$ for any $i\neq j$ and hence G satisfies the ordinary domination principle. Similarly as we derived (4), we obtain

$$\frac{g_{i1}}{g_{ii}}\leq\frac{g_{k1}}{g_{ki}}\qquad(i,\ k\neq 1). \tag{5}$$

If $\gamma_{23}<0$, then

$$\frac{g_{22}}{g_{32}}g_{33}<g_{23}\leq\frac{g_{21}}{g_{31}}g_{33},$$

where the last inequality follows from (5). This yields

$$0<\left(\frac{g_{21}}{g_{31}}-\frac{g_{22}}{g_{32}}\right)g_{33},$$

which contradicts (5). Consequently $\gamma_{23}>0$ and G satisfies the ordinary domination principle by Lemma 3. This completes the proof.

REMARK. When G is finite-valued and non-degenerate and satisfies the weak domination principle, the determinant Δ of the matrix

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} \\ g_{21} & g_{22} & g_{23} \\ g_{31} & g_{32} & g_{33} \end{pmatrix}$$

is positive[9]. In fact, if γ_{12}, γ_{23} and γ_{31} are positive,

$$\begin{aligned}\Delta &= g_{13}(g_{21}g_{32}-g_{22}g_{31})+g_{23}(g_{12}g_{31}-g_{11}g_{32})+g_{33}(g_{11}g_{22}-g_{12}g_{21})\\ &\geq\frac{g_{12}g_{33}}{g_{32}}(g_{21}g_{32}-g_{22}g_{31})+\frac{g_{21}g_{33}}{g_{31}}(g_{12}g_{31}-g_{11}g_{32})+g_{33}(g_{11}g_{22}-g_{12}g_{21})\end{aligned}$$

9) Consequently the non-degeneracy of this kernel G is equivalent to the one of Choquet-Deny (namely $\Delta\neq 0$). In general, it is verified that with respect to a positive finite-valued kernel G on a space consisting of a finite number of points, the non-degeneracy in our sense is equivalent to the one of Choquet-Deny provided that G satisfies the weak domination principle.

$$= \frac{g_{33}}{g_{31}g_{32}}(g_{21}g_{32} - g_{22}g_{31})(g_{12}g_{31} - g_{11}g_{32}) \geq 0.$$

Consequently if $\Delta = 0$ and $g_{12}g_{31} \neq g_{11}g_{32}$, then

$$g_{23}g_{31} = g_{21}g_{33} \quad \text{and} \quad g_{21}g_{32} = g_{22}g_{31}.$$

Hence $g_{22}g_{33} = g_{23}g_{32}$, which contradicts $\gamma_{23} > 0$. If $\Delta = 0$ and $g_{12}g_{31} = g_{11}g_{32}$, then $g_{13}g_{32} = g_{12}g_{33}$ or $g_{21}g_{32} = g_{22}g_{31}$. Hence $g_{11}g_{33} = g_{13}g_{31}$ or $g_{11}g_{22} = g_{12}g_{21}$, which is also a contradiction. Therefore $\Delta > 0$. When $\gamma_{12}, \gamma_{23}, \gamma_{13}$ are negative, similar computations show that $\Delta > 0$.

When G is not necessarily finite-valued, we put

$$G_N(x_i, x_j) = \min\{G(x_i, x_j), N\}.$$

Then for any sufficiently large N, the determinant Δ_N of the matrix corresponding to G_N is positive.

Corollary 1. *Let a positive continuous (in the extented sense) kernel G be non-degenerate and satisfy the weak domination principle. If there exist different points x_1 and x_2 such that $\Gamma(x_1, x_2) > 0$, then for any different points x and y, $\Gamma(x, y) > 0$.*

Proof. It is sufficient to show that $\Gamma(x_1, x_3) > 0$ for any point x_3. Let G' be the restriction of G to $\Omega' = \{x_1, x_2, x_3\}$. Then G' is non-degenerate and satisfies the weak domination principle. Hence by Lemma 4, G' satisfies the ordinary domination principle or the inverse domination principle. Since $\Gamma(x_1, x_2) > 0$, G' satisfies the ordinary domination principle and hence $\Gamma(x_1, x_3) > 0$.

Corollary 2. *Let a positive finite continuous kernel G be non-degenerate and satisfy the weak domination principle. If there exist different points x_1 and x_2 such that $\Gamma(x_1, x_2) < 0$, then for any different points x and y, $\Gamma(x, y) < 0$.*

Now we have

Theorem 7. *Let a positive finite continuous kernel G be non-degenerate and satisfy the weak domination principle. Then G satisfies the ordinary domination principle or the inverse domination principle.*

Proof. Take arbitrary different points x_1 and x_2. Then $\Gamma(x_1, x_2) \neq 0$, since G is non-degenerate and satisfies the weak domination principle. If $\Gamma(x_1, x_2) > 0$, then G satisfies the ordinary domination principle by Lemma 3 and Corollary 1 to Lemma 4. If $\Gamma(x_1, x_2) < 0$, G satisfies the inverse domination principle by Lemma 3′ and Corollary 2 to Lemma 4.

THEOREM 8[10]. *Let $\check{G}$ satisfy the continuity principle and let G satisfy the weak domination principle. If G is non-degenerate and there exists a point x_1 such that $G(x_1, x_1) = +\infty$, then G satisfies the ordinary domination principle.*

Proof. This is an immediate consequence of Lemma 3 and Corollary 1 to Lemma 4, since $\Gamma(x_1, x_2) > 0$ for any point x_2.

Our first main theorem (Theorem 6) follows now from Theorems 7 and 8.

REMARK. Noting that in the proof of Lemma 4 the non-degeneracy of G is effective in the form that $\Gamma(x, y) \neq 0$ for any $x \neq y$ in Ω, we obtain

THEOREM 6′. *Let $\check{G}$ satisfy the continuity principle. If G satisfies the weak domination principle and $\check{G}$ is non-degenerate, G satisfies the ordinary domination principle or the inverse domination principle.*

THEOREM 8′. *Let $\check{G}$ satisfy the continuity principle and let G satisfy the weak domination principle. If $\check{G}$ is non-degenerate and there exists a point x_1 such that $G(x_1, x_1) = +\infty$, then G satisfies the ordinary domination principle.*

§ 5. Second main theorem

By Theorem 6 we obtain our second main theorem which is concerned with the weak balayage principle.

(VIII) *Weak balayage principle.* For any positive measure $\mu \in \mathfrak{M}_0$ and any compact set K, there exists a positive measure μ', supported by K, such that

$$G\mu' = G\mu \qquad G\text{-p.p.p. on } K.$$

THEOREM 9. *Let G be a positive continuous (in the extended sense) kernel such that G satisfies the continuity principle*[11] *and G or $\check{G}$ is non-degenerate. Assume that every open set $\omega \subset \Omega$ is of positive G-capacity*[12]. *If G satisfies the weak balayage principle, it satisfies the ordinary balayage principle or the inverse balayage principle.*

Proof. First we consider the case that $G(x, x) < +\infty$ for any $x \in \Omega$ and hence G is a finite continuous kernel. We suppose that $\check{G}\mu \leq \check{G}\nu$ on $K = S\mu \cup S\nu$ with μ and ν in $\mathfrak{M}_0$, and we take an arbitrary point x_0 in $\Omega - K$. By the weak balayage principle, there exists $\lambda \in \mathfrak{M}_0$ such that

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10) This is sharper than a result obtained in [2] (cf. Theorem II. 13).

11) Notice that we assume the continuity principle for G but not for $\check{G}$.

12) This means that for any open set $\omega \subset \Omega$ there exists a positive measure $\lambda \neq 0$ in $\mathfrak{E}_0$ such that $S\lambda \subset \omega$.

$$G\lambda = G\varepsilon_0 \quad \text{on } K,$$
$$S\lambda \subset K,$$

where ε_0 is the unit measure at x_0. Then

$$\check{G}\mu(x_0) = \int G\varepsilon_0 d\mu = \int G\lambda d\mu = \int \check{G}\mu d\lambda \leq \int \check{G}\nu d\lambda$$
$$= \int G\lambda d\nu = \int G\varepsilon_0 d\nu = \check{G}\nu(x_0).$$

Consequently $\check{G}$ satisfies the weak domination principle. Therefore $\check{G}$ and hence G satisfy the ordinary domination principle or the inverse domination principle by Theorem 7.

Next we consider the case that there exists a point $x' \in \Omega$ such that $G(x', x') = +\infty$. We shall show that

(*) if $\check{G}\mu \leq \check{G}\varepsilon_0$ on $S\mu \cup \{x_0\}$ with $\mu \in \mathfrak{E}_0$ and $x_0 \notin S\mu$,

then the same inequality holds everywhere in Ω.

Let us take a point x_1 in $\Omega - (S\mu \cup \{x_0\})$. It is sufficient to show $\check{G}\mu(x_1) \leq \check{G}\varepsilon_0(x_1)$. When $G(x_0, x_0)$ is finite, we can show the inequality similarly as above. Therefore we suppose that $G(x_0, x_0) = +\infty$. Then there exists a compact neighborhood K of x_0 such that

$$\check{G}\mu \leq \check{G}\varepsilon_0 \quad \text{on } K.$$

Let λ be a weakly balayaged measure of ε_{x_1} on $S\mu \cup K$, i.e.,

$$G\lambda = G\varepsilon_{x_1} \quad G\text{-p.p.p. on } S\mu \cup K,$$
$$S\lambda \subset S\mu \cup K.$$

Then $G\lambda(x_0) \leq G\varepsilon_{x_1}(x_0)$. In fact, if $G\lambda(x_0) > G\varepsilon_{x_1}(x_0)$, there exists an open set $\omega \subset K$ such that $G\lambda > G\varepsilon_{x_1}$ in ω. This is a contradiction, since by our assumption ω is of positive G-capacity. Therefore

$$\check{G}\mu(x_1) = \int G\varepsilon_{x_1} d\mu = \int G\lambda d\mu = \int \check{G}\mu d\lambda$$
$$\leq \int \check{G}\varepsilon_0 d\lambda = G\lambda(x_0) \leq G\varepsilon_{x_1}(x_0) = \check{G}\varepsilon_0(x_1).$$

This proves the implication (*)[13]. By this implication and Corollary 1 to

13) Hence $\check{G}$ satisfies the continuity principle (cf. Lemma 2).

Lemma 4, it is seen that $\Gamma(x, y)$ is positive for any $x \neq y$ in Ω, since $\Gamma(x', y) = +\infty$, where x' is a point such that $G(x', x') = +\infty$. Consequently $\check{G}$ satisfies the elementary domination principle.

Now we shall show that G satisfies the ordinary domination principle. Let $G\mu \leq G\nu$ be true on $S\mu$ for $\mu \in \mathfrak{E}_0$ and $\nu \in \mathfrak{M}_0$, and take a point x_0 in $\Omega - S\mu$. Then by the corollary to Lemma 1, there exists a positive measure τ, supported by $S\mu$, such that

$$\check{G}\tau \geq \check{G}\varepsilon_0 \qquad G\text{-p.p.p. on } S\mu,$$

$$\check{G}\tau \leq \check{G}\varepsilon_0 \qquad \text{on } S\tau \text{ and at } x_0.$$

Then by (*)

$$\check{G}\tau \leq \check{G}\varepsilon_0 \qquad \text{in } \Omega$$

and hence

$$\check{G}\tau = \check{G}\varepsilon_0 \qquad G\text{-p.p.p. on } S\mu.$$

Therefore

$$G\mu(x_0) = \int \check{G}\varepsilon_0 d\nu = \int \check{G}\tau d\mu = \int G\mu d\tau \leq \int G\nu d\tau$$
$$= \int \check{G}\tau d\nu \leq \int \check{G}\varepsilon_0 d\nu = G\nu(x_0).$$

This shows that G satisfies the ordinary domination principle. Consequently G satisfies the ordinary balayage principle by Theorem 1.

Remark. Theorem 9 is not valid unless it is assumed that every open set is of positive G-capacity. In fact, a non-degenerate and non-symmetric kernel G defined by

$$\begin{pmatrix} +\infty & 1 & g_{13} \\ 1 & 1 & g_{23} \\ g_{31} & g_{32} & 1 \end{pmatrix}$$

with $0 < g_{32} < g_{31} < g_{23}^{-1} < g_{13}^{-1} < +\infty$, satisfies the weak balayage principle but not the ordinary balayage principle nor the inverse balayage principle.

Now we consider a weaker principle than the weak balayage principle.

(IX) *Elementary weak balayage principle*[14]. For any compact set K and any point $x_0 \notin K$, there exists a positive measure λ, supported by K, such that

14) This is, in a certain sense, a special case of the light sweeping-out principle of Ohtsuka [5].

$$G\lambda = G\varepsilon_0 \qquad G\text{-p.p.p. on } K.$$

Theorem 9′. *Let G be a positive continuous (in the extended sense) kernel such that G satisfies the continuity principle and G or $\check{G}$ is non-degenerate, and let every open set $\omega \subset \Omega$ be of positive G-capacity. If G satisfies the elementary weak balayage principle, then it satisfies the ordinary balayage principle or the inverse balayage principle.*

This is verified by the proof of Theorem 9, where we used a weakly balayaged measure of a point-mass.

By Theorem 9 we obtain immediately

Theorem 10. *Under the same assumption as in Theorem 9, G satisfies the weak balayage principle if and only if it satisfies the weak domination principle.*

Theorem 11. *Under the same assumption as above, G satisfies the ordinary balayage principle if it satisfies the (elementary) weak balayage principle and if there exists a point x_1 such that $G(x_1, x_1) = +\infty$.*

Remark 1. Let G be symmetric. Then all the theorems in this paper are true without assuming the continuity principle for G, since the existence theorem is, as well-known, true without the continuity principle. Let us remark that if this kernel G is of positive (negative resp.) type satisfying the weak domination principle, then it satisfies the ordinary (inverse resp.) domination principle. In fact, if G is of positive (negative resp.) type,

$$\Gamma(x, y) > 0 \qquad (< 0 \text{ resp.})$$

for any $x \neq y$ in Ω. Therefore by Lemma 3 (Lemma 3′ resp.), G satisfies the ordinary (inverse resp.) domination principle. Similar fact holds concerning the weak balayage principle.

Remark 2. We say that a positive continuous (not necessarily symmetric) kernel G satisfies the *weak equilibrium principle* when for any compact set $K \subset \Omega$, there exists a positive measure μ, supported by K, such that

$$G\mu = 1 \qquad G\text{-p.p.p. on } K.$$

It seems natural to ask whether the analogue to Theorem 9 is valid concerning the weak equilibrium principle. However the answer is negative, in general. In fact, a symmetric kernel G defined by the matrix,

$$\begin{pmatrix} 1 & 2 & \frac{1}{2} \\ 2 & 1 & 1 \\ \frac{1}{2} & 1 & 1 \end{pmatrix},$$

is non-degenerate and satisfies the weak equilibrium principle, but it does not satisfy the ordinary equilibrium principle nor the inverse equilibrium principle. This example also gives an affirmative answer to the question raised by Ninomiya [4]: Is there a positive symmetric kernel satisfying the weak equilibrium principle which is not of positive type nor of negative type?

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