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Extensions of Riemannian Metrics

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1 Introduction

In the present paper, we consider certain problems of extension of a Riemannian metric on a closed submanifold to the whole. Similar problems in metrizable spaces were discussed in [2], [5].

Let N be an n-dimensional submanifold of an m-dimensional, differentiable manifold M. N is said to be a *closed submanifold* if (a) it is set-theoretically closed, and (b) the topology of N as a manifold coincides with the relative topology of N in M. Let h be a Riemannian metric on N. By a *C^s-extension* of h to M is meant a Riemannian metric g of M, of class C^s, if the restriction of g to N is h.

We shall first concern with a general case for extension of a Riemannian metric on a closed submanifold (Section 2) and then prove that if M is separable and connected and N is a connected closed submanifold of M, then there exists a C^s-extension g of h so that N is totally geodesic in a strong sense under g (Section 3).

It is known that each separable, connected differentiable manifold has a bounded (or complete) Riemannian metric [4]. We shall show that if M is connected and a Riemannian metric h of a (not necessarily connected) closed submanifold is bounded (or complete), there exists a bounded (or complete) extension of h (Section 4, 5).

2 General Case

Proposition *Let M be an m-dimensional, separable, differentiable manifold of class $C^r\,(r\geqq 1)$, and let N be a closed submanifold of class $C^{s+1}\,(0\leqq s\leqq r-1)$ with a Riemannian metric h of class C^s. Then there exists a C^s-extension of h to M.*

Proof. The condition (b) of a closed submanifold implies that each point p of N belongs to a coordinate neighborhood U in M with a coordinate system $\{u^1, u^2, \cdots, u^m\}$ such that the set $N\cap U$ is defined by the equations $u^{n+1}=0, \cdots, u^m=0$. (In the following we shall call such a coordinate system a *canonical coordinate system* of M with respect to N.) The restriction of h to $N\cap U$ is expressed by a positive definite symmetric tensor h_{ij} $(i, j=1, 2, \cdots, n)$

of class C^s. Then we define a metric on U by

$$\left(\begin{array}{c:c} h_{ij} & 0 \\ \hdashline 0 & \delta_{\lambda\mu} \end{array}\right)$$

On the other hand, we define a Riemannian metric of class C^s in the open submanifold $M-N$. Smoothly unifying these metrics defined on U's and $M-N$ by a partition of unity (e.g. [1], pp. 104–105), we get a desired C^s-extension. Q. E. D.

We shall slightly generalize Proposition. A subset of a differentiable manifold M is said to be a *piecewise C^s-differentiable linear graph* in M if it is the image of a linear graph L under a homeomorphism which is C^s-diffeomorphic on each edge of L.

Let N be a closed submanifold of M and L_ξ's piecewise C^s-differentiable linear graphs in M. We shall call the set $N\cup(\cup L_\xi)$ a *closed submanifold with branches* if it satisfies the following conditions:

(a) $L_\xi \cap \overline{(\bigcup_{\zeta\neq\xi} L_\zeta)} = \emptyset$ for every ξ,

(b) $L_\xi \cap N$ is the set of endpoints of L_ξ,

(c) the tangent vector of L_ξ at each of its endpoints does not touch N,

(d) the degree of each vertex p of L_ξ is at most three. If p is a branch point, then the tangent vector of an edge at p coincides with one of the tangent vectors of the other edges at p but is linearly independent of the third. If p is an ordinary point, the tangent vectors of the edges at p are linearly independent.

We note here that the condition (a) implies that $\cup(L_\xi\cap N)$ is a discrete set on N.

THEOREM 1 *Let M, N and h be the same as in Proposition and let $N'=N\cup(\cup L_\xi)$ be a closed submanifold with branches L_ξ. If each L_ξ has a metric of class C^s, then there exists a C^s-extension g of the metric h' of N', obtained from h and those metrics of L_ξ's.*

PROOF. We shall first extend h' to a neighborhood of each point $p\in N'$.

(i) The case $p\in N-\cup L_\xi$. Let $\{u^1, u^2, \cdots, u^m\}$ be a canonical coordinate system of M with respect to N on a neighborhood $U(p)$ of p in M such that $N\supset N'\cap U(p)$. We extend $h'|N'\cap U(p)$ to a metric g_p on $U(p)$ by

$$\left(\begin{array}{c:c} h_{ij} & 0 \\ \hdashline 0 & \delta_{\lambda\mu} \end{array}\right)$$

where $\{h_{ij}\}$ is the metric tensor expressing the Riemannian metric h on N.

(ii) The case $p \in N \cap L_\xi$. There exists a local coordinate system $\{u^1, u^2, \cdots, u^m\}$ of M, on a neighborhood $U(p)$ of p in M, such that $U(p) \cap L_\zeta = \emptyset$ for $\zeta \neq \xi$, and

$$N \cap U(p) = \{q \in U(p);\ u^{n+1}(q) = \cdots = u^m(q) = 0\}$$

$$L_\xi \cap U(p) = \{q \in U(p);\ u^{n+1}(q) \geqq 0,\ u^1(q) = \cdots = u^n(q) = u^{n+2}(q) = \cdots = u^m(q) = 0\}.$$

We extend $h' | N' \cap U(p)$ to a metric g_p on $U(p)$ by

$$\left(\begin{array}{cc|c|c} h_{ij} & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & 0 \\ \hline 0 \cdots 0 & h_{n+1n+1} & & \\ \hline & 0 & & \delta_{\lambda\mu} \end{array}\right)$$

where h_{n+1n+1} is the metric of L_ξ.

(iii) The case $p \in L_\xi$ is an ordinary point but not a vertex. There exists a local coordinate system $\{u^1, u^2, \cdots, u^m\}$ of M, on a neighborhood $U(p)$ of p in M, such that $L_\xi \supset N' \cap U(p) = \{q \in U(p);\ u^2(q) = \cdots = u^m(q) = 0\}$. We extend $h' | N' \cap U(p)$ to a metric g_p on $U(p)$ by

$$\left(\begin{array}{c|c} h_{11} & 0 \\ \hline 0 & \delta_{\lambda\mu} \end{array}\right)$$

where h_{11} is the metric of L_ξ.

(iv) The case where $p \in L_\xi$ is a vertex and an ordinary point.

Let H_1, H_2 be the segments of L_ξ with p as the common endpoint. By the condition (d) in the definition of a closed submanifold with branches, the tangent vectors of H_1 and H_2 at p are linearly independent. Hence there exists a local coordinate system $\{u^1, u^2, \cdots, u^m\}$, on a neighborhood $U(p)$ of p in M such that

$$L_\xi \supset N' \cap U(p)$$

$$H_1 \cap U(p) = \{q \in U(p);\ u^1(q) \geqq 0,\ u^2(q) = \cdots = u^m(q) = 0\},$$

$$H_2 \cap U(p) = \{q \in U(p);\ u^2(q) \geqq 0,\ u^1(q) = u^3(q) = \cdots = u^m(q) = 0\}.$$

We extend $h' | N' \cap U(p)$ to a metric g_p on $U(p)$ by

$$\left(\begin{array}{cc|c} h_{11} & 0 & 0 \\ 0 & h_{22} & \\ \hline & 0 & \delta_{\lambda\mu} \end{array}\right),$$

where h_{11} (or h_{22}) is the metric of H_1 (or H_2).

(v) The case where p is a branch point of L_ξ.

Let H_0, H_1, H_2 be the segments of L_ξ with p as the common endpoint. Assume that the tangent vectors of H_0 and H_1 at p coincide with each other. Then there exists a local coordinate system $\{u^1, u^2, \cdots, u^m\}$ on a neighborhood $U(p)$ of p in M such that

$$L_\xi \supset N' \cap U(p),$$

$$(H_0 \cup H_1) \cap U(p) = \{q \in U(p);\ u^2(q) = \cdots = u^m(q) = 0\},$$

and $$H_2 \cap U(p) = \{q \in U(q);\ u^2(q) \geqq 0,\ u^1(q) = u^3(q) = \cdots = u^m(q) = 0\}.$$

We extend $h'|N' \cap U(p)$ to a metric g_p on $U(p)$ by

$$\begin{pmatrix} \begin{matrix} h_{11} & 0 \\ 0 & h_{22} \end{matrix} & 0 \\ 0 & \delta_{\lambda\mu} \end{pmatrix}$$

where h_{11} (or h_{22}) is the metric of H_0 and H_1 (or H_2).

Define a Riemannian metric g_0 on the open submanifold $M-N'$ and then smoothly unify the metric g_0 on $M-N'$ and g_p on $U(p)$ given in (i)$\sim$(v) by a partition of unity. Then we get a desired extension. Q. E. D.

3 Totally Geodesic Case

We shall denote a Riemannian manifold M with the Riemannian metric g by (M, g). A geodesic of (M, g) is called an *M-geodesic*. Let N be a connected submanifold of M with the Riemann structure derived from (M, g). N is called *totally geodesic in a strong sense* if the following two conditions are satisfied:

(a) each N-geodesic is an M-geodesic,

(b) there exists an open set U of M containing N such that each M-geodesic in U joining two points of N is an N-geodesic and such that, for every piecewise differentiable curve α in U joining two points of N and not contained in N, there exists a piecewise differentiable curve in N, which joins the same points and whose length is less than that of α.

THEOREM 2 *Let M be an m-dimensional, separable, differentiable manifold of class $C^r (r \geqq 4)$, and let N be a connected closed submanifold of class $C^{s+1} (1 < s \leqq r-3)$ with a Riemannian metric h of class C^s. Then there exists a C^s-extension g of h such that (N, h) is totally geodesic in a strong sense under g.*

Moreover if N is compact, the extension g above stated can have the more property that, for every pair (p, q) of points of N, there exists at least one

M-geodesic from p to q, whose length is the distance between these points and all such M-geodesics lie completely within N.

PROOF. (i) Let W be a tubular neighborhood (of class C^{r-2}) of N in M (e.g. [3], Theorem 9, p. 73) and let π be the projection of W on N. Then there exists an open covering $\{V_\xi\}$ of N consisting of coordinate neighborhoods of N, open subsets U_ξ of $\pi^{-1}(V_\xi)$ containing V_ξ, and C^{r-2}-diffeomorphisms $f_\xi\colon U_\xi \to V_\xi \times R^{m-n}$ such that the diagram

$$\begin{array}{ccc} U_\xi & \xrightarrow{\quad f_\xi \quad} & V_\xi \times R^{m-n} \\ & \pi \searrow \quad \swarrow \pi' & \\ & V_\xi & \end{array}$$

is commutative, where π' is the natural projection. The local coordinate system $\{u^1, u^2, \cdots, u^n\}$ on V_ξ and the coördinates $u^{n+1}, \cdots, u^m$ of R^{m-n} give a coordinate system to $V_\xi \times R^{m-n}$ and consequently to U_ξ by f_ξ^{-1}.

We extend $h | V_\xi$ to a Riemannian metric to U_ξ by

$$\left(\begin{array}{c:c} h_{ij} & 0 \\ \hdashline 0 & \delta_{\lambda\mu} \end{array} \right).$$

Since M is a normal topological space, there exists an open neighborhood U of N such that $N \subset U \subset \bar{U} \subset \cup U_\xi$. Define a Riemannian metric g_0 on the open manifold $M - \bar{U}$. Then, by a partition of unity, smoothly unify g_0 on $M - \bar{U}$ and those metrics on U_ξ's defined above. It is easily seen that the metric g which results is an extension of h.

(ii) We prove that N is totally geodesic in a strong sense in (M, g). Let $\gamma\colon I \to N$ be an N-geodesic, where I is a closed real interval. Let $\{u^1, u^2, \cdots, u^m\}$ be the local coordinate system on U_ξ, defined in (i). The system is compatible with the fibre structure of W. In terms of a local coordinate system $\{u^1, u^2, \cdots, u^m\}$, $t \to \gamma(t) \in N$ satisfies the second order differential equations

$$\frac{d^2u^l}{dt^2} + \widehat{\left\{ {l \atop ij} \right\}} \frac{du^i}{dt} \frac{du^j}{dt} = 0 \qquad (1 \leqq i, j, l \leqq n),$$

where $\widehat{\left\{ {l \atop ij} \right\}}$ is the Christoffel symbol for h_{ij}. Since

$$\left\{ {\lambda \atop \mu\nu} \right\} = \begin{cases} \widehat{\left\{ {\lambda \atop \mu\nu} \right\}} & \text{if } 1 \leqq \lambda, \mu, \nu \leqq n, \\ 0 & \text{if at least one of } \lambda, \mu, \nu \text{ is larger than } n, \end{cases}$$

γ is also a geodesic in (M, g).

Each M-geodesic in U joining two points of N lies completely within N. For let $\gamma: I\to\gamma(I)\subset U$ be an M-geodesic joining two points of N, where $I=[a, b]$. The geodesic satisfies the equations

$$\frac{d^2u^l}{dt^2}+\left\{ {l \atop ij} \right\}\frac{du^i}{dt}\frac{du^j}{dt}=0 \qquad (1\leqq i, j, l\leqq m).$$

Since $\left\{ {l \atop ij} \right\}=0$ for $l\geqq n+1$,

$$\frac{d^2u^l}{dt^2}=0 \qquad (l\geqq n+1).$$

Therefore, for each $l\geqq n+1$, $\frac{du^l}{dt}=\text{const.}$ On the other hand, since $u^l(\gamma(a))=u^l(\gamma(b))=0$, there exists c such that $a<c<b$ and $\left.\frac{du^l}{dt}\right|_c=0$. Hence $\frac{du^l}{dt}=0$ on I and consequently $u^l(\gamma(t))=0$ $(t\in I)$. Therefore we conclude that $\gamma(I)\subset N$.

For every piecewise differentiable curve $\alpha: I\to U$ joining two points of N but not contained in N, there exists a piecewise differentiable curve on N joining the same points whose length is less than that of α. For since α is not contained in N, for some $i(n+1\leqq i\leqq m)$, there exists c such that $a<c<b$ and $\left.\frac{du^i}{dt}\right|_c\neq 0$. The length l of α with respect to g is

$$\begin{aligned} l&=\int_I\left(h_{ij}\frac{du^i}{dt}\frac{du^j}{dt}+\sum_{k=n+1}^{m}\left(\frac{du^k}{dt}\right)^2\right)^{1/2}dt \\ &>\int_I\left(h_{ij}\frac{du^i}{dt}\frac{du^j}{dt}\right)^{1/2}dt. \end{aligned}$$

The last integral is the length of the curve $\pi(\alpha)\subset N$ obtained by projecting α to N by π.

(iii) Compact case. Let g be the extension of h obtained in (i) and K the diameter of N with respect to g. Let $\pi: U\to N$ be the projection of the tubular neighborhood U of N in (i). Since N is compact, there exists an $\varepsilon>0$ such that, for each point p of N, the spherical neighborhood $S_\varepsilon(p)$ of p with radius ε is contained in U. Then $U'=\bigcup_{p\in N}S_\varepsilon(p)$ is a tubular neighborhood of N contained in U. We define an extension of h to U' by

$$\left(\begin{array}{c|ccc} h_{ij} & & 0 & \\ \hline & (3K/\varepsilon)^2 & & 0 \\ 0 & & \ddots & \\ & 0 & & (3K/\varepsilon)^2 \end{array}\right).$$

On the other hand, we define a Riemannian metric on $M-\bar{U}''$, where $U''=\bigcup_{p\in N} S_{\varepsilon/2}(p)$. The metric g_1, obtained by smoothly unifying the metrics on U' and $M-\bar{U}''$ by a partition of unity, is a desired one.

For, in the same way as in (ii), it can be verified that g_1 is an extension of h such that (N, h) is totally geodesic in a strong sense. Since N is compact and each M-geodesic is an N-geodesic, for every pair (p, q) of points of N there exists at least one M-geodesic on N joining p and q whose length is the distance between p and q with respect to g_1.

Next we shall show that all such M-geodesics lie completely within N. Since (N, h) is totally geodesic in a strong sense, all M-geodesics in U'' from p to q are contained in N. On the other hand, let $\alpha\colon I=[a, b]\to M$ be a parametrized piecewise differentiable curve of M from $p=\alpha(a)$ to $q=\alpha(b)$, not contained in U''. Let $q_1=\alpha(c)$ be the first point at which $\alpha(I)$ meets $M-U''$ as the parameter t runs on I from a to b. Then the length l_1 of the subarc $\alpha(I_1)$, $I_1=[a, c]$, with respect to g_1 is larger than K. For

$$l_1=\int_a^c\left(h_{ij}\frac{du^i}{dt}\frac{du^j}{dt}+\sum_{k=n+1}^{m}\left(\frac{3K}{\varepsilon}\frac{du^k}{dt}\right)^2\right)^{1/2}dt$$

$$\geqq\frac{3K}{\varepsilon}\int_a^c\left(\sum_{k=n+1}^{m}\left(\frac{du^k}{dt}\right)^2\right)^{1/2}dt>K.$$

Hence the length of $\alpha(I)$ is larger than the diameter of N. Q. E. D.

4 Boundedness

LEMMA 1 *Let M be a separable connected differentiable manifold, and N a closed submanifold of M. Then there exists a denumerable collection of piecewise differentiable linear graphs L_ξ in M such that $N\cup(\cup L_\xi)$ is a connected closed submanifold with branches.*

PROOF. Let $\{U_i\}$ be a denumerable increasing sequence of connected open set such that $\bar{U}_i$ are compact, $\bar{U}_i\subset U_{i+1}$ and $\cup U_i=M$. Then each U_i meets at most a finite number of components of N, because $\bar{U}_i$ is compact and N is a closed submanifold.

Let K_{j-1} be the union of the components of N meeting U_{j-1} and assume that we have a connected closed submanifold $K_{j-1}\cup(\cup L_\xi)=K'_{j-1}$ with branches such that L_ξ's are finite in number and $L_\xi\subset U_{j-1}$. Let N_1 be a component of N meeting U_j and not contained in K_{j-1}. Then there exists a piecewise differentiable simple curve L such that $L\cap N_1$ and $L\cap K'_{j-1}$ are the endpoints of L, $K'_{j-1}\cup N_1\cup L$ is a connected closed submanifold with branches and L is contained in U_j-U_k, where k is the largest integer so that we can find such a curve. Therefore it must be noticed that the component of $M-\bar{U}_{k+1}$ meeting L contains no point of K'_{j-1}.

Inductively we can construct a connected closed submanifold K'_j with branches such that $K'_j \supseteq K'_{j-1}$, K'_j contains all components of N meeting U_j and the branches are contained in U_j. The set $\cup K'_j$ is a desired one.

For it is obvious that $\cup K'_j$ satisfies the last three of the conditions (a), (b), (c) and (d) of a closed submanifold with branches. In order to prove that $\cup K'_j$ satisfies (a), it is sufficient to show that each U_j meets at most a finite number of piecewise differentiable linear graphs L_ξ's. Now assume that some U_j meets infinitely many L_ξ's. On the other hand, since $U_{j+1}-\bar{U}_j$ is locally connected and $\bar{U}_{j+1}-U_j$ is compact, the components of $\bar{U}_{j+1}-U_j$ interjecting both $M-U_{j+1}$ and $\bar{U}_j$ are finite in number. Hence one of them, C, and also the component of $M-U_j$ containing C meets at least two, L', L'', of L_ξ's such that, for some integer l, $L' \subset K'_l$ and L'' contains a simple curve joining K'_l and a component N_1 of N meeting U_{l+1}, This contradicts our construction of $\{L_\xi\}$. Q. E. D.

THEOREM 3 *Let M, N and h be the same as in Theorem 2 (except that N is connected). If the sum a of diameters of components of N with respect to h is finite, then for arbitrarily given $\delta > 0$ there exists a C^s-extension g' of h to M such that the diameter of M under g' is less than $a+\delta$.*

PROOF. The proof is similar to [4]. Let $N' = N \cup (\cup L_\xi)$ be the connected closed submanifold with branches, obtained in Lemma 1. In each $L_\xi (\xi = 1, 2, \cdots)$, we define a metric of class C^s so that its diameter is equal to $2^{-\xi-1}\delta$. Thus we get a metric h' of class C^s defined on N' such that $h'|N=h$ and the diameter of N' under h' is less than $a + (\delta/2)$.

Using a tubular neighborhood W of N' (of class C^{r-2}) and U as in the proof of Theorem 2, we have a C^s-extension g of h' to M by Theorem 1. We may choice W so that the distance $\varphi_1(p)=d(p, N')$ $(p \in W)$ is compatible with the fibre structure of W, where d is the distance with respect to g (cf. Section 3). Then $d(p, N')$ $(p \in M-\bar{U})$ is a continuous function on $M-\bar{U}$. Therefore there exists a differentiable function φ_2 of class C^s defined on $M-\bar{U}$ such that

$$\varphi_2(p) > d(p, N') \qquad (p \in M-\bar{U}).$$

We smoothly unify φ_1 on W and φ_2 on $M-\bar{U}$ by a partition of unity and denote the C^s-function which results on M by φ. Here note that, for every point $p \in M$,

$$\varphi(p) \geqq d(p, N') \tag{1}$$

and $\varphi = \varphi_1$ on U.

We can define a C^s-extension g' of h' to M such that, for every point p of M, $d'(p, N') \leqq \delta/4$ where d' is the distance with respect to g'. For let K be a positive number such that $K > 4\pi/\delta^2$ and put $g' = e^{-2K\varphi^2}g$. Then it is a

Riemannian metric on M (conformal to g) and is a C^s-extension of h' to M, since $\varphi(p)=\varphi_1(p)=d(p, N')=0$ for $p \in N'$.

We shall show that, for every $\varepsilon>0$, every point p of M can be joined to N' by a curve of length $<(\delta/4)+\varepsilon$ with respect to g'. There exists a point $q \in N'$ and a piecewise differentiable curve α in M, from p to q, of length l with respect to g such that

$$d(p, N') \leqq d(p, q) \leqq l < d(p, N') + \varepsilon \tag{2}$$

Let $s \to \alpha(s)$ be a parametric representation of α, where s is the length of the subarc of α from $q=\alpha(0)$ to $\alpha(s)$ with respect to g. Then by (1) and (2)

$$s-\varepsilon < d(\alpha(s), N') \leqq \varphi(\alpha(s)).$$

Hence the length l' of α with respect to g' is estimated as follows:

$$l' = \int_0^l e^{-K\{\varphi(\alpha(s))\}^2} ds < \int_0^l e^{-K(s-\varepsilon)^2} ds$$

$$< \int_0^\infty e^{-Ks^2} ds + \int_0^\varepsilon e^{-Ks^2} ds = \frac{\sqrt{\pi}}{2\sqrt{K}} + \int_0^\varepsilon e^{-Ks^2} ds < \frac{\delta}{4} + \int_0^\varepsilon e^{-Ks^2} ds.$$

Since ε is arbitrary, $d'(p, N')$ is not larger than $\delta/4$. Q.E.D.

4 Completeness

LEMMA 2 ([4]) *Let (M, g) be a separable, connected Riemannian manifold. Then there exists a complete Riemannian metric g_0 which is conformal to g and $g_0 \geqq g$.*

PROOF. Let $\{U_i\}$ be a denumerable increasing sequence of connected open sets such that $\bar{U}_i$ are compact, $\bar{U}_i \subset U_{i+1}$ and $\cup U_i = M$. For every $i \geqq 1$, we define a Riemannian metric on the set $U_{2i+1} - \bar{U}_{2i-2}$ $(U_0 = \emptyset)$ by $g_i = g/K_i^2$, where $K_i = \text{Min}\,\{1$, the distance between $\bar{U}_{2i-1}$ and $M - U_{2i}$, with respect to $g\}$. Then we smoothly unify the metrics g_i on $U_{2i+1} - \bar{U}_{2i-2}$ $(i=1, 2, \dots)$ by a partition of unity. Then the Riemannian metric g_0 which results is conformal to g and $g_0 \geqq g$. We note here that $g_0 = g_i$ on $U_{2i} - \bar{U}_{2i-1}$ and $d_0(U_{2i}, U_{2i-1}) \geqq 1$, where d_0 is the distance with respect to g_0.

We shall show that g_0 is complete. Let $Q = \{p_i\}$ be a Cauchy sequence with respect to g_0. Then Q is contained in some U_k. For if not, for every point $p_i \in Q$, there exists an integer l and a point $p_j \in Q$, such that $p_i \in U_{2l-1}$ and $p_j \notin U_{2l}$. Consequently $d_0(p_i, p_j) \geqq d_0(U_{2l}, U_{2l-1}) \geqq 1$. This contradicts the fact that Q is a Cauchy sequence. Thus Q converges to a point of $\bar{U}_k$, because $\bar{U}_k$ is compact. Q.E.D.

THEOREM 4 *Let M, N and h be the same as in Theorem 2. If M is connected and h is complete, then there exists a C^s-extension g of h to M such that (M, g) is complete and N is totally geodesic in a strong sense.*

PROOF. Let g_1 be an extension of h_1 (Theorem 2) and define a complete Riemannian metric g_1' on $M-N$ such that $g_1' \geqq g_1$ (Lemma 2).

Let $\{U_i\}$ be an increasing sequence of open sets such that $\bar{U}_i$ are compact, $\bar{U}_i \subset U_{i+1}$ and $\cup U_i = M$. We now define an open set W containing N as follows: let p be a point in $N \cap (\bar{U}_i - U_{i-1})$ $(U_0 = \emptyset)$ and $W(p)$ a spherical neighborhood of p with radius $<1/i$ (with respect to g_1) whose closure is compact and is contained in a coordinate neighborhood of M on which a canonical coordinate system with respect to N is defined. Since $N \cap (\bar{U}_i - U_{i-1})$ is compact, there exists a finite number of points, $p_1, \cdots, p_k$, in it such that $\bigcup_{j=1}^{k} W(p_j) \supset N \cap (\bar{U}_i - U_{i-1})$. Then the closure of $\bigcup_{j=1}^{k} W(p_j) = W_i$ is compact. Denote $\cup W_i$ by W. $\bar{W}$ is a complete metric space with respect to the natural distance d_1 derived from g_1, because (N, h_1) is complete.

By a partition of unity, we smoothly unify g_1 on W and g_1' on $M-N$. Then we have a Riemannian metric g such that $g \geqq g_1$ on W and $g = g_1'$ on $M - \bar{W}$.

We shall show that g is complete. Let Q be a Cauchy sequence with respect to g. If infinitely many points p_i's of Q are contained in W, $Q_1 = \{p_i\}$ is also a Cauchy sequence with respect to g_1, because $g \geqq g_1$ on W. Since $\bar{W}$ is complete with respect to d_1, Q_1 (and also Q) converges to a point.

Suppose that infinitely many points p_i of Q are contained in $M - \bar{W}$. Let $\alpha(i, j)$ be a piecewise differentiable curve from p_i to p_j, whose length $L(\alpha(i, j))$ with respect to g is less than $d(p_i, p_j) + (1/2^{i+j})$. If $\alpha(i, j)$ is contained in $M - \bar{W}$, $L(\alpha(i, j))$ is equal to the length of $\alpha(i, j)$ with respect to g_1'. Therefore, if infinitely many curves $\alpha(i, j)$ are contained in $M - \bar{W}$, a subsequence of Q is a Cauchy sequence with respect to g_1' and converges to a point, because $(M-N, g_1')$ is complete. If infinitely many curves $\alpha(i, j)$ meet W, for each k we choose a point q_k of $\alpha(i, j) \cap W$. Then $\{q_k\}$ is a Cauchy sequence in W with respect to g, for if $q_k \in \alpha(i, j)$ and $q_{k'} \in \alpha(i', j')$ then

$$\begin{aligned} d(q_k, q_{k'}) &\leqq L(\alpha(i, j)) + d(p_j, p_{i'}) + L(\alpha(i', j')) \\ &\leqq d(p_i, p_j) + d(p_j, p_{i'}) + d(p_{i'}, p_{j'}) + (1/2^{i+j}) + (1/2^{i'+j'}). \end{aligned}$$

Hence the Cauchy sequence $\{q_k\}$ in W, with respect to g, converges to a point. Q.E.D.

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