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On Decompositions of Riemannian Manifolds.

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Introduction. In a homogeneous space G/H, each of its points is a coset of a closed subgroup H of a Lie group G, that is, G/H is based on the decomposition of G into the leaves (maximal integral manifolds) of the Lie algebra of H. Generalizing, from this point of view, the notion of a homogeneous space, we have that of a foliation $M/\mathfrak{M}$ in the sense of R. Palais [4], which consists of leaves of an involutive distribution $\mathfrak{M}$ on a differentiable manifold M, together with the topology induced from M (see p. 82). Foliations in more general spaces have been treated by C. Ehresmann [2], G. Reeb [5] and A. Haefliger [3].

In the present paper we shall investigate the decomposition of a Riemannian manifold M into the leaves of an involutive distribution $\mathfrak{M}$ which has the involutive orthogonal complement $\mathfrak{M}^*$. At first it will be tried to represent the foliation $M/\mathfrak{M}$ with a leaf V^* of $\mathfrak{M}^*$. This requires that the leaf should meet all the leaves of $\mathfrak{M}$. We shall find a sufficient condition of that in terms of certain quantities related to a family of geodesic curvatures (Theorem 2). Under this condition, for any simply connected leaf V^* of $\mathfrak{M}^*$, we have the relation:

$$M/\mathfrak{M} \cong V^*/G(V^*)$$

(Theorem 3), where $G(V^*)$ denotes the group of diffeomorphisms of V^* which make the intersection of V^* and each leaf of $\mathfrak{M}$ invariant. Let H_p be the subgroup of $G(V^*)$ consisting of elements which make a point p invariant. Then, if H_p and H_q are conjugate subgroups, it will be shown that the leaves of $\mathfrak{M}$ through p and q are diffeomorphic. And, if one of the leaves of $\mathfrak{M}$ is simply connected, if the leaves of $\mathfrak{M}$ through p and q are homeomorphic, then H_q and H_p are isomorphic (Theorem 4). Finally we shall show that, when $G(V^*)$ is abelian, its elements can be extended to diffeomorphisms of M which make the decomposition of M invariant (Theorem 5).

1. Let M be an n-dimensional differentiable[1)] manifold with countable base. Let us be given an m-dimensional involutive distribution $\mathfrak{M}$ on M. Since, as is well known, M can be given a Riemannian structure, we have, at any point

1) By "differentiable" we always mean "of class C^∞".

p of M, the orthogonal complement $\mathfrak{M}_p^*$ of $\mathfrak{M}_p$ with respect to the metric. Assigning $\mathfrak{M}_p^*$ to p, we have a differentiable distribution $\mathfrak{M}^*$ on M. We shall assume that $\mathfrak{M}^*$ is involutive. It is the case when $\mathfrak{M}$ is of dimension $n-1$.

In the first place, we shall show the existence of a certain class of coordinate systems on M which will be called *flat* coordinate systems.

PROPOSITION 1. *At any point p of M, there exist a coordinate system $(x^1, \cdots, x^n)$ and a cubic neighborhood W with respect to the system which satisfy the following conditions*: (i) $x^i(p)=0$ $(1\leq i\leq n)$; (ii) *if $\xi^i (1\leq i\leq n)$ are numbers smaller in absolute value than the breadth of W, then the slice of W defined by the equations $x^\lambda=\xi^\lambda$ $(m+1\leq\lambda\leq n)$ is an integral manifold of $\mathfrak{M}$, while the slice defined by $x^a=\xi^a$ $(1\leq a\leq m)$ is an integral manifold of $\mathfrak{M}^*$.*

PROOF. There exists a coordinate system $(y^1,\cdots, y^n)$ with the following properties: $y^i(p)=0$; the system of m vector fields $X_a=\frac{\partial}{\partial y^a}$ $(1\leq a\leq m)$ forms a local base for $\mathfrak{M}$ around p. Let $\left\{Y_\lambda=\sum_{i=1}^n Y_\lambda^i(y)\frac{\partial}{\partial y^i}: m+1\leq\lambda\leq n\right\}$ be a local base for $\mathfrak{M}^*$ around p. Since $\mathfrak{M}^*$ is involutive, the system of differentiable equations

$$Y_\lambda x=0 \quad (m+1\leq\lambda\leq n), \tag{1}$$

has m independent solutions $x^1,\cdots, x^m$ with $x^a(0)=0$. Since $\mathfrak{M}_p$ and $\mathfrak{M}_p^*$ are orthogonal, the vectors $(X_a)_p$, $(Y_\lambda)_p$ are linearly independent and the determinant $|Y_\mu^\lambda(0)|$ does not vanish. This and the equations (1) infer that the functional determinant $\partial(x^1,\cdots, x^m)/\partial(y^1,\cdots, y^m)$ does not vanish at $y=0$. Putting $x^\lambda=y^\lambda$ $(m+1\leq\lambda\leq n)$, we have a coordinate system $(x^1,\cdots, x^n)$ at p with $x^i(p)=0$. In the coordinates, $\mathfrak{M}$ has the system of vector fields $\left\{\frac{\partial}{\partial x^a}: 1\leq a\leq m\right\}$ as its local base around p, while $\mathfrak{M}^*$ has $\left\{\frac{\partial}{\partial x^\lambda}: m+1\leq\lambda\leq n\right\}$ as its local base. Thus the coordinates $x^1,\cdots, x^n$ are proved to have the required properties.

We shall consider mappings Φ of the product space $J_1\times J_2$ of two intervals J_1, J_2 in the real line into M which have the following *property* (D): whenever pairs (u_i, v_j) $(i, j=1, 2)$ are in $J_1\times J_2$ the points $\Phi(u_1, v_j)$ are contained in the same leaf of $\mathfrak{M}$, while $\Phi(u_i, v_1)$ are in the same leaf of $\mathfrak{M}^*$.

LEMMA 1. *Let Φ be a continuous mapping of $J_1\times J_2$ into M with the property (D). Let W be a cubic neighborhood of M with respect to a flat coordinate system. If the points $\Phi(u, v_0)$ $(u\in[u_0, u_1)\subset J_1)$ and $\Phi(u_0, v)$ $(v\in[v_0, v_1)\subset J_2)$ are contained in W, so is the point $\Phi(u, v)$ for every pair (u, v)*

in $[u_0, u_1) \times [v_0, v_1)$.

PROOF. We denote the coordinates by $x^1, \cdots, x^m$. Let v_2 be any number in (v_0, v_1) and u_2 the least upper bound of u such that $\Phi(u', v)$ is in W for any pair $(u', v) \in [u_0, u) \times [v_0, v_2]$. Since Φ is continuous, u_2 is different from u_0. Since M has a countable base, there exist at most countably many slices of W which are contained in a leaf of $\mathfrak{M}$ [1]. Hence $\Phi(u, v)$, for any $v \in [v_0, v_2]$ and a fixed $u \in [u_0, u_2)$, are contained in the same slice of W, that is, $x^\lambda(\Phi(u, v)) = x^\lambda(\Phi(u, v_0))$ $(m+1 \leq \lambda \leq n)$. In the same way we have $x^a(\Phi(u, v)) = x^a(\Phi(u_0, v))$ $(1 \leq a \leq m)$, for $(u, v) \in [u_0, u_2) \times [v_0, v_2]$. From this it follows directly that u_2 is equal to u_1, and our lemma is proved.

The next lemma follows immediately from Lemma 1.

LEMMA 2. *If Φ_1 and Φ_2 are two continuous mappings of $J_1 \times J_2$ into M which have the property (D) and if they coincide on subsets $\{u_0\} \times J_2$ and $J_1 \times \{v_0\}$, then they coincide on the whole $J_1 \times J_2$.*

For convenience, we denote by V_p (resp. V_p^*) the leaf of $\mathfrak{M}$ (resp. $\mathfrak{M}^*$) through a point p of M, and by I the closed interval $[0, 1]$.

LEMMA 3. *Let p be a point of M. Let φ be a way in V_p^* with starting point p (i.e., φ is continuous mapping of I into V_p^* with $\varphi(0)=p$), and ψ a way in V_p with starting point p. Then there exist a positive number u_0 and a continuous mapping Φ of $[0, u_0) \times I$ into M with the property (D) such that $\Phi(u, 0) = \varphi(u)$ and $\Phi(0, v) = \psi(v)$. If ψ_1 is a way from p to $\psi(1)$ and homotopic in V_p to ψ, we have*

$$\Phi_1(u, 1) = \Phi(u, 1) \qquad (0 \leq u < min.\ (u_0, u_1))$$

where Φ_1 is a mapping of $[0, u_1) \times I$ which has the similar property to those of Φ.

PROOF. There exists a family of flat coordinate systems $(x_k^1, \cdots, x_k^n)$ $(1 \leq k \leq N)$ and increasing sequence of numbers $0 = v_0, v_1, \cdots, v_{N-1}, v_N = 1$ such that cubic neighborhoods W_k with respect to these systems cover the way ψ and that $\psi(v)$, for $v \in [v_{k-1}, v_k]$, is contained in W_k.

Since φ is continuous, there exists a positive number u_1 such that $\varphi(u)$, for $u \in [0, u_1)$, is contained in W_1. For any pair (u, v) in $[0, u_1) \times [0, v_1]$ we obtain the point $\Phi(u, v)$ uniquely determined by the equations

$$x_1^a(\Phi(u, v)) = x_1^a(\psi(v)) \qquad (1 \leq a \leq m),$$

$$x_1^\lambda(\Phi(u, v)) = x_1^\lambda(\varphi(u)) \qquad (m+1 \leq \lambda \leq n).$$

Since $\psi(v_1)$ is contained in W_2, there exists a number u_2 in $(0, u_1]$ such that $\Phi(u, v_1)$, for $u \in [0, u_2)$, is in W_2. Hence a point $\Phi(u, v)$, for any pair $(u, v) \in [0, u_2)\times[v_1, v_2]$, is uniquely determined by equations

$$x_2^a(\Phi(u, v)) = x_2^a(\psi(v)) \qquad (1 \leq a \leq m),$$

$$x_2^\lambda(\Phi(u, v)) = x_2^\lambda(\Phi(u, v_1)) \qquad (m+1 \leq \lambda \leq n).$$

After finite steps like this, we obtain the positive number $u_0 = u_N$ and the points $\Phi_{(u,v)}$, for all (u, v) in $[0, u_0)\times I$, which belong to the intersection $V_{\varphi(u)} \cap V^*_{\psi(v)}$. The mapping Φ of $[0, u_0)\times I$ into M defined by $(u, v)\rightarrow\Phi(u, v)$ is clearly continuous and has the required properties. We know from Lemma 2 that Φ is the unique mapping of $[0, u_0)\times I$ into M which has these properties.

Let ψ' be a way from p to $\psi(1)$ lying on V_p such that, for each k, any point $\psi'(v)$ $(v \in [v_{k-1}, v_k])$ belongs to W_k. Let Φ' be the mapping constructed from φ and ψ', in the same way as Φ is done from φ and ψ. Then we have $\Phi'(u, 1) = \Phi(u, 1)$ for any $u \in [0, u_0)$. From this and our assumption that the way ψ_1 is homotopic in V_p to ψ, we have $\Phi_1(u, 1) = \Phi(u, 1)$.

REMARK. Φ is differentiable if φ and ψ are differentiable.

LEMMA 4. *Let Φ be a continuous mapping of $[0, 1)\times I$ into M with the property (D). If $\Phi(u, v)$ converges, for each v in I, when u converges to* 1, *then Φ can be extended uniquely to a continuous mapping $I\times I$ into M with the property (D).*

PROOF. Let q_v denot the limiting point. Let v_0 be any number in I, $(x^1, \cdots, x^n)$ be a flat coordinate system at q_{v_0} and W be a cubic neighborhood with respect to this system. Then there exsists a number u_0 in I and a sufficiently small positive number ε such that the point $\Phi(u, v)$, for any pair (u, v) in $[u_0, 1)\times(v_0-\varepsilon, v_0+\varepsilon)$, is contained in W (Lemma 1). Moreover, we have $x^a(\Phi(u, v)) = x^a(\Phi(u_0, v))$ $(1\leq a\leq m)$ and $x^\lambda(\Phi(u, v)) = x^\lambda(\Phi(u, v_0))$ $(m+1\leq\lambda\leq n)$. Hence the limiting point q_v lies on W for v in $(v_0-\varepsilon, v_0+\varepsilon)$, satisfying the equations $x^a(q_v) = x^a(\Phi(u_0, v))$ and $x^\lambda(q_v) = x^\lambda(q_{v_0})$. This shows that the extension obtained by assigning q_v to $(1, v)$ has the required properties. The uniqueness follows from Lemma 2.

Let $(x^1, \cdots, x^n)$ be a flat coordinate system at a point p of M and W be a cubic neighborhood with respect to the system. Let $X = \sum_{a=1}^{m} X^a(x^1, \cdots, x^m)\frac{\partial}{\partial x^a}$ and $Y = \sum_{\lambda=m+1}^{n} Y^\lambda(x^{m+1}, \cdots, x^n)\frac{\partial}{\partial x^\lambda}$ be differentiable vector fields in W which belong to $\mathfrak{M}$ and $\mathfrak{M}^*$, respectively, and are nowhere zero. Let φ denote the integral curve of Y with $\varphi(0) = p$, ψ that of X with $\psi(0) = p$ and $\eta(u, v)$ that of Y with $\eta(0, v) = \psi(v)$. Then η has the property (D). The surface defined by

$x^i = x^i(\eta(u, v))$ is a two dimensional differentiable surface on which the curves $u=$const. and $v=$const. form an orthogonal net. From this we have

$$\kappa_u = -\|X\|^{-1}\frac{\partial}{\partial v}\log\|Y\|,$$

$$\kappa_v = -\|Y\|^{-1}\frac{\partial}{\partial u}\log\|X\|,$$

where κ_u and κ_v denote the geodesic curvatures, on this surface, of the curves $v=$const. and $u=$const., respectively, and where $\|\ \|$ denotes the length of a vector.

The geodesic curvatures κ_u, κ_v and their derivatives $\|X\|^{-1}\frac{\partial}{\partial v}\kappa_u$, $\|Y\|^{-1}\frac{\partial}{\partial u}\kappa_v$ are independent of the choices of the coordinate system and of the parametrizations of φ and ψ. We put

$$K_{\mathfrak{M},1} = \underset{p\in M,X,Y}{g.\ l.\ b}\,[\kappa_v]_p,$$

$$K_{\mathfrak{M},2} = \underset{p\in M,X,Y}{g.\ l.\ b.}\left[\|Y\|^{-1}\frac{\partial}{\partial u}\kappa_v\right]_p.$$

Then they may be infinite. We define $K_{\mathfrak{M}^*,1}$ and $K_{\mathfrak{M}^*,2}$ in the same way.

Definition. We call $K_{\mathfrak{M},1}$ and $K_{\mathfrak{M},2}$ (resp. $K_{\mathfrak{M}^*,1}$ and $K_{\mathfrak{M}^*,2}$) the first and the second curvature of $\mathfrak{M}$ (resp. $\mathfrak{M}^*$), respectively.

Lemma 5. *Suppose M is complete and either the first curvature of $\mathfrak{M}^*$ is non negative or the second curvature of $\mathfrak{M}^*$ is positive. Let Φ be a differentiable mapping of $[0, 1)\times I$ into M with the property (D). If the mapping Φ_0: $u\to\Phi(u, 0)$ $(u\in[0, 1))$ can be extended to a differentiable mapping of I into $V^*_{\Phi(0,0)}$, then Φ itself can be extented to a differentiable mapping of $I\times I$ into M which has the property (D).*

Proof. We define vector fields X, Y on the image set of Φ by $X_{(u,v)} = \frac{\partial}{\partial v}\Phi(u, v)$ and $Y_{(u,v)} = \frac{\partial}{\partial u}\Phi(u, v)$ respectively. Our assumption says that either κ_u vanishes at any point $\Phi(u, v)$ or $\|X\|^{-1}\frac{\partial}{\partial v}\kappa_u$ is not smaller than $K_{\mathfrak{M}^*,2}(>0)$. In the first case, we have for any u in $[0, 1)$ and any v in I

$$\int_0^u \|Y_{(\sigma,v)}\|\,d\sigma = \int_0^u \|Y_{(\sigma,0)}\|\,d\sigma. \tag{2}$$

In the latter case, we have

$$\log(\|Y_{(u,v)}\|/\|Y_{(u,0)}\|) \leq \int_0^v \|X_{(u,\sigma)}\| \left\{L - K_{\mathfrak{M}^*,2}\int_0^\sigma \|X_{(u,\tau)}\|\, d\tau\right\} d\sigma,$$

where $L = -\underset{0 \leq u < 1}{g.\,l.\,b.}(\kappa_u(u, 0)) < \infty$. We shall show the right side of the inequality is less than $L^2/K_{\mathfrak{M}^*,2}$. We have to consider the following three possible cases:

(i) when $L \leq 0$, it is negative and hence less than $L^2/K_{\mathfrak{M}^*,2}$,

(ii) when L is positive and $\int_0^v \|X_{(u,\sigma)}\| d\sigma$ is less than $L/K_{\mathfrak{M}^*,2}$ then it is less than $L^2/K_{\mathfrak{M}^*,2}$,

(iii) when L is positive and there exists a number v_1 in $[0, v)$ such that $\int_0^{v_1} \|X_{(u,\sigma)}\|\, d\sigma = L/K_{\mathfrak{M}^*,2}$, we have

$$\int_0^v \|X_{(u,\sigma)}\| \left\{L - K_{\mathfrak{M}^*,2}\int_0^\sigma \|X_{(u,\tau)}\|\, d\tau\right\} d\sigma$$

$$< L\int_0^{v_1} \|X_{(u,\sigma)}\|\, d\sigma + \int_{v_1}^v \|X_{(u,\sigma)}\| \left\{L - K_{\mathfrak{M}^*,2}\int_0^\sigma \|X_{(u,\tau)}\|\, d\tau\right\} d\sigma$$

$$< L\int_0^{v_1} \|X_{(u,\sigma)}\|\, d\sigma = L^2/K_{\mathfrak{M}^*,2}.$$

Thus we have always

$$\log(\|Y_{(u,v)}\|/\|Y_{(u,\sigma)}\|) < L^2/K_{\mathfrak{M}^*,2}.$$

By integration we obtain

$$\int_0^u \|Y_{(\sigma,v)}\|\, d\sigma < e^{L^2/K_{\mathfrak{M}^*,2}} \int_0^u \|Y_{(\sigma,0)}\|\, d\sigma. \tag{3}$$

Because Φ_0 can be extended to a differentiable curve of I, it follows from (2) and (3) that in either case the curve: $\sigma(\in[0, u]) \to \Phi(\sigma, v)$ has the length bounded, for any u and v. Since M is complete, $\Phi(u, v)$ converges when v is fiexed and u converges to 1. Defining $\Phi(1, v)$ to be the limiting point, we have an extension which has the property (D) (Lemma 4). It is easily seen as in the proof of Lemma 4 that this extension is differentiable. Thus our assertion is proved.

As an immediate consequence of Lemmas 3 and 5 and of dual of them, we have the following theorem.

THEOREM 1. *Suppose that M is complete, and that either one of the first curvatures of $\mathfrak{M}$ and $\mathfrak{M}^*$ is non negative or one of the second curvatures of $\mathfrak{M}$*

and $\mathfrak{M}^*$ *is positive. Let* φ *and* ψ *be differentiable ways which start from a point* p *of* M *and lie on* V_p^* *and* V_p *respectively. Then there exists unique differentiable mapping* $\Phi_{\varphi,\psi}$ *of* $I\times I$ *into* M *with property* (D) *such that* $\Phi_{\varphi,\psi}(u, 0)=\varphi(u)$ $\Phi_{\varphi,\psi}(0, v)=\psi(v)$, *for any* u, v *in* I.

As a corollary of Theorem 1 we have

Corollary. *Let* V *and* V^* *denote any leaves of* $\mathfrak{M}$ *and* $\mathfrak{M}^*$, *respectively, which have a common point. Under the same assumptions as of Theorem* 1, *every leaf of* $\mathfrak{M}$ *through a point of* V^* *meets any leaf of* $\mathfrak{M}^*$ *which meets* V.

Let p be any point of M, and W_p denote the union $\cup\{V_q\colon q\in V_p^*\}$. Then it is obvious that W_p contains an open neighborhood U_p of V_p^* such that $W_p=\cup\{V_q\colon q\in U_p\}$. This shows that W_p is open set of M [5]. Corollary to Theorem 1 says that, for any pair p and q of points of M, the open sets W_p and W_q either coincide or have no common point. Since M is connected we have

Theorem 2. *Under the same assumptions as of Theorem* 1, *every leaf of* $\mathfrak{M}$ *meets any leaves of* $\mathfrak{M}^*$. *In other words,* M *is represented as the union of leaves of* $\mathfrak{M}$ *(resp.* $\mathfrak{M}^*$*) which meet a fixed leaf of* $\mathfrak{M}^*$ *(resp.* $\mathfrak{M}$*).*

2. In what follows we assume that the manifold M with distributions $\mathfrak{M}$ and $\mathfrak{M}^*$ has the property stated in the conclusion of Theorem 1.

Let φ be a way starting from a point p of M and lying on V_p^*, and ψ be a way in V_p with starting point p. There exists, for any u (resp. v) in I, a differentiable way φ_u in V_p^* (resp. ψ_v in V_p) which is homotopic to the part of φ (resp. ψ) from p to $\varphi(u)$ (resp. $\psi(v)$). In virtue of our assumption, there exists a mapping Φ_{φ_u,ψ_v} such that Φ_{φ_u,ψ_v} (1, 1) is contained in both $V_{\varphi(u)}$ and $V^*_{\psi(v)}$. This point is independent of ways φ_u and ψ_v (Lemma 3). Assigning the point to the pair (u, v), we have a mapping $\Phi_{\varphi,\psi}$ of $I\times I$ into M which is continuous and has the property (D). The point $\Phi_{\varphi,\psi}$ (1, 1) depends only on the homotopy classes $[\varphi]$ and $[\psi]$ which contain φ and ψ respectively. Especially when V_p^* is simply connected, $\Phi_{\varphi,\psi}$ (1, 1) is the point determined by the point $q=\varphi(1)$ and the homotopy class $[\psi]$ which contains ψ. In this case we have a mapping $\Phi_{[\psi]}$ of V_p^* into $V^*_{\psi(1)}$, defined by $\Phi_{[\psi]}(q)=\Phi_{\varphi,\psi}$ (1, 1).

Proposition 2. *Suppose* V_p^* *is simply connected. Then* $\Phi_{[\psi]}$ *is a differentiable covering mapping of* V_p^* *onto* $V^*_{\psi(1)}$. *Hence if* $V^*_{\psi(1)}$ *is also simply connected, it is a diffeomorphism.*

Proof. Let q' be the image of a point q under $\Phi_{[\psi]}$. Let φ be a way in V_p^* from p to q. We take a finite set of flat coordinate systems such that cubic neighborhoods with respect to them cover the way: $v \to \Phi_{\varphi,\psi}(1, v)$ $(v \in I)$. By the coordinate systems we know that $\Phi_{[\psi]}$ is a differentiable local homeomorphism. Let q_1' be any point of $V^*_{\psi(1)}$, φ_1 be a way in $V^*_{\psi(1)}$ from $\psi(1)$ to q_1'. Putting $\varphi_0(u) = \Phi_{\varphi_1,\psi^{-1}}(u, 1)$ and $\varphi_0(1) = q_1$ we have $\Phi_{\varphi_0,\psi}(1, 1) = q_1'$, that is, $\Phi_{[\psi]}(q_1) = q_1'$, which proves that $\Phi_{[\psi]}$ is a surjection. Let U be any simply connected open set in $V^*_{\psi(1)}$, and U_0 be a connected component of the inverse image $\Phi^{-1}_{[\psi]}(U)$. It can be easily seen that the restriction of $\Phi_{[\psi]}$ to U_0 is a univalent mapping onto U and hence homeomorphism. Thus $\Phi_{[\psi]}$ is a covering mapping.

We assume that a leaf V^* of $\mathfrak{M}^*$ is simply connected. Let p be a point of V^* and ψ be a way in V_p from p to a common point of V^* and V_p. In virtue of Proposition 2, we have the diffeomorphism $\Phi_{[\psi]}$ of V^* onto itself which sends any point q into the intersection $V^* \cap V_q$. The set of these diffeomorphisms forms a group. The group is the same for all points p in V^*, and is denoted by $G(V^*)$.

Naturally an equivalence relation in V^* is introduced by the group $G(V^*)$. We introduce the strongest topology in the factor set $V^*/G(V^*)$ that makes the natural mapping Π of V^* into $V^*/G(V^*)$ continuous. Then its open sets are subsets whose inverse images under Π are open in V^*. If U^* is an open set of V^*, then $g(U^*)$ is open in V^* for any member g of $G(V^*)$. Hence $\Pi^{-1}(\Pi(U^*))$ is open, that is, Π is an open mapping.

We introduce the quotient topology [4] in the set $M/\mathfrak{M}$ of leaves of $\mathfrak{M}$ by the family of *open* sets

$$\{U \colon \Pi_0^{-1}(U) \quad \text{is open in } M\}$$

where Π_0 means the natural mapping of M onto $M/\mathfrak{M}$. The mapping Π_0 is an open and continuous mapping [4].

Let V be any leaf of $\mathfrak{M}$, and q be any common point of V and V^* (Theorem 2). The element $\Pi(q)$ of $V^*/G(V^*)$ is independent of the choice of the common point. Assigning $\Pi(q)$ to V, we have a mapping α of $M/\mathfrak{M}$ onto $V^*/G(V^*)$.

THEOREM 3. *Suppose that a differentiable manifold M and two distributions $\mathfrak{M}$ and $\mathfrak{M}^*$ have the property stated in the conclusion of Theorem* 1. *If a leaf V^* of $\mathfrak{M}^*$ is simply connected, the space $M/\mathfrak{M}$ of the leaves of $\mathfrak{M}$ is homeomorphic to the quotient space $V^*/G(V^*)$.*

PROOF. We shall show the mapping α is a homeomorphism. It is obvious that α is a univalent mapping of $M/\mathfrak{M}$ onto $V^*/G(V^*)$.

Let U^* be any open set of V^*_p. Then the union W of leaves of $\mathfrak{M}$ which meet U^* is an open set of M [5]. We have

$$\Pi_0(W) = \alpha^{-1}(\Pi(U^*)).$$

This shows α is continuous, because both Π_0 and Π are open mappings. On the other hand, if W is an open set of M, we have

$$\alpha(\Pi_0(W)) = \Pi(V^* \cap W'),$$

where W' denotes the union of leaves of $\mathfrak{M}$ which meet W. The set W' is open in M and hence $V^* \cap W'$ is open in V^*. It follows from the above equation that α is an open mapping. Our assertion is thereby proved.

We shall consider the subgroup H_q of $G(V^*)$ each of whose elements leaves a point q of V^* fixed. The subgroup consists of such elements $\Phi_{[\psi]}$'s that ψ's are closed way with end points q, and is a homomorphic image of the fundamental group $\pi_1(V_q, q)$ of the leaf V_q. If q_1 is another point of V^* which is also in V_q, H_q and H_{q_1} are conjugate subgroups of $G(V^*)$.

Let L_q denote the kernel of the homomorphism: $\pi_1(V_q, q) \to H_q$. Then L_q, for any points q of V^*, are isomorphic to one another. In fact, if q' is any other point of V^* and φ is a way from q to q', then we have for any class $[\psi]$ in L_q

$$\Phi_{[\psi_1]} = \Phi_{[\psi]}$$

where ψ_1 denote the way defined by $\psi_1(V) = \Phi_{\varphi,\psi}(1, v)$. The class $[\psi]$ being in L_q, $\Phi_{[\psi_1]}$ is the identity mapping of V^*, and hence the class $[\psi_1]$ is an element of L_{q_1}. We assign $[\psi_1]$ to $[\psi]$ to have an isomorphism of L_q onto L_{q_1}.

If follows immediately from above property of the L_q's that, if there exists at least a simply connected one among the leaves of $\mathfrak{M}$, L_q for any point q of V^* is the trivial subgroup of $\pi_1(V_q, q)$ and H_q is isomorphic to $\pi_1(V_q, q)$. This proves the first half of the following theorem.

THEOREM 4. *Let assumptions be as in Theorem 3. If there exists at least one leaf of $\mathfrak{M}$ which is simply connected and if two leaves V_q and $V_{q'}$ $(q, q' \in V^*)$ of $\mathfrak{M}$ are homeomorphic, then the groups H_p and $H_{q'}$ are isomorphic to each other. On the other hand, if H_q and $H_{q'}$ are conjugate subgroups of $G(V^*)$, the leaves V_q and $V_{q'}$ of $\mathfrak{M}$ are diffeomorphic to each other.*

PROOF. We have to prove the latter half. Our assumption says that there exists an element g of $G(V^*)$ such that $H_q = g^{-1}H_{q'}g$. Since the point $q'' = g^{-1}(q')$ is contained in $V_{q'}$ and $H_{q''}$ coincides with $g^{-1}H_{q'}g$, it is sufficient to prove this when H_q and $H_{q'}$ coincide.

Let φ be a way in V^* from q to q', q_1 be any point of V_q and ψ_1 be a way in V_q from q to q_1. Then the point $\Phi_{\varphi,\psi_1}(1, 1)$ is independent of the chice of the way φ. We shall show that it is independent also of the way ψ_1. Let ψ_2 be another way. We have, from ψ_1 and ψ_2^{-1}, a closed way ψ in V_q. It follows from our assumption $\Phi_{[\psi]}$ leaves the point q' invariant. This shows

$\Phi_{\varphi,\psi_1}(1, 1)=\Phi_{\varphi_1,\psi_2}(1, 1)$ and proves our assertion. It is easy to show that the mapping: $q\to\Phi_{\varphi,\psi}(1, 1)$ is a diffeomorphism of V_q onto $V_{q'}$.

One of the next examples shows that the conclusion of Theorem 4 is not true when there exists no simply connected leaf of $\mathfrak{M}$. The other shows that there occurs the case where subgroups H_q and $H_{q'}$ are isomorphic but not conjugate even when leaves V_q and $V_{q'}$ are homeomorphic.

EXAMPLE. In the two dimensional Euclidean space R^2 we introduce the equivalence relation ρ: $\rho(x, y)=\rho(x', y')$ if and only if $x\equiv x'$ (mod 1) and $y'=(-1)^{x-x'}y$. Let M denote the quotient space R^2/ρ. Let U_1 denote the domain $\{\rho(x_1, y_1)\colon 0<x_1<1\}$ of M and h_1 the homeomorphism: $\rho(x_1, y_1)\to(x_1, y_1)$. Let U_2 denote the domain $\{\rho(x_2, y_2)\colon \frac{1}{2}<x_2<\frac{3}{2}\}$ and h_2 the homeomorphism: $\rho(x_2, y_2)\to(x_2, y_2)$. Then (U_i, h_i) $(i=1, 2)$ are two charts in M related analytically. Thus M is an analytic manifold. The manifold M has the Riemannian metric $ds^2=(dx_1)^2+(dy_1)^2$ in (x_1, y_1) and $ds^2=(dx_2)^2+(dy_2)^2$ in (x_2, y_2).

Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be one dimensional distributions with the base vector fields X_1 and X_2, respectively, which are defined as follows:

$$X_1=\begin{cases}\dfrac{\partial}{\partial x_1}\\[2mm] \dfrac{\partial}{\partial x_2}\end{cases}, \qquad X_2=\begin{cases}\dfrac{\partial}{\partial x_1}+\xi_1\dfrac{\partial}{\partial y_1}\\[2mm] \dfrac{\partial}{\partial x_2}+\xi_2\dfrac{\partial}{\partial y_2}\end{cases}$$

where the functions ξ_i $(i=1, 2)$ are given by

$$\xi_i=\begin{cases}0 & y_i=0,\quad |y_i|\geqq 1\\[2mm] exp\left(-\dfrac{1}{y_i+1}+\dfrac{1}{y_i}\right) & -1<y_i<0\\[2mm] -exp\left(-\dfrac{1}{y_i}+\dfrac{1}{y_i-1}\right) & 0<y_i<1.\end{cases}$$

The leaves of the distribution $\mathfrak{M}_1^*$ $(\perp\mathfrak{M}_1)$ are the geodecics $x_1=$const. and $x_2=1$, and every leaf of $\mathfrak{M}_1$ is a closed curve. Let V^* be the geodesic defined by $x_2=1$, q_i $(i=0, 1)$ be the point with coordinates $x_2(q_0)=x_2(q_1)=1$, $y_2(q_0)=0$, $y_2(q_1)=1$. Then V_{q_0} is homeomorphic to V_{q_1}. But H_{q_0} is isomorphic to the group of order two, and H_{q_1} is the trivial.

The distribution $\mathfrak{M}_2^*$ has, in U_i $(i=1, 2)$, a local base Y_i:

$$Y_i=-\xi_i\frac{\partial}{\partial x_i}+\frac{\partial}{\partial y_i}$$

All of the leaves of $\mathfrak{M}_2^*$ are homeomorphic to a line. Let V_1^* denote the leaf of $\mathfrak{M}_2^*$ through the point p_0 $(x_1(p_0)=\frac{1}{2}, y_1(p_0)=0)$. If p_1 is a common point of V_1^* and the leaf of $\mathfrak{M}_2$ through the point $(x_1, y_1)=(\frac{1}{2}, 1)$, then the group H_{p_0} and H_{p_1} are isomorphic to the additive group of integers, but they do not coincide.

We do not know whether every member of $G(V^*)$ can be extended to an automorphism of M which leaves invariant the decomposition of M by the leaves of $\mathfrak{M}$ and $\mathfrak{M}^*$. But we have the following theorem.

THEOREM 5. *Let assumptions be as in Theorem* 3. *If an element of* $G(V^*)$ *belongs to the center of the group, it can be extended to a diffeomorphism of* M *which sends any point of* M *into the intersection of the leaves of* $\mathfrak{M}$ *and* $\mathfrak{M}^*$ *which contains the point.*

PROOF. Let g_0 be an element of the center of $G(V^*)$. Let q' be a point of M and q be a point of V^* which lies on $V_{q'}$. Let ψ be a way in $V_{q'}$ from q to q' and φ a way in V^* from q to $g_0(q)$. We shall show that the point $\Phi_{\varphi,\psi}(1, 1)$ is independent of the choices of q, φ and ψ. In fact, let q_1 be another point in $V_{q'}\cap V^*$, ψ_1 be a way in $V_{q'}$ from q_1 to q' and φ_1 be a way in V^* from q_1 to $g_0(q_1)$. The ways ψ and ψ_1^{-1} define an element g of $G(V^*)$. The diffeomorphism g_0 being in the center of $G(V^*)$, we have

$$g_0(q_1)=g_0(g(q))=g(g_0(q)).$$

This shows that $\Phi_{\varphi_1,\psi_1}(1, 1)=\Phi_{\varphi,\psi}(1, 1)$, which proves our assertion. Hence we can define a mapping $\tilde{g}_0$ of M into M by $\tilde{g}_0(q)=\Phi_{\varphi,\psi}(1, 1)$. We know easily that $\tilde{g}_0$ is differentiable and $\tilde{g}_0(q')$ is contained in the intersection $V_{q'}\cap V_{q'}^*$. Considering the inverse element g_0^{-1}, we know $\tilde{g}_0$ is an diffeomorphism of M onto M. Our theorem is thereby proved.

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