J. Sci. Hiroshima Univ. Ser. A-I
29 (1965), 51-74

On the Multiplicative Products of Distributions

Mitsuyuki Itano

(Received March 1, 1965)

The present paper is a continuation of the previous paper [12] of the same title collaborated with R. Shiraishi. We investigated there multiplication between distributions centering around the two definitions of multiplicative product of two distributions; one is due to Y. Hirata and H. Ogata [3] and the other J. Mikusiński [8]. We have shown that these two definitions are entirely equivalent. In the sequel the multiplicative product of two distributions S, $T \in \mathcal{D}'(R^N)$, if it exists, will be denoted by ST. It has been pointed out there that multiplication under consideration has the following properties:

(1) if ST exists, then $(\alpha S)T$, $S(\alpha T)$ also exist for any $\alpha \in \mathcal{E}$ and

$$(\alpha S)T = S(\alpha T) = \alpha(ST),$$

(2) if $\dfrac{\partial S}{\partial x_j}T$, $j=1, 2, \cdots, N$, exist, then ST, $S\dfrac{\partial T}{\partial x_j}$, $j=1, 2, \cdots, N$, also exist and

$$\frac{\partial}{\partial x_j}(ST) = \frac{\partial S}{\partial x_j}T + S\frac{\partial T}{\partial x_j}.$$

With necessary modifications, our treatments will also hold for distributions defined on an open subset of R^N. The multiplication is of local character. For the case $N=1$, the first part of (1) and (2) are postulated by H. König [4] as fundamental in his axiomatic approach to a multiplication theory for distributions. It might as well be said that these properties together with local considerations express a precise statement of Schwartz's observation that the multiplicative product of two distributions is well defined if locally one is "more regular" than the other is "irregular".

On the other hand, H. G. Tillmann [13, 14] has investigated the representation theory of distributions by the boundary distributions of locally analytic functions with certain properties. In accordance with the idea of H. J. Bremermann and L. Durand [1], he suggested another approach of defining multiplication between distributions when $N=1$. Let $\hat{S}(z)$ and $\hat{T}(z)$ be locally analytic functions corresponding to S and T respectively. Putting $\hat{S}_\varepsilon(x) = \hat{S}(x+i\varepsilon) - \hat{S}(x-i\varepsilon)$ and $\hat{T}_\varepsilon(x) = \hat{T}(x+i\varepsilon) - \hat{T}(x-i\varepsilon)$, $\varepsilon > 0$, he defined the product $S\cdot T$ to be $\lim_{\varepsilon\to 0} \hat{S}_\varepsilon \hat{T}_\varepsilon$ if it exists, or more generally the finite part of $\hat{S}_\varepsilon \hat{T}_\varepsilon$ (in Hadamard's sense) if it exists. We shall concern ourselves mainly with the case when $\lim_{\varepsilon\to 0} \hat{S}_\varepsilon \hat{T}_\varepsilon$ exists, and write $S\bigcirc T=\lim_{\varepsilon\to 0} \hat{S}_\varepsilon \hat{T}_\varepsilon$ which is referred to in this paper as the multiplicative product of S and T in Tillmann's sense.

Our main purpose of this paper is to investigate this multiplication by making a comparison with the former one.

Section 1 is concerned with a supplement to our previous paper [12]. We shall show that $(S_1\otimes S_2)(T_1\otimes T_2)$ exists and coincides with $S_1S_2\otimes T_1T_2$ if S_1S_2 and T_1T_2 exist (Proposition 3). In Section 2, after giving a short discussion on Tillmann's product which is confined to the case $N=1$, we shall remark that a natural extension of his definition to the case $N\geq 2$ presents serious difficulties. This will be shown by examples. However, in a special case, where distributions are of compact support and certain conditions are satisfied, the existence of ST implies the existence of the multiplicative product of S and T in Tillmann's sense and both products coincide with each other (Proposition 4). Section 3 is devoted to a comparison of multiplication between two distributions in accordance with the two definitions mentioned above. Hereafter we confine ourselves to the case $N=1$. It is shown that multiplication in Tillmann's sense has a really wider range of application than the former one, and that both products coincide with each other when the product in the former sense exists (Theorem 1). However, the multiplication $S\bigcirc T$ in Tillmann's sense fails to satisfy the properties (1) and (2) cited above. In the final Section 4 the two definitions are compared from another standpoint, by considering scalar product of two distributions in a fairly general sense. This concept of scalar product was introduced by S. Łojasiewicz [5] by making use of the concept of the value of a distribution at a point. We have shown in [12] that ST exists if and only if $S*(\alpha T)^\vee$, $\alpha\in\mathcal{D}$, when restricting it to a neighbourhood of 0 which may depend on α, is a bounded function continuous at 0. Thus $S*(\alpha T)^\vee$, $\alpha\in\mathcal{D}$, has the value at 0, which we denote by $(S*(\alpha T)^\vee)(0)$. But, if we require only that the value $(S*(\alpha T)^\vee)(0)$, $\alpha\in\mathcal{D}$, exists, the multiplicative product ST need not exist. However, if we then define after S. Łojasiewicz the scalar product $<S, \alpha T>$ by the equation

$$<S, \alpha T>=(S*(\alpha T)^\vee)(0),$$

then since the linear form $\alpha\to<S, \alpha T>$ is continuous on $\mathcal{D}$, it will be natural to define a multiplicative product $S\times T$ by the equation

$$<S\times T, \alpha>=<S, \alpha T>, \qquad \alpha\in\mathcal{D},$$

where the left side of the equation denotes the scalar product between the spaces $\mathcal{D}'$ and $\mathcal{D}$. We can show that in this case $S \times T$ coincides with $S \bigcirc T$. This is a special instance of a more general result (see Theorem 2). A necessary and sufficient condition in order that $S \bigcirc \alpha T$ may exist for every $\alpha \in \mathcal{E}$ is given (Theorem 2 and Proposition 9).

§ 1. Supplement to the previous paper [12].

We denote by $\mathcal{D}'(R^N)$, or simply by $\mathcal{D}'$, the space of distributions S, T, $\cdots$ defined on N-dimensional Euclidean space R^N. $\mathcal{D}'$ is the strong dual of $\mathcal{D}$, the space of infinitely differentiable functions with compact support in R^N. We denote by $\mathcal{E}'$ the space of distributions on R^N with compact support. $\mathcal{E}'$ is the strong dual of the space $\mathcal{E}$ of infinitely differentiable functions on R^N equipped with the usual topology. Unless otherwise stated, the symbol $< \ , \ >$ is used to denote the scalar product between $\mathcal{D}'$ and $\mathcal{D}$, or between $\mathcal{E}'$ and $\mathcal{E}$. We understand by ST the multiplicative product of S and T in the sense of Y. Hirata and H. Ogata ([3], p. 151), or equivalently in the sense of J. Mikusiński ([8], p. 254). We have shown in [12] (p. 229) that ST exists if and only if there exists for any given $\alpha \in \mathcal{D}$ a neighbourhood U of 0 in R^N on which $\alpha S * \check{T}$ is a bounded function continuous at 0, or more precisely speaking, the restriction of $\alpha S * \check{T}$ to U is equivalent to a bounded measurable function continuous at 0. U may depend on α, and $<ST, \alpha> = (\alpha S * \check{T})(0)$, where the right side denotes the value of distribution $\alpha S * \check{T}$ at 0 in the sense of S. Łojasiewicz [5]. Let us denote by $L^\infty(U)$ the Banach space of bounded measurable functions defined on U with the usual norm. If K is a compact subset of R^N, then $\mathcal{D}_K$ will stand for the space of functions in $\mathcal{D}$ with supports contained in the same K. We note that $\mathcal{D}_K$ is a space of type (**F**).

For our later purpose we show

Proposition 1. *If ST exists and K is a compact subset of R^N, then there exists a neighbourhood U of 0 in R^N such that the restriction of $\alpha S * \check{T}$, $\alpha \in \mathcal{D}_K$, to U is an element of $L^\infty(U)$. The linear map $\alpha \to \alpha S * \check{T}$ of $\mathcal{D}_K$ into $L^\infty(U)$ is then continuous.*

Proposition 2. *If ST exists and K is a compact subset of R^N, then there exists a neighbourhood U of 0 in R^N such that the restriction of $\alpha S * (\beta T)^\vee$, $\alpha, \beta \in \mathcal{D}_K$, to U is an element of $L^\infty(U)$. For such a U the bilinear map $(\alpha, \beta) \to \alpha S * (\beta T)^\vee$ of $\mathcal{D}_K \times \mathcal{D}_K$ into $L^\infty(U)$ is then continuous.*

We shall here give the proof of Proposition 2 since the proof of Proposition 1 will be carried out along the same line.

Proof of Proposition 2. Let us denote by $E_{k,\beta}$ for a positive integer k and $\beta \in \mathcal{D}_K$ the set of all $\alpha \in \mathcal{D}_K$ such that the restriction of $\alpha S * (\beta T)^\vee$ to

$B(0, \frac{1}{k})$ is an element of $L^{\infty}\left(B(0, \frac{1}{k})\right)$ whose norm does not exceed k, where generally $B(x, r)$ denotes the open ball with center x and radius $r>0$. $E_{k,\beta}$ is a closed convex subset of $\mathcal{D}_K$. Evidently it is a convex subset of $\mathcal{D}_K$. We shall show that it is closed. Consider any sequence $\{\alpha_n\}$, $\alpha_n \in E_{k,\beta}$, converging in $\mathcal{D}_K$ to α. Then for any $\phi \in \mathcal{D}_{\bar{B}(0,\frac{1}{k})}$, we have because of the definition of $E_{k,\beta}$ just mentioned

$$|<\alpha_n S*(\beta T)^{\vee}, \phi>| \leq k\int_{R^N}|\phi|dx.$$

Passing to the limit as $n\to\infty$, we obtain

$$|<\alpha S*(\beta T)^{\vee}, \phi>| \leq k\int_{R^N}|\phi|dx \quad \text{for any} \quad \phi \in \mathcal{D}_{\bar{B}(0,\frac{1}{k})}.$$

Consequently, the restriction of $\alpha S*(\beta T)^{\vee}$ to $B(0, \frac{1}{k})$ is an element of $L^{\infty}\left(B(0, \frac{1}{k})\right)$ whose norm does not exceed k, that is, $\alpha \in E_{k,\beta}$, as desired. It follows since ST exists that any $\alpha \in \mathcal{D}_K$ is an element of some $E_{k,\beta}$, that is, $\mathcal{D}_K = \bigcup_1^{\infty} E_{k,\beta}$. $\mathcal{D}_K$ is a space of type (**F**). Owing to Baire's category theorem some $E_{k,\beta}$ must be a zero neighbourhood of $\mathcal{D}_K$.

Let $\{\mathcal{U}_k\}$ be a fundamental system of zero neighbourhoods of $\mathcal{D}_K$. Let us denote by F_k the set of all $\beta \in \mathcal{D}_K$ such that the restriction of $\alpha S*(\beta T)^{\vee}$ to $B(0, \frac{1}{k})$ is for every $\alpha \in \mathcal{U}_k$ an element of $L^{\infty}\left(B(0, \frac{1}{k})\right)$ with norm $\leq k$. From the first part of the proof we see that any $\beta \in \mathcal{D}_K$ is an element of some F_k. As before we can show that F_k is a closed convex subset of $\mathcal{D}_K$. Therefore, applying again the category theorem, we can conclude that there exists a neighbourhood U of 0 in R^N such that for some k the restriction of $\alpha S*(\beta T)^{\vee}$, α, $\beta \in \mathcal{U}_k$, to U is an element of $L^{\infty}(U)$ with norm $\leq k$. Consequently, we see that the first statement of Proposition 2 holds.

If there exists a U stated in the first part of Proposition 2, the bilinear map considered there is always continuous. This is an easy consequence of the closed graph theorem. Thus the proof is complete.

From the preceding proposition we have immediately

COROLLARY. *If $S, T \in \mathcal{E}'$ and ST exists, then there exists a neighbourhood U of 0 in R^N such that the restriction of $\alpha S*(\beta T)^{\vee}$, α, $\beta \in \mathcal{E}$, to U is an element of $L^{\infty}(U)$ and the map $(\alpha, \beta)\to\alpha S*(\beta T)^{\vee}$ from $\mathcal{E}\times\mathcal{E}$ into $L^{\infty}(U)$ is continuous. There exists for any positive ε a positive δ such that $\alpha S*(\beta T)^{\vee}$ is for any α, $\beta \in \mathcal{D}_{\bar{B}(x,\delta)}$ an element of $L^{\infty}(B(0, \varepsilon))$.*

If $S_1 \in \mathcal{D}'(R^{N_1})$ and $S_2 \in \mathcal{D}'(R^{N_2})$, where $N_1+N_2=N$, we denote by $S_1 \otimes S_2$ the tensor product of S_1 and S_2, a distribution on R^N. By making use of Proposition 1 we show

PROPOSITION 3. *Let* $S_1, T_1 \in \mathcal{D}'(R^{N_1})$ *and* $S_2, T_2 \in \mathcal{D}'(R^{N_2})$, *where* $N=N_1+N_2$. *If* S_1T_1 *and* S_2T_2 *exist, then* $(S_1 \otimes S_2)(T_1 \otimes T_2)$ *does exist and coincide with* $S_1T_1 \otimes S_2T_2$.

PROOF. For any $\phi \in \mathcal{D}(R^N)$, we can write

$$\phi = \sum_1^\infty c_n(\alpha_n \otimes \beta_n),$$

where $\sum |c_n| \leq 1$ and $\{\alpha_n\}$, $\{\beta_n\}$ are bounded subsets of $\mathcal{D}(R^{N_1})$ and $\mathcal{D}(R^{N_2})$ respectively ([2], p. 51, [9], p. 98).

By virtue of Proposition 1 we can choose zero neighbourhoods $U_1 \subset R^{N_1}$ and $U_2 \subset R^{N_2}$ such that $\{(\alpha_n S_1) * \check{T}_1\}$ and $\{(\beta_n S_2) * \check{T}_2\}$ are uniformly bounded functions on U_1 and U_2 respectively. Since

$$\big((\alpha_n \otimes \beta_n)(S_1 \otimes S_2)\big) * (T_1 \otimes T_2) = (\alpha_n S_1 * T_1) \otimes (\beta_n S_2 * T_2),$$

we obtain for any sequence $\{\rho_j\}$, $\rho_j \in \mathcal{D}(R^N)$, of regularizations

$$\text{(1)} \quad \langle (S_1 \otimes S_2)\big((T_1 \otimes T_2) * \rho_j\big), \phi \rangle = \sum_1^\infty c_n \langle (\alpha_n S_1 * \check{T}_1) \otimes (\beta_n S_2 * \check{T}_2), \rho_j \rangle,$$

where we may assume that $\operatorname{supp} \rho_j \subset U_1 \times U_2$.

Now there exists a positive constant M independent of n, j such that

$$|\langle (\alpha_n S_1 * \check{T}_1) \otimes (\beta_n S_2 * \check{T}_2), \rho_j \rangle| \leq M \int \rho_j dx = M.$$

And

$$\begin{aligned} \lim_{j\to\infty} \langle (\alpha_n S_1 * \check{T}_1) \otimes (\beta_n S_2 * \check{T}_2), \rho_j \rangle &= \big((\alpha_n S_1 * \check{T}_1)(0)\big)\big((\beta_n S_2 * \check{T}_2)(0)\big) \\ &= \langle S_1T_1, \alpha_n \rangle \langle S_2T_2, \beta_n \rangle \\ &= \langle S_1T_1 \otimes S_2T_2, \alpha_n \otimes \beta_n \rangle, \end{aligned}$$

since $(\alpha_n S_1 * \check{T}_1) \otimes (\beta_n S_2 * \check{T}_2)$ is bounded on $U_1 \times U_2$ and is continuous at 0 in R^N. Therefore from (1) we have

$$\lim_{j\to\infty} <(S_1\otimes S_2)\,((T_1\otimes T_2)*\rho_j),\ \phi>$$

$$=\sum_1^\infty c_n<S_1T_1\otimes S_2T_2,\ \alpha_n\otimes\beta_n>$$

$$=<S_1T_1\otimes S_2T_2,\ \phi>,$$

which was to be proved.

§ 2. Multiplication in the sense of Tillmann.

Let C be the complex number field with usual topology. According to H. G. Tillmann ([14], p. 122) we shall denote by $H_*(C\setminus R)$ the space of locally analytic functions $\hat{g}(z)$ on $C\setminus R$ satisfying an inequality $|\hat{g}(z)|\leq M(|z|)|y|^{-n(|z|)}$ for all $z=x+iy$, where $0<|y|\leq 1$ and $M(r)$, $n(r)$ are continuous functions which may depend on g. Let us denote by $\hat{g}_\varepsilon(x)$ the difference $\hat{g}(x+i\varepsilon)-\hat{g}(x-i\varepsilon)$, where ε is any positive number. Tillmann proved that $\hat{g}_\varepsilon$ converges in $\mathcal{D}'(R)$ to a distribution g as $\varepsilon\to 0$ and the map $\hat{g}\to g$ of $H_*(C\setminus R)$ into $\mathcal{D}'(R)$ is an epimorphism whose kernel is the space of entire functions. It is to be noticed that $\hat{g}(z)$ is analytic in $C\setminus \operatorname{supp} g$. The fact is not explicitly stated in his paper [14], but is really shown in the proof of Satz 3.3 there. In accordance with the idea of H. J. Bremermann and L. Durand ([1], p. 251) he has suggested the definition of multiplication between distributions of $\mathcal{D}'(R)$ as follows: Let $\hat{S}(z)$ and $\hat{T}(z)$ be locally analytic functions of $H_*(C\setminus R)$ which correspond to S and T respectively by the process just mentioned. A distribution $W\in\mathcal{D}'(R)$, denoted by $S\cdot T$, is by definition a multiplicative product of S and T when W is the distributional limit of $\hat{S}_\varepsilon\hat{T}_\varepsilon$ as $\varepsilon\to 0$, or more generally when the finite part of $\hat{S}_\varepsilon\hat{T}_\varepsilon$ (in the sense of Hadamard) exists and equals W. W, if it exists, does not depend on the choice of $\hat{S}$ and $\hat{T}$ corresponding to S and T respectively. We shall concern ourselves mainly with the case where the distributional limit of $\hat{S}_\varepsilon\hat{T}_\varepsilon$ exists and we denote the limit by $S\circ T$ instead of $S\cdot T$ to distinguish the case. In the sequel $S\circ T$, if it exists, will be called simply a multiplicative product of S and T in the sense of Tillmann. If $S\in\mathcal{E}'(R)$, we define $\tilde{S}(z)$ by the equation:

$$\tilde{S}(z)=\frac{1}{2\pi i}<S_x,\ \frac{1}{x-z}>.$$

It is shown in [13] (p. 77) that $\tilde{S}$ is an element of $H_*(C\setminus R)$ corresponding to S.

The generic point of the product space C^N is denoted by $z=(z_1, z_2, \cdots, z_N)$, $z_j\in C$. Similarly the generic point of R^N is denoted by $x=(x_1, x_2, \cdots, x_N)$, $x_j\in R$. We use an abbreviation $\dfrac{1}{x-z}$ for $\prod_1^N \dfrac{1}{x_j-z_j}$. Let $\sigma=(\sigma_1, \sigma_2, \cdots, \sigma_N)$ be any vector, where $\sigma_j = \pm 1$. Then $(C\setminus R)^N$ is the open subset with 2^N components $G_\sigma = \{z : \sigma_j \operatorname{Im} z_j > 0\}$.

For any distribution $S \in \mathcal{E}'(R^N)$, we put

$$\tilde{S}(z) = \frac{1}{(2\pi i)^N} < S_x, \frac{1}{x-z} >$$

as before. Let $\tilde{S}^\sigma(z)$ denote the component of $\tilde{S}(z)$ in G_σ. We define for any $\varepsilon > 0$ the function $\tilde{S}_\varepsilon(x)$ by the equation:

$$\tilde{S}_\varepsilon(x) = \sum\left(\prod_1^N \sigma_j\right) \tilde{S}^\sigma(x + i\varepsilon\sigma).$$

We can write $\tilde{S}_\varepsilon = S * h_\varepsilon$, where $h_\varepsilon(x) = \prod_1^N \frac{\varepsilon}{\pi} \frac{1}{x_j^2 + \varepsilon^2}$. $\tilde{S}_\varepsilon$ converges in $\mathcal{D}'(R^N)$ to S as $\varepsilon \to 0$ ([13], p. 77).

We can define $S \bigcirc T$ for S, $T \in \mathcal{E}'(R^N)$ as in the 1-dimensional case: $S \bigcirc T = \lim_{\varepsilon \to 0} \tilde{S}_\varepsilon \tilde{T}_\varepsilon$ if the distributional limit exists. However, the fact that, generally, $\tilde{S}_\varepsilon$ does not converge in $\mathcal{E}(R^N \setminus \operatorname{supp} S)$ to zero as $\varepsilon \to 0$, causes some difficulties. For it is possible that $\tilde{S}_\varepsilon \tilde{T}_\varepsilon$ does not converge to zero in the case $\operatorname{supp} S \cap \operatorname{supp} T = \emptyset$. The example will be given later on. We note that the analogous statement to Proposition 3 concerning tensor products holds also for the multiplicative product under discussion. This follows easily from the definition of multiplication just given.

H. G. Tillmann has developed the representation theory for distributions on R^N of finite orders ([14], p. 117), where locally analytic function $\hat{S}(z)$ with certain properties determines the distribution S of finite order as the boundary distribution as mentioned in the case $S \in \mathcal{E}'$. Here the map $S \to \hat{S}$ is not unique as in the 1-dimensional case. If we attempt to define $S \bigcirc T$ in the same way as in the case S, $T \in \mathcal{E}'$, we shall be led to some serious difficulties. It occurs that $\hat{S}_\varepsilon \hat{T}_\varepsilon$ does not converge to zero even in the case $S = 0$.

Examples. Consider the case $N = 2$. Let S, T be the Dirac measures concentrated at point $(1, 0)$ and $(0, 0)$ respectively. We obtain

$$\tilde{S}_\varepsilon(x) = \frac{1}{\pi^2} \frac{\varepsilon}{(x_1 - 1)^2 + \varepsilon^2} \frac{\varepsilon}{x_2^2 + \varepsilon^2},$$

$$\tilde{T}_\varepsilon(x) = \frac{1}{\pi^2} \frac{\varepsilon}{x_1^2 + \varepsilon^2} \frac{\varepsilon}{x_2^2 + \varepsilon^2}.$$

To carry out the estimation of $\tilde{S}_\varepsilon \tilde{T}_\varepsilon$ it is convenient to note that, in the case $N = 1$, we have for the Dirac measure δ and for any $\phi \in \mathcal{D}(R)$

(1) $$\int \tilde{\delta}_\varepsilon(x)\tilde{\delta}_\varepsilon(x)\phi(x)dx = \frac{1}{2\pi\varepsilon}<\delta, \phi> + o(1),$$

(2) $$\int \tilde{\delta}_\varepsilon(x-1)\tilde{\delta}_\varepsilon(x)\phi(x)dx = \frac{\varepsilon}{\pi}<\delta, \phi> + \frac{\varepsilon}{\pi}<\delta_1, \phi> + o(\varepsilon)$$

as $\varepsilon \to 0$.

By making use of these relations we obtain

$$\tilde{S}_\varepsilon \tilde{T}_\varepsilon = \frac{1}{2\pi^2}(\delta\otimes\delta + \delta_1\otimes\delta) + o(1), \qquad \text{as } \varepsilon\to 0.$$

This is an example which shows that $\tilde{S}_\varepsilon \tilde{T}_\varepsilon$ does not converge in $\mathcal{D}'$ to zero as $\varepsilon\to 0$ in case that $\operatorname{supp} S \cap \operatorname{supp} T = \emptyset$.

Now let $S = 0 = 0\otimes\delta$ and let T be the same as above. Then $\hat{S}(z) = -\frac{z_1}{2\pi i\, z_2}$. Therefore $\hat{S}_\varepsilon(x) = \frac{2\, i\, \varepsilon^2}{\pi(x_2^2+\varepsilon^2)}$. Then we obtain because of (1)

$$\hat{S}_\varepsilon \tilde{T}_\varepsilon = \frac{i}{\pi}\delta\otimes\delta + o(1), \qquad \text{as } \varepsilon\to 0.$$

This is an example which shows that $\hat{S}_\varepsilon \hat{T}_\varepsilon$ does not converge in $\mathcal{D}'$ to zero even when $S=0$.

The first example shows also that, for $S, T\in\mathcal{E}'(R^N)$, $N>1$, even if ST and $S\bigcirc T$ exist, they may not coincide. Under certain conditions we can show that the existence of ST guarantees the existence of $S\bigcirc T$ and $ST = S\bigcirc T$. From our previous paper ([12], p. 229), we know that if $S(\tau_h T)$ exists for $S, T\in\mathcal{E}'(R^N)$ and for every translation τ_h of R^N, then $\alpha S*(\beta T)^\vee$ is a continuous function on R^N for any $\alpha, \beta\in\mathcal{E}$. In such a case the existence of ST will imply the existence of $S\bigcirc T$ as the following proposition shows.

PROPOSITION 4. *Let $S, T\in\mathcal{E}'(R^N)$. If ST exists and $S*(\beta T)^\vee$, $\beta\in\mathcal{E}$, is a bounded function on R^N, then $S\bigcirc T$ does exist and coincide with ST.*

PROOF. From the assumptions it follows that $S*\check{T}$ is a bounded function with compact support and is continuous at 0. Let ϕ be any element of $\mathcal{D}$. Let $\alpha\in\mathcal{D}$ be equal to 1 on a sufficiently large ball with center origin in such a way that $\alpha=\check{\alpha}$, $0\le\alpha\le 1$ and $((\operatorname{supp} S\cup\operatorname{supp} T)+\operatorname{supp}(1-\alpha))\cap\operatorname{supp}\phi = \emptyset$. We then obtain, because of the fact that $(S*(1-\alpha)h_\varepsilon)\phi = (T*(1-\alpha)h_\varepsilon)\phi = 0$,

$$<\tilde{S}_\varepsilon\tilde{T}_\varepsilon, \phi> = <(S*\alpha h_\varepsilon)\,(T*\alpha h_\varepsilon), \phi>.$$

Our aim is to show that $\lim_{\varepsilon\to 0}<\tilde{S}_\varepsilon\tilde{T}_\varepsilon, \phi> = <ST, \phi>$. To this end we evaluate the right side of the equation. Since $S*\alpha h_\varepsilon$, $T*\alpha h_\varepsilon$ are of compact support we may assume that ϕ is periodic with period $2l$ for each coordinate with a sufficiently large l. Then we can write

$$\phi(x)=\sum c_m e^{i\frac{\pi}{l}<m,x>},$$

where $\sum|c_m|(1+|m|)^k<\infty$ for any integer $k\geq 0$.

Now we obtain

$$\begin{aligned}&<(S*\alpha h_\varepsilon)\,(T*\alpha h_\varepsilon),\,\phi>\\ &\qquad=\sum c_m<(S*\alpha h_\varepsilon)\,(T*\alpha h_\varepsilon),\,e^{i\frac{\pi}{l}<m,x>}>\\ &\qquad=\sum c_m<(S*\alpha h_\varepsilon*e^{-i\frac{\pi}{l}<m,x>}\alpha h_\varepsilon)\,T,\,e^{i\frac{\pi}{l}<m,x>}>\\ &\qquad=\sum c_m<(S*g_{m,\varepsilon})\,T,\,e^{i\frac{\pi}{l}<m,x>}>,\end{aligned}$$

where we put $g_{m,\varepsilon}=\alpha h_\varepsilon*e^{-i\frac{\pi}{l}<m,x>}\alpha h_\varepsilon$.

We shall show that (i) the set $\{(S*g_{m,\varepsilon})T\}_{m,\varepsilon}$ is bounded in $\mathcal{E}'$ and that (ii) $(S*g_{m,\varepsilon})T$ converges in $\mathcal{E}'$ to ST as $\varepsilon\to 0$. Suppose these are true. We can then find an integer $k\geq 0$ and a constant M such that

$$|<(S*g_{m,\varepsilon})\,T,\,e^{i\frac{\pi}{l}<m,x>}>|\leq M(1+|m|)^k.$$

Consequently, since $\sum|c_m|(1+|m|^k)<\infty$, the series

$$\sum c_m<(S*g_{m,\varepsilon})\,T,\,e^{i\frac{\pi}{l}<m,x>}>$$

is normally convergent where each term is considered as a function of ε. And

$$\lim_{\varepsilon\to 0}<(S*g_{m,\varepsilon})\,T,\,e^{i\frac{\pi}{l}<m,x>}>=<ST,\,e^{i\frac{\pi}{l}<m,x>}>.$$

Therefore we have

$$\begin{aligned}&\lim_{\varepsilon\to 0}<(S*h_\varepsilon)\,(T*h_\varepsilon),\,\phi>\\ &\qquad=\sum c_m<ST,\,e^{i\frac{\pi}{l}<m,x>}>\\ &\qquad=<ST,\,\phi>,\end{aligned}$$

as desired.

Now we shall turn to the proof of (i) and (ii). To begin with, we note that

$$\int |g_{m,\varepsilon}|\,dx \leq 1,$$

$$\lim_{\varepsilon\to 0}\int g_{m,\varepsilon}\,dx = 1,$$

$$\lim_{\varepsilon\to 0}\int_{|x|>\delta} |g_{m,\varepsilon}|\,dx = 0 \qquad \text{for any positive } \delta.$$

Let β be any element of $\mathcal{E}$. Then

$$<(S*g_{m,\varepsilon})T,\ \beta> = <S*(\beta T)^{\vee},\ g_{-m,\varepsilon}>.$$

If we remember that $S*(\beta T)^{\vee}$ is a bounded function and is continuous at origin, we can confer with the aid of the properties of $g_{m,\varepsilon}$ noted above that

$$|<(S*g_{m,\varepsilon})T,\ \beta>| \leq M_1 \int |g_{m,\varepsilon}|\,dx \leq M_1,$$

where the constant M_1 depends on β, but not on m, ε, and that

$$\lim_{\varepsilon\to 0}<(S*g_{m,\varepsilon})T,\ \beta> = \big(S*(\beta T)^{\vee}\big)(0) = <ST,\ \beta>.$$

Thus the proof of Proposition 5 is complete.

In the statement of Proposition 4, if we take $N=1$, the condition imposed on $S*(\beta T)^{\vee}$ may be omitted. This follows from the fact that $\tilde{S}(z)$ is analytic in $C\setminus \operatorname{supp} S$ as the proof of the following proposition shows.

PROPOSITION 5. *Let $S,\ T \in \mathcal{E}'(R)$. If ST exists, then $S \bigcirc T$ does exist and equal ST.*

PROOF. With the aid of a decomposition of unity we can confer that it is sufficient to show that there exists a neighbourhood $B(x,\delta)$ for every $x \in R$ such that $\lim_{\varepsilon\to 0}<\tilde{S}_{\varepsilon}\tilde{T}_{\varepsilon},\ \phi> = <ST,\ \phi>$ for any $\phi \in \mathcal{D}_{\bar{B}(x,\delta)}$.

Owing to Corollary to Proposition 2 there exists a positive number δ such that $\alpha S*(\beta T)^{\vee}$ is a bounded function continuous at 0 for any α, $\beta \in \mathcal{D}_{\bar{B}(x,3\delta)}$. Let $\phi \in \mathcal{D}_{\bar{B}(x,\delta)}$. We take an $\alpha \in \mathcal{D}_{\bar{B}(x,3\delta)}$ such that $\alpha=1$ on $B(x,2\delta)$. Then $(1-\alpha)S*h_{\varepsilon}$ and $(1-\alpha)T*h_{\varepsilon}$ converge in $\mathcal{E}(B(x,2\delta))$ to zero as $\varepsilon\to 0$. We can write

$$\begin{aligned}<\tilde{S}_{\varepsilon}\tilde{T}_{\varepsilon},\ \phi> &= <(\alpha S*h_{\varepsilon})\,(\alpha T*h_{\varepsilon}),\ \phi> \\ &\quad + <\alpha S*h_{\varepsilon},\ \big((1-\alpha)T*h_{\varepsilon}\big)\phi> \\ &\quad + <T*h_{\varepsilon},\ \big((1-\alpha)S*h_{\varepsilon}\big)\phi>.\end{aligned}$$

Consequently, we can confer that $\lim_{\varepsilon\to 0} <\tilde{S}_\varepsilon \tilde{T}_\varepsilon, \phi> = <ST, \phi>$ if we can show that $\lim_{\varepsilon\to 0} <(\alpha S * h_\varepsilon)(\alpha T * h_\varepsilon), \phi> = <ST, \phi>$. But α is taken so that αS and αT satisfy the conditions of Proposition 4. Therefore

$$\begin{aligned}\lim_{\varepsilon\to 0} &<(\alpha S * h_\varepsilon)(\alpha T * h_\varepsilon), \phi> \\ &= <(\alpha S)(\alpha T), \phi> = <\alpha^2 ST, \phi> \\ &= <ST, \alpha^2\phi> = <ST, \phi>,\end{aligned}$$

which was to be proved.

§ 3. Comparison between the two definitions on multiplication.

Hereafter we assume that $N=1$.

PROPOSITION 6. *If $S, T \in \mathcal{D}'$ and ST exists, then* $ST = \lim_{\varepsilon\to 0} \hat{S}_\varepsilon T = \lim_{\varepsilon\to 0} S\hat{T}_\varepsilon$.

PROOF. First assume that S is of compact support, that is, $S \in \mathcal{E}'$. Then we have for any $\phi \in \mathcal{D}$

$$\begin{aligned}\lim_{\varepsilon\to 0} <\tilde{S}_\varepsilon T, \phi> &= \lim_{\varepsilon\to 0} <(S*h_\varepsilon) T, \phi> \\ &= \lim_{\varepsilon\to 0} <S*(T\phi)^\vee, h_\varepsilon> \\ &= (S*(T\phi)^\vee)(0) \\ &= <ST, \phi>,\end{aligned}$$

where we used the fact that the restriction of $S*(\phi T)^\vee$ to a neighbourhood of 0 in R is a bounded function continuous at 0.

Next we turn to the general case. Let I be any open interval. We choose $\alpha \in \mathcal{D}$ so that α takes the value 1 on I. Put $S_1 = \alpha S$ and $S_2 = S - S_1$. Then we can write for any $\hat{S}$ corresponding to S

$$\hat{S}(z) = \hat{S}_1(z) + \hat{S}_2(z),$$

where $\hat{S}_2(z)$ is analytic in $C\setminus(R\setminus I)$.

Consequently, $(\hat{S}_2)_\varepsilon T$ converges in $\mathcal{D}'(I)$ to zero as $\varepsilon \to 0$. Now using the first part of the proof we obtain for any $\phi \in \mathcal{D}(I)$

$$\begin{aligned}\lim_{\varepsilon\to 0} <\hat{S}_\varepsilon T, \phi> &= \lim_{\varepsilon\to 0} <(\tilde{S}_1)_\varepsilon T, \phi> \\ &= <\alpha ST, \phi> = <ST, \phi>.\end{aligned}$$

Since I is arbitrary, we have $\lim_{\varepsilon\to 0}\hat{S}_\varepsilon T=ST$. Similarly for $\lim_{\varepsilon\to 0}S\hat{T}_\varepsilon=ST$, completing the proof.

In the preceding proposition, if we do not assume that ST exists, it may occur that $\lim_{\varepsilon\to 0}\hat{S}_\varepsilon T$, $\lim_{\varepsilon\to 0}S\hat{T}_\varepsilon$ and $\lim_{\varepsilon\to 0}\hat{S}_\varepsilon\hat{T}_\varepsilon$ exist, but differ from each other. For example, let $S=\delta$ and $T=\mathrm{Pf}\frac{1}{x}$. Then $\hat{S}(z)=-\frac{1}{2\pi iz}$, $\hat{T}(z)=\frac{1}{2z}$ for $\mathrm{Im}\, z>0$ and $=-\frac{1}{2z}$ for $\mathrm{Im}\, z<0$. By calculation we shall have $\lim_{\varepsilon\to 0}\hat{S}_\varepsilon T=-\delta'$, $\lim_{\varepsilon\to 0}S\hat{T}_\varepsilon=0$ and $\lim_{\varepsilon\to 0}\hat{S}_\varepsilon\hat{T}_\varepsilon=-\frac{1}{2}\delta'$.

With the aid of Proposition 5, and by proceeding in the same way as in the proof of the Proposition 6, we have

THEOREM 1. *If $S, T\in\mathcal{D}'$ and ST exists, then $S\odot T$ does exist and coincide with ST.*

PROOF. Let I be any open interval. Choose $\alpha\in\mathcal{D}$ as in the proof of Proposition 6. Put $S_1=\alpha S$, $S_2=S-S_1$, $T_1=\alpha T$ and $T_2=T-T_1$. Since each of $(\hat{S}_2)_\varepsilon(\hat{T}_1)_\varepsilon$, $(\hat{S}_2)_\varepsilon(\hat{T}_2)_\varepsilon$ and $(\hat{S}_1)_\varepsilon(\hat{T}_2)_\varepsilon$ converges in $\mathcal{D}'(I)$ to zero as $\varepsilon\to 0$, it follows from Proposition 5 that $\hat{S}_\varepsilon\hat{T}_\varepsilon$ converges in $\mathcal{D}'(I)$ to ST as $\varepsilon\to 0$. Now I is arbitrary. Therefore we obtain $S\odot T=\lim_{\varepsilon\to 0}\hat{S}_\varepsilon\hat{T}_\varepsilon=ST$, which was to be proved.

REMARK 1. S is an element of $\mathcal{E}$ if and only if $S\odot T$ exists for every $T\in\mathcal{D}'$, or more generally for every $T\in\mathcal{E}'$. Indeed, if $S\in\mathcal{E}$, then ST exists for every $T\in\mathcal{D}'$, and therefore $S\odot T$ does exist by Theorem 1. Conversely, assume that $S\odot T$ exists for every $T\in\mathcal{E}'$, then for a fixed $\hat{S}$ corresponding to S, we have

$$S\odot T=\lim_{\varepsilon\to 0}(\hat{S})_\varepsilon(T*h_\varepsilon).$$

Since the map $T\to(\hat{S})_\varepsilon(T*h_\varepsilon)$ from $\mathcal{E}'$ into $\mathcal{D}'$ is continuous and $\mathcal{E}'$ is barrelled, it follows from a theorem of Banach-Steinhaus that the map $T\to S\odot T$ of $\mathcal{E}'$ into $\mathcal{D}'$ is continuous. Then for any $\phi\in\mathcal{D}$, there exists an element $f(\phi)\in\mathcal{E}$ such that for every $T\in\mathcal{E}'$

$$<S\odot T, \phi>=<f(\phi), T>.$$

If T happens to be an element $\alpha\in\mathcal{D}$, then from the equation we have

$$<\phi S, \alpha>=<S\odot\alpha, \phi>=<f(\phi), \alpha>.$$

Consequently, $\phi S=f(\phi)\in\mathcal{E}$, which shows that $S\in\mathcal{E}$, as desired.

REMARK 2. $\hat{S}(z)$ and $\hat{T}(z)$ are analytic in $C\setminus \operatorname{supp} S$ and in $C\setminus \operatorname{supp} T$ respectively, so $\hat{S}_\varepsilon \hat{T}_\varepsilon$ converges in $\mathcal{D}'(R\setminus(\operatorname{supp} S\cap \operatorname{supp} T))$ to 0. Therefore, if $S\bigcirc T$ exists, then $\operatorname{supp} S\bigcirc T \subset \operatorname{supp} S\cap \operatorname{supp} T$. This support theorem is also valid for ST as easily shown from the relation $\alpha(ST)=(\alpha S)T=S(\alpha T)$, $\alpha\in\mathcal{E}$. Assume that $x\bigcirc T=0$ and $S\bigcirc T$ exists for every $S\in L^1$. Then T must be zero. In fact, since $xT=x\bigcirc T=0$, it follows that $\operatorname{supp} T\subset\{0\}$. Consider the map $S\to S\bigcirc T$ of L^1 into $\mathcal{D}'$ which will be continuous as easily shown. From the support theorem just noted, $S\bigcirc T$ is of the form $S\bigcirc T=a_0(S)\delta+a_1(S)\delta'+\cdots+a_n(S)\delta^{(n)}$. Then a_j being a continuous linear form on L^1, we can write $a_j(S)=\int g_j(x)S(x)dx$, where $g_j\in L^\infty$, $j=1, 2, \cdots, n$. If we take S for any $\phi\in\mathcal{D}$ such that $0\notin \operatorname{supp}\phi$, then $a_j(\phi)=0$. This implies that $g_j=0$, that is, $S\bigcirc T=0$ for every $S\in L^1$. Putting $S=1$, we see that $T=0$, as desired. As a result we see that if $x\bigcirc T=0$ and $S\bigcirc T$ exists for every S with the form $S=F''$, F being a continuous function, then T must be zero (compare with Satz of [4], p. 392).

The following remarks are related to the axioms of H. König [4] as explained in the introduction.

REMARK 3. We have shown in [12] (p. 225) that if ST exists, then $(\alpha S)T$ and $S(\alpha T)$ exist also for any $\alpha\in\mathcal{E}$ and the relation $(\alpha S)T=S(\alpha T)=\alpha(ST)$ holds. The statement is not true in general for $S\bigcirc T$. In fact, if we take $S=\mathrm{Pf}\frac{1}{x^2}$, $T=\delta'$, then $S\bigcirc T=\frac{1}{2\pi i}\mathrm{Pf}\frac{1}{x^4}$. But $(xS)\bigcirc T$ and $S\bigcirc(xT)$ do not exist. On the other hand, if we take $S=\delta$ and $T=\mathrm{Pf}\frac{1}{x}$, then $S\bigcirc T=-\frac{1}{2}\delta'$ ([1], p. 251). But, for any $\alpha\in\mathcal{E}$, $(\alpha S)\bigcirc T=-\frac{\alpha(0)}{2}\delta'$ and $\alpha(S\bigcirc T)=\frac{1}{2}\alpha'(0)\delta-\frac{1}{2}\alpha(0)\delta'$, so that $(\alpha S)\bigcirc T\neq\alpha(S\bigcirc T)$ for $\alpha\in\mathcal{E}$ such that $\alpha'(0)\neq 0$.

REMARK 4. We have shown in [12] (p. 229) that if $S\frac{dT}{dx}$ exists, then ST, $\frac{dS}{dx}T$ exist and $\frac{d}{dx}(ST)=\frac{dS}{dx}T+S\frac{dT}{dx}$. The statement is not true in general for the multiplication in the sense of Tillmann. In fact, if we take $S=\mathrm{Pf}\frac{1}{x}$ and $T=Y$ (Heaviside function), then $S\bigcirc\frac{dT}{dx}=-\frac{1}{2}\delta'$ but $\frac{dS}{dx}\bigcirc T$ does not exist. However, it is easily seen from the definition that if $S\bigcirc T$ and $S\bigcirc\frac{dT}{dx}$ exist, then $\frac{dS}{dx}\bigcirc T$ exists and $\frac{d}{dx}(S\bigcirc T)=\frac{dS}{dx}\bigcirc T+S\bigcirc\frac{dT}{dx}$ holds.

When S is a tempered distribution, we use the notation $\hat{S}$ to denote the Fourier transform $\mathcal{F}(S)$:

$$\hat{S}=\int e^{ix\xi}S_x dx.$$

Now, let S and T be $\mathscr{S}'$-composable tempered distributions. It is known ([3], p. 151) that $\hat{S}\hat{T}$ exists and $\mathfrak{F}(S*T)=\hat{S}\hat{T}$. Therefore, by Theorem 1, we can also write

$$\mathfrak{F}(S*T)=\hat{S}\bigcirc\hat{T}.$$

LEMMA 1. *If S and T are tempered distributions with supports in the positive real axis, then S and T are $\mathscr{S}'$-composable.*

PROOF. Let ϕ, ψ be any elements of $\mathscr{D}$, and χ be any element of $\mathscr{S}$. Consider the expression

$$(S*\phi)(x)(T*\psi)(y)\chi(x+y). \tag{1}$$

Since $S*\phi$, $T*\psi$ belong to O_M, there is a positive integer k such that we have for a constant M

$$|(S*\phi)(x)|,\ |(T*\psi)(x)|\leq M(1+|x|)^k. \tag{2}$$

On the other hand, since χ is an element of $\mathscr{S}$, there is a constant M_1 such that we have

$$(1+|x|)^{2k+4}|\chi(x)|\leq M_1. \tag{3}$$

Since the supports of $S*\phi$ and $T*\psi$ are limited on the left side of the real axis, we can find a constant M_2 such that for any $x\in\operatorname{supp}S*\phi$ and $y\in\operatorname{supp}T*\psi$

$$(1+|x|)(1+|y|)\leq M_2(1+|x+y|)^2. \tag{4}$$

Combining (2), (3) and (4), we can easily find that (1) is an integrable function in x and y. Therefore $S*\phi$, $T*\psi$ are $\mathscr{S}'$-composable, so S and T are also ([11], p. 27), completing the proof.

Let S be a tempered distribution with support in the positive real axis. Consider the complex Fourier transform

$$\hat{S}(\zeta)=\int e^{i\zeta x}S(x)dx,\quad \operatorname{Im}\zeta>0,\ \zeta=\xi+i\eta.$$

$\hat{S}(\zeta)$ is analytic in the upper open half plane and $\hat{S}(\xi+i\eta)\to\hat{S}$ in $\mathscr{S}'$ as $\eta\to 0$. It is known ([10], p. 75) that $\hat{S}(\zeta)$ is slowly increasing in the half plane

$\operatorname{Im} \zeta > 0$ in the sense that for a positive integer k and a constant M

$$|\hat{S}(\zeta)| \leq \frac{M(1+|\zeta|^2)^k}{|\operatorname{Im} \zeta|^{k+1}}.$$

Therefore if we define $\hat{S}(\zeta)=0$ for $\operatorname{Im} \zeta < 0$. Then the locally analytic function $\hat{S}(\zeta)$ belongs to $H_*(C \setminus R)$ and $\hat{S}_\varepsilon(\xi) = \hat{S}(\xi + i\varepsilon)$ converges in $\mathscr{S}'$ to $\hat{S}$. From these considerations we see that if S and T are tempered with support in the positive axis, then, by Lemma 1, S and T are $\mathscr{S}'$-composable and

$$\mathcal{F}(S*T) = \lim_{\varepsilon \to 0} \hat{S}_\varepsilon \hat{T}_\varepsilon = \hat{S} \bigcirc \hat{T}.$$

Any tempered distribution S is written in the form

$$S = S_+ - S_-,$$

where the supports of S_+ and S_- lie in the positive and negative real axis respectively. Consider the complex Fourier transforms $\hat{S}_+(\zeta)$, $\operatorname{Im} \zeta > 0$ and $\hat{S}_-(\zeta)$, $\operatorname{Im} \zeta < 0$. The pair $(\hat{S}_+(\zeta), \hat{S}_-(\zeta))$ is a locally analytic function $\in H_*(C \setminus R)$ and determines $\hat{S}$ as a boundary distribution. If we put

$$\hat{S}_\varepsilon(\xi) = \hat{S}_+(\xi + i\varepsilon) - \hat{S}_-(\xi - i\varepsilon)$$

then $\hat{S}_\varepsilon \to \hat{S}$ in $\mathscr{S}'$ as $\varepsilon \to 0$. Therefore if S and T are two $\mathscr{S}'$-composable tempered distributions, then

$$\mathcal{F}(S*T) = \lim_{\varepsilon \to 0} \hat{S}_\varepsilon \hat{T}_\varepsilon = \hat{S} \bigcirc \hat{T}.$$

It is open to us whether we can conclude that S and T are $\mathscr{S}'$-composable when $\hat{S}\hat{T}$ exists, or more generally when $\hat{S} \bigcirc \hat{T}$ exists.

§ 4. The value of a distribution at a point. Scalar product and multiplication.

Let us recall some definitions and results concerning the value of a distribution at a point.

After S. Łojasiewicz ([5], p. 239, [6], p. 2), the value of a distribution $S \in \mathscr{D}'(R)$ at a point x_0 is defined as the distributional limit

(1) $$\lim_{\lambda \to 0} S(\lambda \hat{x} + x_0)$$

provided that such limit exists. If it exists it is always a constant function

([7], p. 479, [15], p. 519). The value $S(x_0)$ of S at x_0 is defined as the value of this constant function.

S has the value c at x_0 if and only if there exists an open interval I containing x_0 such that the restriction S to I is written as $D^nF=S$ and

$$\lim_{x\to x_0}\frac{F(x)}{(x-x_0)^n}=\frac{c}{n!}, \tag{2}$$

where n is a non-negative integer and F is a continuous function on I ([6], p. 5).

A distribution S is called to be bounded at x_0 if the family of distributions $S(\lambda\hat{x}+x_0)$, $0<\lambda<1$, is bounded ([16], p. 28). This is equivalent to requiring instead of (2) that $\frac{F(x)}{(x-x_0)^n}$ is bounded ([16], p. 29).

Lojasiewicz has also defined the right (resp. left) hand limit of S at x_0 ([6], p. 3). S is said to have the right hand limit c for $x\to x_0+$ if the distributional limit $\lim_{\lambda\to 0} S(\lambda x+x_0)$ exists in a neighbourhood of x_0 for $x>x_0$ and if it is a constant distribution c. We write $\lim_{x\to x_0+} S=c$. He proved in [6] (p. 5) that for the existence of $\lim_{x\to x_0+} S=c$ it is necessary and sufficient that there exist a non-negative integer n and a continuous function F in a neighbourhood of x_0 for $x>x_0$ such that $S=D^nF$ and

$$\lim_{x\to x_0+}\frac{F(x)}{(x-x_0)^n}=\frac{c}{n!}$$

Similarly for the left hand limit of S.

Proposition 7. *If $S\in\mathcal{D}'(R)$ has the value c at the point $x_0\in R$, then $S\bigcirc\delta_{x_0}$ exists and equals $c\delta_{x_0}$, where δ_{x_0} is the Dirac measure concentrated at x_0.*

Proof. We may assume that $x_0=0$. $\tilde{\delta}_\varepsilon=h_\varepsilon(x)=\frac{1}{\pi}\frac{\varepsilon}{x^2+\varepsilon^2}$. Let ϕ be any given element of $\mathcal{D}$. We take $\alpha\in\mathcal{D}$ with value 1 in an open interval I containing $\operatorname{supp}\phi$ and 0. Let $S_1=\alpha S$ and $S_2=(1-\alpha)S$. Then S_1 has the value c at 0. We can write

$$\hat{S}(z)=\hat{S}_1(z)+\hat{S}_2(z),$$

where $\hat{S}_2$ is analytic in $C\setminus(R\setminus I)$.

It is clear that $(\hat{S}_2)_\varepsilon$ converges in $\mathcal{E}(I)$ to zero as $\varepsilon\to 0$, so that $<(\hat{S}_2)_\varepsilon\tilde{\delta}_\varepsilon, \phi>\to 0$ as $\varepsilon\to 0$. Therefore it remains to show that $<(\tilde{S}_1)_\varepsilon\tilde{\delta}_\varepsilon, \phi>\to c\phi(0)$ as $\varepsilon\to 0$. Now we write S_1 in the form

$$S_1 = T + W, \qquad T,\ W \in \mathcal{E}',$$

where $\operatorname{supp} W \cap I = \emptyset$ and T is the n-th derivative of a continuous function F such that $\lim\limits_{x\to 0} \dfrac{F(x)}{x^n} = \dfrac{c}{n!}$. As before $<\tilde{W}_\varepsilon \tilde{\delta}_\varepsilon, \phi> \to 0$ as $\varepsilon \to 0$. Therefore we have only to show that $\lim\limits_{\varepsilon\to 0} <\tilde{T}_\varepsilon \tilde{\delta}_\varepsilon, \phi> = c\phi(0)$.

$$<\tilde{T}_\varepsilon \tilde{\delta}_\varepsilon, \phi> = <(T * h_\varepsilon) h_\varepsilon, \phi> = <T, h_\varepsilon \phi * h_\varepsilon>.$$

After a change of variable $x \to \varepsilon x$, we can write with $h(x) = \dfrac{1}{\pi} \dfrac{1}{x^2+1}$

$$<T, h_\varepsilon \phi * h_\varepsilon> = (-1)^n \int \frac{F(\varepsilon x)}{\varepsilon^n} \left(h(x)\phi(\varepsilon x) * h(x)\right)^{(n)} dx$$

$$= (-1)^n \int \frac{F(\varepsilon x)}{\varepsilon^n x^n} x^n \left(h(x)\phi(\varepsilon x) * h(x)\right)^{(n)} dx.$$

On the other hand

$$x^n \left(h(x)\phi(\varepsilon x) * h(x)\right)^{(n)}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} (n-k)! \binom{n}{k}^2 \left(x^k (h(x)\phi(\varepsilon x) * h(x))\right)^{(k)}$$

$$= \sum_{k=0}^{n} (-1)^{n-k} (n-k)! \binom{n}{k}^2 \sum_{p+q=k \cdot p, q \geq 0} \binom{p+q}{p} \left((x^p h(x)\phi(\varepsilon x))^{(p)} * (x^q h)^{(q)}\right).$$

Now

$$\left(x^p h(x)\phi(\varepsilon x)\right)^{(p)}$$

$$= \sum_{r=0}^{p} \binom{p}{r} \frac{p!}{(p-r)!} x^{p-r} \left(h(x)\phi(\varepsilon x)\right)^{(p-r)}$$

$$= \sum_{r=0}^{p} \binom{p}{r} \frac{p!}{(p-r)!} x^{p-r} \sum_{s=0}^{p-r} \binom{p-r}{s} h^{(s)}(x) \varepsilon^{p-r-s} \phi^{(p-r-s)}(\varepsilon x)$$

$$= \sum_{r=0}^{p} \binom{p}{r} \frac{p!}{(p-r)!} \sum_{s=0}^{p-r} \binom{p-r}{s} x^s h^{(s)}(x) (\varepsilon x)^{p-r-s} \phi^{(p-r-s)}(\varepsilon x)$$

Similarly for $\left(x^q h(x)\right)^{(q)}$ we have

$$\left(x^q h(x)\right)^{(q)} = \sum_{t=0}^{q} \binom{q}{t} \frac{q!}{t!} x^t h^{(t)}(x).$$

Since $(\varepsilon x)^k \phi^{(k)}(\varepsilon x)$ is bounded and tends to 0 for $k \geq 1$ as $\varepsilon \to 0$, $|x^k h^{(k)}(x)|$

$\leq C_k h(x)$ for a constant C_k. Using Lebesgue's theorem concerning dominated convergence we obtain

$$\lim_{\varepsilon\to 0} < T,\ h_\varepsilon \phi * h_\varepsilon > = (-1)^n \frac{c}{n!} \phi(0) \int x^n (h*h)^{(n)} dx$$

$$= \frac{c}{n!} \phi(0) \int n!(h*h)\, dx$$

$$= c\phi(0),$$

which was to be proved.

For example let S be a locally summable function $f(x)$. If the indefinite integral $\int_0^x f(t)dt$ has the ordinary derivative c at the point x_0, then f, as a distribution, has the value c at x_0. Then Proposition 7 shows that $f \bigcirc \delta_{x_0} = c\delta_{x_0}$. On the other hand $S\delta_{x_0}$ exists if and only if S is a bounded function continuous at x_0 in a neighbourhood of x_0. Of course, there may occur the case where S has not the value at a point x_0, but $S \bigcirc \delta_{x_0}$ exists: $Y \bigcirc \delta = \frac{1}{2}\delta$.

LEMMA 2. *$\delta^{(n)} \cdot \delta = 0$ for $n = 0, 1, \cdots$. $\delta^{(n)} \bigcirc \delta$ does not exist. $(a_0\delta + a_1\delta' + \cdots + a_n\delta^{(n)}) \bigcirc \delta$ exists if and only if $a_0 = a_1 = \cdots = a_n = 0$.*

PROOF. We have $\tilde{\delta}_\varepsilon = h_\varepsilon$ and $(\widetilde{\delta^{(n)}})_\varepsilon = h_\varepsilon^{(n)}$. Since $\int_{-1}^1 h_\varepsilon^{(n)} h_\varepsilon x^{n+1} dx = 0$, we have for any $\phi \in \mathcal{D}$

$$< h_\varepsilon^{(n)} h_\varepsilon,\ \phi > = \int_{-\infty}^{\infty} h_\varepsilon^{(n)} h_\varepsilon \phi\, dx$$

$$= \int_{-1}^{1} h_\varepsilon^{(n)} h_\varepsilon \left(\phi - \sum_{k=0}^{n+1} \frac{\phi^{(k)}(0)}{k!} x^k \right) dx$$

$$+ \sum_{k=0}^{n} \frac{\phi^{(k)}(0)}{k!} \int_{-1}^{1} h_\varepsilon^{(n)} h_\varepsilon x^k\, dx + o(1)$$

as $\varepsilon \to 0$. The first term of the right side of the equation tends to zero as $\varepsilon \to 0$ since $x^{n+1} h_\varepsilon^{(n)}$ and xh_ε are bounded. After a change of variable $x \to \varepsilon x$, we can write

$$\int_{-1}^{1} h_\varepsilon^{(n)} h_\varepsilon x^k\, dx = \frac{1}{\varepsilon^{n-k+1}} \left\{ \left(\int_{-\infty}^{\infty} - \int_{|x|>\frac{1}{\varepsilon}} \right) h^{(n)} h x^k\, dx \right\}$$

$$= \frac{1}{\varepsilon^{n-k+1}} \int_{-\infty}^{\infty} h^{(n)} h x^k\, dx + o(1)$$

as $\varepsilon \to 0$. If n is an even number $2m$, we have

$$<h_\varepsilon{}^{(2m)}h_\varepsilon,\ \phi>$$

$$=\frac{\phi(0)}{\varepsilon^{2m+1}}\int_{-\infty}^{\infty} h^{(2m)}h\,dx+\cdots+\frac{\phi^{(2m)}(0)}{(2m)!\,\varepsilon}\int_{-\infty}^{\infty} h^{(2m)}hx^{2m}\,dx+o(1)$$

$$=(-1)^m\frac{\phi(0)}{\varepsilon^{2m+1}}\int_{-\infty}^{\infty}(h^{(m)})^2dx+\cdots+\frac{\phi^{(2m)}(0)}{(2m)!\,\varepsilon}\int_{-\infty}^{\infty} h^{(2m)}hx^{2m}\,dx+o(1)$$

as $\varepsilon\to 0$. If n is an odd number $2m-1$, we have

$$<h_\varepsilon{}^{(2m-1)}h_\varepsilon,\ \phi>$$

$$=\frac{\phi'(0)}{\varepsilon^{2m-1}}\int_{-\infty}^{\infty} h^{(2m-1)}hx\,dx+\cdots$$

$$+\frac{\phi^{(2m-1)}(0)}{(2m-1)!\,\varepsilon}\int_{-\infty}^{\infty} h^{(2m-1)}hx^{2m-1}\,dx+o(1)$$

$$=(-1)^m\frac{(2m-1)\phi'(0)}{2\varepsilon^{2m-1}}\int_{-\infty}^{\infty}(h^{(m-1)})^2\,dx+\cdots$$

$$+\frac{\phi^{(2m-1)}(0)}{(2m-1)!\,\varepsilon}\int_{-\infty}^{\infty} h^{(2m-1)}hx^{2m-1}\,dx+o(1)$$

as $\varepsilon\to 0$. Since $\int_{-\infty}^{\infty}(h^{(m)})^2dx$, $\int_{-\infty}^{\infty}(h^{(m-1)})^2dx\neq 0$, it follows that $\delta^{(n)}\bigcirc\delta$ does not exist but $\delta^{(n)}\cdot\delta$ exists and equals 0.

Next we show that if $(a_0\delta+a_1\delta'+\cdots+a_n\delta^{(n)})\bigcirc\delta$ exists then $a_0=a_1=\cdots=a_n=0$. Suppose the contrary. Assuming that $a_n\neq 0$, we shall deduce a contradiction. If n is an even number, we choose $\phi\in\mathcal{D}$ such that $\phi(0)\neq 0$ and if n is an odd number, we choose $\phi\in\mathcal{D}$ such that $\phi(0)=0$ and $\phi'(0)\neq 0$. Then $\lim_{\varepsilon\to 0}|<(a_0h_\varepsilon+\cdots+a_nh_\varepsilon{}^{(n)})h_\varepsilon,\ \phi>|=\infty$, which is a contradiction.

By making use of Lemma 2 we shall show

Proposition 8. *Let T be a distribution such that $\lim_{x\to+0}T=c_+$, $\lim_{x\to-0}T=c_-$. Then $T\cdot\delta$ exists and equals $\frac{1}{2}(c_++c_-)\delta$. $T\bigcirc\delta$ exists if and only if T is bounded at 0, and we then have $T\bigcirc\delta=\frac{1}{2}(c_++c_-)\delta$.*

Proof. If we put $S=T-(c_+-c_-)Y$ then $\lim_{x\to+0}S=c_-=\lim_{x\to-0}S$. Therefore there exists a distribution H such that H coincides with S for $x\neq 0$ and $H(0)=c_-$ ([6], p. 15). Then we can write $S-H=a_0\delta+a_1\delta'+\cdots+a_n\delta^{(n)}$ with some constants a_i and we have

$$T=(c_+-c_-)Y+H+a_0\delta+a_1\delta'+\cdots+a_n\delta^{(n)}.$$

Since $\delta^{(n)}\cdot\delta=0$ by Lemma 2, $Y\bigcirc\delta=\frac{1}{2}\delta$ and $H\bigcirc\delta=c_-\delta$ by Proposition 7, it follows that $T\cdot\delta$ exists and

$$T\cdot\delta=\frac{1}{2}(c_+-c_-)\delta+c_-\delta=\frac{1}{2}(c_++c_-)\delta.$$

Moreover if T is bounded at 0, then $a_i=0$ for $i=1, 2, \cdots, n$. Therefore we obtain

$$T\bigcirc\delta=\frac{1}{2}(c_++c_-)\delta.$$

Conversely, if $T\bigcirc\delta$ exists then $(a_0\delta+a_1\delta'+\cdots+a_n\delta^{(n)})\bigcirc\delta$ exists, so that by Lemma 2 we have $a_i=0$ for $i=0, 1, \cdots, n$. Consequently, T is bounded at 0. Thus the proof is complete.

When $S*\check{T}$ exists and has the value at 0, S. Łojasiewicz ([5], p. 241) has suggested the way of defining the scalar product $<S, T>$ as follows:

$$<S, T>=(S*\check{T})(0).$$

This mode of defining the scalar product will lead us to define a product $S\times T$ by the equation

$$<S\times T, \phi>=<S, \phi T>, \qquad \phi\in\mathcal{D},$$

provided that $<S, \phi T>$ exists in the above sense. We note that the linear form $\phi\to<S, \phi T>$ is continuous on $\mathcal{D}$. In fact, we can write $(S*(\phi T)^\vee)\bigcirc\delta$ $=((S*(\phi T)^\vee)(0))\delta$. Now let ϕ run through any $\mathcal{D}_K$, K being a compact subset of R. We can take $\alpha\in\mathcal{D}$ in such a way that α takes the value 1 in a neighbourhood of K such that $\alpha S*(\phi T)^\vee$ coincides with $S*(\phi T)^\vee$ in a neighbourhood of 0. Then

$$(S*(\phi T)^\vee)\bigcirc\delta=(\alpha S*(\phi T)^\vee)\bigcirc\delta$$
$$=\lim_{\varepsilon\to 0}((\alpha S*(\phi T)^\vee)*h_\varepsilon)h_\varepsilon.$$

Then by virtue of the Banach-Steinhaus theorem, we see that the map $\phi\to(S*(\phi T)^\vee)(0)$ is a continuous linear form on $\mathcal{D}$.

Consider the case where ST exists. We know that $S*(\phi T)^\vee$ is a bounded function continuous at 0 in a neighbourhood of 0 and $<ST, \phi>=(S*(\phi T)^\vee)(0)$. Consequently, $S\times T$ does exist and coincide with ST. Conversely, when one requires only that $S*(\phi T)^\vee$ has the value at 0 for every $\phi\in\mathcal{D}$, we can not conclude that ST exists. However, $S\bigcirc T$ exists and coincides with $S\times T$.

This follows from the following Theorem 2, which contains Theorem 1 as a special case. The following lemma will be needed for the proof of Theorem 2.

LEMMA 3. *Let E and F be spaces of type* (**F**). *Let G be a locally convex space. If a family of separately continuous bilinear maps u_α, $\alpha \in A$, of $E\times F$ into G is bounded at each point of $E\times F$, then $\{u_\alpha\}_{\alpha\in A}$ is equicontinuous.*

PROOF: Let W be an absolutely convex closed zero neighbourhood in G. Let $F_x=\{y \in F;\ u_\alpha(x, y) \in W,\ \alpha \in A\}$. Then clearly F_x is a barrel in F, therefore a zero neighbourhood in F. Let $\{\mathcal{V}_n\}$ be a fundamental sequence of zero neighbourhoods in F. Put $E_n=\{x \in E;\ u_\alpha(x, y) \in W,\ \alpha \in A,$ for any $y \in \mathcal{V}_n\}$. Then E_n is an absolutely convex closed subset of E and $E=\cup E_n$. Therefore an E_n is a zero neighbourhood in E. Thus we see that there exist zero neighbourhoods U in E and V in F, such that $u_\alpha(U, V)\subset W$, $\alpha \in A$.

If $S\bigcirc\delta$ exists and equals $c\delta$, we shall define c as the *generalized value* of S at 0 and denote it by $S[0]$. Consequently, if S has the value c at 0, or if S is bounded at 0 and has the right and left hand limits c_+ and c_- at 0 respectively and if we put $c=\frac{1}{2}(c_+ + c_-)$, then c is also the generalized value of S at 0.

THEOREM 2. *If $(S*(\alpha T)^\vee)\bigcirc\delta$ exists for every $\alpha \in \mathcal{D}$, then $S\bigcirc T$ exists. In particular, if $S*(\alpha T)^\vee$ has the generalized value at* 0, *then $S\bigcirc T$ exists and $<S\bigcirc T, \alpha>=(S*(\alpha T)^\vee)\,[0]$, $\alpha \in \mathcal{D}$.*

PROOF. Let K be any compact subset of R and ϕ be any element of $\mathcal{D}_K$. We choose compact subsets $K_1, K_2\subset R$ so that K (resp. K_2) lies in the interior of K_2 (resp. K_1). Let $\alpha_1 \in \mathcal{D}$ be chosen equal to 1 on K_1 so that $\alpha_1 S*(\psi T)^\vee$, $\psi \in \mathcal{D}_{K_2}$, coincides with $S*(\psi T)^\vee$ in a neighbourhood of 0. Then $(\alpha_1 S*(\psi T)^\vee)\bigcirc\delta$ $=(S*(\psi T)^\vee)\bigcirc\delta$ for every $\psi \in \mathcal{D}_{K_2}$. Let $\alpha_2 \in \mathcal{D}_{K_2}$ be chosen equal to 1 on K. Further, we choose $\beta \in \mathcal{D}$ so that $\beta=\check{\beta}$ and equals 1 in a neighbourhood of 0. As in the proof of Proposition 5 we have

$$\lim_{\varepsilon\to 0} <\hat{S}_\varepsilon \hat{T}_\varepsilon, \phi> = \lim_{\varepsilon\to 0} <(\alpha_1 S*\beta h_\varepsilon)\,(\alpha_2 T*\beta h_\varepsilon), \phi>$$

if the right hand side exists. To estimate the right hand side of the equation we may assume that ϕ is a periodic function with period $2l$, where l is taken large enough. We can write

$$\phi=\sum c_m e^{i\frac{\pi}{l}mx}=\sum c_m e(m)$$

where $\sum|c_m|(1+|m|)^k<\infty$ for any integer $k\geq 0$.

Then we obtain

$$(1)\quad <(\alpha_1 S*\beta h_\varepsilon)(\alpha_2 T*\beta h_\varepsilon), \phi>$$
$$=\sum c_m <(\alpha_1 S*\beta h_\varepsilon)(\alpha_2 T*\beta h_\varepsilon), e(m)>$$
$$=\sum c_m <(\alpha_1 S*\beta h_\varepsilon * e(-m)\beta h_\varepsilon)\alpha_2 T, e(m)>$$
$$=\sum c_m <\alpha_1 S*(e(m)\alpha_2 T)^\vee, \beta h_\varepsilon * e(m)\beta h_\varepsilon>.$$

On the other hand, it follows from the existence of $(\alpha_1 S*(\psi T)^\vee)\bigcirc\delta$, $\psi\in\mathcal{D}_{K_2}$, that we have for any $\chi\in\mathcal{B}$

$$<(\alpha_1 S*(\psi T)^\vee)\bigcirc\delta, \chi>$$
$$=\lim_{\varepsilon\to 0} <((\alpha_1 S*(\psi T)^\vee)*\beta h_\varepsilon)\beta h_\varepsilon, \chi>$$
$$=\lim_{\varepsilon\to 0} <\alpha_1 S*(\psi T)^\vee, \beta h_\varepsilon * \chi\beta h_\varepsilon>.$$

Consider the family of maps

$$(\psi, \chi)\to <\alpha_1 S*(\psi T)^\vee, \beta h_\varepsilon * \chi\beta h_\varepsilon>$$

of $\mathcal{D}_{K_2}\times\mathcal{B}$ into the complex number field. By virtue of Lemma 3 this family of maps is equicontinuous. Therefore we have for some positive integer k and a constant M

$$|<\alpha_1 S*(\psi T)^\vee, \beta h_\varepsilon * \chi\beta h_\varepsilon>|\leq M\sup_{p\leq k}|D^p\psi|\sup_{p\leq k}|D^p\chi|$$

Consequently, we have for a constant M_1

$$|<\alpha_1 S*(e(m)\alpha_2 T)^\vee, \beta h_\varepsilon * e(m)\beta h_\varepsilon>|\leq M_1(1+|m|)^{2k}$$

Therefore the series $\sum c_m <\alpha_1 S*(e(m)\alpha_2 T)^\vee, \beta h_\varepsilon * e(m)\beta h_\varepsilon>$ is normally convergent and each term has a limit as $\varepsilon\to 0$. Consequently, it follows from (1) that $<(\alpha_1 S*\beta h_\varepsilon)(\alpha_2 T*\beta h_\varepsilon), \phi>$ converges as $\varepsilon\to 0$, that is, $S\bigcirc T$ exists.

In particular, if $S*(\alpha T)^\vee$ has a generalized value at 0 for every $\alpha\in\mathcal{D}$, then $S\bigcirc T$ exists by the first part of our theorem. The linear form defined on $\mathcal{B}: \chi\to(\alpha_1 S*(\chi\alpha_2 T)^\vee)[0]$ is continuous. Therefore we obtain for $\phi\in\mathcal{D}_K$

$$<S\bigcirc T, \phi>=\lim_{\varepsilon\to 0} <(\alpha_1 S*\beta h_\varepsilon)(\alpha_2 T*\beta h_\varepsilon), \phi>$$
$$=\sum c_m (\alpha_1 S*(e(m)\alpha_2 T)^\vee)[0]$$
$$=(\alpha_1 S*(\phi\alpha_2 T)^\vee)[0]$$
$$=(S*(\phi T)^\vee)[0].$$

Thus the proof is complete.

The existence of $S\bigcirc T$ does not imply the existence of $S\bigcirc\alpha T$, $\alpha\in\mathcal{E}$. However if this is the case, the condition of Theorem 2 is necessary and sufficient in order that $S\bigcirc T$ may exist. This follows from Theorem 2 and the following proposition.

PROPOSITION 9. *If $S\bigcirc\beta T$ exists for every $\beta\in\mathcal{E}$ then $(S*(\alpha T)^{\vee})\bigcirc\delta$ exists for every $\alpha\in\mathcal{D}$.*

PROOF. Let K be any compact subset of R and $\beta\in\mathcal{D}$ be chosen equal to 1 on K so that $\beta S*(\alpha T)^{\vee}$, $\alpha\in\mathcal{D}_K$, may coincide with $S*(\alpha T)^{\vee}$ in a neighbourhood of 0. Then $\beta S\bigcirc\alpha T$ exists for any $\alpha\in\mathcal{D}_K$ and coincides with $S\bigcirc\alpha T$. Further, we choose $\beta_0\in\mathcal{D}$ so that $\beta_0=\check{\beta}_0$ and equals 1 in a neighbourhood of 0. We have for any $\phi\in\mathcal{D}$

$$<(S*(\alpha T)^{\vee})\bigcirc\delta,\ \phi>=\lim_{\varepsilon\to0}<(\beta S*(\alpha T)^{\vee}*\beta_0h_\varepsilon)\,\beta_0h_\varepsilon,\ \phi>$$

if the right hand side exists. We may assume that ϕ is a periodic function with the same form as in the proof of Theorem 2. Then we obtain

$$\begin{aligned}&<(\beta S*(\alpha T)^{\vee}*\beta_0h_\varepsilon)\,\beta_0h_\varepsilon,\ \phi>\\&\qquad=\sum c_m<\beta S*(\alpha T)^{\vee},\ \beta_0h_\varepsilon*e(m)\beta_0h_\varepsilon>\\&\qquad=\sum c_m<(\beta S*\beta_0h_\varepsilon)\,(e(-m)\alpha T*\beta_0h_\varepsilon),\ e(m)>.\end{aligned}$$

On the other hand, as in the proof of Theorem 2, it follows from the existence of $\beta S\bigcirc\chi T$ for any $\chi\in\mathcal{D}_K$, that we have for any $\psi\in\mathcal{B}$

$$|<(\beta S*\beta_0h_\varepsilon)\,(\chi T*\beta_0h_\varepsilon),\ \psi>|\leq M\sup_{p\leq k}|D^p\chi|\sup_{p\leq k}|D^p\psi|$$

with some positive integer k and a constant M. Consequently, we have for a constant M_1

$$|<(\beta S*\beta_0h_\varepsilon)\,(e(-m)\alpha T*\beta_0h_\varepsilon),\ e(m)>|\leq M_1(1+|m|)^{2k}.$$

Therefore $\sum c_m<(\beta S*\beta_0h_\varepsilon)\,(e(-m)\alpha T*\beta_0h_\varepsilon),\ e(m)>$ is normally convergent and $\lim_{\varepsilon\to0}<(\beta S*\beta_0h_\varepsilon)\,(e(-m)\alpha T*\beta_0h_\varepsilon),\ e(m)>=<S\bigcirc e(-m)\alpha T,\ e(m)>$. Consequently, we see that $\lim_{\varepsilon\to0}<(\beta S*(\alpha T)^{\vee}*\beta_0h_\varepsilon)\,\beta_0h_\varepsilon,\ \phi>$ exists. Therefore $(S*(\alpha T)^{\vee})\bigcirc\delta$ exists for every $\alpha\in\mathcal{D}$. Thus the proof is complete.

Reference

[1] H. J. Bremermann and L. Durand, *On analytic continuation, multiplication, and Fourier transformations of Schwartz distributions*, J. Math. Physics, **2** (1961), 240-258.

[2] A. Grothendieck, *Produits tensoriels topologiques et espaces nucléaires*, Mem. Amer. Math. Soc., **16** (1955).

[3] Y. Hirata and H. Ogata, *On the exchange formula for distributions*, this Journal, **22** (1958), 147-152.

[4] H. König, *Multiplikation und Variablentransformation in der Theorie der Distributionen*, Arch. Math., **6** (1955), 391-396.

[5] S. Łojasiewicz, *Sur la valeur d'une distribution dans un point*, Bull. Acad. Polon. Sci., **4** (1956), 239-242.

[6] ______, *Sur la valeur et la limite d'une distribution en un point*, Studia Math., **16** (1957), 1-36.

[7] ______, J. Wloka und Z. Zieleźny, *Über eine Definition des Wertes einer Distribution*, Bull. Acad. Polon. Sci., **3** (1955), 479-481.

[8] J. Mikusiński, *Criteria of the existence and of the associativity of the product of distributions*, Studia Math., **21** (1962) 253-259.

[9] L. Schwartz, *Théorie des distributions à valeurs vectorielles*, Chap. I, Ann. Inst. Fourier, **7** (1957), 1-141.

[10] J. Sebastião e Silva, *Les fonctions analytiques comme ultra-distributions dans le calcul opérationnel*, Math. Ann., **136** (1958), 58-96.

[11] R. Shiraishi, *On the definition of convolutions for distributions*, this Journal, **23** (1959), 19-32.

[12] ______, and M. Itano, *On the multiplicative products of distributions*, this Journal, **28** (1964), 223-235.

[13] H. G. Tillmann, *Randverteilungen analytischer Funktionen und Distributionen*, Math. Z., **59** (1953), 61-83.

[14] ______, *Darstellung der Schwartzschen Distributionen durch analytische Funktionen*, Math. Z., **77** (1961), 106-124.

[15] Z. Zieleźny, *Sur la définition de Łojasiewicz de la valeur d'une distribution dans un point*, Bull. Acad. Polon. Sci., **3** (1955), 519-520.

[16] ______, *Über die Mengen der regulären und singulären Punkte einer Distribution*, Studia Math., **19** (1960), 27-52.

Faculty of Education and Liberal Arts
Kagawa University