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On Loop Extensions of Groups and M-cohomology Groups. II

Noboru Nishigôri

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Introduction

In the previous paper [5][1], we discussed the problem of BM-extensions of a group by a group, that is, for given two groups G and Γ, the problem to determine all Bol-Moufang loop L's with the following properties[2]: (i) L has a normal subgroup G' which is isomorphic to G, (ii) $L/G' \cong \Gamma$, (iii) G' is contained in the nucleus of L. When we consider the case where L is a Bol-Moufang loop, it seems natural to consider the case where Γ is also a Bol-Moufang loop. In this paper we shall investigate the classification of all *BM*-extensions of a group G by a Bol-Moufang loop Γ. In this case, we shall modify the *M*-cohomology groups defined in the previous paper and classify all *BM*-extensions, using this new cohomology groups.

§1 will be devoted to the construction of the *M*-cohomology groups of a Bol-Moufang loop Γ over an abelian group G, and in §2, we shall first obtain the necessary and sufficient conditions for the existence of the *BM*-extension L of a group G by a Bol-Moufang loop Γ by making use of a *M*-factor set and a system of automorphisms of G, and next, using this result and the new *M*-cohomology groups we shall classify the set of all *BM*-extensions. The methods used in this paper are the same as those of the previous, and the results obtaind in this paper are as follows:

(i) *For a given group G with the center C, a Bol-Moufang loop Γ and a homomorphism $\theta: \Gamma \rightarrow Aut\, G/In\, G$*[3]*, the BM-extension of G by Γ exists if and only if an element of $H^{*3}(\Gamma, C)$ determined by G, Γ and θ is zero (Theorem 2). Especially in the case G is abelian, this element is always zero.*

(ii) *If the BM-extension exists for assigned G, Γ and θ, all non-equivalent BM-extensions are in one-to-one correspondence with the elements of the second M-cohomology group $H^{*2}(\Gamma, C)$ (Theorem 3, 4).*

§ 1. M-cohomology groups of a Bol-Moufang loop over an abelian group

In this section we shall extend the previous *M*-cohomology group of a

1) The number in the bracket referes to the references at the end of this paper.
2) A loop which satisfies the condition $a[b(ac)] = [a(ba)]c$ is called a Bol-Moufang loop.
3) Aut G means the group of all automorphisms of G and In G is the group of all inner automorphisms of G.

group Γ over an abelian group G to the case that Γ is a Bol-Moufang loop. Let G be an abelian group and Γ be a Bol-Moufang loop. Further, suppose that to each element α of Γ there corresponds an automorphism $\bar{\alpha}$ of G which satisfies the following conditions: $(g\bar{\alpha})\bar{\beta}=g(\bar{\alpha}\bar{\beta})=g(\overline{\alpha\beta})$, $g \in G$, α, $\beta \in \Gamma$; $g\bar{\varepsilon}=g$ (ε is the identity element of Γ).

Every function $f(\alpha_1, \alpha_2, \cdots, \alpha_n)$ of n elements of Γ, with its value in G, is called an n-dimensional cochain and the set of these n-dimensional cochains is a group $C^n(\Gamma, G)$ under the ordinary addition. With every n-dimensional cochain f, we associate an $(n+1)$-dimensional cochain ∂f called the M-coboundary of the cochain f and defined as follows:

$$(1)\quad \begin{cases} \partial f(\alpha) = a - a\bar{\alpha}, \\ \partial f(\alpha_1, \alpha_2) = f(\alpha_2) - f(\alpha_1\, \alpha_2) + f(\alpha_1)\bar{\alpha}_2, \\ \partial f(\alpha_1, \alpha_2, \cdots, \alpha_{n+1}) = u(f;\, \alpha_1, \alpha_2, \cdots, \alpha_{n+1}) - u(f;\, \alpha_1, \alpha_2, \cdots, \alpha_n, \varepsilon)\bar{\alpha}_{n+1} \quad (n \geqq 2), \\ \text{where}^{4)}\ u(f;\, \alpha_1, \alpha_2, \cdots, \alpha_{n+1}) \\ \quad = f(\alpha_2, [\alpha_1\alpha_3\alpha_1], [\alpha_1\alpha_4\alpha_1], \cdots, [\alpha_1\alpha_n\alpha_1], \alpha_1\alpha_{n+1}) \\ \quad + \sum_{i=2}^{n-1} (-1)^i f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, [\alpha_i \cdots \alpha_{i+2} \cdots \alpha_i], \cdots, [\alpha_i \cdots \alpha_n \cdots \alpha_i], \\ \qquad\qquad [\alpha_i \cdots \alpha_{n+1}]) \\ \quad + (-1)^n f(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, \alpha_{n+1}) \\ \quad + \sum_{i=1}^{n-1} (-1)^i f(\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, [\alpha_i \cdots \alpha_{i+1} \cdots \alpha_i], \alpha_{i+2}, \cdots, \alpha_{n+1}) \\ \quad + (-1)^n f(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}, [\alpha_n \cdots \alpha_{n+1}]). \end{cases}$$

In the above definition (1), the product $[\alpha_i \cdots \alpha_j \cdots \alpha_i]$ $(i<j)$ means the product of $\alpha_1, \alpha_2, \cdots, \alpha_i, \alpha_j$ which is obtained by arranging these letters and multiplying them as follows:

(i) We put α_i at the left end and α_j at the right end on a line, (ii) α_{i-1} in the middle between α_i and α_j, (iii) α_{i-2} in the middles respectively both between α_i and α_{i-1}, and between α_{i-1} and α_j. (iv) After the arrangement of $\alpha_{k+1}, \alpha_{k+2}, \cdots, \alpha_i, \alpha_j$ $(k+1<i<j)$ by the above processes we put α_k in the middles between all adjacent elements respectively in this sequence of the letters. (v) Continuing these processes until we put α_1, we get the arrangement of letters in the left half part of the product. (vi) The arrangement of α_k's in the right half part $\alpha_j \cdots \alpha_i$ of the product $[\alpha_i \cdots \alpha_j \cdots \alpha_i]$ is symmetric to the left half part $\alpha_i \cdots \alpha_j$ with respect to α_j. (vii) We multiply the letters of the above constructed sequence one by one from the right end to the left. For example, in the case $i=4$ and $j=6$,

4) In the right side of the definition of $u(f;\, \alpha_1, \alpha_2, \ldots, \alpha_{n+1})$ we take the form $(-1)^{n-1}f(\alpha_1, \alpha_2, \ldots, \alpha_{n-2}, \alpha_n, [\alpha_{n-1} \ldots \alpha_{n+1}])$ when i equals $n-1$ in the second line.

$$[\alpha_4\cdots\alpha_6\cdots\alpha_4]=\alpha_4(\alpha_1(\alpha_2(\alpha_1(\alpha_3(\alpha_1(\alpha_2(\alpha_1(\alpha_6(\alpha_1(\alpha_2(\alpha_1(\alpha_3(\alpha_1(\alpha_2(\alpha_1\alpha_4))))\cdots).$$

When $j=n+1$, the product $[\alpha_i\cdots\cdots\alpha_{n+1}]$ is the left half part of the above product.

We explain some lemmas concerning the arguments which appear in the terms of the formula (1).

LEMMA 1. *If we denote* $\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, \alpha_{i+1}, [\alpha_i\cdots\alpha_{i+2}\cdots\alpha_i], \cdots, [\alpha_i\cdots\alpha_n\cdots\alpha_i], [\alpha_i\cdots\alpha_{n+1}]$ *by* $\beta_1, \beta_2, \cdots, \beta_n$ *respectively, then it holds that*

$$[\beta_j\cdots\beta_l\cdots\beta_j]=\begin{cases}[\alpha_j\cdots\alpha_l\cdots\alpha_j] & (j<l<i\leqq n),\\ [\alpha_j\cdots\alpha_{i+1}\cdots\alpha_j] & (j<i,\ i=l<n),\\ [\alpha_j\cdots\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i\cdots\alpha_j] & (j<i,\ i+1\leqq l<n),\\ [\alpha_{i+1}\cdots\alpha_{l+1}\cdots\alpha_{i+1}] & (j=i,\ i+1\leqq l<n),\\ [\alpha_i\cdots\alpha_{j+1}\cdots\alpha_{l+1}\cdots\alpha_{j+1}\cdots\alpha_i] & (i+1\leqq j<l<n),\end{cases}$$

where the product $[\alpha_j\cdots\alpha_k\cdots\alpha_l\cdots\alpha_k\cdots\alpha_j]$ $(j<k<l)$ *is made as follows*: (i) *first, the middle part* $\alpha_k\cdots\alpha_l\cdots\alpha_k$ *is arranged by the method explained above*, (ii) *next, the part* $\alpha_j\cdots\alpha_k$ *at the left end is arranged by the above method*, (iii) *the part* $\alpha_k\cdots\alpha_j$ *at the right end is arranged in the symmetric position to* $\alpha_j\cdots\alpha_k$ *with respect to* α_l, (iv) *finally these letters are multiplied one by one from the right end to the left.*

PROOF. We prove this lemma by dividing into five cases. In the cases 1 and 2: $j<l<i$ and $j<l=i$, the lemma is evident. Case 3: $j<i$, $l\geqq i+1$. By the definition of β_i $(1\leqq i<n+1)$ it is sufficient to prove the following: $[\beta_j\cdots\beta_l\cdots\beta_j]=[\alpha_j\cdots[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]\cdots\alpha_j]=[\alpha_j\cdots\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i\cdots\alpha_j]$. Since we can easily see that the arrangement of the letters α_k's is the same in both sides, we show that the two products equal in the Bol-Moufang loop Γ. To prove it, it is sufficient to show that $[[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]\cdots\alpha_j]=[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i\cdots\alpha_j]$. We prove this by dividing into few steps. We prove that $[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]=((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))$, where $((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))$ is the product in which the arrangement of α_k's is the same as that of $[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]$ and which is obtained by multiplying α_k's from the right and from the left alternatively beginning with the multiplication of α_{l+1} and α_1 at the middle of this product, i.e., $\alpha_i((\alpha_1\cdots((\alpha_1(\alpha_{l+1}\alpha_1))\cdots\alpha_i)$. If we use the Bol-Moufang condition for the product obtained by taking away α_i from the left end of $((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))$ we have:

$$(\alpha_1((\alpha_2\cdots((\alpha_1(\alpha_{l+1}\alpha_1))\cdots\alpha_2))\alpha_1))\alpha_i=\alpha_1\{(\alpha_2(\cdots((\alpha_1(\alpha_{l+1}\alpha_1))\cdots\alpha_2))(\alpha_1\alpha_i)\}.$$

If we use again the Bol-Moufang condition for the part in parenthesises

$\{(\alpha_2((\alpha_1\cdots((\alpha_1(\alpha_{l+1}\alpha_1))\cdots\alpha_2))(\alpha_1\alpha_i)\}$ of the right side of the above equation, we obtain

$$(\alpha_2((\alpha_1\cdots((\alpha_1(\alpha_{l+1}\alpha_1))\cdots\alpha_2))(\alpha_1\alpha_i)=\alpha_2\{(\alpha_1((\cdots((\alpha_1(\alpha_{l+1}\alpha_1))\cdots\alpha_1))[\alpha_2\alpha_1\alpha_i]\}.$$

Continuing the same processes we get $[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]=((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))$. We now proceed to prove that $[[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]\cdots\alpha_j]=[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i\cdots\alpha_j]$. Since $[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]=((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))$, it holds that

$$[[\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i]\cdots\alpha_j]=((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))(\alpha_1\cdots(\alpha_1(\alpha_2(\alpha_1\alpha_j)))\cdots).$$

In the same way as the above, taking into account to two α_i's at the both ends of $((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))$, if we use the Bol-Moufang condition on the right side of this equation, we have

$$((\alpha_i\cdots\alpha_{l+1}\cdots\alpha_i))(\alpha_1\cdots(\alpha_1(\alpha_2(\alpha_1\alpha_j)))\cdots)$$
$$=\alpha_i\{((\alpha_1\cdots\alpha_{l+1}\cdots\alpha_1))(\alpha_i(\alpha_1\cdots(\alpha_2(\alpha_1\alpha_j))\cdots))\}.$$

Further, if we use again the Bol-Moufang condition for the part $\{((\alpha_1\cdots\alpha_{l+1}\cdots\alpha_1))(\alpha_i(\alpha_1\cdots(\alpha_1\alpha_j))\cdots)\}$ on the right side of the above, we obtain

$$\alpha_i\{\alpha_1\{((\alpha_2\cdots\alpha_{l+1}\cdots\alpha_2))(\alpha_1(\alpha_i(\alpha_1(\cdots(\alpha_2(\alpha_1\alpha_j))\cdots)))))\}\}.$$

Hence we have the required result by repeating the same processes.

Case 4: $j=i,\ l\geqq i+1$: We may prove this case in the same way as that of the case 3.

Case 5: $i+1\leqq j<l$: We show that when we rewrite β_i by α_k's the arrangement of the letters in $[\beta_j\cdots\beta_l\cdots\beta_j]$ coincides with that of α_k's in $[\alpha_i\cdots\alpha_{j+1}\cdots\alpha_{l+1}\cdots\alpha_{j+1}\cdots\alpha_i]$. It is sufficient to prove it about the left half product. Since $\beta_k(k=i+1, i+2,\cdots, j)$ contains only one $\alpha_{k+1}(k=i+1, i+2,\cdots, j)$ respectively, only one α_j appears between α_{j+1} and α_{l+1} and only one α_{j-1} appears between α_{j+1} and α_j, and between α_j and α_{l+1} respectively in the sequence of $\beta_l, \beta_j, \beta_{j-1},\cdots, \beta_{i+1}$ in the course of the construction of the product $[\beta_j\cdots\beta_l]$. Continuing the same considerations we may see that the arrangement and numbers of $\alpha_{l+1}, \alpha_{j+1}, \alpha_j,\cdots, \alpha_{i+2}$ in $[\beta_j\cdots\beta_l]$ coincide with those of them in the part $\alpha_{j+1}\cdots\alpha_{l+1}$ of $[\alpha_i\cdots\alpha_{j+1}\cdots\alpha_{l+1}]$. Since each of $\beta_l, \beta_j,\cdots, \beta_{i+1}$ does not contain α_{i+1}, when we put $\beta_i=\alpha_{i+1}$ in the middle of each adjacent pair of letters in the sequence constructed by $\beta_l, \beta_j,\cdots, \beta_{i+1}$, only one α_{i+1} appears in the middle of each adjacent pair of letters in the sequence of α_{l+1}, $\alpha_{j+1}, \alpha_j,\cdots, \alpha_{i+2}$ in $[\beta_j\cdots\beta_l]$. Further, since each of $\beta_l, \beta_j,\cdots, \beta_{i+1}$ contains α_i's on both ends and each of $\beta_{i-1}, \beta_{i-2},\cdots, \beta_1$ does not contain α_i, the arrangement of $\alpha_{l+1}, \alpha_{j+1},\cdots, \alpha_i$ in $[\beta_j\cdots\beta_l]$ is the same as that of $\alpha_{l+1}, \alpha_{j+1},\cdots, \alpha_i$ in $[\alpha_i\cdots\alpha_{j+1}\cdots\alpha_{l+1}]$. Moreover, since $\beta_{i-1}=\alpha_{i-1},\cdots, \beta_1=\alpha_1$ and the arrangement

of α_k's in each of $\beta_l, \cdots, \beta_{i+1}$ is the same as that of α_k's in the construction of $[\alpha_i \cdots \alpha_{j+1} \cdots \alpha_{l+1}]$, we may see that the arrangement of α_k's in $[\beta_j \cdots \beta_l]$ is the same as that of $[\alpha_i \cdots \alpha_{j+1} \cdots \alpha_{l+1}]$. Therefore the arrangement of α_k's in $[\beta_j \cdots \beta_l \cdots \beta_j]$ is the same as that of α_k's in $[\alpha_i \cdots \alpha_{j+1} \cdots \alpha_{l+1} \cdots \alpha_{j+1} \cdots \alpha_i]$.

We prove that $[\beta_j \cdots \beta_l \cdots \beta_j] = [\alpha_i \cdots \alpha_{j+1} \cdots \alpha_{l+1} \cdots \alpha_{j+1} \cdots \alpha_i]$ in the Bol-Moufang loop Γ. First, in the same way as the case 3, we have that $[\beta_{i+1} \cdots \beta_j]$ at the right end of $[\beta_j \cdots \beta_l \cdots \beta_j]$ is equal to $[\alpha_i \cdots \alpha_{i+2} \cdots \alpha_{j+1} \cdots \alpha_i]$. Next, we can prove $[\beta_{i+2} \cdots \beta_j] = [\alpha_i \cdots \alpha_{i+3} \cdots \alpha_{j+1} \cdots \alpha_i]$, where $[\beta_{i+2} \cdots \beta_j]$ is the part of the right end of $[\beta_j \cdots \beta_l \cdots \beta_j]$. Continuing these processes as often as β_s $(s \geqq i+1)$ appears, we obtain $[\beta_j \cdots \beta_l \cdots \beta_j] = [\alpha_i \cdots \alpha_{j+1} \cdots \alpha_{l+1} \cdots \alpha_{j+1} \cdots \alpha_i]$.

In the same way as the above, we may prove that the following lemma.

Lemma 2. *If we denote* $\alpha_1, \alpha_2, \cdots, \alpha_{i-1}, [\alpha_i \cdots \alpha_{i+1} \cdots \alpha_i], \alpha_{i+2}, \alpha_{i+3}, \cdots, \alpha_{n+1}$ *by* $\beta_1, \beta_2, \cdots, \beta_n$ *respectively, then it holds that*

$$[\beta_j \cdots \beta_l \cdots \beta_j] = \begin{cases} [\alpha_j \cdots \alpha_l \cdots \alpha_j] & (j < l < i \leqq n), \\ [\alpha_j \cdots \alpha_i \cdots \alpha_{i+1} \cdots \alpha_i \cdots \alpha_j] & (j < i,\ l = i < n), \\ [\alpha_j \cdots \alpha_{l+1} \cdots \alpha_j] & (j < i,\ i+1 \leqq l < n), \\ [\alpha_i \cdots \alpha_{i+1} \cdots \alpha_{l+1} \cdots \alpha_{i+1} \cdots \alpha_i] & (j = i,\ i+1 \leqq l < n), \\ [\alpha_{j+1} \cdots \alpha_{l+1} \cdots \alpha_{j+1}] & (i+1 \leqq j < l < n). \end{cases}$$

Note. By the method of the above proof, we may see that the similar lemmas, concerning the half product $[\beta_j \cdots \beta_n]$ as the lemmas 1 and 2, hold.

Under these preparations, we shall construct the M-cohomology group of a Bol-Moufang loop Γ over an abelian group G.

In the following, we shall prove the theorem:

Theorem 1. *If f is any cochain, then* $\partial(\partial f) = 0$.

Proof. In the case where $n=0$ and $n=1$, we may prove this by simple calculations. So, we assume $n \geqq 2$. If f is an n-dimensional cochain, then $\partial(\partial f)$ is an $(n+2)$-dimensional cochain. When we express $\partial(\partial f)(\alpha_1, \alpha_2, \cdots\cdots, \alpha_{n+2})$ in terms of the values of ∂f, using the definition (1), we obtain

$$\partial(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{n+2}) = u(\partial f; \alpha_1, \alpha_2, \cdots, \alpha_{n+2}) - u(\partial f; \alpha_1, \alpha_2, \cdots, \alpha_{n+1}, \varepsilon)\bar{\alpha}_{n+2}.$$

Further, we express each term in $u(\partial f; \alpha_1, \alpha_2, \cdots, \alpha_{n+2})$ and $u(\partial f; \alpha_1, \cdots, \alpha_{n+1}, \varepsilon)$ in terms of the values of f, we have:

$$\partial(\partial f)(\alpha_1, \alpha_2, \cdots, \alpha_{n+2})$$

$$= \sum_{i=1}^{2(n+1)} \{u(f; \beta_{i1}, \beta_{i2}, \cdots, \beta_{i\ n+1}) - u(f; \beta_{i1}, \beta_{i2}, \cdots, \beta_{in}, \varepsilon)\bar{\beta}_{i\ n+1}\}$$

$$- \sum_{i=1}^{2(n+1)} \{u(f; \beta_{i1}, \beta_{i2}, \cdots, \beta_{in}, \beta'_{i\ n+1})\bar{\alpha}_{n+2} - u(f; \beta_{i1}, \beta_{i2}, \cdots, \beta_{in}, \varepsilon)\bar{\beta}'_{i\ n+1}\bar{\alpha}_{n+2}\},$$

where $u(f; \beta_{i1}, \beta_{i2}, \cdots, \beta_{i\ n+1}) - u(f; \beta_{i1}, \cdots, \beta_{in}, \varepsilon)\bar{\beta}_{i\ n+1}$ is the expression obtained by expressing the i term of $u(\partial f; \alpha_1, \cdots, \alpha_{n+2})$ in terms of the values of f and $\beta'_{i\ n+1}$ is the argument obtained by putting $\alpha_{n+2}=\varepsilon$ in $\beta_{i\ n+1}$. If we combine each of the terms in $\sum_{i=1}^{2(n+1)} u(f; \beta_{i1}, \cdots, \beta_{i\ n+1})$ and $\sum_{i=1}^{2(n+1)} u(f; \beta_{i1}, \cdots, \beta_{in}, \beta'_{i\ n+1})$ with the other whose sign only differs from each other as we did in [5], we obtain that $\sum_{i=1}^{2(n+1)} u(f; \beta_{i1}, \cdots, \beta_{i\ n+1})=0$ and $\sum_{i=1}^{2(n+1)} u(f; \beta_{i1}, \cdots, \beta_{in}, \beta'_{i\ n+1})=0$ (cf. [5], pp. 156–158). Further, from $\bar{\beta}'_{i n+1}\bar{\alpha}_{n+2}=\bar{\beta}_{i n+1}$, it follows that $-\sum_{i=1}^{2(n+1)} u(f; \beta_{i1}, \beta_{i2}, \cdots, \beta_{in}, \varepsilon)\bar{\beta}_{i n+1} + \sum_{i=1}^{2(n+1)} u(f; \beta_{i1}, \cdots, \beta_{in}, \varepsilon)\bar{\beta}'_{i\ n+1}\bar{\alpha}_{n+2}=0$. Therefore we obtain $\partial(\partial f)=0$.

We call an n-dimensional cochain f an n-dimensional M-cocycle if $\partial f=0$. All n-dimensional M-cocycles form a subgroup of $C^n(\Gamma, G)$, which we denote by $Z^{*n}(\Gamma, G)$. For $n>0$ the n-dimensional cochains that are M-coboundaries of some $(n-1)$-dimensional cochains form also a subgroup of $C^n(\Gamma, G)$, which we denote by $B^{*n}(\Gamma, G)$. Since $\partial(\partial f)=0$, we have $B^{*n}(\Gamma, G) \subset Z^{*n}(\Gamma, G)$. The factor group $H^{*n}(\Gamma, G)=Z^{*n}(\Gamma, G)/B^{*n}(\Gamma, G)$ is called the n-th M-cohomology group of a Bol-Moufang loop Γ over an abelian group G.

In the following, we assume that $C^1(\Gamma, G)$ and $C^2(\Gamma, G)$ are the groups of the normalized cochains f, that is, $f(\varepsilon)=0$ and $f(\alpha, \varepsilon)=0=f(\varepsilon, \beta)$.

§ 2. Extensions of a group by a Bol-Moufang loop

We shall proceed to classify all BM-extensions of a group G by a Bol-Moufang loop Γ by making use of the 2nd and 3rd M-cohomology groups constructed in §1.

A loop L is called a BM-extension of G by Γ if it satisfies the following conditions: (i) L is a Bol-Moufang loop, (ii) L contains a normal subgroup G' which is isomorphic to G, (iii) $L/G' \cong \Gamma$, (iv) G' is contained in the nucleus of L, where the nucleus is a subgroup consisted of elements a which satisfies the conditions: $(ax)y=a(xy)$, $(xa)y=x(ay)$ and $(xy)a=x(ya)$. (Usually we identify G' with G). Further, we define the equivalence of two BM-extensions of G by Γ exactly as in the case Γ is a group (cf. [5], pp. 153). Then we can prove the following propositions by the same methods as those where Γ is a group (cf. [5], pp. 152–154).

Proposition 1. *For a given BM-extension of a group G by a Bol-Moufang loop Γ, there exists a system of elements $f(\alpha, \beta)$ of G and a system of automorphisms T_α which satisfy the conditions*:

$$aT_\alpha T_\beta = aT_{\alpha\beta}T_{f(\alpha,\beta)} \qquad a \in G,$$

$$f(\alpha, [\beta\alpha\gamma])f(\beta, \alpha\gamma)f(\alpha, \gamma) = f([\alpha\beta\alpha], \gamma)\,(f(\alpha, \beta\alpha)T_\gamma)\,(f(\beta, \alpha)T_\gamma),$$

$$f(\alpha, \varepsilon) = e = f(\varepsilon, \beta).$$

Conversely, to every system of elements $f(\alpha, \beta)$ and every system of automorphisms T_α of G which satisfy the above conditions, there corresponds a BM-extension of G by Γ.

A set of elements $f(\alpha, \beta)$ of G which satisfy the above conditions is called a *M-factor set.*

Proposition 2. *Two BM-extensions L and L′ of a group G by a Bol-Moufang loop Γ which are given by the M-factor sets $f(\alpha, \beta)$ and $f'(\alpha, \beta)$, and automorphisms T_α and T'_α respectively, are equivalent if and only if every element α of Γ can be associated with an element $c_\alpha (c_\varepsilon = e)$ of G in such a way that the following conditions are satisfied*:

$$f'(\alpha, \beta) = c_{\alpha\beta}^{-1} f(\alpha, \beta)\,(c_\alpha T_\beta)\,c_\beta,$$

$$T'_\alpha = T_\alpha T_{c_\alpha}.$$

We prepare some lemmas to investigate the set of all BM-extensions of G by Γ. In the same way as in the previous paper, for a given BM-extension L of G by Γ there exists a homomorphism θ on Γ into $\mathrm{Aut}\, G/\mathrm{In}\, G$ defined by $\alpha \to T_\alpha(\mathrm{In}\, G)$, which is called the homomorphism associated with this BM-extension L.

Let now G, Γ and a homomorphism $\theta\colon \Gamma \to \mathrm{Aut}\, G/\mathrm{In}\, G$ be given. Then the homomorphism θ induces a homomorphism $\theta_0\colon \Gamma \to \mathrm{Aut}\, C$. So, we may regard Γ as an operator set of the center C of G. Therefore, we may construct the M-cohomology group $H^{*n}(\Gamma, C)$, using the methods in §1. If in every coset $\theta(\alpha)$ of $\mathrm{In}\, G$ in $\mathrm{Aut}\, G$, we choose a representative φ_α, where φ_ε is the identity automorphism, then there exist the elements $h(\alpha, \beta)$ of G such that $\varphi_\alpha\varphi_\beta = \varphi_{\alpha\beta}T_{h(\alpha,\beta)}$, where $h(\alpha, \varepsilon) = e = h(\varepsilon, \beta)$. Using the Bol-Moufang condition to the representatives φ_α, φ_β and φ_γ and taking into account that for $a \in G$, $\varphi \in \mathrm{Aut}\, G$ it holds that $\varphi^{-1}T_a\varphi = T_{(a\varphi)}$, we can see that there exists an element $z^*(\alpha, \beta, \gamma)$ of C such that

(2) $h(\alpha, [\beta\alpha\gamma])h(\beta, \alpha\gamma)h(\alpha, \gamma) = z^*(\alpha, \beta, \gamma)h([\alpha\beta\alpha], \gamma)\,(\{h(\alpha, \beta\alpha)h(\beta, \alpha)\}\,\varphi_\gamma).$

So, for given G, Γ and θ, there exists an element $z^*(\alpha, \beta, \gamma)$ of $C^3(\Gamma, C)$. We can prove that in the case where Γ is a Bol-Moufang loop, the following lemmas concerning $z^*(\alpha, \beta, \gamma)$, which are similar to those in the previous paper, also hold.

LEMMA 3. *A 3-dimensional cochain* $z^*(\alpha, \beta, \gamma)$ *is an element of* $z^{*3}(\Gamma, C)$.

PROOF. We calculate the expression:

$$J = h(\alpha, [\beta\alpha\gamma\alpha\beta\alpha\delta])h(\beta, [\alpha\gamma\alpha\beta\alpha\delta])h(\alpha, [\gamma\alpha\beta\alpha\delta])h(\gamma, [\alpha\beta\alpha\delta])\cdot$$
$$\cdot h(\alpha, [\beta\alpha\delta])h(\beta, \alpha\delta)h(\alpha, \delta)$$

in two ways. First, we begin with the calculations of the first three factors and the last three factors, using (2). Then we have:

$$J = z^*(\alpha, \beta, [\gamma\alpha\beta\alpha\delta])z^*(\alpha, \beta, \delta)h([\alpha\beta\alpha], [\gamma\alpha\beta\alpha\delta])h(\gamma, [\alpha\beta\alpha\delta])h([\alpha\beta\alpha], \delta)\cdot$$
$$\cdot((h(\alpha, \beta\alpha)h(\beta, \alpha))\varphi_{[\gamma\alpha\beta\alpha\delta]}T_{h(\gamma,[\alpha\beta\alpha\delta])h([\alpha\beta\alpha],\delta)})\,((h(\alpha, \beta\alpha)h(\beta, \alpha))\varphi_\delta)$$
$$= z^*(\alpha, \beta, [\gamma\alpha\beta\alpha\delta])z^*(\alpha, \beta, \delta)z^*([\alpha\beta\alpha], \gamma, \delta)h([\alpha\beta\alpha\gamma\alpha\beta\alpha], \delta)\,((h[\alpha\beta\alpha], [\gamma\alpha\beta\alpha])\cdot$$
$$\cdot h(\gamma, [\alpha\beta\alpha]))\varphi_\delta)\,((h(\alpha, \beta\alpha)h(\beta, \alpha))\varphi_\gamma\varphi_{[\alpha\beta\alpha]}\varphi_\delta)\,((h(\alpha, \beta\alpha)h(\beta, \alpha))\varphi_\delta)$$
$$= z^*(\alpha,\beta,[\gamma\alpha\beta\alpha\delta])z^*(\alpha,\beta,\delta)z^*([\alpha\beta\alpha],\gamma,\delta)h([\alpha\beta\alpha\gamma\alpha\beta\alpha],\delta)(h([\alpha\beta\alpha],[\gamma\alpha\beta\alpha])\varphi_\delta)\cdot$$
$$\cdot(\{(h(\alpha, \beta\alpha)h(\beta, \alpha))\varphi_{[\gamma\alpha\beta\alpha]})h(\gamma, [\alpha\beta\alpha])\}\,\varphi_\delta)\,((h(\alpha, \beta\alpha)h(\beta, \alpha))\varphi_\delta).$$

Next, we begin with the calculation of the middle three factors by applying (2). Then we obtain:

$$J = z^*(\alpha, \gamma, [\beta\alpha\delta])h(\alpha, [\beta\alpha\gamma\alpha\beta\alpha\delta])h(\beta, [\alpha\gamma\alpha\beta\alpha\delta])h([\alpha\gamma\alpha], [\beta\alpha\delta])h(\beta, \alpha\delta)\cdot$$
$$\cdot((h(\alpha, \gamma\alpha)h(\gamma, \alpha))\varphi_\beta\varphi_{\alpha\delta})h(\alpha, \delta)$$
$$= z^*(\alpha, \gamma, [\beta\alpha\delta])z^*(\beta, [\alpha\gamma\alpha], \alpha\delta)h(\alpha, [\beta\alpha\gamma\alpha\beta]\,(\alpha\delta))h([\beta\alpha\gamma\alpha\beta], \alpha\delta)h(\alpha, \delta)\cdot$$
$$\cdot(\{h(\beta, [\alpha\gamma\alpha\beta])h([\alpha\gamma\alpha], \beta)\}\,\varphi_\alpha\varphi_\delta)\,(\{h(\alpha, \gamma\alpha)h(\gamma, \alpha)\}\,\varphi_\beta\varphi_\alpha\varphi_\delta)$$
$$= z^*(\alpha, \gamma, [\beta\alpha\delta])z^*(\beta, [\alpha\gamma\alpha], \alpha\delta)z^*(\alpha, [\beta\alpha\gamma\alpha\beta], \delta)h([\alpha\beta\alpha\gamma\alpha\beta\alpha], \delta)\cdot$$
$$\cdot(h(\alpha, [\beta\alpha\gamma\alpha\beta\alpha])\varphi_\delta)\,(\{h(((\beta\alpha\gamma\alpha\beta)), \alpha)(\{h(\beta, (\alpha(\gamma\alpha))\beta)h([\alpha\gamma\alpha], \beta)\}\,\varphi_\alpha)\}\varphi_\delta)\cdot$$
$$\cdot(\{h(\alpha, \gamma\alpha)h(\gamma, \alpha)\}\,\varphi_\beta\varphi_\alpha\varphi_\delta)$$
$$= z^*(\alpha, \gamma, [\beta\alpha\delta])z^*(\beta, [\alpha\gamma\alpha], \alpha\delta)z^*(\alpha, [\beta\alpha\gamma\alpha\beta], \delta)\cdot$$
$$\cdot(z^{*-1}(\beta, [\alpha\gamma\alpha], \alpha)\varphi_\delta)h([\alpha\beta\alpha\gamma\alpha\beta\alpha], \delta)\,(\{h(\alpha, [\beta\alpha\gamma\alpha\beta\alpha])h(\beta, [\alpha\gamma\alpha\beta\alpha])\}\varphi_\delta)\cdot$$
$$\cdot(\{h([\alpha\gamma\alpha], \beta\alpha)\,((h(\alpha, \gamma\alpha)h(\gamma, \alpha))\varphi_{\beta\alpha})\}\,\varphi_\delta)\,(h(\beta, \alpha)\varphi_\delta)$$

$$= z^*(\alpha, \gamma, [\beta\alpha\delta])z^*(\beta, [\alpha\gamma\alpha], \alpha\delta)z^*(\alpha, [\beta\alpha\gamma\alpha\beta], \delta)\,(z^{*-1}(\beta, [\alpha\gamma\alpha], \alpha)\varphi_\delta)\cdot$$

$$\cdot(z^{*-1}(\alpha, \gamma, \beta\alpha)\varphi_\delta)h([\alpha\beta\alpha\gamma\alpha\beta\alpha], \delta)\,(\{h(\alpha, [\beta\alpha\gamma\alpha\beta\alpha])h(\beta, [\alpha\gamma\alpha\beta\alpha])\cdot$$

$$\cdot h(\alpha, [\gamma\alpha\beta\alpha])h(\gamma, [\alpha\beta\alpha])h(\alpha, \beta\alpha)h(\beta, \alpha)\}\,\varphi_\delta)$$

$$= z^*(\alpha, \gamma, [\beta\alpha\delta])z^*(\beta, [\alpha\gamma\alpha], \alpha\delta)z^*(\alpha, [\beta\alpha\gamma\alpha\beta], \delta)\,(z^{*-1}(\beta, [\alpha\gamma\alpha], \alpha)\varphi_\delta)\cdot$$

$$\cdot(Z^{*-1}(\alpha, \gamma, \beta\alpha)\varphi_\delta)\,(z^*(\alpha, \beta, [\gamma\alpha\beta\alpha])\varphi_\delta)h([\alpha\beta\alpha\gamma\alpha\beta\alpha], \delta)\,(\{h([\alpha\beta\alpha], [\gamma\alpha\beta\alpha])\cdot$$

$$\cdot((h(\alpha, \beta\alpha)h(\beta, \alpha))\varphi_{[\gamma\alpha\beta\alpha]})h(\gamma, [\alpha\beta\alpha])\}\,\varphi_\delta)\,(\{h(\alpha, \beta\alpha)h(\beta, \alpha)\}\,\varphi_\delta).$$

Comparing the above two calculations, we have $\partial z^*(\alpha, \beta, \gamma, \delta) = 0$.

The M-cocycle $z^*(\alpha, \beta, \gamma)$ depends on the choice of the representatives φ_α and of the elements $h(\alpha, \beta)$. In the following we investigate the change of $z^*(\alpha, \beta, \gamma)$ for different choices of $h(\alpha, \beta)$ and φ_α. Taking into account that we must consider what order to multiply the letters in Γ as we did in the above lemma, we have the following lemmas by the same methods as used in the previous paper (cf. [5], pp. 161–162).

LEMMA 4. *If the choice of* $h(\alpha, \beta)$ *is changed, then* $z^*(\alpha, \beta, \gamma)$ *is changed to a cohomologous M-cocycle. By suitably changing the choice of* $h(\alpha, \beta)$, $z^*(\alpha, \beta, \gamma)$ *may be changed to any M-cohomologous cocycle.*

Using the expression

$$M = c([\alpha\beta\alpha\gamma])z^*(\alpha, \beta, \gamma)h'([\alpha\beta\alpha], \gamma)\,(\{h'(\alpha, \beta\alpha)h'(\beta, \alpha)\}\,\varphi'_\gamma),$$

we have the following:

LEMMA 5. *If the automorphisms* φ_α *are changed, then with a suitable new choice of* $h(\alpha, \beta)$ *the* 3*-dimensional M-cocycle* $z^*(\alpha, \beta, \gamma)$ *remains unchanged.*

Thus, we have proved that only one element of $H^{*3}(\Gamma, C)$ corresponds to the given group G, the Bol-Moufang loop Γ and the homomorphism θ. After S. MacLane, we call a pair of a Bol-Moufang loop Γ and a group G together with a homomorphism $\theta: \Gamma \to \mathrm{Aut}\, G/\mathrm{In}\, G$ an *abstract kernel* and denote by (Γ, G, θ). The unique element of $H^{*3}(\Gamma, C)$ determined by a given abstract kernel (Γ, G, θ) is called an obstraction of it and denoted by $\mathrm{Obs}(\Gamma, G, \theta)$.

Then, we have the following theorem in the similar way as that where Γ is a group (cf. [5] pp. 162–163).

THEOREM 2. *The abstract kernel* (Γ, G, θ) *has a BM-extension if and only if* $Obs(\Gamma, G, \theta) = 0$.

We now give a survey of the non-equivalent BM-extensions of G by Γ.

In the case that Γ is a Bol-Moufang loop, we can obtain similar results to those in the case that Γ is a group.

When G is an abelian group and Γ is a Bol-Moufang loop, we have the following theorem:

THEOREM 3. *If G, Γ and a homomorphism $\theta: \Gamma \rightarrow Aut\, G$ are given, there always exists a BM-extension of G by Γ and all non-equivalent BM-extensions correspond one-to-one to the elements of the second M-cohomology group $H^{*2}(\Gamma, G)$.*

PROOF. Since G is an abelian group, the 3-dimensional M-cocycle $z^*(\alpha, \beta, \gamma)$ corresponding to the abstract kernel (Γ, G, θ) is an M-coboundary from the definition (2). Hence, from the theorem 2, there exists a BM-extension of G by Γ.

On account of the 2nd and 3rd conditions of the proposition 1, to a given BM-extension of G by Γ there corresponds a 2-dimensional M-cocycle, i.e., M-factor set. Conversely, for every 2-dimensional M-cocycle there exists a BM-extension of G by Γ from the proposition 1. Further, the proposition 2 shows that the two M-factor sets which correspond to two equivalent BM-extensions are cohomologous. Hence we have proved the theorem 3.

When G is a non-abelian group, taking into account the theorem 2, the following theorem is proved in the same way as that where Γ is a group (cf. [5], pp. 162–163).

THEOREM 4. *Let a non-abelian group G with the center C, a Bol-Moufang loop Γ and a homomorphism $\theta: \Gamma \rightarrow Aut\, G/In\, G$ be given. If the obstraction of the abstract kernel (Γ, G, θ) is zero, there exists a BM-extension of G by Γ and all non-equivalent BM-extensions of G by Γ are in one-to-one correspondence with the elements of the second M-cohomology group $H^{*2}(\Gamma, C)$.*

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Shinonome Bunkô,
Hiroshima University